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FUNDAMENTAL STUDY ON DESIGN AND STABILITY OF TUNNEL STRUCTURES

December 2005

Chanvanichskul Chatawut
A dissertation submitted in partial fulfillment of

the requirements of the degree of

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Kyoto University

Examination Committees:  Professor Takeshi TAMURA (Chairman)
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Prof. Fusao OKA
A number of individuals have contributed, directly or indirectly, toward making this thesis a reality. I feel extremely honored and wishes to express my deepest gratitude to my supervisor, Professor Takeshi Tamura for his inspiration, support, friendship, kindness as well as suggestions on how to tackle a problem guided me towards my research goals during 6 years in Kyoto University.

I also would like to express my gratitude to the members of my committee, Professor Fusao Oka and Toshihiro Asakura for their valuable suggestions on this research work.

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Nowadays, the underground space has been utilized more frequently due to high demand of the infrastructures such as subways, sewerage systems, underground parking area, power supply communication. In response to the increasing demand for underground space, tunneling has gained importance as one of the underground construction strategies. The most popular methods for constructing tunnels in soil are the shield tunneling method and the New Austrian Tunneling Method (NATM). Therefore, the studies on both tunneling methods are essential.

Various topics regarding the shield tunneling method have been examined. The examples of such topics are the development of new shield tunneling method, the study on shield machine, the design of the shield tunnel lining. In this dissertation, the design of the shield tunnel lining is focused. The new structural model, called ‘Rigid Segment Spring Model’, to analyze the shield tunnel segments was proposed. This model can improve various problematic aspects of the existing design methods of shield tunnel lining. For example, fewer unknowns are required to be solved for each segment, comparing to those required for the existing models. The three-dimensional analysis can also be carried out more conveniently using this proposed model. In this study, the full-scale loading test of the three segmental rings of the Kyoto Subway Tozai Line Rokujizo-Kita Construction Section was simulated. By comparing the computed deformation and the observed deformation, the appropriate spring parameters can be
obtained. Then, the application of the model to the actual segmental rings in the construction site was conducted using the appropriate spring parameters. Both the short-term and long-term loading conditions were considered. The deformation of segmental rings for the short-term loading condition agreed well with the measured values from construction site. The agreement indicates that the rigid segment spring model can be used for analyzing the segmental rings when the appropriate spring parameters are available.

The evaluation of the stability of tunnel face or heading in soft clay is important for both the New Austrian Tunneling Method and the shield tunneling method. The numerical analysis of the tunnel stability in soft clay using the rigid-plastic finite element method was carried out. The numerical results were compared to the experimental results obtained from the Cambridge University centrifugal modeling test done by Mair. The results agreed well in the 2D cross-section unlined tunnel case. For 3D tunnel heading, some consistent differences between the numerical analysis of 3D tunnel heading and the observed values were found. For example, the stability ratios obtained from the numerical analysis were 20% larger than those of experiments. Due to the consistency in the difference, the numerical analysis can provide a good prediction of the real stability of tunnel in soft clay if the 20% smaller soil cohesion is considered.

In urban areas, tunnels are normally excavated under roads as well as near existing buildings. Therefore, the effect of the building loads during tunneling should be accessed. The fundamental study on the tunnel stability in soft clay affected by surface loads (building loads) was done by using the rigid-plastic finite element method. The uniformly distributed load was applied on the ground surface at various distances apart from the face in order to determine the most critical distance. When the tunnel face collapses, soil behind the face flows towards the tunnel and the failure zone is formed between the tunnel face and the ground surface. Within trough of the failure zone at the ground surface, a critical location of the surface load(or surface structures) was determined. Furthermore, even without surface load, the vertical velocity of the ground surface reached the maximum value at the critical location. This suggests that the critical location is the position at which the tunneling stability is affected by the surface load the most and vice versa. This critical location of surface load depends on the cover depth of the tunnel but is independent of the width of surface load.
For the tunnel construction under groundwater table, the stability of tunnel face is reduced and the effect of seepage forces is required to be evaluated. The fundamental studies on the effect of groundwater on the tunnel stability were carried out using the rigid-plastic finite element method with consideration of groundwater. The two-dimensional cross-section tunnel embedded in the cohesive-frictional ground with the horizontal groundwater table (no flow) were considered. The effect of groundwater on the stability of the unlined tunnel was examined first. Then, the effect of groundwater on the lined tunnel stability was determined through the various lining patterns. The results suggest that the groundwater table affects the unlined tunnel stability significantly, especially in frictional soil. If the groundwater table is higher than the springline of the tunnel, the stability of the unlined tunnel in frictional soil is significantly reduced. For the lined tunnel, particularly in frictional soil, it is found that the reinforcement at tunnel invert plays very important role if the groundwater table is high.
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1.1 Background

In the past few years, with a sensitive infrastructure and the high rate of utilization of the ground surface in urban area, the underground space development is a better option to be considered in the long run. Subsurface construction is technically viable and has fewer problems with land expropriation and environmental impacts, which are the major points of considerations in any development projects today. Especially, construction by means of tunneling has rapidly become popular in the urban area to build infrastructure such as railroads, road, power supply, communication, water, and sewer. In general, nowadays there are several tunneling methods but the most popular methods are shield tunneling method and the New Austrian Tunneling Method (NATM). Hence, the researches on both tunneling methods are necessary to be carried out.

1.1.1 Shield tunneling method

The shield tunneling method was invented by Sir Marc Isambard Brunel in the United Kingdom in the early 19th century. The first use of the shield tunneling method is to construct a tunnel under the Thames River. Shield tunneling allows the subsurface construction of longitudinal underground structures also with the little cover, in unstable ground and groundwater, without causing disturbances on the surface or major settlements. Application is possible in very friable
or in high pressure ground such as non-cohesive loose soil, just as in soft plastic or running ground. In general, shield tunneling should not and cannot replace other methods. But it can be a technically sensible and also economical alternative to other tunneling methods in unfavorable ground conditions for long drives, when high advance rates need to be achieved or when there are strict regulations concerning surface settlements (Maidl et al., 1996). Fig. 1.1 shows the construction sequence of the shield tunneling method. The first step is to excavate ahead for a distance equivalent to the length of one segmental lining while support the tunnel face by jacks and face supporting pressure. Then, the shield machine is advanced for the distance of a segmental ring length by applying the jack thrust against the lining which is already erected. The next step is to install the segments to complete one ring after retracting the jacks. Finally, the tail void, which is the space between the lining and the opening ground, is filled by grouting.

The advantages of the shield tunneling method are mechanization and high advance rates,
1.1. Background

exact tunnel profile, least impact possible on surface structures and environmentally friendly construction method, etc. However, in the same conditions, the overall construction cost of the shield tunneling method is approximately 1.5~2.0 times of that of NATM (Koyama, 2000). The main reason is the cost of shield tunnel segment is very expensive and approximately 20~40% of overall construction cost. In this sense, the rational and economical design of shield tunnel segment must be expected.

1.1.2 New Austrian Tunneling Method (NATM)

Not only shield tunneling method but also NATM is very popular method to construct the tunnels. The term 'New Austrian Tunneling Method' was first published in English by Rabcewicz in 1964 (Rabcewicz, 1964). He described the fundamental principles of ground mechanics and the interaction of the ground with a tunnel lining. He stated that the sprayed concrete lining must be designed in both shape and material properties so that it was capable of moving with the ground to develop a stable condition with adequate factors of safety. This means that the ground is allowed to move towards the tunnel in a controlled manner, thereby developing shear stresses which act in conjunction with the tunnel lining. By mobilizing the strength of ground in this way, the cost of tunnel lining or support systems can be reduced. However, according the growing of this method, some confusion about the definition of NATM developed. In 1980, due to the conflict existing in the definition of NATM, the Austrian National Committee on 'Underground Construction' of International Tunneling Association (ITA) published an official definition of the New Austrian Method in ten languages which runs as follows: (Health and Safety Executive (HSE), 1996)

Austrian Definition

“The New Austrian Tunneling Method constituted a method where the surrounding rock or soil formations of a tunnel are integrated into an overall ring-like support structure. Thus the formations will themselves be part of this supporting structure”.

However, this definition was ultimately criticized by Kovari (Kovari, 1994). He stated that 'In reality, tunneling without the structural action of the ground is inconceivable ... and ... the idea of the ground as a structural element is inherent in the concept of tunnel.’ In other words, there is nothing new in this definition of NATM. Despite criticism, the word 'NATM' (de-
Figure 1.2: Construction sequence of NATM tunnel proposed by Rabcewicz (Rabcewicz, 1965)

noted bold italicized and pronounced 'Natam’) was introduced by Health and Safety Executive (HSE) (Health and Safety Executive (HSE), 1996). The definition of NATM which goes beyond the Austrian definition, is written as

**HSE Definition**

“A tunnel constructed using open face excavation techniques and with a lining constructed within the tunnel from sprayed concrete to provide ground support often with the additional use of ground anchors, bolts and dowels as appropriate”.

A number of different NATM tunnel sizes, geometry and excavation patterns have been adopted in a range of geological conditions. The most cases, especially in soft ground, it is not applicable to excavate the full tunnel face. Hence, the excavation face is usually divided into small cells that will help the ground stand until completion of the lining. The fundamental construction sequence of NATM was proposed by Rabcewicz (Rabcewicz, 1965). According Fig. 1.2, the first step is the excavation of the top heading (I), leaving the central part to support tunnel face. Then, the auxiliary lining II is formed and followed by removing the top central portion (III) subsequently excavation of left and right walls (IV). The fifth step is the application of shotcrete with additional reinforcements (V) followed by excavation of a bench (VI). Finally, the invert is closed with concrete (VII) following the installation of a waterproof membrane (VIII) and concreting of the inside lining (IX). In this dissertation, the excavation face or the area at the front of the tunnel construction beyond the completed NATM ring is denoted by the
1.1. Background

There are several failure incidents for NATM throughout the world. HSE summarized the NATM incidents occurred before 1994 in Table 1.1 and classified the causes of the tunnel collapse into three principal locations (see Fig. 1.3):

- **Category A - Heading collapses** which have occurred, initially at least, in the area of the NATM tunnel heading in front of the first completed ring;

- **Category B - Completed lining collapse** i.e., in the area where the sprayed concrete lining is complete; and

- **Category C - Other collapse** locations.

