Quantization of Open Systems and the Quantum–Classical Transition

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開いた系において，どのように量子–古典挙動間の転移が生起しているかは理論物理学の未解決な問題のひとつである．ここでは Markov 的に時間発展する開いた系を扱う最も一般的な枠組みのひとつ Gorini–Kossakowski–Sudarshan と Lindblad (GKSL) のマスター方程式及びそれを unravel した quantun state diffusion (stochastic Schrödinger equation) に準拠しながら，Lindblad 演算子の分類，その物理的由来に触れる．量子測定，散逸について，モデルによりながら，その量子–古典挙動の具体例を提示する．

1 Introduction

The quantum–classical correspondence (QCC) is a fundamental problem in quantum mechanics, but particularly, in a chaotic system, this correspondence is still unclear. While various studied have been done in Hamiltonian chaotic systems fruitfully, dissipative quantum chaos, where a definite Hamiltonian does not exist, has scarcely been analysed. Recently, we [1] studied the QCC for the quantum version of the Duffing oscillator, which is a nonlinear dynamical system with dissipation and external periodic force, assuming that the system evolves according to the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) master equation [2, 3]. The GKSL master equation is a general expression for nonunitary Markovian dynamics. Both the positivity and the trace of the density matrix are preserved as long as the system evolves according to it. The effect of dissipation is represented by a set of operators, Lindblad operators. However, it is difficult to derive the relationship between the form of Lindblad operators and a dissipative phenomenon in the microscopic model in which the system concerned is a nonlinear dynamical one and the interaction between the system and the environment is complex. In this report, we discuss the classification of Lindblad operators through an effective model for open quantum systems with nonlinear dynamics.

2 The QCC in the Duffing oscillator

First, we review the QCC in the Duffing oscillator described by a following classical equation of motion: $m\ddot{x} + 2\gamma m\dot{x} + m\omega_0^2 x^3/l^2 - m\omega_0^2 x = m\omega_0^2 lg\cos(\omega t)$, where l characterizes a size of the system. It is known that if one choose a set of dimensionless parameters, $(\Gamma,\, g,\, \Omega) \equiv (\gamma/\omega_0,\, g,\, \omega/\omega_0) = (0.125,\, 0.3,\, 1.00)$, a classical chaotic motion occurs in the Poincaré surface. In the following, we discus the quantum version of Duffing oscillator. We assume that the reduced density matrix ρ evolves by the GKSL master equation: $\dot{\rho} = -\frac{i}{\hbar}[\hat{H},\rho] + \hat{L}\rho\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\rho - \frac{1}{2}\rho\hat{L}^\dagger\hat{L}$. We guess the Hamiltonian $\hat{H}$ ($\hat{H}^\dagger = \hat{H}$) and the Lindblad operator $\hat{L}$ phenomenologically and analyze the system by the use of an unraveled form of the master equation, the quantum state diffusion (QSD) [4], for discussing the QCC. It is convenient to define a scaling parameter

