Cluster Distributions in the Long-Run: Two-Parameter Models

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Abstract

Long-run behavior of Poisson-Dirichlet two-parameter models are summarized. It is shown that the non-self averaging behavior in physics has a counterpart in the Poisson-Dirichlet two-parameter model. Implications to economic and financial modeling are mentioned.

1 Introduction

In economics we look for stationary distributions of some variables of interest. The Ewens distribution is one such example. In this sense we look for conditions under which distributions are invariant under mixing conditions such as size-biased sampling. A primitive version of the size-biased sampling has been used in the old industrial organization, and we put it in modern context, using mostly contributions by Pitman (2002), and Yamato and Sibuya (2000).

In the Poisson-Dirichlet distibutions with two parameters, $PD(\alpha, \theta)$, as in its one parameter version, $PD(0, \theta)$ denoted usually in an abbreviated way as $PD(\theta)$, exchangeable random partitions are used to model divisions of a large number of agents (firms, goods) into a number of types. In this paper we use the two-parameter models to discuss long-run behavior of the number of clusters, and sizes of clusters. We find non-vanishing of their variances, or equivalently their coefficients of variations of these variables. These behaviors in economics or in finance correspond to the phenomena of non-self averaging in physics.

2 Size-biased Permutations

Let (P_i) be the fractions of agents of type i, assumed to be positive a.s, ans sum to 1.

The first size-biased pick from (P_n) is

$$Pr(\tilde{P}_1 = P_i | P_1, P_2, \ldots) = P_i,\ i \geq 1)$$

, and

$$Pr(\tilde{P}_{i+1} = P_j | \tilde{P}_1, \ldots, \tilde{P}_i, ; P_1, P_2, \ldots,) = \frac{P_j}{1 - \tilde{P}_1 - \cdots \tilde{P}_j}.$$

If $\tilde{P}_n) =^d (P_n)$, then we say that (P_n) are invariant under size-biased permutation or sampling. The idea behind this operation is that distributions are stationary and do not change distributions under this mixing.

In the old industrial organization literature, market shares are ordered in decreasing order, $x_1 > x_2 > \cdots)$, where x_i is the fraction of share by firm i or sector i, and they sum to 1. Then, in order to discuss how dominant the leading share is, they constructed the ratio $x_2/(1-x_1), x_3/(1-x_1-x_2)$ etc., as some measures of how leading firms are monopolistic or not. In other words, the idea of size-biased sampling was used without naming the procedure as such.

3 Residual Allocation (Stick-breaking) Models

Pitman (1996) gave a necessary and sufficient conditions for ISBP. It is that (P_n) are distributed as follows:

$$P_1 = W_1; P_i = W_i(1 - W_1 - \cdots W_{i-1}, i \geq 2,$$

where Ws are distributed as $W_i \tilde{B}e(1-\alpha, \theta i \alpha)$, where Be denotes beta distribution. (This condition may appear rather special. It comes from the assumption of Moran subordinator, see Kingman (1993), for example, and not contrived as it may appear on first sight.)

4 Long-run Behavior

Let K_n denote the number of clusters formed by n agents (sectors, firms, etc.,). It is given by

$$K_n = \sum_{j=1}^{n} a_j,$$

where a_j is the number of clusters of size j, that is, j agents. From this definition, $n = \sum j a_j$. These are the elements of the partition vector in Aoki (2002), for example.

Pitman (1997), and Yamato and Sibuya (2000) have established that

$$K_n / n^\alpha \to^d L,$$

where the random variable L is distributed as Mittag-Leffler(α,θ)distribution. It is also known that

$$limE(\frac{K_n}{n^\alpha}) = \frac{\theta\Gamma(\theta)}{\alpha\Gamma(\theta+alpha)}.$$

This limit is the same as the mean of

$$\frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)}x^{\theta/\alpha}g_\alpha(x),$$

where g_α is the probability density of the Mittag-Leffler (α) distribution.

It is known that it is uniquely determined by its moments

$$\int_0^\infty x^p g_\alpha(x)dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)},$$

for $p > -1$.

From these the non-vanishing conditions are explicitly verified by calculating the means and variances of K_n/n^α and that of ja_j/n^α for example.

5 references

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