### Table 1.1: Summary of NATM incidents by HSE (Health and Safety Executive (HSE), 1996)

<table>
<thead>
<tr>
<th>No</th>
<th>Date</th>
<th>Location</th>
<th>Reported cause</th>
<th>Urban or rural</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Oct 1973</td>
<td>Paris, France</td>
<td>A</td>
<td>?</td>
</tr>
<tr>
<td>2</td>
<td>Dec 1981</td>
<td>Sao Paulo metro, Brazil</td>
<td>A</td>
<td>Urban</td>
</tr>
<tr>
<td>3</td>
<td>1983</td>
<td>Santana Underground Railway, Brazil</td>
<td>?</td>
<td>Urban</td>
</tr>
<tr>
<td>4</td>
<td>Nov 1984</td>
<td>Landrucken Tunnel, Germany</td>
<td>A</td>
<td>Rural</td>
</tr>
<tr>
<td>5</td>
<td>1984</td>
<td>Bochum Metro, Germany</td>
<td>A, B</td>
<td>Urban</td>
</tr>
</tbody>
</table>

*continued on next page*
According to Table 1.1, almost the incidents of NATM tunnel collapse occur in the area be-
1.2. Objectives of the research

According to the above-mentioned background, the studies in this dissertation are divided into two parts; shield tunneling method and New Austrain Tunneling Method. For shield tunneling method, to solve some problems in other design methods which will be discussed in the subsequent chapter, the new structural model for segmental lining is proposed and some applications of the model is shown. For NATM, the numerical analysis of the tunnel stability in soft ground is carried out and the effect of groundwater on the tunnel stability will be also studied. The main objectives of this dissertation are summarized as follows:

**Shield tunneling method:**

1. To propose the new structural model, called *rigid segment spring model*, for the shield tunnel segment design;

2. To evaluate the application of the model to the actual tunnel (the rectangular tunnel is considered in this dissertation).
New Austrian Tunneling Method:

1. To investigate the stability of tunnel heading in soft clay (purely cohesive soil) considering the ground as green field;

2. To investigate the effects of the existing super structure on the surface on the tunnel heading stability;

3. To evaluate the effects of the groundwater on the tunnel stability.

1.3 Scope and organization of the dissertation

Chapter 1 introduces the background of both shield tunneling method and New Austrian Tunneling Method, objectives, scope and organization of the dissertation. In Chapter 2, for shield tunneling method, the guideline for design shield tunnel lining is described and some existing design methods of shield tunnel lining are reviewed. For the NATM, previous researches on the evaluation of the stability of tunnel face (heading) both in cohesive and in frictional soils are reviewed. Then, previous studies on the effect of groundwater or seepage force on the tunnel stability are also described.

In this dissertation, the researches are divided into two parts: Part I: Shield tunneling method; Part II: New Austrian Tunneling Method (NATM).

Part I deals with the studies relatively to the shield tunneling method and consists of Chapter 3 and Chapter 4.

In Chapter 3, the rigid segment spring model firstly proposed by Tamura et al. (Tamura et al., 2000) is described. The model is based on the assumption that segments are perfectly rigid and connected with three dimensional paired-springs at all joints to which the total deformation of the tunnel is attributed. The formulation of the analysis method using this model for both circular and rectangular tunnels is described in detail. For more reality on the joint behavior, the methodology for the consideration of the nonlinear joint spring model is proposed. In general, the shield tunnel lining is formed by assembling two different segmental rings alternately to build the staggered joint arrangement. It is apparent that, for long tunnel, the behavior of
1.3. Scope and organization of the dissertation

every two adjacent segmental rings which locate far enough away from the boundary ends of the overall tunnel span, is not much different. The methodology for infinite ring analysis is, therefore, introduced.

The simulation of the actual shield tunnel segments using the rigid segment spring model is illustrated in Chapter 4. The rectangular ring of the Kyoto Subway Tozai Line Rokujizo-Kita Construction Section is selected to be simulated. First, the outline of this construction project is reviewed. In order to fundamentally check the correctness of the analysis model, the preparation analysis considering the long tunnel as the cantilever beam is carried out. The spring parameters for this model are unknown hence the parametric study is carried out by simulating the segmental rings in the full scale segment loading test. The appropriate spring parameters are obtained by compared the deformation of segmental ring obtained from the simulation with the observed deformation by test. Finally, the simulation of the in-situ segmental rings using the appropriate spring parameters is carried out. Both short-term and long-term loading conditions which measured in construction site are considered. The comparison between the results obtained from the simulation and observed data can indicate that the rigid segment spring model can be expected to be one effective tool to simulate the shield tunnel segments.

Part II consists of three chapters, namely Chapters 5, 6 and 7. This part illustrates the studies concerned with the New Austrian Tunneling Method; specifically, stability of tunnel heading in soft clay and effect of groundwater on the tunnel stability.

Chapter 5 describes the rigid-plastic finite element method (RPFEM) proposed by Tamura (Tamura et al., 1984; Tamura et al., 1987; Tamura, 1990). The rigid-plastic finite method is recently considered as the effective tool to determine the critical strength of soil or rock structures such as dams, tunnels and the foundations for the large structures. Less number of material parameters is required and solution is independent of the initial condition in this analysis method. The chapter starts from describing the theoretical basis for plasticity. The stress-strain rate relations and fundamental equation and boundary conditions for rigid-plastic analysis are subsequently described. Upper-bound theorem for rigid-plastic analysis is also explained. Then, the formulation of the RPFEM is described in detail. The methodology to include the groundwater into the rigid-plastic finite element method is proposed and described.
Chapter 1. Introduction

The numerical analysis of the tunnel in soft clay using RPFEM is illustrated in Chapter 6. Both of two and three dimensional analysis for determining the critical state of tunnel are carried out and the numerical analysis results are taken to compare with the observed data obtained from the series of centrifugal modeling tests done by Mair (Mair, 1979). The very good agreement is achieved especially in the two dimensional analysis. In the later section of this chapter, the study on the influence of the surface load on the stability of plane strain tunnel heading in soft clay is described.

Chapter 7 studies the effects of the groundwater on the tunnel stability. The RPFEM with consideration of groundwater which is proposed in Chapter 5 is used to analyze the two dimensional cross-section tunnel in various groundwater table. Both of tunnels in clay and sandy soil are considered. The effect of groundwater on the stability of the unlined tunnel is studied first. Then, the effect of groundwater on the lined tunnel stability is studied through the various lining patterns. According to the studies, it is found that the groundwater table affects the unlined tunnel stability significantly, especially in sandy soil. For lined tunnel, the tunnel invert reinforcement is very important if the groundwater table is so high.

Finally, the results presented in each chapter are summarized in Chapter 8.
In this chapter, previous studies on tunneling are reviewed. In the design of the shield tunnel lining, the outline of the design of shield tunnel lining is described first, followed by existing design methods for shield tunnel lining. The studies on the evaluation of the tunnel stability are subsequently reviewed. Finally, the studies on the effect of seepage force or groundwater on the tunnel stability are reviewed.

2.1 Design of shield tunneling method

2.1.1 Outline of the design of shield tunnel lining

In most cases, the shield tunnel lining consists of the primary lining and the secondary lining, and is required to maintain a specified internal space by directly supporting the ground. Particularly, the primary lining is generally formed by assembling the box type or the flat type factory-manufactured segments which connect to each other by bolt joints in the directions of tunnel axis and tunnel cross section.

According to Japanese standard for shield tunneling (JSCE, 1996), the basis of the design of shield tunnel lining is “Design of lining shall based on the confirmation of the safety in compliance with the purpose of tunnel usage and shall be carried out by the allowable stress design method on condition that adequate and proper construction works are executed in using good quality materials” and working group no.2 of International Tunneling Association(ITA)
Chapter 2. Literature review

 issu a report titled *Guidelines for the Design of Shield Tunnel Lining* (ITA, 2000). This report presents a flow chart of shield tunnel lining design as shown in Fig. 2.1 and summarizes the procedure of the shield tunnel lining design as the following sequence:

1. **Adherence to specification, code or standard.** The tunnel to constructed should be designed according to the appropriate specification standard, code which are determined by the persons in charge of the project.

2. **Decision on inner dimension of tunnel.** The inner diameter of the tunnel to be designed should be decided in consideration of the space that is demanded by the functions of the tunnel.
3. **Determination of load condition.** The following loads should be considered in the design of shield tunnel lining: earth pressure and water pressure, dead load, surcharge, soil reaction, internal loads, construction loads and effect of earthquake, etc. The designer should select the cases critical to the design lining.

4. **Determination of lining conditions.** The lining conditions that should be designed are dimension of the lining (thickness), strength of material, arrangement of the reinforcement, etc.

5. **Computational of member forces.** The designer should compute member forces such as bending moment, axial force and shear force of the lining, by using appropriate models and design methods.

6. **Safety check.** The safety can be checked by considering the computed member forces.

7. **Review.** If the designed lining is not safe against design loads, the designer should change the lining conditions and design lining. If the designed lining safe but not economical, the designer should change the lining conditions and redesign the lining.

8. **Approval of the design.** After the designer judges that the designed lining is safe, economical and optimally designed, a document of design should be approved by the persons in charge of the project.