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$\beta = \sqrt{\hbar/S_0}$, where $S_0 = ml^2\omega_0$ is a characteristic action of the system. We can suppose that the region of $\beta \sim 0$ corresponds to a classical case and the region of $\beta \sim 1$, to a quantum case. What phenomena can occur between classical- and quantum-region? We investigate the system as β goes from 0 to 1 with fixed S_0. To analyze the difference in temporal behavior for two different initial conditions, we calculate the following quantity: $\Delta(\tau) = \frac{1}{N}\sum_{\{1,2\}}\{\delta\overline{Q}_{12}(\tau)^2 + \delta\overline{P}_{12}(\tau)^2\}^{\frac{1}{2}}$, where $\delta\overline{Q}_{12}(\tau) = \mathrm{Tr}\{\hat{Q}\rho_1(\tau) - \mathrm{Tr}\{\hat{Q}\rho_2(\tau)\}$, $\delta\overline{P}_{12}(\tau) = \mathrm{Tr}\{\hat{P}\rho_1(\tau)\} - \mathrm{Tr}\{\hat{P}\rho_2(\tau)\}$ and dimensionless variables $\hat{Q} = \hat{x}/l$, $\hat{P} = \hat{p}/ml\omega_0$, and $\tau = \omega_0 t$. The canonical commutation relation is $[\hat{Q}, \hat{P}] = i\beta^2$ for the dimensionless variables. The two matrices $\rho_1(\tau)$ and $\rho_2(\tau)$ evolve from different initial states $\rho_1(0)$ and $\rho_2(0)$, respectively. Hereafter, we assume that $\rho_i(0) = |\alpha_i\rangle\langle\alpha_i|$ $(i = 1, 2)$, where $\alpha_i = \sqrt{2}(\langle\hat{Q}\rangle_i(0) + i\,\langle\hat{P}\rangle_i(0)) = \sqrt{2}(\mathrm{Tr}\{\hat{Q}\rho_i(0)\} + i\,\mathrm{Tr}\{\hat{P}\rho_i(0)\})$. The calculation of $\Delta(\tau)$ is similar to the derivation of a Lyapunov exponent in classical mechanics. In order to determine $\hat{H}$ and $\hat{L}$, we require a reproduction of the equation of motion with respect to the expectation values for $\hat{Q}$ and $\hat{P}$: $\hat{H} = \hat{H}_D + \hat{H}_R + \hat{H}_{ex}$, $\hat{L} = \sqrt{\Gamma}(\hat{Q} + i\,\hat{P})$ where $\hat{H}_D = \hat{P}^2/2 + \beta^2\hat{Q}^4/4 - \hat{Q}^2/2$, $\hat{H}_R = \Gamma(\hat{Q}\hat{P} + \hat{P}\hat{Q})/2$, $\hat{H}_{ex} = -g\hat{Q}\cos(\Omega t)/\beta$. We choose the parameters, $(\Gamma, g, \Omega) = (0.125, 0.3, 1.00)$. Before performing simulation, we have to determine a suitable value of $\epsilon \equiv \Delta(\tau = 0)$. If two points in the phase space coexist inside the same Planck cell, they are not *distinguishable* each other. The size of the Planck cell is determined from the commutation relation $[\hat{Q}, \hat{P}] = [\hat{x}, \hat{p}]/S_0 = i\beta^2$. The Planck cell has a volume $\Delta Q\Delta P = \frac{1}{2}\beta^2$ in the scaled phase space and has a linear size of effective Planck cell β in $(\mathrm{Tr}(\hat{Q}\rho), \mathrm{Tr}(\hat{P}\rho))$–plane. With the fixed S_0 the smaller β^2 corresponds to the smaller $\hbar(=\beta^2 S_0)$. We adopt two choices: (1) $\epsilon = 0.01$ (two points are distinguishable only for classical region ($\beta = 0.01$)) and (2) $\epsilon \sim \beta$ (they are distinguishable for all βs).

We reproduced results similar to those of Ref. [5] for the constant phase maps and verified the existence of the strange attractor in the Poincaré surface. We show the results of simulation of $\Delta(\tau)$ in case(1) (Fig. 1) In Fig. 1.(a), we see an exponential increase of $\Delta(\tau)$, which is a charac-

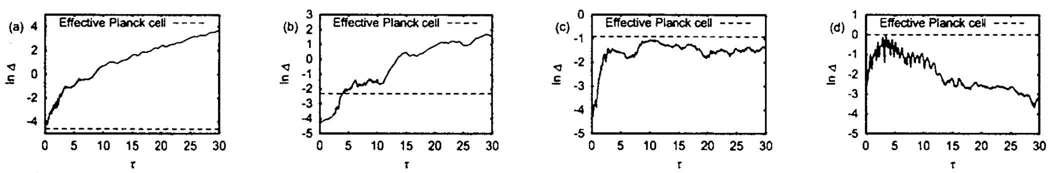

FIG. 1: The time evolution of $\Delta(\tau)$ with ϵ fixed as 0.01. The quantities plotted are dimensionless by definition. The complex Wiener process is used in the QSD. Figures (a) and (b) are obtained with a single realization of the complex Wiener process for each initial condition (20 samples). Figures (c)–(d) are obtained by averaging over 100 realizations of the complex Wiener process for each initial condition (10 samples). Figures (a), (b), (c), and (d) are for $\beta = 0.01$, 0.10, 0.40, and 1.00, respectively.

