Modified Fast-Sample/Fast-Hold Approximation and γ-Independent H_∞-Discretisation for General Sampled-Data Systems by Fast-Lifting

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Abstract

This paper is concerned with the fast-lifting approach to H_∞ analysis and design of sampled-data systems, and extends our preceding study on modified fast-sample/fast-hold (FSFH) approximation, in which the direct feedthrough matrix D_{11} from the disturbance w to the controlled output z was assumed to be zero. More precisely, this paper removes this assumption and shows that a γ-independent H_∞ discretization is still possible in a nontrivial fashion by applying what we call quasi-finite-rank approximation of an infinite-rank operator and then the loop-shifting technique. As in the case of $D_{11} = 0$, the modified FSFH approach retains the feature that both the upper and lower bounds of the H_∞-norm or the frequency response gain can be computed, where the gap between the upper and lower bounds can be bounded with the approximation parameter N and is independent of the discrete-time controller. This feature is significant in applying the new method especially to control system design, and this study indeed has a very close relationship to the recent progress in the study of control system analysis/design via noncausal linear periodically time-varying scaling. The significance of a key lemma pertinent to the fast-lifting approach is suggested in connection with such a relationship, and also with its application to time-delay systems.

Key words: fast-lifting, H_∞ discretization, quasi-finite-rank approximation, loop-shifting.

1 Introduction

It is essential for the analysis and design of sampled-data systems that we deal with the intersample behavior of continuous-time signals as it is. There exist studies on the techniques for such treatment, e.g., the lifting technique (Bamieh and Pearson, 1992; Tadmor, 1992; Toivonen, 1992; Yamamoto, 1994; Yamamoto and Khargonekar, 1996), the FR-operator technique (Araki et al., 1996), the parametric transfer function approach (Rosenwasser and Lampe, 2000). These techniques can be regarded as methods for manipulating infinite-dimensional operators in the definitions of the H_∞-norm and the frequency response gain of sampled-data systems and then reducing the infinite-dimensional analysis or design problems to finite-dimensional ones in an exact fashion.

On the other hand, an approximation approach called fast-sample/fast-hold (FSFH) approximation (Yamamoto et al., 1999) was also proposed, in which the approximation error is assured to converge to zero as the approximation parameter N tends to infinity. A somewhat similar approach

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called modified FSFH approximation was also proposed in Hagiwara and Umeda (2008) based on what is called the fast-lifting technique (Hagiwara, 2006). This latter approach also discretizes the continuous-time generalized plant in an approximate but γ-independent fashion as in the former conventional FSFH approximation approach[1] and leads to a discrete-time generalized plant with a similar structure to what is obtained by the former. In contrast to the former, however, the latter allows us to obtain both the upper and lower bounds of the H_∞-norm or the frequency response gain of sampled-data systems. The study by Hara et al. (1995) also possesses the same feature and advantage, but the upper bound and the lower bound are obtained by different computations, and the upper bound seems rather loose in general. Thus, it is not very suitable for control system design. In the modified FSFH approximation approach, on the other hand, the gap between the upper and lower bounds can be evaluated in advance for each fixed approximation parameter N. This feature is very important particularly in control system design, and thus modified FSFH approximation can be said to provide useful features that are not present in the conventional FSFH approximation and in the method by Hara et al. (1995). In other words, the modified FSFH approximation approach provides a promising direction toward a rigorous and less conservative study on robustness of sampled-data systems. This observation is particularly supported by the close relationship, suggested in the recent study (Hagiwara and Umeda, 2007), between modified FSFH approximation and the novel technique for robustness studies called noncausal linear-periodically time-varying (LPTV) scaling. Both techniques depend heavily on the fast-lifting technique, which enables us to go far beyond the theoretical results that are attained by the conventional FSFH approximation technique.

As opposed to the conventional FSFH approximation, however, the arguments about the modified FSFH approximation developed in Hagiwara and Umeda (2008) was based on the assumption that the direct feedthrough matrix from the disturbance input w to the controlled output z, denoted by D_{11}, in the sampled-data system is zero. This leads to restriction on the admissible class of systems in H_∞ analysis and design, and moreover, the admissible class of uncertainties when we extend our arguments on modified FSFH approximation to sampled-data system analysis and design with respect to uncertainties. The novel study on noncausal LPTV scaling (Hagiwara and Umeda, 2007) is indeed intended for dealing with such uncertainties in a less conservative fashion with solid theoretical bases. Thus, removing the assumption $D_{11} = 0$ in the arguments on modified FSFH approximation is definitely an important research topic with a significant extended research direction, yet it is not straightforward in view of the arguments developed in the preceding study (Hagiwara and Umeda, 2008).

To get around the difficulty, this paper applies the well-known loop-shifting technique, but the arguments are nontrivial. This is because the loop-shifting generally leads to a γ-dependent generalized plant, so that simply applying the loop-shifting technique on the continuous-time generalized plant leads to a loss of one of the most important features of the modified FSFH approximation. Moreover, such γ-dependency will make it hard to extend the technique to the context of noncausal LPTV scaling. Thus we develop a method for circumventing the problem by working on what we call fast-lifted frequency response operators (Hagiwara and Umeda, 2008) and then carrying out some special factorizations of matrices represented as operator compositions (a key lemma in Section 3.2). The significance of this lemma is suggested also in connection with noncausal LPTV scaling (Hagiwara and Umeda, 2007) and time-delay system analysis/design (Hagiwara, 2008),

[1] γ-independent discretization is such a discretization method that is required to be carried out only once independently of the H_∞ performance level γ. On the other hand, γ-dependent discretization is a standard method (e.g., Bamieh and Pearson (1992)), which is required to be carried out every time γ changes in the so-called γ-iteration process.

which shows that the scope of its possible applications is not limited to the problem dealt with in this paper.

The contents of this paper are as follows. Section 2 reviews the lifting-based transfer operators and frequency response operators of sampled-data systems. In Section 3, we introduce a key technique for the modified FSFH approximation called fast-lifting, and give an extension of the H_∞-discretization method by taking nonzero D_{11} into consideration. Here, a key lemma is introduced to support the arguments, and its significance on other problems is also suggested. In Section 4, we give a numerical example and demonstrate the effectiveness of the new method, and Section 5 concludes the paper.

2 Lifting-Based Transfer Operators and Frequency Response Operators

We collect in this section some definitions and fundamental results pertinent to the lifting technique (Bamieh and Pearson, 1992; Tadmor, 1992; Toivonen, 1992; Yamamoto, 1994; Yamamoto and Khargonekar, 1996).