According to the above-mentioned procedure of the design of shield tunnel lining, the step 5 is mainly considered in this dissertation (see the hatched box of flow chart in Fig. 2.1). This step is the extremely important step of the design of shield tunnel lining. If the appropriate models or design methods are selected, the computed member forces are close to the actual behavior of the lining. This leads the obtained lining conditions to be economical and optimal.

### 2.1.2 Existing segment design methods

There are several design models or design methods of shield tunnel lining so far. The structural models which are classified by joint evaluation for design of segmental ring are shown in Fig. 2.2. 

1. The first model is called *Usual Calculation Method*, as shown in Fig. 2.2(a), the segmental ring is assumed as a ring with uniform bending rigidity without joint is subjected
to loading and soil reaction. Since the circumferential joint (segment joint) is assumed to have the same rigidity as that of the segment, the computed bending moment for the design of joint is overestimated and bending moment at main section of segment is not calculated correctly also (Koyama, 2003). Generally, a segmental ring is composed of several segments assembled by bolts, its deformation tends to be larger than a ring with uniform bending rigidity. This is because the rigidity of joints is less rigid than a main section of segment. Thus, it is important to consider the decrease of rigidity at joints for calculating member forces. Moreover because the staggered joint arrangement is widely used in Japan, the effect of adjacent rings must be also evaluated. (2) The Modified Usual Calculation Method has been introduced to solve the above-mentioned problem. The decrease of rigidity at segment joints is evaluated by considering the bending rigidity of segmental ring to be less than that of a ring with uniform bending rigidity by means of an effective ratio of bending rigidity \( \eta \). This means, similarly to the Usual Calculation Method, the bending rigidity of a ring is assumed to be uniform and equal to \( \eta EI \) \( (\eta \leq 1) \).

In staggered joint arrangement, Some bending moment is transferred to the adjacent ring as represented by \( M_2 \) in Fig. 2.3. This redistribution of bending moments are expressed by introducing a transfer ratio of bending moment \( \zeta \) which is a ratio of \( M_2/M \). \( M \) is the bending moment calculated as a uniform ring of \( \eta EI \) in rigidity. However, the coefficient \( \eta \) depends on ground conditions as well as a type of segments, structural character of the segment joint, how staggered, and structural properties of staggered segmental ring, of which value is determined by experiences in consideration of joint performance test results and past project records (JSCE, 1996). Moreover, the transfer ratio of bending moment \( \zeta \) and the coefficient \( \eta \) are interrelated each other. Thus, there is no basis for the determination of these two parameters and it is then
2.1. Design of shield tunneling method

impossible to calculate actual distribution of the bending moment by using this model. In the countries where ground condition is good, (3) the Ring with multiple hinged joint calculation method is used. Segment joints are treated as hinges as shown in Fig. 2.2(b). A ring with multiple hinged joints is an unstable structure by itself but it becomes stable once a ground supports it. Thus, it is very important to evaluate load distributions acting on the ring and soil reactions.

For the above reason, In Japan, (4) the beam-spring model calculation method is applied nowadays. Each segment is considered as a curved beam connected by rotational springs on segment joints. For the ring joints, for the beam-spring model I, it is assumed that the displacement of ring beam is equal to that of the adjacent ring beam as shown in Fig. 2.2(c). This means there is no gap caused by shear stress. shear springs on ring joints. However, the relative displacement between two adjacent rings also occurs. Therefore, the beam spring model II is also applied. This relative displacement is expressed by means of the shear spring at ring joints.

According to the above design methods, even though several experiments and analyses have been carried, but they are only based on 2-dimensional model in cross section of segmental ring. For 3-dimensional analysis which is required the large scale calculation, the configuration of boundary condition is very complicated in finite element method. Also, in the beam-spring model calculation method, even though it can be utilized to analyze the longitudinal direction of tunnel, it still cannot present the actual three dimensional behavior of tunnel lining.

To solve some problems proposed above, the Rigid Segment Spring Model which is first proposed by Tamura et al. (Tamura et al., 2000) is introduced into consideration. This model is based on the assumption that segments are perfectly rigid and connected with three dimensional paired-springs at all joints to which the total deformation of the tunnel is attributed. The model can simulate the tunnel lining in three dimensional space and does not have many degree of
freedom. Also, the offset length between neighboring segments which impossible to determine by other methods, can be determined by this method. The detail of this model will be described in the subsequent chapter. Tamura et al. (Tamura et al., 2000) and Hiromatsu (Hiromatsu, 1999) studied the effect of the staggered joint arrangement by using the rigid segment spring model. But, the fact that the two adjacent segments cannot overlapped each other was not considered. In this dissertation, the rigid segment spring model is modified considering the above fact and improved so that the model can be applied to the actual segmental ring.

2.2 Researches on evaluation of tunnel stability

The safety of the tunnel construction is usually considered into two problems. Firstly, the stability of tunnel heading (area near the face) is necessary to be evaluated for the safety of the construction. Secondly, the ground deformation due to the tunneling is also required to be determined to prevent damage to the existing surface or subsurface structures. Only the first problem is studied in this dissertation.

Fig. 2.4(a) and 2.4(b) show the layout of tunnel heading problem, where a circular tunnel of diameter $D$ is constructed with a depth of cover $C$. The non-supported heading length is defined by $P$. The tunnel stability is evaluated by determining the tunnel pressure $\sigma_T$ that are necessary to maintain the stability of the tunnel heading. In addition, the acceleration factor of weight of soil mass that causes the collapse of tunnel, in accordance with the centrifugal modeling test concept, can also evaluate the tunnel stability. This section reviews the previous works on the evaluation of the tunnel stability both in cohesive and frictional ground and some studies on the effects of groundwater in tunneling are also reviewed.

The stability of tunnel heading in cohesive material (clay) has been studied by several authors using both experimental and theoretical approaches. Broms and Bennermark (Broms and Bennermark, 1967) conducted the laboratory extrusion tests on soft clay to investigate the stability of tunnel heading. They indicated that the stability of tunnel heading depends on the term called stability ratio $N$ which equals to the difference between the total overburden pressure at the tunnel axis and the tunnel pressure divided by the undrained shear strength $c_u$ as shown in Eq. (2.1):
2.2. Researches on evaluation of tunnel stability

\[ N = \frac{\gamma \left( C + \frac{D}{2} \right) + \sigma_s - \sigma_T}{c_u} \]  \hspace{1cm} (2.1)

where \(\sigma_s\) and \(\gamma\) denote the surface surcharge and the unit weight of soil, respectively. At the critical state, \(N\) and \(\sigma_T\) are substituted by \(N_C\) and \(\sigma_{TC}\), respectively. They concluded that the tunnel face would be stable if the stability ratio \(N\) does not exceed \(6 \sim 7\). Peck (Peck, 1969) collected several case histories on the stability of tunnels in saturated plastic clays and concluded that \(N\) should not exceed \(5 \sim 7\) for the stable tunnel face.

A number of centrifugal modeling tests to investigate the stability of tunnel in clay have been carried out. Mair (Mair, 1979) conducted the centrifugal modeling tests to model both 2D cross-section tunnels and 3D tunnel headings in clay. Mair also did the numerical analysis using the Cam-clay model finite element method to analyze some of his experimental results but the comparison between two approaches does not agree well. Schofield (Schofield, 1980) summarized the experimental data obtained from several three dimensional series of the centrifugal modeling tests done by Cambridge University into the graph shown in Fig. 2.5.

Takemura et al. (Takemura et al., 1990) and Lee et al. (Lee et al., 1999) carried out the centrifugal modeling tests to model the 2D cross-section tunnels and also proposed solutions based on the upper and lower bound theorems to compare with their experimental results. Also, Davis et al. (Davis et al., 1980) presented a range of solutions based on the upper and lower bound theorems for unlined plane strain tunnels, circular headings as well as the plane strain headings. Yu and Rowe (Yu and Rowe, 1999) proposed the plasticity solution using the cavity
expansion method for the unlined plane strain tunnel.

In the case of the cohesive-frictional ground, Leca and Dormieux (Leca and Dormieux, 1990) presented the upper bound solution which represented fairly well the actual behavior of the tunnel face. Mashimo (Mashimo, 1998) conducted several 2D and 3D tests to investigate the stability of tunnel face in sandy grounds and concluded that if the dimensionless cohesion \( c/(\gamma D) \) of sandy ground with the angle of friction \( \phi = 30^\circ \) is approximately not below the range 0.13 \( \sim \) 0.17, the tunnel face is stable.

Murayama (Expressway Technology Center, 1997) proposed the method to evaluate the two dimensional stability of tunnel heading in sandy ground by using the equilibrium of the moment acting on the failure soil wedge as shown in Fig. 2.6. This failure line of soil wedge is assumed to be log-spiral curve and expressed as

\[
r = r_0 \exp(\theta \tan \phi).
\]  

(2.2)

The resultant force \( P \) to maintain the tunnel face can be determined by using the equilibrium equation of the moment, caused by the 5 forces acting on soil wedge, around the center \( O \) of the spiral curve. The resultant force \( P \) is written as

\[
P = \frac{1}{l_p} \left\{ W \times l_w + q \times B \left( l_a + \frac{B}{2} \right) - \frac{c}{2 \tan \phi} \left( r_e^2 - r_0^2 \right) \right\}
\]  

(2.3)

in which the Terzaghi’s loosening earth pressure \( q \) acting on the width \( \alpha B \) of the top of wedge...
is
\[ q = \frac{\alpha B (\gamma - 2c/\alpha B)}{2K \tan \phi} \left[ 1 - \exp\left(\frac{-2KC}{\alpha B}\right) \tan \phi \right] \]  
(2.4)

or, if the overburden \( C \) is much larger than \( B \), approximately \( C > 1.5B \), Eq. (2.4) can be reduced to
\[ q = \frac{\alpha B (\gamma - 2c/\alpha B)}{2K \tan \phi} \]
(2.5)

and some notations are introduced as
- \( \gamma \): unit weight of soil
- \( \phi \): angle of internal friction
- \( c \): cohesion of soil
- \( K \): the ratio of the horizontal earth pressure to vertical earth pressure
  \[ (K = 1 \sim 1.5) \]
- \( r_0, r_c \): the radius of the log-spiral curve measured from point \( b \) and \( c \), respectively
- \( \alpha \): coefficient to indicate the loosening width (according to experiments, \( \alpha \approx 1.8 \)).