teristic behavior of chaos and corresponds to the fact that the maximum Lyapunov exponent is positive in classical chaotic systems. We find very different behavior between (b) and (c)–(d), where each pair of initial two points is within the same Planck cell and they are indistinguishable. However, there remains the remnant of chaotic dynamics for $\beta = 0.10$(b) and more or less up to $\beta = 0.40$(c). On the otherhand, chaotic dynamics is completely lost for $\beta = 0.40$, and 1.00. This observation suggests that the crossover from classical to quantum behavior exists around ~ 0.40. Let us show the results for case(2) $\epsilon \sim \beta$, where initial two points are separated by the effective Planck cell size: $\Delta(\tau)$ for $\beta = 0.10$, 0.40, 0.60, 1.00, 1.50, and 2.00. We also find an exponentially increasing behavior in Fig. 2(a) for $\beta = 0.10$. While in Figs. 2(b)–(f) except for a very short period after the starting time, $\Delta(\tau)$ for each β decreases for some duration and tends to approach Δ_{asymp}. This observation also suggests our issue that the crossover from classical to quantum behavior exists around ~ 0.40. Now let us call the region of $\beta > 1$ the deep quantum region. We analyze the behavior of $\Delta(\tau)$ in this region. First we estimate τ_0

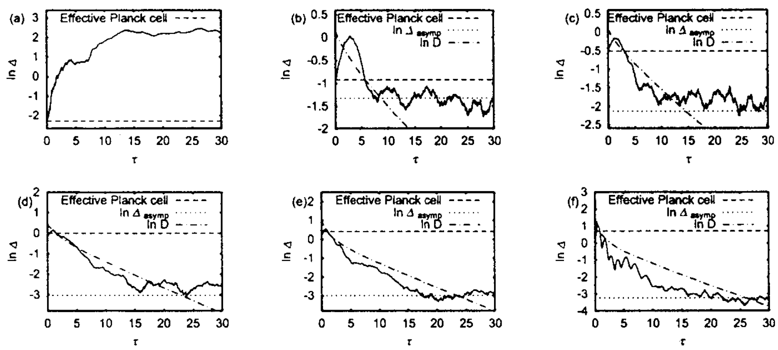

FIG. 2: The time evolution of $\Delta(\tau)$ with $\epsilon = \beta$. The quantities plotted are dimensionless by definition. The complex Wiener process is used in the QSD. The asymptotic value of $\Delta(\tau)$, Δ_{asymp}, is indicated by the dotted line. The right–hand side of Eq. (??), $D(\tau)$, is expressed by the broken dotted line. Figure (a) is obtained with a single realization of the complex Wiener process for each initial condition (20 samples). Figures (b)–(f) are obtained by averaging over 100 realizations of the complex Wiener process for each initial condition (10 samples). Figures (a), (b), (c), (d), (e), and (f) are for $\beta = 0.10$, 0.40, 0.60, 1.00, 1.50, and 2.00, respectively.

which is the time for when the value of $\Delta(\tau)$ becomes smaller than the size of effective Planck cell. Using these data, the upper bound for $\Delta(\tau)$ is expressed by a equation derived from QSD, $\Delta(\tau - \tau_0) \leq \left\{ \left(1 + \frac{1}{\beta^2}\right) e^{2\Gamma(\tau - \tau_0)} - 1 \right\}^{-\frac{1}{2}}$, which is shown by the broken dotted line in Fig. 2. We find in the case of $\beta \geq 1$ this inequality gives a good approximation of the upper bound.

Finally we remark that while, in case of $S_0 \gg \hbar$, the existence of dissipation is very important for occurrence of chaotic dynamics, in case of $S_0 \sim / \leq \hbar$ (in quantum and deep quantum cases) effect of dissipation suppresses even the occurrence of chaotic behavior.

