

Fidelity recovery in the Gaussian ensembles

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(2003N110)

Using supersymmetry techniques analytical expressions for the average of the fidelity amplitude $f_\epsilon(\tau) = \frac{1}{N} \text{Tr} [\exp(2\pi i H_\phi \tau) \exp(-2\pi i H_0 \tau)]$ are obtained, where $H_\phi = H_0 \cos \phi + H_1 \sin \phi$, and H_0 and H_1 are matrices of rank N , taken from the Gaussian unitary ensemble (GUE) or the Gaussian orthogonal ensemble (GOE), respectively. For small perturbation strengths $\epsilon = 4N\phi^2$ a Gaussian decay of the fidelity amplitude is observed, whereas for stronger perturbations a change to a single-exponential decay takes place. Close to the Heisenberg time $\tau = 1$, however, a partial revival of the fidelity is found. It can be interpreted in terms of an spectral analogue of the Debye-Waller factor, describing in X-ray spectroscopy the decrease of von Laue reflexes with increasing temperature.

1 Introduction

The concept of fidelity has been developed by Peres as a tool to characterize the stability of a quantum-mechanical system against perturbations [1]. It was introduced as the squared modulus of the overlap integral of a wave packet with itself after developing forth and back under the influence of two slightly different Hamiltonians. The recent interest in the topic results from the idea to realize quantum computers by means of spin systems, where stability against quantum-mechanical perturbations obviously is of vital importance.

Roughly speaking there are three regimes. In the perturbative regime, where the strength of the perturbation is small compared to the mean level spacing, the decay of the fidelity is Gaussian. As soon as the perturbation strength becomes of the order of the mean level spacing, exponential decay starts to dominate, with a decay constant obtained from Fermi's golden rule [2]. For very strong perturbations the decay becomes independent of the strength of the perturbation. Here, in the Loschmidt regime, the decay is still exponential, but now the decay constant is given by the classical Lyapunov exponent [3].

It will be shown here that this scenario is only part of the truth, and that for chaotic systems there is a partial recovery of the fidelity at the Heisenberg time. Using the Brownian-motion model for the eigenvalues of random matrices introduced by Dyson many years ago [4], it will

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be shown that this behaviour has its direct analogue in the Debye-Waller factor of solid state physics.

2 Exact results

In the present paper we consider the Hamiltonian

$$H_\phi = H_0 \cos \phi + H_1 \sin \phi \quad (1)$$

where H_0 and H_1 are of rank N and are taken either from the Gaussian orthogonal ensemble (GOE) or the Gaussian unitary one (GUE) with a mean level spacing of one. To allow for a proper limit $N \rightarrow \infty$ we have to assume that ϕ is of $\mathcal{O}(\frac{1}{\sqrt{N}})$ (see Ref. [5]). We introduce a perturbation parameter $\epsilon = 4N\phi^2$, which is well-defined in the limit $N \rightarrow \infty$. This particular form of the Hamiltonian has been chosen, since the mean density of states does not change with the perturbation.

For chaotic system the fidelity amplitude may be written as

$$f_\epsilon(\tau) = \frac{1}{N} \text{Tr} [\exp(2\pi i H_\phi \tau) \exp(-2\pi i H_0 \tau)] , \quad (2)$$

The Gaussian average over $f_\epsilon(\tau)$ can be performed using standard supersymmetry techniques. Details can be found in Ref. [5]. In the limit $N \rightarrow \infty$ all integrations can be performed for the GUE and lead to the particularly simple result

$$f_\epsilon(\tau) = \begin{cases} e^{-\frac{1}{2}\epsilon\tau} \left[s(\frac{1}{2}\epsilon\tau^2) - \tau s'(\frac{1}{2}\epsilon\tau^2) \right] , & \tau \leq 1 \\ e^{-\frac{1}{2}\epsilon\tau^2} \left[s(\frac{1}{2}\epsilon\tau) - \frac{1}{\tau} s'(\frac{1}{2}\epsilon\tau) \right] , & \tau > 1 \end{cases} , \quad (3)$$

where

$$s(x) = \frac{\sinh(x)}{x} . \quad (4)$$

We have thus obtained an analytic expression for the GUE average of the fidelity amplitude for arbitrary perturbation strengths. In the limit of small perturbations it is in complete accordance with linear response results obtained by Gorin et al. [6]. For the GOE average of the fidelity amplitude we get

$$\begin{aligned} f_\epsilon(\tau) &= 2 \int_{\text{Max}(0, \tau-1)}^{\tau} du \int_0^u \frac{v dv}{\sqrt{[u^2 - v^2][(u+1)^2 - v^2]}} \\ &\quad \times \frac{(\tau - u)(1 - \tau + u)}{(v^2 - \tau^2)^2} \\ &\quad \times [(2u + 1)\tau - \tau^2 + v^2] e^{-\frac{1}{2}\epsilon[(2u+1)\tau - \tau^2 + v^2]} . \end{aligned} \quad (5)$$