Let us consider the sampled-data system $\mathit{\Sigma}$ shown in Fig. 1, in which P represents the continuous-time linear time-invariant (LTI) generalized plant, while $\mathit{\Psi}$, $\mathcal{S}$ and $\mathcal{H}$ represent the discrete-time LTI controller, the ideal sampler and the zero-order hold, respectively, all operating at the sampling period h. Suppose that P and $\mathit{\Psi}$ are described by

$$\frac{dx}{dt} = Ax + B_1 w + B_2 u, \quad z = C_1 x + D_{11} w + D_{12} u, \quad y = C_2 x \tag{1}$$

and

$$\psi_{k+1} = A_{\mathit{\Psi}} \psi_k + B_{\mathit{\Psi}} y_k, \quad u_k = C_{\mathit{\Psi}} \psi_k + D_{\mathit{\Psi}} y_k \tag{2}$$

respectively, where $y_k := y(kh)$, $u(t) = u_k$ $(kh \le t < (k+1)h)$. We assume that $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^l$ and $z(t) \in \mathbb{R}^p$, and that $\mathit{\Sigma}$ is internally stable. Let us define $x_k := x(kh)$ and denote by $\{\widehat{w}_k\}_{k=0}^\infty$ and $\{\widehat{z}_k\}_{k=0}^\infty$ the lifted representations of $w(t)$ and $z(t)$, respectively, with the sampling period h (i.e., $\widehat{w}_k(\theta) = w(kh+\theta)$). Now, let us denote by $\mathcal{K}_\mu$, or sometimes just by $\mathcal{K}$ for simplicity, the Hilbert space $(L_2[0,h))^\mu$ of square integrable μ-dimensional vector functions over the time interval $[0,h)$ with the standard inner product. We assume that $\widehat{w}_k \in \mathcal{K}_l$ and thus $\widehat{z}_k \in \mathcal{K}_p$. The lifted representation of the system $\mathit{\Sigma}$ is given by

$$\xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k, \quad \widehat{z}_k = \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \tag{3}$$

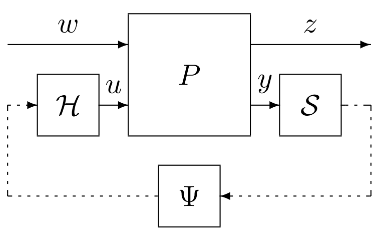

Figure 1: Sampled-data system $\mathit{\Sigma}$.

with the matrix $\mathcal{A}$ and the operators $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ defined appropriately, where $\xi_k := [x_k^T,\ \psi_k^T]^T$. Based on this representation, the lifting-based transfer operator of the sampled-data system Σ is defined by

$$\widehat{G}(\zeta) = \mathcal{C}(\zeta I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} \tag{4}$$

and the frequency response operator is defined as $\widehat{G}(e^{j\varphi h})$, $\varphi \in \mathcal{I}_0 := (-\omega_s/2, \omega_s/2]$, where $\omega_s := 2\pi/h$. Furthermore, the frequency response gain and the H_∞-norm of Σ are defined respectively as

$$\|\widehat{G}(e^{j\varphi h})\| = \sup_{\widehat{w}\in\mathcal{K}} \frac{\|\widehat{G}(e^{j\varphi h})\widehat{w}\|_{\mathcal{K}}}{\|\widehat{w}\|_{\mathcal{K}}}, \quad \|\widehat{G}(\zeta)\|_\infty = \max_{\varphi\in\mathcal{I}_0} \|\widehat{G}(e^{j\varphi h})\| \tag{5}$$

where $\|\cdot\|_{\mathcal{K}}$ denotes the norm on $\mathcal{K}$.

The operator $\mathcal{D}$ in (4) can be represented as $\mathcal{D} = \mathbf{D}_{110} + D_{11} (= \mathbf{D}_{11})$, where the first term on the right-hand side is the Hilbert-Schmidt operator given by

$$\mathbf{D}_{110} : \mathcal{K}_l \ni w \mapsto z \in \mathcal{K}_p, \quad z(\theta) = \int_0^\theta C_1 \mathrm{exp}\{A(\theta-\sigma)\} B_1 w(\sigma) d\sigma \tag{6}$$

and the second term is the operator of multiplication by the matrix D_{11}; in this paper, we use the same symbol for the underlying matrix and the associated operator of multiplication for notational simplicity, but they can be easily distinguished from the context. The definitions of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are omitted due to limited space; they are found in Bamieh and Pearson (1992); Tadmor (1992); Toivonen (1992); Yamamoto (1994); Yamamoto and Khargonekar (1996) but are not required explicitly in the following, and we just mention that $\mathcal{A}$ involves the matrices

$$A_d := \exp(Ah), \quad B_{2d} := \int_0^h \exp(A\sigma) B_2 d\sigma, \quad C_{2d} := C_2 \tag{7}$$

3 Modified Fast-Sample/Fast-Hold Approximation for $D_{11} \neq 0$

In this section, we give an extension of the modified FSFH approximation method (Hagiwara and Umeda, 2008) with nonzero D_{11} taken into account, and show that the frequency response gain and the H_∞-norm can still be evaluated to any degree of accuracy with a discretized generalized plant that is derived in a γ-independent fashion. A lemma on operator compositions provided in this section plays a significant role in the derivation, and it is suggested that this lemma is quite useful in the sense that it also plays a crucial role in the extended arguments on noncausal LPTV scaling of sampled-data systems (Hagiwara and Umeda, 2007), and also in time-delay system analysis/design with the fast-lifted monodromy operator approach (Hagiwara, 2008).

3.1 Application of the Fast-Lifting Technique and Quasi-Finite-Rank Approximation

We first introduce the fast-lifting operator $\mathbf{L}_N$ (Hagiwara, 2006; Hagiwara and Umeda, 2008), which plays a key role in modified FSFH approximation. For positive integers N and μ, let us define $h' := h/N$ and $(L_2[0,h'))^\mu =: \mathcal{K}'_\mu$ (which we sometimes denote $\mathcal{K}'$ for simplicity). For $x \in \mathcal{K}_\mu$, we define $x^{(i)} \in \mathcal{K}'_\mu$ $(i = 1, \cdots, N)$ by

$$x^{(i)}(\theta') := x((i-1)h' + \theta') \quad (0 \le \theta' < h') \tag{8}$$

Then, we define $\breve{x} := [(x^{(1)})^T \ \cdots \ (x^{(N)})^T]^T$, and refer to the mapping from $x \in \mathcal{K}_\mu$ to $\breve{x} \in (\mathcal{K}'_\mu)^N$ as fast lifting. We denote it by $\breve{x} = \mathbf{L}_N x$. It obviously follows from the definition of $\mathbf{L}_N$ that

$$\|\mathbf{L}_N \widehat{G}(e^{j\varphi h})\mathbf{L}_N^{-1}\| = \|\widehat{G}(e^{j\varphi h})\| \tag{9}$$

where the left-hand side of (9) is defined as the induced norm on $\mathcal{K}'$ in a parallel fashion to (5). We call $\mathbf{L}_N \widehat{G}(e^{j\varphi h})\mathbf{L}_N^{-1}$ the fast-lifted frequency operator, and we study how to compute its norm, as suggested by (9). To that end, we first recall that an explicit representation of the fast-lifted frequency operator has been shown in Hagiwara and Umeda (2008) for the case of $D_{11} = 0$, which we briefly review as follows.