3 Some consideration of Lindblad operators

In this section, we show some trials to inquire the source of Lindblad operator, based on the method of Ref. [6]. We assume that a total system is closed and consists of (system)+(environment) with coupling between two systems represented by an extra stochastic force. This force is modeled by a bosonic field.

First, we consider a particle in a one–dimensional potential $\Phi(x)$. The classical equation of motion is $\dot{p}(t) = -\Phi'(x) - \hat{f}(t)$ and $\dot{x}(t) = \frac{1}{m}p(t)$, where $\hat{f}(t) = i\hbar\sqrt{\frac{\lambda}{2}}(\hat{a}(t) - \hat{a}^\dagger(t))$, $[\hat{a}(t), \hat{a}^\dagger(t')] = \delta(t - t')$, and $[\hat{a}(t), \hat{a}(t')] = 0$. We quantize $x(t)$ and $p(t)$ by the canonical commutation relation, $[\hat{x}(t), \hat{p}(t)] = i\hbar$. Then, the Heisenberg equation becomes $d\hat{p}(t) = -\frac{\partial \Phi}{\partial x}|_{\hat{x}} dt - i\hbar\sqrt{\frac{\lambda}{2}}(d\hat{A} - d\hat{A}^\dagger)$, $d\hat{A} = \hat{a}dt$, and $d\hat{x}(t) = \frac{1}{m}\hat{p}(t)dt$. It is easy to check that $\langle (d\hat{p})^2 \rangle_0 = \frac{\lambda\hbar^2}{2}dt$, where $\langle\ \rangle_0$ means an expectation value with respect to a vacuum state of operator $\hat{a}$. If an operator $\hat{z} = \hat{z}(\hat{x}(t), \hat{p}(t))$ is an arbitrary polynomial of $\hat{x}$ and $\hat{p}$, then we obtain the Heisenberg equation of $\hat{z}$ as $d\hat{z} = \left\{ -\frac{i}{\hbar}\left[\hat{z},\ \frac{\hat{p}^2}{2m} + \Phi\right] - \frac{\lambda}{4}[[\hat{z},\ \hat{x}],\ \hat{x}] \right\} dt - \sqrt{2\lambda}\mathrm{Re}\{[\hat{z},\ \hat{x}]d\hat{A}\}$. We can change it to the Schrödinger picture $\mathrm{Tr}(\hat{z}\hat{\rho}(t)) = \mathrm{Tr}(\hat{z}(t)\hat{\rho}(0))$, using a density matrix $\hat{\rho}(t)$, $d\hat{\rho}(t) = \left\{ -\frac{i}{\hbar}\left[\hat{\rho}(t),\ \frac{\hat{p}^2}{2m} + \Phi\right] - \frac{\lambda}{4}[[\hat{\rho}(t),\ \hat{x}],\ \hat{x}] \right\} dt$. This is the GKSL master equation of continuous measurement process with Lindblad operator $\hat{x}$.

Next, we consider the Hamiltonian, $H_{\mathrm{int}} = -x_+ f(t)$, with $x_+ = x + ip$. From the Heisenberg equation, $d\hat{p}(t) = -\frac{\partial \Phi}{\partial x}|_{\hat{x}} - i\hbar\sqrt{\frac{\lambda}{2}}(d\hat{A} - d\hat{A}^\dagger)$, $d\hat{A} = \hat{a}dt$, and $d\hat{x}(t) = \frac{1}{m}\hat{p}(t) + \hbar\sqrt{\frac{\lambda}{2}}(d\hat{A} - d\hat{A}^\dagger)$.

We obtain the time evolution of operator $\hat{z}(t)$, $d\hat{z} = \left\{-\frac{i}{\hbar}\left[\hat{z},\ \frac{\hat{p}^2}{2m}+\Phi\right] - \frac{\lambda}{4}[[\hat{z},\ \hat{x}_+],\ \hat{x}_+]\right\}dt - \sqrt{2\lambda}\mathrm{Re}\{[\hat{z},\ \hat{x}_+]d\hat{A}\}$ and master equation of this system, $d\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{\rho}(t),\ \frac{\hat{p}^2}{2m}+\Phi]dt - \frac{\lambda}{4} \times [[\hat{\rho}(t),\ \hat{x}_+],\ \hat{x}_+]dt$. In this case the Lindblad operator is $\hat{x}_+$.