The results are shown in Figure 1. For small perturbation strengths ϵ the decay of the fidelity is predominantly Gaussian changing to an exponential decay with increasing ϵ , in accordance

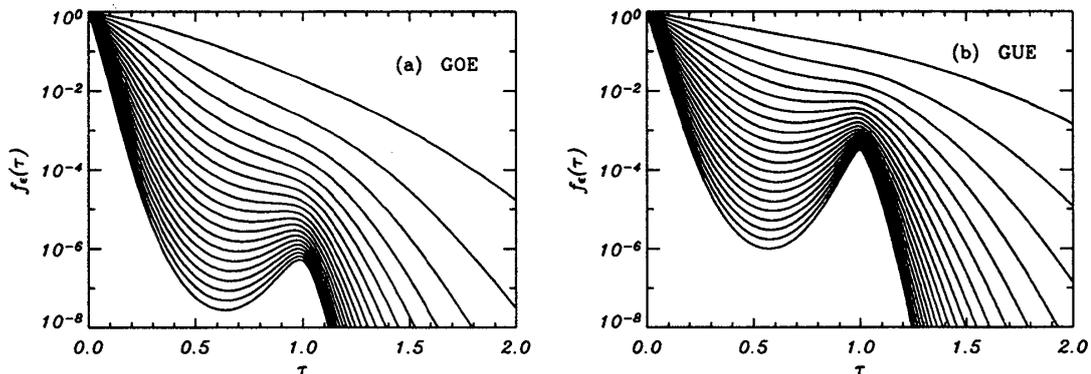


Figure 1: Fidelity amplitude $f_\epsilon(\tau)$ for perturbation strengths $\epsilon = 2, 4, \dots, 40$ for (a) the GOE and (b) the GUE.

with literature. The most conspicuous result, however, is the partial recovery of the fidelity at the Heisenberg time $\tau = 1$ which has not been reported previously to the best of our knowledge.

3 Debye-Waller factor

What is the origin of the surprising recovery? We believe that there is a simple intuitive explanation in terms of Dyson's Brownian motion model [4]. To illustrate this, we write expression (2) for the fidelity amplitude as

$$\begin{aligned}
 f_\epsilon(\tau) &= \frac{1}{N} \left\langle \text{Tr} \left[R_\phi e^{2\pi i H_\phi^D \tau} R_\phi^{-1} R_0 e^{-2\pi i H_0^D \tau} R_0^{-1} \right] \right\rangle \\
 &= \frac{1}{N} \left\langle \text{Tr} \left[e^{2\pi i H_\phi^D \tau} R e^{-2\pi i H_0^D \tau} R^{-1} \right] \right\rangle \\
 &= \frac{1}{N} \left\langle \sum_{kl} e^{2\pi i \tau (E_k^{(\phi)} - E_l^{(0)})} |R_{lk}|^2 \right\rangle, \tag{6}
 \end{aligned}$$

where $H_0^D = R_0^{-1} H_0 R_0$ and $H_\phi^D = R_\phi^{-1} H_\phi R_\phi$ are diagonal, and $R = R_\phi^{-1} R_0$. The $E_k^{(\phi)}$ are the eigenvalues of H_ϕ .

Since the mean density of states is kept constant during the parameter change, the eigenvalues may be written as $E_k^{(\phi)} = k + \delta_k^{(\phi)}$, where $\delta_k^{(\phi)}$ fluctuates about zero. Let us assume for the sake of simplicity that for strong perturbations the eigenvectors of the perturbed and unperturbed system are uncorrelated. In this case we obtain from equation (6) for the ensemble average of the fidelity amplitude

$$f_\epsilon(\tau) \sim \frac{1}{N} \sum_{kl} \langle |R_{lk}|^2 \rangle e^{2\pi i \tau (k-l)} W, \tag{7}$$

where W is given by

$$W = \left\langle e^{2\pi i \tau (\delta_k^{(\phi)} - \delta_l^{(0)})} \right\rangle \approx e^{-(2\pi \tau)^2 \langle \delta^2 \rangle}. \tag{8}$$