First, as a result of applying fast-lifting to $\widehat{G}(e^{j\varphi h})$ and thus to $\mathcal{D} = \mathbf{D}_{110}$, there arises the operator $\mathbf{D}'_{110}$, which is nothing but $\mathbf{D}_{110}$ given by (6) with the underlying horizon $[0, h)$ replaced by $[0, h')$ ($'$ is used for the same meaning in the following). Then, to get around the difficulty stemming from the infinite-rank nature of $\mathbf{D}'_{110}$ and reduce the problem to finite-dimensional computations, this operator was approximated by the finite-rank operator of the form $\mathbf{M}'_1 X \mathbf{B}'_1$, where $\mathbf{B}'_1$ and $\mathbf{M}'_1$ are the operators defined by

$$\mathbf{B}'_1 : w \mapsto \int_0^{h'} \exp\{A(h' - \sigma)\} B_1 w(\sigma) d\sigma \tag{10}$$

$$\mathbf{M}'_1 : \begin{bmatrix} x \\ u \end{bmatrix} \mapsto z', \quad z'(\theta') = [C_1 \ \ D_{12}] \exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \theta' \right) \begin{bmatrix} x \\ u \end{bmatrix} \tag{11}$$

and X is a matrix introduced for the approximation purpose, which we determine later. We denote the approximation error by

$$\mathbf{E}' = \mathbf{D}'_{110} - \mathbf{M}'_1 X \mathbf{B}'_1 \tag{12}$$

Then, the fast-lifted frequency response operator was shown to be represented by

$$\mathbf{L}_N \widehat{G}(e^{j\varphi h})\mathbf{L}_N^{-1} = \overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} + \overline{\mathbf{E}'} \tag{13}$$

where $\overline{\mathbf{E}'}$ is defined as $\overline{\mathbf{E}'} = \mathrm{diag}[\mathbf{E}', \cdots, \mathbf{E}']$ consisting of N copies of $\mathbf{E}'$ and the operators $\overline{\mathbf{B}'_1}$ and $\overline{\mathbf{M}'_1}$ are also defined in a parallel way; the notation $\overline{(\cdot)}$ will be used in the same meaning throughout the paper, not only for operators but also for matrices. The matrix $Z_N(\zeta)$ in (13), on the other hand, is given by

$$Z_N(\zeta) := \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{N-1} \end{bmatrix} Z(\zeta)[\ (A'_d)^{N-1} \ \cdots \ I\ ] + \begin{bmatrix} X & 0 & \cdots & \cdots & 0 \\ J & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ (A'_{2d})^{N-2}J & \cdots & \cdots & J & X \end{bmatrix} \tag{14}$$

with X in (12), $J := [I, 0]^T$ and A'_d, A'_{2d}, $Z(\zeta)$ defined by

$$A'_d := \exp\left(Ah'\right),\ A'_{2d} := \exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} h' \right),\ Z(\zeta) := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} (e^{j\varphi h} I - \mathcal{A})^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{15}$$

Now, let us return to the case with $D_{11} \neq 0$. In this case, we still apply the same approximation to $\mathbf{D}'_{110} = \mathbf{D}'_{11} - D_{11}$. This leads to approximating the infinite-rank operator $\mathbf{D}'_{11}$ by $\mathbf{M}'_1 X \mathbf{B}'_1 + D_{11}$, which we call quasi-finite-rank approximation of $\mathbf{D}'_{11}$, since D_{11} is generally of infinite rank. Then, (13) only changes by $\overline{D_{11}}$ so that

$$\mathbf{L}_N\widehat{G}(e^{j\varphi h})\mathbf{L}_N^{-1} = \overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}} + \overline{\mathbf{E}'} \tag{16}$$

Applying the triangle inequality to (16), it follows that

$$\|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\| - \gamma_N \le \|\widehat{G}(e^{j\varphi h})\| \le \|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\| + \gamma_N \tag{17}$$

with γ_N given by

$$\gamma_N := \|\overline{\mathbf{E}'}\| = \|\mathbf{E}'\| \tag{18}$$

Since $\gamma_N = \|\mathbf{E}'\| \le \|\mathbf{E}'\|_{\mathrm{HS}}$, we also have similar inequalities with γ_N replaced by $\|\mathbf{E}'\|_{\mathrm{HS}}$, where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. There exist methods for finding the matrix X in (12) minimizing $\|\mathbf{E}'\|$ or $\|\mathbf{E}'\|_{\mathrm{HS}}$ (Hagiwara et al., 2001; Mirkin and Palmor, 2002; Hagiwara and Umeda, 2008), together with the resulting norm of $\mathbf{E}'$. In the minimization, dealing with $\|\mathbf{E}'\|_{\mathrm{HS}}$ is much simpler and seems numerically more reliable, and this is why we also consider $\|\mathbf{E}'\|_{\mathrm{HS}}$. In any case, it is shown in Hagiwara et al. (2001) and Hagiwara and Umeda (2008) that $\|\mathbf{E}'\| \to 0$ and $\|\mathbf{E}'\|_{\mathrm{HS}} \to 0$ as $N \to \infty$ ($h' \to 0$) under optimal approximation. This implies that $\|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\|$ gives a value that is close enough to the frequency response gain if N is large enough.

3.2 Computation of $\|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\|$

The key result of the present paper is to show that $\|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\|$ can be computed exactly if we introduce an appropriate discretized system. When $D_{11} = 0$, it follows immediately that $\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}$ is a finite-rank operator, and the computation reduces to a finite-dimensional problem by a standard technique, leading to a discretized system. This essentially was the contribution of our preceding paper (Hagiwara and Umeda, 2008). When $D_{11} \neq 0$, however, the computation becomes nontrivial. To get around the difficulty, we begin with a preliminary result on operator compositions, which plays a crucial role in this paper.