Toki and Tamura et al. (Toki et al., 1994) also proposed the simplified method to evaluate the 2D tunnel face stability. They assumed the failure wedge as the triangular wedge and its linear failure line connects point \( b \) and point \( c \) as shown in Fig. 2.7. The resultant force \( P \) to maintain the tunnel face is calculated by equating all forces acting on the wedge both in horizontal and
Chapter 2. Literature review

Figure 2.7: Force equilibrium on soil wedge in method for evaluating tunnel face stability proposed by Tamura et al.

vertical directions. The force $P$ is expressed as

$$P = \left\{ \frac{1}{2} \gamma D \tan \beta + q (\tan \beta - \tan \phi) - 3c \right\} D \tan \beta$$

(2.6)

where $\beta = 45^\circ - \phi/2$ and the Terzaghi’s loosening earth pressure $q$ is the same as that of Eq. (2.4) but $\alpha B$ is substituted by $D \tan \beta$.

According to both of Murayama’s method and Tamura’s method, the tunnel can be stable by itself if the resultant force $P$ is not positive.

For cohesionless soil, Atkinson and Potts (Atkinson and Potts, 1977) investigated theoretically and experimentally the stability of a circular tunnel in a cohesive soil. They conducted small-scale model tests and used the centrifuge of the Cambridge University and also derived some solutions, based on the upper and lower bound theorems, to predict the tunnel pressure at collapse.

The finite element method is also used to investigate stability of tunnel heading by many authors. Eisenstein and Samarasekara (Eisenstein and Samarasekera, 1992) examined the stability of tunnels in clay using a combination of the finite element method and the limit equilibrium theory. Sloan et al. and Augarde et al. (Sloan and Assadi, 1994; Augarde et al., 2003) presented the finite element limit analysis method considering the upper bound analysis and the lower bound analysis independently by using nonlinear programming. However, these works were based on merely the two-dimensional analysis. The rigid-plastic finite element method
2.3 Effects of groundwater on the tunnel stability

(RPFEM) based on upper bound theorem, developed by Tamura (Tamura et al., 1984; Tamura et al., 1987; Tamura, 1990), is one of the effective approaches for evaluation of the bearing capacity and the stability of slopes and tunnel faces. Tamura et al. (Tamura et al., 1999) firstly presented the use of RPFEM to simulate the tunnel face stability problems, mainly in sandy ground. Konishi (Konishi, 2000) studied the stability of tunnel face in the ground of alternate layers (sandy and cohesive soils) by conducting the 3D experimental tests and comparing the test results with RPFEM both in two and three dimensional analysis. He found that RPFEM can simulate the stability of tunnel face in the ground of alternate layers effectively.

2.3 Effects of groundwater on the tunnel stability

The studies on the evaluation of the stability of tunnel face are done by several authors in the last thirty years. But there are very few studies that consider the effects of groundwater on the stability of the tunnel face. Pollet et al. (Pellet et al., 1993) carried out a three-dimensional finite element analysis of groundwater flow and concluded that the head losses are concentrated in the close vicinity of the tunnel face i.e., seepage forces acting on the tunnel face exists. They also conducted several three-dimensional tests to investigate the mechanical behavior and stability of the microtunnel face in soft ground. The observed supporting pressures at tunnel face were compared with the supporting pressures calculated from the elasto-plastic finite element method in plane strain condition. Buhan et al. (Buhan et al., 1999) determined the tunnel pressure that stabilize the tunnel face by comparing the equilibrium of the conical volume, suggested by Leca and Dormieux (Leca and Dormieux, 1990), of soil subjected to the combined effect of gravity and seepage forces and the Mohr-Coulomb yield criterion. The seepage forces were determined by the 3D seepage finite element analysis. Lee and Nam (Lee and Nam, 2001) calculated the minimum supporting pressure at the tunnel face by summing up the upper bound solution proposed by Leca and Dormieux (Leca and Dormieux, 1990), based on effective stress analysis, and seepage forces acting on the tunnel face, calculated from the 3D seepage finite element analysis. Also, Lee et al. (Lee et al., 2003) verified the calculated results of the numerical analysis by comparing them with the observed values of the model tests. They found the very good agreement between the two approaches. Droniuc et al. (Droniuc et al., 2004) studied the effect of groundwater on the stability of tunnel
face in cohesive-frictional soil by using the regularized kinematical approach of limit analysis, developed at LCPC (Laboratoire Central des Ponts et Chausées), formulated by finite element method. The results calculated from the numerical analysis were agreed well with those obtained from the method proposed by Lee and Nam (Lee and Nam, 2001).

Konishi and Tamura (Konishi and Tamura, 2002; Konishi et al., 2003) proposed the method to include the groundwater into the rigid-plastic finite element method and carried out some fundamental studies on the effect of the groundwater on the simple geotechnical problems and simple tunnel problems. But, this method can analyze only the problem with non-dilatant condition in soil (non-associated flow rule).

In this dissertation, the new method which can analyze both associated and non-associated flow rules, to include the groundwater into the RPFEM is proposed and then some fundamental studies of the effect of groundwater on the tunnel stability will be carried out.
Part I

Shield tunneling method
Chapter 3

Rigid segment spring model

3.1 Introduction

Generally, the shield tunnel lining is composed of the primary lining and secondary lining. Particularly, the primary lining is often formed by assembling the box type or the flat type factory-manufactured segments which connect to each other by bolt joints in the directions of tunnel axis and tunnel cross section. There are several methods to design the shield tunnel segment such as the usual calculation method, the beam-spring model calculation method and the finite element method. In this chapter, new alternative design method, so called rigid segment spring model, is proposed and described in detail. Not only the conventional circular ring but also the rectangular ring of the Kyoto Subway Tozai line(Rokujizo-Kita Construction Section Project), can be analyzed by this model. Therefore, the methodology to analyze the circular rings by this model is explained as well as the rectangular rings.

In this chapter, the description of the model is explained first. Then, some main governing equations including the matrix formulation of the model are described. Moreover, for more reality in the application of the model to the actual rings, the algorithm for considering the nonlinear joint-spring model is described.
Chapter 3. Rigid segment spring model

Figure 3.1: Staggered joint arrangement

Figure 3.2: Two types of the arrangement of segment joints

3.2 Description of the model

The rigid segment spring model is the model such that segments are perfectly rigid and connected with three dimensional paired-springs at all joints to which the total deformation of the tunnel is attributed.

The model is based on the three dimensional analysis of the shield tunnel with the staggered joint arrangement shown in Fig. 3.1. The so called staggered joint arrangement can be formed by combining the two types of the arrangement of segment joints, as shown in Fig. 3.2, alternately. This model has some advantages as follows: (Tamura et al., 2000)

- For three dimensional analysis, there are a few unknowns (six unknowns per one segment i.e., three translational components of the segment center and three rotational components of the segment);

- Able to express the mechanical behavior of the longitudinal direction of tunnel conveniently;

- Able to show the effect of the staggered joint arrangement and joint displacement easily;

- Able to determine the offset length between the adjacent segments which cannot be determined by any other methods.

At joints, there are two types of joint: segment joint and ring joint. At each segment joint, two paired-segment springs connect the two adjacent segments together to form the segmental ring, as illustrated in Fig. 3.3. At ring joint, as shown in Fig. 3.4, four paired-ring springs connect the adjacent segmental rings to form the primary lining of tunnel. Either segment spring or ring spring is three dimensional spring composed of one normal spring and two shear springs. Models of normal and shear springs are illustrated in Fig. 3.5. In addition,
3.3. *Coordinate systems*

**Figure 3.3:** Description of segment springs

**Figure 3.4:** Description of ring springs

**Figure 3.5:** Normal and shear spring models

For each segment, three dimensional ground reaction springs connect six external nodes to the surrounding ground and six internal nodes to some supports inside the tunnel as shown in Fig. 3.6(a). The three components of each three dimensional ground reaction spring are all normal spring as illustrated in Fig. 3.6(b).

**Figure 3.6:** Description of ground reaction springs: (a) Positions of springs, (b) three directions of each spring
3.3 Coordinate systems

3.3.1 Local and global coordinates

There are two coordinate systems in this analysis: local coordinate system and global coordinate system. For coordinate systems of simple circular ring analysis, all of segments have totally the same size thus, local coordinate system of each segment is also the same as illustrate in Fig. 3.7(a) and global coordinate system is illustrated in Fig. 3.7(b). Segment \( j \) of ring \( i \) is defined as segment \((i, j)\). The segment number of each ring is defined increasingly clockwise as shown in Fig. 3.2. Ring number is defined increasingly in the positive direction of tunnel axis (global \( Y \) axis) as shown in Fig. 3.7(b). If total number of rings is more than two, the arrangement of segment numbers in each ring is given by Fig. 3.8.

For the rectangular ring analysis, dimension of each segment is different, thus it is necessary to define specifically the local coordinate systems. The local coordinate systems of segments in ring 1 and ring 2 are illustrated in Figs. 3.9(a) and 3.9(b), respectively. Global coordinate system of rectangular ring case is shown in Fig. 3.10. It should be noted that local coordinate system of each segment is defined on the concept that \( Z \)-axis of segment \((i, j)\) is parallel to edge of segment of neighboring rings as shown in Fig. 3.11. As the same as in the circular ring analysis, if total number of rings is more than two, the setting of segment numbers surrounding segment no. \((i, j)\) used in programming is illustrated in Fig. 3.8.
3.3. Coordinate systems

![Diagram]

**Figure 3.8**: Arrangement of numbers of segments in analysis if total number of rings > 2

### 3.3.2 Coordinate transformation

Since this analysis is carried out by considering both of global coordinate system and local coordinate system of each segment, the transformation between these coordinate systems is described here.

