The path integral formulation of a polymer chain with stiffness and its phase transitions

— Analytic theory of DNA condensation —

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As simple models of DNA condensation [1], semiflexible homopolymers in a poor solvent condition have been much investigated by simulations using Monte Carlo, Langevin approaches or Gaussian variational method for its phase diagram [2], and by theoretical works estimating such toroidal properties [3]. It becomes increasingly probable that toroid is the ground state. However in the theoretical side, they assume a priori toroidal geometry as the stable lowest energy state. Compared to the theory of coil-globule transition of flexible chains [4, 5] which are well described by field theoretical formalism [4], there is no simple microscopic theory, which demonstrates whip-toroid transition of the semiflexible polymer [6]. In this talk, we show such a transition using path integral method and the nonlinear sigma model on a line segment.

In the continuum limit, the Green function of a stiff polymer chain with attractive interactions can be given by the path integral with a constraint $|\vec{u}|^2 = 1$ [6]:

$$G(\vec{0}, \vec{R}; \vec{u}_i, \vec{u}_f; L, W) = N^{-1} \int_{\vec{r}(0)=\vec{0}, \vec{u}(0)=\vec{u}_i} \mathcal{D}[\vec{r}(s)] e^{-\mathcal{H}[\vec{r}, \vec{u}, W]}.$$  

$s$ is the proper time along the chain of length $L$. $\vec{r}(s)$ is the 3-d pointing vector at $s$ while $\vec{u}(s) \equiv \frac{\partial \vec{r}(s)}{\partial s}$ is the unit orientation vector at $s$. $N$ is the normalisation constant. The dimensionless Hamiltonian can be written by $\mathcal{H}[\vec{r}, \vec{u}, W] = \int_0^L ds \left[ H(s) + V_{AT}(s) \right]$ with $H(s) = \frac{1}{2} \left| \frac{\partial}{\partial s} \vec{u}(s) \right|^2$, $V_{AT}(s) = -W \int_0^s ds' \delta (\vec{r}(s) - \vec{r}(s'))$. $l$ is the persistence length and $W$ is a positive coupling constant of attraction between polymer segments. The Boltzmann weight is implicit.

Our Hamiltonian with the constraint $|\vec{u}(s)|^2 = 1$ can be interpreted as the low energy theory of the $O(3)$ nonlinear sigma model on a line segment. The constraint $|\vec{u}|^2 = 1$ in $\mathcal{H}(\vec{u}) = \int_0^L ds \left[ H(s) + V_{AT}(s) \right]$. 

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\[ \frac{1}{2} \int_0^L ds \, |\partial \bar{u}(s)|^2 \] in the polar coordinates gives

\[ \mathcal{H}_{bending} = \frac{l}{2} \int_0^L ds \, \left[ (\partial \theta u)^2 + \sin^2 \theta u (\partial \varphi u)^2 \right]. \tag{2} \]

By solving the classical equations of motion, we obtain two types of classical solutions such as \( \bar{u}(s) = const. \), or \( \theta = \frac{\pi}{2} \) and \( \varphi_u = a s + b \) where \( a, b \) are constants. Introducing the winding number of the solutions \( N(s) \equiv [as/2\pi] \) with Gauss’ symbol, we obtain their Hamiltonian:

\[ \mathcal{H}_{cl}(a, l, L, W) = \frac{LL}{2} (\frac{a^2}{l}) + \pi W a N(L) (N(L) + 1) - WL \cdot N(L). \tag{3} \]

The non-zero winding number of the classical solution in the \( \bar{u} \) space means a “Toroid state” of radius \( 1/a \) which stabilises itself by attracting neighbouring segments. When \( 0 < a \leq \frac{2\pi}{l} \), as long as the total energy of the chain does not exceed the bending energy of \( \frac{2\pi a^2}{l} \), they can whip with zero winding number. We call such low-energy states “Whip states.”

By rewriting the above form as follow, one can easily see that the functional shape of the energy level is governed by the value \( c \equiv \frac{W}{2l} \left( \frac{L}{2\pi} \right)^2 \)

\[ \mathcal{H}_{cl}(a, l, L, W) = \frac{WL}{2} \mathcal{H}(c, x) \text{ where } \mathcal{H}(c, x) = \frac{x^2}{2c} + \frac{1}{x} [x]([x] + 1) - 2[x], \tag{4} \]

with a new variable \( x \equiv \frac{aL}{2\pi} \), \( [x] = N(L) \). By plotting this with different values of \( c \), one can find that there is a critical value \( c = \frac{1}{2} \) where the meta-stable state with \( N(L) = 1 \) vanishes. This is a critical point where the phase transition from a Toroid state to a Whip state is accomplished.

References