Lemma 1 *Let $F_{ll} \in \mathbb{R}^{l\times l}$, $F_{lp} \in \mathbb{R}^{l\times p}$ and $F_{pp} \in \mathbb{R}^{p\times p}$ be arbitrary matrices, and let us consider the matrices $\mathbf{B}'_1 F_{ll}(\mathbf{B}'_1)^*$, $\mathbf{B}'_1 F_{lp}\mathbf{M}'_1$ and $(\mathbf{M}'_1)^* F_{pp}\mathbf{M}'_1$ defined as the operator compositions with the operators $\mathbf{B}'_1$ and $\mathbf{M}'_1$ together with the operators of multiplication by the matrices F_{ll}, F_{lp} and F_{pp}. Then, these matrices can be equivalently represented as matrix products in such a way that the underlying matrices F_{ll}, F_{lp} and F_{pp} are left explicitly. More specifically, we have*

$$\begin{aligned} \mathbf{B}'_1 F_{ll}(\mathbf{B}'_1)^* &= W'(F_{ll}\otimes I_s)(W')^T \\ \mathbf{B}'_1 F_{lp}\mathbf{M}'_1 &= W'(F_{lp}\otimes I_s)V' \\ (\mathbf{M}'_1)^* F_{pp}\mathbf{M}'_1 &= (V')^T(F_{pp}\otimes I_s)V' \end{aligned} \tag{19}$$

In the above, $\otimes$ denotes the Kronecker product, and the matrices $W' := [W'_1, \cdots, W'_l]$ and $V' := [(V'_1)^T, \cdots, (V'_p)^T]^T$ and the positive integer s are defined from the factorization

$$\begin{bmatrix} K' & J' \\ (J')^T & L' \end{bmatrix} =: \begin{bmatrix} W'_1 \\ \vdots \\ W'_l \\ (V'_1)^T \\ \vdots \\ (V'_p)^T \end{bmatrix} \begin{bmatrix} (W'_1)^T & \cdots & (W'_l)^T & V'_1 & \cdots & V'_p \end{bmatrix},$$

$$W'_\alpha \in \mathbb{R}^{n\times s}\ (\alpha = 1,\cdots,l), \quad V'_\beta \in \mathbb{R}^{s\times(n+m)}\ (\beta = 1,\cdots,p) \tag{20}$$

where

$$K' := \int_0^{h'} \exp\left\{\underline{A}\left(h'-\sigma\right)\right\} \underline{b}_1\, \underline{b}_1^T \exp\left\{\underline{A}^T\left(h'-\sigma\right)\right\} d\sigma \tag{21}$$

$$L' := \int_0^{h'} \exp\left(\underline{A}_2{}^T\sigma\right) \underline{m}_1^T \underline{m}_1 \exp\left(\underline{A}_2\sigma\right) d\sigma \tag{22}$$

$$J' := \int_0^{h'} \exp\left\{\underline{A}\left(h'-\sigma\right)\right\} \underline{b}_1\ \underline{m}_1\ \exp\left(\underline{A}_2\sigma\right) d\sigma \tag{23}$$

with $\underline{A}$ *and* $\underline{A}_2$ *defined as* $\underline{A} := I_l \otimes A$ *and* $\underline{A}_2 := I_p \otimes \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}$*, respectively, and* $\underline{b}_1$ *and* $\underline{m}_1$ *defined as the column and row expansions of* B_1 *and* $[C_1\ D_{12}]$*, respectively. That is,*

$$\underline{b}_1 := \begin{bmatrix} b_{11} \\ \vdots \\ b_{1l} \end{bmatrix}, \quad \underline{m}_1 := [\, c_{11} \quad d_{121} \quad \cdots \quad c_{1p} \quad d_{12p} \,] \tag{24}$$

where

$$B_1 =: [\, b_{11}\ \cdots\ b_{1l} \,], \quad [\, C_1\ \ D_{12} \,] =: \begin{bmatrix} c_{11} & d_{121} \\ \vdots & \vdots \\ c_{1p} & d_{12p} \end{bmatrix} \tag{25}$$

Proof. It is easy to see that $\begin{bmatrix} K' & J' \\ (J')^T & L' \end{bmatrix}$ can be represented as an integral of a nonnegative definite matrix function, so that the factorization (20) is feasible.

Now, we only prove the first equation in (19); the other two equations can be proved similarly. Let us denote the (α, β) entry of F_{ll} by $f_{\alpha\beta}^{(ll)}$ where $\alpha, \beta = 1, \cdots, l$. Then it follows from (10) and (25) that

$$\begin{aligned} \mathbf{B}_1' F_{ll} (\mathbf{B}_1')^* &= \int_0^{h'} \exp\left\{A(h'-\sigma)\right\} \left(\sum_{\alpha=1}^{l} \sum_{\beta=1}^{l} f_{\alpha\beta}^{(ll)} b_{1\alpha} b_{1\beta}^T \right) \cdot \exp\left\{A^T(h'-\sigma)\right\} d\sigma \\ &= \sum_{\alpha=1}^{l} \sum_{\beta=1}^{l} f_{\alpha\beta}^{(ll)} K'_{\alpha\beta} \end{aligned} \tag{26}$$

with $K'_{\alpha\beta}$ defined by

$$K'_{\alpha\beta} = \int_0^{h'} \exp\left\{A(h'-\sigma)\right\} b_{1\alpha} b_{1\beta}^T \exp\left\{A^T(h'-\sigma)\right\} d\sigma \tag{27}$$

Since $K'_{\alpha\beta}$ is the submatrix at the α-th block column and β-th block row of K' given in (21), it follows from (20) that $K'_{\alpha\beta} = W'_\alpha (W'_\beta)^T$ and hence (26) leads to

$$\mathbf{B}_1' F_{11} (\mathbf{B}_1')^* = \left[\, W_1' \ \cdots \ W_l' \,\right] \begin{bmatrix} f_{11}^{(ll)} I_s & \cdots & f_{1l}^{(ll)} I_s \\ \vdots & \ddots & \vdots \\ f_{l1}^{(ll)} I_s & \cdots & f_{ll}^{(ll)} I_s \end{bmatrix} \begin{bmatrix} (W_1')^T \\ \vdots \\ (W_l')^T \end{bmatrix} = W' \left(F_{ll} \otimes I_s\right) (W')^T \tag{28}$$

This is nothing but the first equation in (19). Q.E.D.