**Relationship between Local coordinate systems**

Firstly, consider the relationship between local coordinates of segment \((i, j)\) and its neighboring segments. As shown in Fig. 3.12, the \(X\)-axis and \(Z\)-axis of segments \((i - 1, j)\) and \((i + 1, j - 1)\) can be obtained by rotating \(X\) and \(Z\) axis of segment \((i, j)\) counterclockwise around \(Y\)-axis through the angle \(\theta_{(i,j)}^{(i-1,j)}\) and \(\theta_{(i,j)}^{(i+1,j-1)}\), respectively. On the other hand, \(X\)-axis and \(Z\)-axis of segments \((i - 1, j + 1)\) and \((i + 1, j)\) can be obtained by rotating \(X\) and \(Z\) clockwise through the angle \(\theta_{(i,j)}^{(i-1,j+1)}\) and \(\theta_{(i,j)}^{(i+1,j)}\), respectively.

**Transformation matrix for two vectors**

Firstly, rotation of coordinates in this transformation is supposed as:

Consider two sets of rectangular Cartesian frames of reference \(o - x_1x_3\) and new \(o' - x'_1x'_3\) on a plane. If origin remains fixed, and the new axes \(o' - x'_1x'_3\) are obtained by rotating \(ox_1\) and \(ox_3\), around \(ox_2\) axis, through an angle \(\theta\), as shown in Fig 3.13. If the rotation is done in the **clockwise direction**, and point \(P\) has coordinates \((x_1, x_1, x_3)\) and \((x'_1, x'_2, x'_3)\) with respect to
Chapter 3. Rigid segment spring model

Note:  
1. segment no. i  
2. node no. 2  
3. x-axis through that point

Figure 3.9: Global coordinates of rectangular tunnel

Figure 3.10: Global coordinates of rectangular tunnel
3.3. Coordinate systems

Figure 3.11: Concept of local coordinate system setting

The old and new frames of reference, respectively, then

\[
\begin{align*}
\begin{bmatrix}
    x'_1 \\
    x'_2 \\
    x'_3 \\
\end{bmatrix} &=
\begin{bmatrix}
    \cos \theta & 0 & -\sin \theta \\
    0 & 1 & 0 \\
    \sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} \\
&= \begin{bmatrix}
    x'_1 \\
    x'_2 \\
    x'_3
\end{bmatrix}
\end{align*}
\]

or, in the another form:

\[
x' = Rx.
\]

It can be seen that we can transform \( \mathbf{x} \) to \( \mathbf{x}' \) by using an orthogonal tensor \( \mathbf{R} \). Similarly, displacement vector and force vector can be transformed to the new one by using \( \mathbf{R} \) as follows:

\[
\begin{align*}
\begin{bmatrix}
    u'_1 \\
    u'_2 \\
    u'_3
\end{bmatrix} &= \begin{bmatrix}
    \cos \theta & 0 & \sin \theta \\
    0 & 1 & 0 \\
    -\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3
\end{bmatrix} \\
&= \begin{bmatrix}
    u'_1 \\
    u'_2 \\
    u'_3
\end{bmatrix}
\end{align*}
\]  

\[
\begin{align*}
\begin{bmatrix}
    f'_1 \\
    f'_2 \\
    f'_3
\end{bmatrix} &= \begin{bmatrix}
    \cos \theta & 0 & \sin \theta \\
    0 & 1 & 0 \\
    -\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{bmatrix} \\
&= \begin{bmatrix}
    f'_1 \\
    f'_2 \\
    f'_3
\end{bmatrix}
\end{align*}
\]

Inversely, if the rotation is done in the counterclockwise direction, we can get

\[
\begin{align*}
\begin{bmatrix}
    x'_1 \\
    x'_2 \\
    x'_3
\end{bmatrix} &=
\begin{bmatrix}
    \cos \theta & 0 & \sin \theta \\
    0 & 1 & 0 \\
    -\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} \\
&= \begin{bmatrix}
    x'_1 \\
    x'_2 \\
    x'_3
\end{bmatrix}
\end{align*}
\]
Chapter 3. Rigid segment spring model

Figure 3.13: Rotation of coordinate axis: counterclockwise and clockwise direction

or, in the another form:

\[ x' = R^{-1}x. \] (3.6)

Since \( R \) is orthogonal tensor which has property that

\[ R^{-1} = R^T, \] (3.7)

\( R^T \) can be used instead of \( R^{-1} \). In this sense, \( R \) and \( R^T \) can be used to transform between segment coordinates. For example, as seen in Fig 3.12, in order to transform the local coordinates of segment \((i - 1, j)\) to the local coordinates of segment \((i, j)\) the orthogonal tensor \( R_1 \) can be used as a transformation tensor as shown symbolically as:

\[ (i - 1, j) \xrightarrow{R_1} (i, j) \] (3.8)

where, \( R_1 \) is defined as

\[
R_1 = \begin{pmatrix}
\cos \left( \theta_{(i-1,j)} \right) & 0 & \sin \left( \theta_{(i-1,j)} \right) \\
0 & 1 & 0 \\
-\sin \left( \theta_{(i,j)} \right) & 0 & \cos \left( \theta_{(i,j)} \right)
\end{pmatrix}.
\] (3.9)

3.3.3 Matrix of spring constants

As mentioned previously, all of springs are 3-dimensional springs and then directions of spring constants are supposed as \( k_1 \), \( k_2 \) and \( k_3 \) with respect to each direction, respectively. From
3.3. Coordinate systems

Hooke’s laws:

\[ f_i = k_i u_i \quad (3.10) \]

where, \( f_i \) is force in direction \( i \) and \( u_i \) is displacement of spring in direction \( i \).

Therefore, the relation between spring force vector and spring displacement can be expressed as follows:

\[ f = K u \quad (3.11) \]

or, by showing their components;

\[
\begin{align*}
\begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{bmatrix} &=
\begin{bmatrix}
  k_1 & 0 & 0 \\
  0 & k_2 & 0 \\
  0 & 0 & k_3
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix}.
\end{align*}
\]

(3.12)

According to Eq. (3.11), \( K \) is called spring constant matrix.

Now, suppose that force vector \( f \) is transformed to \( f' \) of new frame of reference by rotating in clockwise direction through an angle \( \theta_1 \) and; also, displacement vector \( u \) is transformed to \( u'' \) of new frame of reference by rotating in clockwise direction through an angle \( \theta_2 \), as illustrated in Fig 3.14. Thus, from Eqs. (3.4) and (3.11) by using different matrix \( R \), we can obtain

\[ f' = R_1 f \quad \text{or} \quad f = R_1^T f' \quad (3.13) \]

\[ u'' = R_2 u \quad \text{or} \quad u = R_2^T u''. \quad (3.14) \]
After the transformation, suppose that we will get the following equation:

\[ f' = K^f u''. \]  

(3.15)

Combine Eq. (3.11) and Eq. (3.14) then substitute it into Eq. (3.13); and compare the result to Eq. (3.15), then

\[ K^f = R_1 K R_2^T \]  

(3.16)

is obtained.

### 3.4 Spring position setting

The position of each spring will be explained in this section. Each type of set of springs connects to different set of node numbers as described below.
### Figure 3.18: Numbers of nodes connected with each type of springs in rectangular ring 1

<table>
<thead>
<tr>
<th>seg No.</th>
<th>ground reaction</th>
<th>segment joint</th>
<th>ring joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td><img src="image2.png" alt="Diagram 2" /></td>
<td><img src="image3.png" alt="Diagram 3" /></td>
</tr>
<tr>
<td>2 (Key segment)</td>
<td><img src="image4.png" alt="Diagram 4" /></td>
<td><img src="image5.png" alt="Diagram 5" /></td>
<td><img src="image6.png" alt="Diagram 6" /></td>
</tr>
<tr>
<td>3, 6</td>
<td><img src="image7.png" alt="Diagram 7" /></td>
<td><img src="image8.png" alt="Diagram 8" /></td>
<td><img src="image9.png" alt="Diagram 9" /></td>
</tr>
<tr>
<td>4, 7</td>
<td><img src="image10.png" alt="Diagram 10" /></td>
<td><img src="image11.png" alt="Diagram 11" /></td>
<td><img src="image12.png" alt="Diagram 12" /></td>
</tr>
<tr>
<td>5</td>
<td><img src="image13.png" alt="Diagram 13" /></td>
<td><img src="image14.png" alt="Diagram 14" /></td>
<td><img src="image15.png" alt="Diagram 15" /></td>
</tr>
</tbody>
</table>
### Table 3.1: Numbers of nodes connected with each type of springs in rectangular ring 2

<table>
<thead>
<tr>
<th>seg No.</th>
<th>ground reaction</th>
<th>segment joint</th>
<th>ring joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>2, 5</td>
<td><img src="image4" alt="Diagram" /></td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
<tr>
<td>3, 6</td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
<td><img src="image9" alt="Diagram" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
</tr>
<tr>
<td>7 (Key segment)</td>
<td><img src="image13" alt="Diagram" /></td>
<td><img src="image14" alt="Diagram" /></td>
<td><img src="image15" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Figure 3.19**: Numbers of nodes connected with each type of springs in rectangular ring 2.
3.4. Spring position setting

3.4.1 Ground reaction spring

Ground reaction springs connect each segment to surrounding support at the nodes no.1 to no. 12, as illustrated in Fig. 3.15 for circular ring analysis or Fig. 3.18 and 3.19 for rectangular ring analysis. At each connecting node, ground reaction spring has 3 directions; $k_1$, $k_2$ and $k_3$, as shown in Fig. 3.6(b).