Furthermore, we consider A certain type of kicked top with dissipation. The Hamiltonian is $\hat{H} = \alpha\hat{J}_z + \frac{\lambda}{2j}\hat{J}_x^2 F(t)$, $\quad F(t) = \sum_{n=-\infty}^{\infty}\delta(t-n)$. Let us bosonize this system using the following correspondence: $\frac{1}{4}(\hat{a}^2+\hat{a}^{\dagger 2}) \leftrightarrow \hat{J}_x$, $\frac{1}{4i}(\hat{a}^2-\hat{a}^{\dagger 2}) \leftrightarrow \hat{J}_y$, $\frac{1}{4}(2\hat{a}^{\dagger 2}\hat{a}^2+1) \leftrightarrow \hat{J}_z$, with the usual commutation relations, $[\hat{a},\ \hat{a}^\dagger]=1$, $[\hat{x},\ \hat{p}]=i$, where $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger)$ $\hat{p} = \frac{1}{\sqrt{2}i}(\hat{a}-\hat{a}^\dagger)$. Substituting $\hat{a}$ and $\hat{a}^\dagger$ for $\hat{J}_i$ and $\hat{x}$, $\hat{p}$ by order, we obtain a Hamiltonian, $\hat{H} = \frac{\alpha}{2}\left(\frac{\hat{p}^2}{2}+\frac{\hat{x}^2}{2}\right) + \frac{\lambda}{2j}\hat{K}(\hat{x},\hat{p})F(t)$, $\quad \hat{K}(\hat{x},\hat{p}) = \left\{\frac{1}{4}(\hat{x}^2-\hat{p}^2)\right\}^2$. Based on the Heisenberg equations derived from the Hamiltonian, we can assume stachastic equations, $d\hat{p} = -\frac{\alpha}{2}\hat{x}\,dt - \frac{\lambda}{2j}\left(\frac{\partial K}{\partial x}\right)_{\hat{x},\hat{p},W}F(t)dt + i\sqrt{\frac{\eta}{2}}(d\hat{A}-d\hat{A}^\dagger)$, $d\hat{x} = \frac{\alpha}{2}\hat{p}\,dt + \frac{\lambda}{2j}\left(\frac{\partial K}{\partial p}\right)_{\hat{x},\hat{p},W}F(t)dt$. Then the time evolution of arbitrary operator $\hat{z}$ is determined by the equation, $d\hat{z} = -i[\hat{z},\hat{H}]dt - \frac{\eta}{4}[[\hat{z},\ \hat{x}],\ \hat{x}]dt - \sqrt{\frac{\eta}{2}}[\hat{z},\ \hat{x}](d\hat{A}-d\hat{A}^\dagger)$. Finally we obtain a continuous measurement of kicked quantum top, $\frac{d\hat{\rho}}{dt} = -i[\hat{H},\ \hat{\rho}] + \frac{\eta}{4}(2\hat{x}\hat{\rho}\hat{x} - \hat{\rho}\hat{x}^2 - \hat{x}^2\hat{\rho})$. Mapping the variables back to top, the Hamiltonian is expressed as follows: $\hat{H} = \alpha\hat{J}_z + \frac{\lambda}{2j}\hat{J}_x^2F(t)$, $\hat{x}^2 = \frac{1}{2}(\hat{a}+\hat{a}^\dagger)^2 = \hat{J}_+ + \hat{J}_- + 2\hat{J}_z$, $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger) = \sqrt{\hat{J}_+ + \hat{J}_- + 2\hat{J}_z}$.

4 Concluding remarks

First we simulated the quantum version of a dissipative chaotic system using the QSD (stochastic Schrödinger eq) and found the crossover from classical to quantum behavior in the quantum version of the Duffing oscillator through the analysis of $\Delta(\tau)$. This analysis is expected to be an effective one for investigating the QCC in the dissipative chaotic systems. The QSD is powerful to analyse quantum open systems.

Secondarily we presented some trials to make physical explanation of Lindblad operators.

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