By applying the loop-shifting technique and using Lemma 1, we can obtain the following result about the computation of $\|\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}+\overline{D_{11}}\|$. It is somewhat related to the results in Braslavsky et al. (1998) but is much more general and entirely different in that a general disturbance w and a general controlled output z are considered and thus Lemma 1 plays a crucial role, apart from the fast-lifting context here.

Proposition 1 *Let us define*

$$\Phi_N(\zeta) := \overline{V'}Z_N(\zeta)\overline{W'} + \overline{D_{11}\otimes I_s} \tag{29}$$

Then, we have

$$\|\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}+\overline{D_{11}}\| = \max\Big(\|D_{11}\|, \|\Phi_N(e^{j\varphi h})\|\Big) \tag{30}$$

Proof. We first show that for any γ such that $\gamma > \|D_{11}\|$, the condition

$$\|\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}+\overline{D_{11}}\| < \gamma \tag{31}$$

is equivalent to the condition $\|\Phi_N(e^{j\varphi h})\| < \gamma$. Once this claim is established, the proposition follows readily from the well-known fact (Yamamoto and Khargonekar, 1996) that $\|\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}+\overline{D_{11}}\| \geq \|D_{11}\|,\ \forall\varphi\in\mathcal{I}_0$.

To establish the above claim, we first note that (31) is equivalent to the condition

$$\gamma^2 I - (\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}+\overline{D_{11}})^*(\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}+\overline{D_{11}}) > 0 \tag{32}$$

Here, we define the Hermitian matrix E (>0) as follows.

$$E := \gamma^2\left(\gamma^2 I - D_{11}^T D_{11}\right)^{-1} \tag{33}$$

Following the well-known technique of the loop-shifting, we multiply $\overline{E^{1/2}}$ from left and right of (32), which leads to the equivalent condition

$$\begin{aligned}\gamma^2 I - \overline{E^{1/2}}\Big\{\Big(\overline{D_{11}}\Big)^*\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'} + \Big(\overline{\mathbf{B}_1'}\Big)^* Z_N(e^{j\varphi h})^*\Big(\overline{\mathbf{M}_1'}\Big)^*\overline{D_{11}}\\ + \Big(\overline{\mathbf{B}_1'}\Big)^* Z_N(e^{j\varphi h})^*\Big(\overline{\mathbf{M}_1'}\Big)^*\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}\Big\}\overline{E^{1/2}} > 0\end{aligned} \tag{34}$$

or equivalently,

$$\gamma^2 I - \mathbf{Y}_1\mathbf{Y}_2 > 0 \tag{35}$$

with

$$\begin{aligned}\mathbf{Y}_1 &:= \begin{bmatrix} \overline{E^{1/2}D_{11}^*}\,\overline{\mathbf{M}_1'} & \overline{E^{1/2}}\Big(\overline{\mathbf{B}_1'}\Big)^* & \overline{E^{1/2}}\Big(\overline{\mathbf{B}_1'}\Big)^* \end{bmatrix}\\ \mathbf{Y}_2 &:= \begin{bmatrix} Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}\,\overline{E^{1/2}} \\ Z_N(e^{j\varphi h})^*\Big(\overline{\mathbf{M}_1'}\Big)^*\overline{D_{11}E^{1/2}} \\ Z_N(e^{j\varphi h})^*\Big(\overline{\mathbf{M}_1'}\Big)^*\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}\,\overline{E^{1/2}} \end{bmatrix}\end{aligned} \tag{36}$$

Since $\mathbf{Y}_1\mathbf{Y}_2$ is obviously a compact operator (in fact, a finite-rank operator), the condition (35) is equivalent to the condition that the eigenvalues of $\gamma^2 I - \mathbf{Y}_1\mathbf{Y}_2$ are all positive (e.g., Ito et al. (2001)). They are all positive if and only if the eigenvalues of $\gamma^2 I - \mathbf{Y}_2\mathbf{Y}_1$ are, and thus we consider $\mathbf{Y}_2\mathbf{Y}_1$ instead; $\mathbf{Y}_2\mathbf{Y}_1$ is actually a matrix and can be computed by applying Lemma 1. In fact, since we have

$$\overline{\mathbf{B}_1'}\,\overline{E}\left(\overline{\mathbf{B}_1'}\right)^* = \left(\overline{W'}\,\overline{E^{1/2}\otimes I_s}\right)\left(\overline{E^{1/2}\otimes I_s}\,\left(\overline{W'}\right)^T\right) \tag{37}$$

$$\overline{\mathbf{B}_1'}\,\overline{ED_{11}^*}\,\overline{\mathbf{M}_1'} = \left(\overline{W'}\,\overline{E^{1/2}\otimes I_s}\right)\left(\overline{E^{1/2}D_{11}^T\otimes I_s}\,\overline{V'}\right) \tag{38}$$

$$\left(\overline{\mathbf{M}_1'}\right)^*\overline{D_{11}ED_{11}^*}\,\overline{\mathbf{M}_1'} = \left(\left(\overline{V'}\right)^T\,\overline{D_{11}E^{1/2}\otimes I_s}\right)\left(\overline{E^{1/2}D_{11}^T\otimes I_s}\,\overline{V'}\right) \tag{39}$$

$$\left(\overline{\mathbf{M}_1'}\right)^*\,\overline{\mathbf{M}_1'} = \left(\overline{V'}\right)^T\overline{V'} \tag{40}$$

by Lemma 1, we see that

$$\mathbf{Y}_2\mathbf{Y}_1 = Y_2Y_1 \tag{41}$$

with the matrices

$$\begin{aligned} Y_1 &:= \begin{bmatrix} \overline{E^{1/2}D_{11}^T\otimes I_s}\,\overline{V'} & \overline{E^{1/2}\otimes I_s}\,\left(\overline{W'}\right)^T & \overline{E^{1/2}\otimes I_s}\,\left(\overline{W'}\right)^T \end{bmatrix} \\ Y_2 &:= \begin{bmatrix} Z_N(e^{j\varphi h})\overline{W'}\,\overline{E^{1/2}\otimes I_s} \\ Z_N(e^{j\varphi h})^*\left(\overline{V'}\right)^T\overline{D_{11}E^{1/2}\otimes I_s} \\ Z_N(e^{j\varphi h})^*\left(\overline{V'}\right)^T\overline{V'}Z_N(e^{j\varphi h})\overline{W'}\,\overline{E^{1/2}\otimes I_s} \end{bmatrix} \end{aligned} \tag{42}$$