Ground reaction spring constant matrices of node no. $k = 1 \sim 12$ of segment $(i, j)$ are denoted by $iK_j^1 \sim iK_j^{12}$.

3.4.2 Segment joint spring

Segment springs are placed at the nodes no.13 to no. 20, as illustrated in Fig. 3.16 for circular ring analysis or Figs. 3.18 and 3.19 for rectangular ring analysis. At each connecting node, segment spring has 3 directions: one normal spring constant; two shear spring constants.

Segment spring constant matrices of node $k = 17 \sim 20$ of segment $(i, j)$ are denoted by $iK_j^{13} \sim iK_j^{16}$. These four segment springs connect node $k = 17 \sim 20$ of segment $(i, j)$ to node no. $k = 13 \sim 16$ of segment $(i, j + 1)$, individually.

3.4.3 Ring joint spring

Ring springs are placed at the nodes no.21 to no. 36, as illustrated in Fig. 3.17 for circular ring analysis or Figs. 3.18 and 3.19 for rectangular ring analysis. At each connecting node, similar to segment spring, ring spring has 3 directions: one normal spring constant and two shear spring constants.

Segment spring constant matrices of node $k = 29 \sim 36$ of segment $(i, j)$ are denoted by $iK_j^{17} \sim iK_j^{24}$. These eight ring springs are divided into 2 parts equally; first four ring springs connect node $k = 29 \sim 32$ of segment $(i, j)$ to node $k = 25 \sim 28$ of segment $(i + 1, j - 1)$ and the another four ring springs connect node $k = 33 \sim 36$ of segment $(i, j)$ to node $k = 21 \sim 24$ of segment $(i + 1, j)$.
3.5 Reaction force of spring

The equilibrium equation of each segment is used corresponding in local coordinate system of segment. In any segment \((i, j)\), the reaction forces of springs at all nodes are brought into the equilibrium equation and, finally, the unknown vector \(x\) will be obtained by using Eq. (A.38).

Here, some symbols used in this section are explained. Firstly, the definitions of subscripts and superscripts of symbol \(i, j, l, A, k, m, k\) is shown as follows:

- Left superscript \((i, j)\) means quantity \(A\) is considered relatively to the local coordinate of segment \((i, j)\);
- Left subscript \(l\) means quantity \(A\) is the quantity of ring no. \(l\);
- Right subscript \(m\) means quantity \(A\) is the quantity of segment no. \(m\);
- Right superscript \(k\) means quantity \(A\) is the quantity of node no. \(k\).

Conclusively, \(i, j, l, A, m, k\) means the quantity \(A\) of node no. \(k\) of segment \((l, m)\) relative to the local coordinates of segment \((i, j)\). Next, \(R_{(i,j)}^{(m,n)}\) and \(\theta_{(i,j)}^{(m,n)}\) are the orthogonal tensor and the difference of angles between local coordinates of segment \((i, j)\) and local coordinates of segment \((m, n)\), respectively. \(R_{(i,j)}^{(m,n)}\) can be represented corresponding to \(\theta_{(i,j)}^{(m,n)}\) as

\[
R_{(i,j)}^{(m,n)} = \begin{pmatrix}
\cos \theta_{(i,j)}^{(m,n)} & 0 & -\sin \theta_{(i,j)}^{(m,n)} \\
0 & 1 & 0 \\
\sin \theta_{(i,j)}^{(m,n)} & 0 & \cos \theta_{(i,j)}^{(m,n)}
\end{pmatrix}.
\]

(3.17)

3.5.1 Ground reaction spring

\(k = 1, 4, 7, 10\)

According to Figs. 3.15 and 3.8, 3 directions of ground reaction spring are parallel to 3 directions of \((i + 1, j - 1)\) coordinate axes. Thus, by referring to coordinate system \((i + 1, j - 1)\), Eq. (3.11) can be written as

\[
(i+1,j-1) f^k_i = -i K_j^k (i+1,j-1) u^k_j.
\]

(3.18)
3.5. Reaction force of spring

Next, the transformation \((i + 1, j - 1) \rightarrow (i, j)\) will be carried out. Suppose \(iQ_j^k\) be tensor \(Q\) of Eq. (A.38), Eq. (A.38) can be written in this form:

\[
^{(i,j)}i u_j^k = iQ_j^k i x_j.
\] (3.19)

By using the transformation \((i + 1, j - 1) \rightarrow (i, j)\) and Eq. (3.18),

\[
^{(i,j)}i f_j^k = - (R_{(i+1,j-1)}^{(i,j)}) iK_j^k (R_{(i+1,j-1)}^{(i,j)})^T iQ_j^k i x_j
\] (3.20)

where,

\[
iD_j^k = - (R_{(i+1,j-1)}^{(i,j)}) iK_j^k (R_{(i+1,j-1)}^{(i,j)})^T iQ_j^k
\] (3.21)

is obtained.

\(k = 2, 5, 8, 11\)

Three directions of ground reaction spring are parallel to 3 directions of \((i, j)\) coordinate axes. Thus, by referring to coordinate system \((i, j)\), Eq. (3.11) can be written as

\[
^{(i,j)}i f_j^k = - iK_j^k (^{(i,j)}i u_j^k).
\] (3.22)

Suppose \(iQ_j^k\) be tensor \(Q\) of Eq. (A.38), Eq. (A.38) can be written in the following form:

\[
^{(i,j)}i u_j^k = iQ_j^k i x_j.
\] (3.23)

Then, Eq. (3.22) can be rewritten as

\[
^{(i,j)}i f_j^k = - iK_j^k iQ_j^k i x_j = - iD_j^k i x_j
\] (3.24)

where,

\[
iD_j^k = iK_j^k iQ_j^k.
\] (3.25)

\(k = 3, 6, 9, 12\)
Three directions of ground reaction spring are parallel to 3 directions of \((i + 1, j)\) coordinate axes. Thus, by referring to coordinate system \((i + 1, j)\), Eq. (3.11) can be written as

\[
(i+1,j) f^k_j = - i K^k_j (i+1,j) u^k_j. \tag{3.26}
\]

Next, the transformation \((i + 1, j) \rightarrow (i, j)\) will be carried out. Suppose \(i Q^k_j\) be tensor \(Q\) of Eq. (A.38), Eq. (A.38) can be written in the form:

\[
(i,j) u^k_j = i Q^k_j i x^k_j. \tag{3.27}
\]

By using the transformation \((i + 1, j) \rightarrow (i, j)\) and Eq. (3.26), the following equation is obtained.

\[
(i,j) f^k_j = -(R^k_j (R^k_{i+1,j})^T i K^k_j (R^k_{i+1,j}) i Q^k_j i x^k_j
= - i D^k_j i x^k_j \tag{3.28}
\]

where,

\[
i D^k_j = -(R^k_j (R^k_{i+1,j})^T i K^k_j (R^k_{i+1,j}) i Q^k_j. \tag{3.29}
\]

### 3.5.2 Segment spring

In one segment, there are two edges which connected by segment springs to previous and next segments, respectively. Thus following explanation will be divided into two cases:

**Connecting to the previous segment \((k = 13, 14, 15, 16)\)**

According to Figs. 3.16 and 3.8, node \((i, j, 13) \sim (i, j, 16)\) are connected by springs no. \((i, j - 1, 13) \sim (i, j - 1, 16)\) to node \((i, j - 1, 17) \sim (i, j - 1, 20)\). 3 directions of segment spring are parallel to 3 directions of \((i + 1, j - 1)\) coordinate axes. Thus, by respecting to coordinate system \((i + 1, j - 1)\), Eq. (3.11) can be written as

\[
(i+1,j-1) f^k_j = i K^k_{j-1} \left[ (i+1,j-1) u^k_{j-1} - (i+1,j-1) u^k_j \right]. \tag{3.30}
\]
3.5. Reaction force of spring

For \( (i+1,j-1)^i f_j^k \) and \( (i+1,j-1)^i u_j^k \), the transformation \((i + 1, j - 1) \) is carried out and, for \( (i+1,j-1)^i u_j^{k+4} \), the transformation \((i + 1, j - 1) \) is carried out. Then, Eq. (3.30) can be expressed in following form:

\[
(i,j)^i f_j^k = (R_{(i+1,j-1)}^{i,j})^i K_j^{k-1} \left( (R_{(i+1,j-1)}^{i,j-1})^i u_j^{k-4} - (R_{(i+1,j-1)}^{i,j})^i u_j^k \right). \tag{3.31}
\]

Next, Eq. (A.38) can be written in the following forms:

\[
(i,j)^i u_j^{k} = i Q_j^{k} x_j, \tag{3.32}
\]
\[
(i,j)^i u_j^{k+4} = i Q_j^{k+4} x_j. \tag{3.33}
\]

Substitute Eqs. (3.32) and (3.33) into Eq. (3.31), obtain

\[
(i,j)^i f_j^k = (R_{(i+1,j-1)}^{i,j})^i K_j^{k-1} \left( (R_{(i+1,j-1)}^{i,j-1})^i u_j^{k-4} - (R_{(i+1,j-1)}^{i,j})^i u_j^k \right)
- (R_{(i+1,j-1)}^{i,j})^i K_j^{k-1} (R_{(i+1,j-1)}^{i,j})^i Q_j^{k} x_j
= i C_j^{k} x_j - i D_j^{k} x_j \tag{3.34}
\]

where,

\[
i C_j^{k} = (R_{(i+1,j-1)}^{i,j})^i K_j^{k-1} (R_{(i+1,j-1)}^{i,j-1})^i Q_j^{k+4}, \tag{3.35}
\]
\[
i D_j^{k} = (R_{(i+1,j-1)}^{i,j})^i K_j^{k-1} (R_{(i+1,j-1)}^{i,j})^i Q_j^{k}, \tag{3.36}
\]