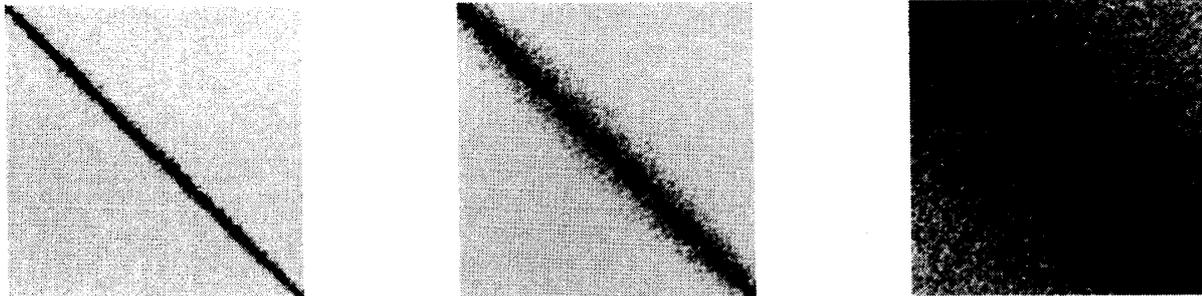


Figure 2: Visualization of the matrix R for $\epsilon = 10, 100,$ and 1000 . The grey-scale of the plots corresponds to the values of $|R_{lk}|^2$, white for zero and black for the maximal value.

In the second step a Gaussian approximation was applied. Within the framework of the Brownian motion model $\langle \delta^2 \rangle$ is interpreted as the mean squared displacement of an eigenvalue from its equilibrium position. It is proportional to “temperature” T , which is just the reciprocal universality factor β , whence follows

$$W = e^{-\alpha \tau^2 T} \quad (9)$$

with some constant α . It follows from equation (7) that there is a revival of the fidelity at the Heisenberg time $\tau = 1$ decreasing with “temperature” proportional to $e^{-\alpha T}$. This is exactly the behavior illustrated in Figure 1.

There is a perfect analogy to the temperature dependence of X-ray and neutron diffraction patterns in solid state physics. Caused by lattice vibrations the von Laue interference maxima decrease with increasing temperature according to the Debye-Waller factor

$$W_{\text{DW}} = e^{-\beta g^2 T}, \quad (10)$$

where β is another constant, and g is the modulus of the reciprocal lattice vector characterizing the reflex (see e. g. appendix A of reference [7]). This is our justification to call W a spectral Debye-Waller factor.

4 Perturbation of eigenvectors

In section 3 we gave a simplified expression for the fidelity in the case of strong perturbations (see equation (7)). However, we had to make the assumption that the eigenvectors are so strongly affected by the perturbation that the matrix R as defined in equation (6) is basically a full random matrix. Only in this case we can average separately over the $|R_{lk}|^2$ in equation (6).

To study under which conditions this assumption is valid, we calculated the matrix R for different perturbation strengths. The dimension of the Hamiltonians was $N = 200$.

Figure 2 shows a visualization of the results for $\epsilon = 10, 100, \text{ and } 1000$. The grey-scale of the plots corresponds to the values of $|R_{ik}|^2$, white for zero and black for the maximal value.

In the perturbative regime the eigenvectors are only weakly affected by the perturbation, and thus the matrix R is close to the unit matrix. In this regime equation (7) does not describe the fidelity amplitude, but instead yields the form factor, multiplied by a Gaussian decay due to the fluctuations of the spectrum. For larger perturbation strengths the matrix R becomes a banded matrix, which is illustrated in figure 2. For a given finite dimension N of the Hamiltonians, we can indeed reach the point, where the matrix R becomes a full matrix.

Acknowledgments

Numerous discussions with T. Gorin, Dresden, Germany, T. Seligman, Cuernavaca, Mexico, and T. Prosen, Ljubljana, Slovenia are gratefully acknowledged. This work had been funded by the Deutsche Forschungsgemeinschaft.

References

- [1] A. Peres. Stability of quantum motion in chaotic and regular systems. *Phys. Rev. A*, 30:1610, 1984.
- [2] N. R. Cerruti and S. Tomsovic. Sensitivity of wave field evolution and manifold stability in chaotic systems. *Phys. Rev. Lett.*, 88:054103, 2002.
- [3] R. A. Jalabert and H. M. Pastawski. Environment-independent decoherence rate in classically chaotic systems. *Phys. Rev. Lett.*, 86:2490, 2001.
- [4] F. J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.*, 3:1191, 1962.
- [5] H.-J. Stöckmann and R. Schäfer. Recovery of the fidelity amplitude for the Gaussian ensembles. *New J. of Physics*, 6:199, 2004.
- [6] T. Gorin, T. Prosen, and T. H. Seligman. A random matrix formulation of fidelity decay. *New J. of Physics*, 6:20, 2004.
- [7] C. Kittel. *Introduction to solid state physics*. Wiley, New York, 1996.