Note that Y_1 and Y_2 are nothing but $\mathbf{Y}_1$ and $\mathbf{Y}_2$ with the operators $\mathbf{B}_1'$ and $\mathbf{M}_1'$ replaced by the matrices W' and V' respectively and the operators $E^{1/2}$ and $D_{11}E^{1/2}$ replaced by the matrices $E^{1/2}\otimes I_s$ and $D_{11}E^{1/2}\otimes I_s$ respectively. Since the eigenvalues of $\gamma^2 I - \mathbf{Y}_2\mathbf{Y}_1 = \gamma^2 I - Y_2Y_1$ are all positive if and only if those of $\gamma^2 I - Y_1Y_2$ are and since Y_1Y_2 is a Hermitian matrix, we readily have the equivalent condition $\gamma^2 I - Y_1Y_2 > 0$. If we write down Y_1Y_2 explicitly, it is easy to see that this condition is nothing but (34) with the same replacement as above. Hence, it is easy to see that multiplying $\overline{E^{-1/2}\otimes I_s}$ from left and right leads to the equivalent condition

$$\gamma^2 I - \left(\overline{V'}Z_N(e^{j\varphi h})\overline{W'} + \overline{D_{11}\otimes I_s}\right)^*\left(\overline{V'}Z_N(e^{j\varphi h})\overline{W'} + \overline{D_{11}\otimes I_s}\right) > 0 \tag{43}$$

which naturally has a form of (32) with the same replacement of $\mathbf{B}_1'$ and $\mathbf{M}_1'$ as above, together with the replacement of the operator D_{11} with the matrix $\overline{D_{11}\otimes I_s}$. Hence, by the definition of $\Phi_N(\zeta)$, the claim has been established. Q.E.D.

3.3 γ-Independent H_∞-Discretization

We are now ready to give a γ-independent H_∞-discretization method via modified FSFH approximation; the following arguments are mostly the same as those in the case of $D_{11} = 0$ (Hagiwara and Umeda, 2008), but are given explicitly to make the discussions clearer.

It follows by (14) and the definitions of $\overline{W'}$ and $\overline{V'}$ that $\Phi_N(\zeta)$ in (29) can be rewritten as

$$\Phi_N(\zeta) = \begin{bmatrix} V_{1N} & V_{2N} \end{bmatrix} Z(\zeta) W_N + \Delta_{ND} \tag{44}$$

with W_N, V_{1N}, V_{2N} and Δ_{ND} given by

$$W_N := \begin{bmatrix} (A'_d)^{N-1} W' & \cdots & W' \end{bmatrix}, \quad \begin{bmatrix} V_{1N} & V_{2N} \end{bmatrix} := \begin{bmatrix} V' \\ \vdots \\ V' \, (A'_{2d})^{N-1} \end{bmatrix} \tag{45}$$

$$\Delta_{ND} := \Delta_N + \overline{D_{11} \otimes I_s} \tag{46}$$

respectively; in the above, Δ_N is given by

$$\Delta_N := \begin{bmatrix} V'XW' & 0 & \cdots & \cdots & 0 \\ V'_A W' & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ V'_A \, (A'_d)^{N-2} W' & \cdots & \cdots & V'_A W' & V'XW' \end{bmatrix} \tag{47}$$

with V'_A defined by partitioning V' into $V' = [V'_A, V'_B]$ according to the partitioning of A'_{2d} in (15).

Now, let us consider the discrete-time system shown in Fig. 2 with the discrete-time generalized plant Π_N given by

$$x_{k+1} = A_d x_k + W_N \rho_k + B_{2d} u_k, \quad \upsilon_k = V_{1N} x_k + \Delta_{ND} \rho_k + V_{2N} u_k, \quad y_k = C_{2d} x_k \tag{48}$$

Then, it can be seen that the discrete-time transfer matrix from ρ to υ is equal to $\Phi_N(\zeta)$ defined in (29). Thus, $\|\overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} + \overline{D_{11}}\|$ can be evaluated exactly with the discrete-time frequency response gain $\|\Phi_N(e^{j\varphi h})\|$ by Proposition 1. Taking account of the inequality (17), together with the fact that $\|D_{11}\| \le \|\widehat{G}(e^{j\varphi h})\|, \forall \varphi \in \mathcal{I}_0$ (Yamamoto and Khargonekar, 1996), we readily obtain the following main result that gives the γ-independent H_∞-discretization method for the case of $D_{11} \neq 0$ with modified FSFH approximation.

Theorem 1 *Consider the discrete-time system shown in Fig. 2, where Π_N is given by (48) with X determined appropriately, and let γ_N be defined by (12) and (18). Then, with the closed-loop transfer matrix $\Phi_N(\zeta)$ from ρ to υ, we have the following inequalities for the frequency response gain and H_∞ norm of the sampled-data system Σ in Fig. 1.*

$$\max\left(\|D_{11}\|, \|\Phi_N(e^{j\varphi h})\| - \gamma_N\right) \le \|\widehat{G}(e^{j\varphi h})\| \le \max\left(\|D_{11}\|, \|\Phi_N(e^{j\varphi h})\|\right) + \gamma_N, \quad \forall \varphi \in \mathcal{I}_0 \tag{49}$$

$$\max\left(\|D_{11}\|, \|\Phi_N(\zeta)\|_\infty - \gamma_N\right) \le \|\widehat{G}(\zeta)\|_\infty \le \max\left(\|D_{11}\|, \|\Phi_N(\zeta)\|_\infty\right) + \gamma_N \tag{50}$$

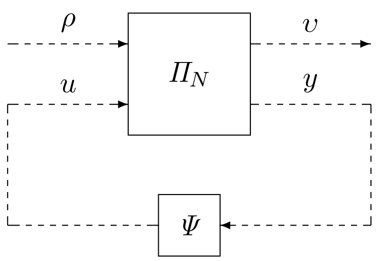

Figure 2: Discrete-time system Σ_d.