Connecting to the next segment \((k = 17, 18, 19, 20)\)

According to Figs. 3.16 and 3.8, node \((i, j, 17) \sim (i, j, 20)\) are connected by springs no. \((i, j, 13) \sim (i, j, 16)\) to node \((i, j + 1, 13) \sim (i, j + 1, 16)\). 3 directions of segment spring are parallel to 3 directions of \((i + 1, j)\) coordinate axes. Thus, by referring to coordinate system \((i + 1, j)\), Eq. (3.11) can be written as

\[
(i+1,j)^i f_j^k = i K_j^{k-4} \left[ (i+1,j)^i u_j^{k-4} - (i+1,j)^i u_j^k \right]. \tag{3.37}
\]

For \( (i+1,j)^i f_j^k \) and \( (i+1,j)^i u_j^k \), the transformation \((i + 1, j) \) is carried out and,
for \( \mathbf{u}^{k-4}_{j+1} \), the transformation \((i+1,j) \rightarrow (i,j+1)\) is carried out. Then, Eq. (3.37) can be expressed in following form:

\[
\begin{align*}
\mathbf{f}^k_i &= (\mathbf{R}^{(i,j)}_{(i+1,j)})^T \mathbf{K}^{k-4}_j \left[ (\mathbf{R}^{(i,j+1)}_{(i+1,j)})^T \mathbf{u}^{k-4}_{j+1} - (\mathbf{R}^{(i,j)}_{(i+1,j)})^{(i,j)} \mathbf{u}^k_i \right].
\end{align*}
\] (3.38)

Next, Eq. (A.38) can be written in the following forms:

\[
\begin{align*}
\mathbf{u}^{k-4}_{j+1} &= i \mathbf{Q}^{k-4}_j \mathbf{x}_{j+1} \quad (3.39) \\
\mathbf{u}^k_j &= i \mathbf{Q}^k_j \mathbf{x}_j \quad (3.40)
\end{align*}
\]

Substitute Eqs. (3.39) and (3.40) into Eq. (3.38), then we obtain

\[
\begin{align*}
\mathbf{f}^k_i &= - (\mathbf{R}^{(i,j)}_{(i+1,j)})^T \mathbf{K}^{k-4}_j (\mathbf{R}^{(i,j+1)}_{(i+1,j)})^T i \mathbf{Q}^k_j \mathbf{x}_j \\
&\quad + (\mathbf{R}^{(i,j)}_{(i+1,j)})^T \mathbf{K}^{k-4}_j (\mathbf{R}^{(i,j+1)}_{(i+1,j)}) i \mathbf{Q}^{k-4}_{j+1} \mathbf{x}_{j+1} \\
&= - \mathbf{D}^k_j \mathbf{x}_j + \mathbf{E}^k_j \mathbf{x}_{j+1}
\end{align*}
\] (3.41)

where,

\[
\begin{align*}
\mathbf{D}^k_j &= (\mathbf{R}^{(i,j)}_{(i+1,j)})^T \mathbf{K}^{k-4}_j (\mathbf{R}^{(i,j+1)}_{(i+1,j)})^T \mathbf{Q}_j^k \quad (3.42) \\
\mathbf{E}^k_j &= (\mathbf{R}^{(i,j)}_{(i+1,j)})^T \mathbf{K}^{k-4}_j (\mathbf{R}^{(i,j+1)}_{(i+1,j)}) i \mathbf{Q}^{k-4}_{j+1} \quad (3.43)
\end{align*}
\]

### 3.5.3 Ring spring

At the ring spring connections, we can divided them into 4 parts: \( k = 21 \sim 24 \); \( k = 25 \sim 28 \); \( k = 29 \sim 32 \) and \( k = 33 \sim 36 \) (see Fig. 3.17). Each part will be discussed as follows:

\( k = 21, 22, 23, 24 \)

According to Figs. 3.17 and 3.8, node \((i,j,21) \sim (i,j,24)\) are connected by springs no. \((i-1,j,21) \sim (i-1,j,24)\) to node \((i-1,j,33) \sim (i-1,j,36)\). By the assumption that isotropic shear spring, spring force acting at node \((i,j,k)\), with respect to coordinate system
3.5. Reaction force of spring

\((i, j)\), can be expressed as

\[
\begin{align*}
    (i, j) f^k_j &= i-1 K^{k}_{j} \left[ (i, j) u^{k+12}_{j+1} - (i, j) u^k_j \right].
\end{align*}
\] (3.44)

For \((i, j) u^{k+12}_{j+1}\), the transformation \((i - 1, j) \rightarrow (i, j)\) is carried out. Then, Eq. (3.44) can be expressed in following form:

\[
\begin{align*}
    (i, j) f^k_j &= i-1 K^{k}_{j} \left[ (R(i, j))_i-1(i, j) u^{k+12}_{j+1} - (i, j) u^k_j \right].
\end{align*}
\] (3.45)

Next, Eq. (A.38) can be written in the following forms:

\[
\begin{align*}
    (i-1, j) u^{k+12}_{j+1} &= i-1 Q^{k+12}_{j+1} i-1 x_j, \quad (i, j) u^k_j = i Q^k_j x_j.
\end{align*}
\] (3.46) (3.47)

Substitute Eqs. (3.46) and (3.47) into Eq. (3.45), then we obtain

\[
\begin{align*}
    (i, j) f^k_j &= i-1 K^{k}_{j} \left[ (R(i, j))_i-1(i-1, j) u^{k+12}_{j+1} - i-1 K^{k}_{j_1} Q^k_j x_j \right] \\
    &= i A^k_j x_j - i D^k_j x_j
\end{align*}
\] (3.48)

where,

\[
\begin{align*}
    A^k_j &= i-1 K^{k}_{j} \left( R(i, j) \right)_i-1(i-1, j) u^{k+12}_{j+1} \\
    D^k_j &= i-1 K^{k}_{j} Q^k_j.
\end{align*}
\] (3.49) (3.50)

\(k = 25, 26, 27, 28\)

According to Figs. 3.17 and 3.8, node \((i, j, 25) \sim (i, j, 28)\) are connected by springs no. \((i - 1, j + 1, 17) \sim (i - 1, j + 1, 20)\) to node \((i - 1, j + 1, 29) \sim (i - 1, j + 1, 32)\). By the assumption that isotropic shear spring, spring force acting at node \((i, j, k)\), with respect to coordinate system \((i, j)\), can be expressed as

\[
\begin{align*}
    (i, j) f^k_j &= i-1 K^{k}_{j+1} \left[ (i, j) u^{k+4}_{j+1} - (i, j) u^k_j \right].
\end{align*}
\] (3.51)
For \((i,j)_{k}^{k+4}\), the transformation \((i - 1, j + 1) \rightarrow (i, j)\) is carried out. Then, Eq. (3.51) can be expressed in following form:

\[
(i,j)_{k}^{k} f_{j}^{k} = (i-1) K_{j+1}^{k-8}\left[ (R_{(i-1, j+1)}^{(i,j)}) T (i-1, j+1) u_{j+1}^{k-4} - (i,j) u_{j}^{k} \right].
\] (3.52)

Next, Eq. (A.38) can be written in the following forms:

\[
(i_{-1}, j+1)_{k}^{k+4} = (i-1) Q_{j+1}^{k+4} - (i,j) k x_{j+1}.
\] (3.53)

\[
(i,j)_{k}^{k} u_{j}^{k} = (i,j) k x_{j}.
\] (3.54)

Substitute Eqs. (3.53) and (3.54) into Eq. (3.52), then we obtain

\[
(i,j)_{k}^{k} f_{j}^{k} = (i-1) K_{j+1}^{k-8}(R_{(i-1, j+1)}^{(i,j)}) T (i-1, j+1) Q_{j+1}^{k+4} - (i-1) K_{j+1}^{k-8} Q_{j}^{k} x_{j}.
\] (3.55)

where,

\[
i B_{j}^{k} = (i-1) K_{j+1}^{k-8} (R_{(i-1, j+1)}^{(i,j)}) T (i-1, j+1) Q_{j+1}^{k+4}
\] (3.56)

\[
i D_{j}^{k} = (i-1) K_{j+1}^{k-8} Q_{j}^{k}.
\] (3.57)

\[k = 29, 30, 31, 32\]

According to Figs. 3.17 and 3.8, node \((i, j, 29) \sim (i, j, 32)\) are connected by springs no. \((i, j, 17) \sim (i, j, 20)\) to node \((i + 1, j - 1, 25) \sim (i + 1, j - 1, 28)\). By the assumption that isotropic shear spring, spring force acting at node \((i, j, k)\), with respect to coordinate system \((i, j)\), can be expressed as

\[
(i,j)_{k}^{k} f_{j}^{k} = (i+1) K_{j-1}^{k-12}\left[ (i,j)_{k-4}^{k-4} u_{j-1}^{k-4} - (i,j) u_{j}^{k} \right].
\] (3.58)

For \((i,j)_{k}^{k-4}\), the transformation \((i + 1, j - 1) \rightarrow (i, j)\) is carried out. Then, Eq. (3.58) can be expressed in following form:

\[
(i,j)_{k}^{k} f_{j}^{k} = (i+1) K_{j-1}^{k-12}\left[ (R_{(i+1, j-1)}^{(i,j)}) (i+1, j-1) u_{j-1}^{k-4} - (i,j) u_{j}^{k} \right].
\] (3.59)
3.5. Reaction force of spring

Next, Eq. (A.38) can be written in the following forms:

\[
(i+1, j-1)_{i+1} u_{j-1}^{k-4} = i+1 Q_{j-1}^{k-4} (i+1 x_{j-1}) \quad (3.60)
\]

\[
(i, j) i u_j^k = i Q_j^k x_j. \quad (3.61)
\]