Remark 1 *The matrix X is usually chosen to minimize either $\|\mathbf{E}'\|$ or $\|\mathbf{E}'\|_{\mathrm{HS}}$. See the last paragraph of Section 3.1 (also recall that $\gamma_N \leq \|\mathbf{E}'\|_{\mathrm{HS}}$); the above inequalities ensure that the method is asymptotically exact in the sense that the H_∞-norm and the frequency response gain can be computed to any degree of accuracy by choosing N that is large enough. In the context of designing the H_∞ controller Ψ for the sampled-data system Σ, on the other hand, the above theorem still implies that we can simply deal with the H_∞ controller design problem for the discrete-time system Σ_d in Fig. 2. This is because the H_∞ norm of the sampled-data system Σ cannot be less than $\|D_{11}\|$ whatever Ψ we may take, so that we always assume that $\gamma > \|D_{11}\|$ in the H_∞ design $\|\widehat{G}(\zeta)\|_\infty < \gamma$. Hence, it follows from the proof of Proposition 1 that the H_∞ controller design minimizing the H_∞-norm of Σ_d is equivalent to minimizing $\max_{\varphi\in\mathcal{I}_0} \|\overline{\mathbf{M}_1'} Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'}+\overline{D_{11}}\|$, which in turn is equivalent to minimizing the upper bound of $\|\widehat{G}(\zeta)\|_\infty$ that follows readily from (17). Since γ_N is independent of the controller Ψ, the minimization of $\|\widehat{G}(\zeta)\|_\infty$ can be carried out within the error by γ_N with the discrete-time system Σ_d, where the only point is that the necessary condition $\gamma > \|D_{11}\|$ must be imposed explicitly in the γ-iteration process with Σ_d.*

Remark 2 *When X is determined to minimize $\|\mathbf{E}'\|_{\mathrm{HS}}$, we have*

$$\|\varPhi_N(e^{j\varphi h})\| \leq \|\widehat{G}(e^{j\varphi h})\| \leq \Big(\|\varPhi_N(e^{j\varphi h})\|^2 + \|\mathbf{E}'\|_{\mathrm{HS}}^2\Big)^{1/2}, \quad \forall\varphi \in \mathcal{I}_0 \tag{51}$$

provided that $D_{11} = 0$ (Hagiwara and Umeda, 2008), which gives sharper evaluation than (49) with γ_N replaced by $\|\mathbf{E}'\|_{\mathrm{HS}}$. A parallel result, however, seems hard to derive when $D_{11} \neq 0$ since the existence of nonzero D_{11} prevents us from developing an orthogonality argument, which plays a crucial role in the derivation of (51) under $D_{11} = 0$.

Remark 3 *The discretized generalized plant (48) is similar to that given in Hagiwara and Umeda (2008) under the assumption $D_{11} = 0$, and at a glance, the appearance of the second term on the right-hand side of (46) might look the only difference. This, however, is not the case; when $D_{11} = 0$, the matrices W' and V' are given simply by the Cholesky factors of the matrices $\mathbf{B}_1'(\mathbf{B}_1')^*$ and $(\mathbf{M}_1')^*\mathbf{M}_1'$, respectively, so that we do not have to consider the coupling between the operators $\mathbf{B}_1'$ and $\mathbf{M}_1'$ and thus Lemma 1 is irrelevant. The existence of $D_{11} \neq 0$, on the other hand, leads to such coupling as well as other more involved operator compositions, for which Lemma 1 plays a crucial role. The resulting W' and V' are thus different from those in the case $D_{11} = 0$ in spite of the same notations.*

The operator $D_{11} \neq 0$ is noncompact and whatever sort of finite-rank approximation one may apply to D_{11} alone, the approximation error cannot be less than $\|D_{11}\|$. Hence, such an approach always fails to give an asymptotically exact result. In this sense, no simple interpretation will be possible even as to the reason why D_{11} appears in Π_N only in the second term on the right-hand side of (46); at least, this term is not a result of some independent finite-rank approximation of the operator D_{11} alone. As such, the treatment of $D_{11} \neq 0$ in this paper is a nontrivial extension of the previous result under the assumption $D_{11} = 0$ (Hagiwara and Umeda, 2008).

3.4 Significance of Lemma 1 in Other Problems

The idea of employing such key relations as in Lemma 1 is actually closely related to the technique employed in the recent studies on analysis and design of sampled-data systems (Hagiwara and Umeda, 2007) called noncausal linear periodically time-varying (LPTV) scaling (Hagiwara, 2006), which is developed under the framework of the fast-lifting technique. Indeed, roughly speaking, we can establish that under the general setting $D_{11} \neq 0$, the optimization problem of noncausal LPTV scaling for less conservative treatment of uncertainties can be reduced to that of conventional LTI scaling applied to the discrete-time system Σ_d in Fig. 2 with the same discretized generalized plant Π_N as that derived in this paper (the details of the arguments will be reported elsewhere independently). This implies that the use of Lemma 1 leads to a more general result than our preceding study (Hagiwara and Umeda, 2008) under the assumption $D_{11} = 0$ on modified FSFH approximation, in the sense that the generalized result can be viewed as a sort of unified result on unscaled treatment and noncausally scaled treatment. In arriving at such a unified result, Lemma 1 plays a key role since, when $D_{11} = 0$ under which Lemma 1 can be dispensed with (see Remark 3), the resulting H_∞-discretization based on modified FSFH approximation (Hagiwara and Umeda, 2008) is different from the discretized generalized plant used for optimizing noncausal LPTV scaling (Hagiwara and Umeda, 2007), except for a special case (see Remark 1 of Hagiwara and Umeda (2008) for details). In other words, applying the conventional LTI scaling on the H_∞-discretization based on modified FSFH approximation *derived under the assumption* $D_{11} = 0$ (Hagiwara and Umeda, 2008) cannot lead to any theoretically rigorous treatment about robustness studies. The unified result mentioned above derived through Lemma 1 can be regarded as successfully filling such a gap (possibly at a sacrifice of increasing computational load when $D_{11} = 0$).

As a side remark, we mention that Lemma 1 is very important also in the fast-lifted monodromy operator approach to time-delay systems recently developed in Hagiwara (2008) when an infinite-dimensional operator Lyapunov inequality is reduced to a finite-dimensional LMI problem via fast-lifting. Various types of operator compositions arise in the reduction process, for which Lemma 1 plays a crucial role.

4 Numerical Example

In this section, we give a numerical example of H_∞ analysis with modified FSFH approximation, and demonstrate its effectiveness in comparison with the conventional FSFH approximation (Yamamoto et al., 1999). For sampled-data systems with $D_{11} = 0$, however, the new method introduced in this paper is essentially equivalent to the one proposed in Hagiwara and Umeda (2008) (except for those differences described in Remarks 2 and 3), and the effectiveness has already been verified there. Thus, we consider a slightly modified numerical example of Hagiwara and Umeda (2008) so that D_{11} becomes nonzero. More precisely, let us consider the continuous-time system shown in Fig. 3 (Anderson and Moore, 1990) so that we have $z = [w - u,\ y]^T$ and thus $D_{11} = [1, 0]^T$, where the plant $G(s)$ and the controller $C_r(s)$ are given respectively by

$$G(s) = \frac{1}{s^2} \cdot \frac{(s/a + 1)\prod_{i=0}^{1}\{(s/\omega_i)^2 + 2\zeta_i\,(s/\omega_i) + 1\}}{\prod_{i=2}^{4}\{(s/\omega_i)^2 + 2\zeta_i\,(s/\omega_i) + 1\}}$$

$$a = 4.84,\ \zeta_0 = 0.02,\ \ \zeta_1 = -0.4,\ \zeta_2 = \zeta_3 = \zeta_4 = 0.02,$$

$$\omega_0 = 1,\ \omega_1 = 5.65,\ \omega_2 = 0.765,\ \omega_3 = 1.41,\ \omega_4 = 1.85$$

$$C_r(s) = \frac{0.0513s^3 + 0.00424s^2 + 0.0296s + 0.00157}{s^4 + 0.693s^3 + 0.779s^2 + 0.293s + 0.0739}$$

We then discretize the controller $C_r(s)$ by the Tustin transformation with $h = 8$, and consider the sampled-data system shown in Fig. 4. We analyze its H_∞-norm from w to z. As in Hagiwara and Umeda (2008), we determine the matrix X minimizing the Hilbert-Schmidt norm $\|\mathbf{E}'\|_{\mathrm{HS}}$ with the

Table 1: H_∞-norm analysis.

N	1	2	3	4	5
conv. FSFH	102.8916	111.2174	110.8725	111.1307	111.4949
mod. FSFH (upper)	112.0104	111.9774	111.9771	111.9771	111.9771
mod. FSFH (lower)	111.9437	111.9768	111.9770	111.9771	111.9771
$\|\|\mathbf{E}'\|\|_{\mathrm{HS}}$	0.0334	2.7891×10^{-4}	3.1299×10^{-5}	1.0757×10^{-5}	5.4312×10^{-6}

Table 2: Comparison of the computation time.

N	1	2	3	4	5
conv. FSFH	0.060	0.070	0.070	0.070	0.070
mod. FSFH	0.080	0.090	0.100	0.110	0.110

method of Hagiwara et al. (2001) and evaluate the H_∞-norm based on (50) with γ_N replaced by $\|\mathbf{E}'\|_{\mathrm{HS}}$. All computations are executed with MATLAB on a PC with Pentium 4, 3.0GHz.

Table 1 shows the results of H_∞ analysis by the conventional and modified FSFH approximation. The exact value of the H_∞-norm obtained by the γ-dependent exact discretization method is 111.9771, so the modified FSFH approximation can be seen to give accurate enough upper and lower bounds of the H_∞-norm at $N = 4$, while the conventional FSFH approximation (Yamamoto et al., 1999) gives 111.9757, which is not satisfactorily accurate, even at $N = 100$.

Table 2 shows the computation time in seconds required for the computations about Table 1. For the same value of N, the modified FSFH approximation method takes much more time because the resulting discrete-time system has larger numbers of input and output than in the conventional FSFH approximation method. However, the conventional method takes about three times as much time (i.e., 0.32 seconds) even at $N = 100$ that is still small for accurate computations.

From the above arguments, we can see that the modified FSFH approximation method is a more effective method for H_∞ analysis of sampled-data systems in the sense of accuracy and efficient computation, compared with the conventional FSFH approximation method. Our experience with other examples also supports this observation as in the case with $D_{11} = 0$ (Hagiwara and Umeda, 2008).

5 Conclusion

In this paper, we gave a method for H_∞-discretization through the fast-lifting technique, which we call modified FSFH approximation. This method can lead to a γ-independent discretized general-

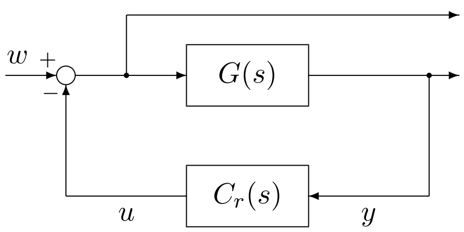

Figure 3: Continuous-time control system.

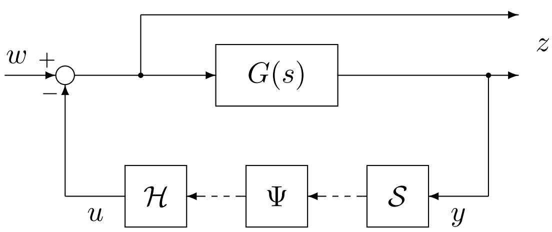

Figure 4: Sampled-data system with controller discretization.

ized plant even for the case with nonzero D_{11}, while the preceding study in Hagiwara and Umeda (2008) only dealt with the case of $D_{11} = 0$. The method developed in this paper is a nontrivial generalization of the previous result as discussed in Remark 3 and the paragraphs that follow this remark, and special factorizations of matrices defined as operator compositions (Lemma 1) played a crucial role in the derivation, together with other techniques such as quasi-finite-rank approximation of an infinite-rank operator and the loop-shifting technique.

The method given in this paper still possesses similarity to the conventional FSFH approximation method (Yamamoto et al., 1999) in the structure of the resulting discretized generalized plant and in the respect that the discretization is ensured to be asymptotically exact as the approximation parameter N is made larger. A distinctive advantage of the modified FSFH approximation method over the conventional FSFH method, however, is that the former can give both the upper and lower bounds of the approximation error in terms of N. Since these bounds are independent of the discrete-time controller, the modified FSFH method is more suitable for control system design with guaranteed performance. In this respect, some relationship of the arguments of this paper to the recent study on control system analysis/design via noncausal linear periodically time-varying (LPTV) scaling (Hagiwara, 2006; Hagiwara and Umeda, 2007) was also suggested. Simply speaking, the discretized generalized plant derived with our modified FSFH approximation technique allows us to apply a discrete-time scaling on it. This corresponds exactly to applying noncausal LPTV scaling on the original sampled-data system, which leads to a technique for reducing conservativeness in robust stability analysis and design with respect to uncertainties, provided that some appropriate modified error analysis is combined with it. Even though our technique leads to increase in the number of inputs and outputs as in the conventional FSFH technique, ours leads to such increase in such a way that we can benefit from theoretically rigorous and practically effective results in analysis and design to a quite essential extent in the above sense. Such reduction in conservativeness seems quite hard to achieve without resorting to the fast-lifting approach developed in this paper. Regarding such further theoretical benefit from the fast-lifting approach, it was also mentioned that a key result, Lemma 1, of this paper is quite useful in different research topics such as time-delay systems (Hagiwara, 2008).

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