Substitute Eqs. (3.60) and (3.61) into Eq. (3.59), then we obtain

\[
(i, j) f_j^k = -i K_j^{k-12} Q_j^k x_j + i K_j^{k-12} (R^{(i, j)}_{(i+1, j-1)}) \cdot i+1 Q_j^{k-4} (i+1 x_{j-1}) \quad (3.62)
\]

where,

\[
i D_j^k = i K_j^{k-12} Q_j^k \quad (3.63)
\]

\[
i F_j^k = i K_j^{k-12} (R^{(i, j)}_{(i+1, j-1)}) \cdot i+1 Q_j^{k-4}. \quad (3.64)
\]

\[k = 33, 34, 35, 36\]

According to Figs. 3.17 and 3.8, node \((i, j, 33) \sim (i, j, 36)\) are connected by springs no. \((i, j, 21) \sim (i, j, 24)\) to node \((i + 1, j, 21) \sim (i + 1, j, 24)\). By the assumption that isotropic shear spring, spring force acting at node \((i, j, k)\), with respect to coordinate system \((i, j)\), can be expressed as

\[
(i, j) f_j^k = i K_j^{k-12} \left[ (i+1) u_j^{k-12} - (i, j) u_j^k \right]. \quad (3.65)
\]

For \((i, j) u_j^{k-12}\), the transformation \(\left((i + 1, j) \xrightarrow{R^{(i, j)}_{(i+1, j-1)}} (i, j)\right)\) is carried out. Then, Eq. (3.65)) can be expressed in following form:

\[
(i, j) f_j^k = i K_j^{k-12} \left[ (R^{(i, j)}_{(i+1, j-1)})^T (i+1) u_j^{k-12} - (i, j) u_j^k \right]. \quad (3.66)
\]

Next, Eq. (A.38) can be written in the following forms:

\[
(i+1, j) u_{j}^{k-12} = i+1 Q_j^{k-12} i+1 x_j \quad (3.67)
\]

\[
(i, j) u_j^k = i Q_j^k x_j. \quad (3.68)
\]
Chapter 3. Rigid segment spring model

Substitute Eqs. (3.67) and (3.68) into Eq. (3.66), then we obtain

\[ f^{(i,j)}_j = -K^{k-12}_j Q^k_j x_j + K^{k-12}_j (R^{(i,j)}_{(i+1,j)})^T i+1 Q^{k-12}_j i+1 x_j \]

(3.69)

where,

\[ D^k_j = K^{k-12}_j Q^k_j \]

(3.70)

\[ G^k_j = K^{k-12}_j (R^{(i,j)}_{(i+1,j)})^T i+1 Q^{k-12}_j \]

(3.71)

3.6 Equilibrium equations

In this section, the equilibrium equations are introduced as the main conditions for solving problems. Each segment is subject to external forces and spring reaction forces. The equilibrium equations are applied to each segment and its local coordinates by means of force equilibrium equations and moment equilibrium equations. The six equilibrium equations, composed of three equilibrium equations of force and three equilibrium equations of moment, correspond to six unknowns of one segment (\( x \) of Eq. (A.38)). Thus, the number of equations is matched with the number of the unknowns for one segment. First, the equilibrium equations of one segment are built. Secondly, the equilibrium equations of one ring are created by combining the equilibrium equations of each segment together. Finally, the equilibrium equations of total rings are built and then overall system of linear equations is solved. The final solution is the unknown vector \( x \) of Eq. (A.38) of each segment.

3.6.1 Stiffness matrix of segment

Referring to local coordinates of segment, the stiffness matrix of segment is determined by means of the force equilibrium equations and the moment equilibrium equations. Force equilibrium is shown by following equation:

\[ \sum_{k=1}^{36} (i,j) f^{k}_j + \sum_{n} (i,j) p^{n}_j = 0 \]

(3.72)
where, \( k \) is number of node as described in the preceding section. The equilibrium of moment is expressed by following equation:

\[
\sum_{k=1}^{36} \left\{ \mathbf{X}^k \times \left( \mathbf{f}_j^k \right) \right\} + \left\{ \mathbf{X}^n \times \sum_{n} \left( \mathbf{p}_j^n \right) \right\} = 0
\]  

(3.73)

in which, \( \mathbf{X}^k \) and \( \mathbf{X}^n \) are position vector of node \( k \) and position vector of external load \( \left( \mathbf{p}_j^n \right) \) relative to local coordinates of segment \( (i, j) \), respectively. \( n \) is number of external load.

By using Eq. (A.29)), Eq. (3.73) can be written as follows:

\[
\sum_{k=1}^{36} \left\{ \mathbf{W}^k \left( \mathbf{f}_j^k \right) \right\} + \sum_{n} \left\{ \mathbf{W}^n \left( \mathbf{p}_j^n \right) \right\} = 0
\]  

(3.74)

where, \( \mathbf{W}^k \), \( \mathbf{W}^n \) are antisymmetric matrices of \( \mathbf{X}^k \) and \( \mathbf{X}^n \) vectors, respectively. Then, by combining Eq. (3.72) and Eq. (3.74), obtain

\[
\sum_{k=1}^{36} \left\{ \left( i, j \right) \mathbf{f}_j^k \right\} + \sum_{n} \left\{ \left( i, j \right) \mathbf{p}_j^n \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  

(3.75)

According to the previous section, following equation can be obtained.

\[
\sum_{k=1}^{36} \left( i, j \right) \mathbf{f}_j^k = \left( \sum_{k=21}^{24} \mathbf{A}_j^k \right) \sum_{k=25}^{28} \mathbf{B}_j^k \sum_{k=13}^{16} \mathbf{C}_j^k - \sum_{k=1}^{36} \mathbf{D}_j^k \sum_{k=17}^{20} \mathbf{E}_j^k \sum_{k=29}^{32} \mathbf{F}_j^k \sum_{k=33}^{36} \mathbf{G}_j^k \right) \begin{bmatrix} i-1 \mathbf{x}_j \\ i-1 \mathbf{x}_{j+1} \\ i \mathbf{x}_j-1 \\ i \mathbf{x}_{j} \\ i+1 \mathbf{x}_j-1 \\ i+1 \mathbf{x}_{j} \end{bmatrix}
\]  

(3.76)
Suppose that

\[
S_{ij} = \begin{bmatrix}
  x_{i-1 j} \\
  x_{i-1 j+1} \\
  x_{i j-1} \\
  x_{i j} \\
  x_{i j+1} \\
  x_{i+1 j-1} \\
  x_{i+1 j}
\end{bmatrix}, \quad S_i p_j^n = \sum_n \begin{bmatrix} \epsilon_{ij} p_j^n \end{bmatrix} \quad (3.77)
\]

where, \( S_{ij} \) components are unknown vectors of the neighboring segments of segment \((i, j)\) as shown in Fig. 3.8. Then, Eq. (3.75) can be written by following equation:

\[
- S_j K_j S_{ij} x_j + S_i p_j = 0. \quad (3.78)
\]

That is

\[
S_j K_j S_{ij} x_j = S_i p_j \quad (3.79)
\]

in which, \( S_j K_j \) is called stiffness matrix of segment. Stiffness matrix of segment is written by

\[
S_i K_j = \begin{bmatrix}
  S_{ij} K_j & K_j & S_{ij} K_j & S_{ij} K_j & S_{ij} K_j & S_{ij} K_j & S_{ij} K_j
\end{bmatrix}
\]

where

\[
S_{ij} A_j = - \sum_{k=21}^{24} \begin{bmatrix} \epsilon_{ij} k \end{bmatrix}, \quad S_{ij} B_j = - \sum_{k=25}^{28} \begin{bmatrix} \epsilon_{ij} k \end{bmatrix}, \\
S_{ij} C_j = - \sum_{k=13}^{16} \begin{bmatrix} \epsilon_{ij} k \end{bmatrix}, \quad S_{ij} D_j = - \sum_{k=1}^{17} \begin{bmatrix} \epsilon_{ij} k \end{bmatrix}, \\
S_{ij} E_j = - \sum_{k=17}^{20} \begin{bmatrix} \epsilon_{ij} k \end{bmatrix}, \quad S_{ij} F_j = - \sum_{k=29}^{32} \begin{bmatrix} \epsilon_{ij} k \end{bmatrix}, \\
S_{ij} G_j = - \sum_{k=33}^{36} \begin{bmatrix} \epsilon_{ij} k \end{bmatrix}.
\]

\[
(3.81)
\]
3.6. Equilibrium equations

3.6.2 Stiffness matrix of ring

Following shows the equilibrium equations of ring in which built from the stiffness matrices of each segment in that ring. Suppose \( N_s \) is the total numbers of segments in that ring. Next, the equilibrium equations of ring is shown as follows:

\[
\begin{pmatrix}
R_{1i}^r K & R_{2i}^r K & R_{3i}^r K
\end{pmatrix}
\begin{pmatrix}
R_{i-1} x \\
R_i x \\
R_{i+1} x
\end{pmatrix}
= 
\begin{pmatrix}
R p
\end{pmatrix}
\]

(3.82)

where

\[
R_{1i}^r K = 
\begin{pmatrix}
S_{A1}^i K_1 & S_{B1}^i K_1 \\
S_{A2}^i K_2 & S_{B1}^i K_2 \\
& & & \ddots & \ddots \\
& & & & S_{B(N_s-2)}^i K_{N_s-2} \\
&S_{A(N_s-1)}^i K_{N_s-1} & S_{B(N_s-1)}^i K_{N_s-1} & & & S_{A(N_s)}^i K_{N_s}
\end{pmatrix}
\]

(3.83)

\[
R_{2i}^r K = 
\begin{pmatrix}
S_{D1}^i K_1 & S_{E1}^i K_1 \\
S_{C1}^i K_2 & S_{D1}^i K_2 & S_{E1}^i K_2 \\
& & & \ddots & \ddots \\
& & & & S_{E(N_s-2)}^i K_{N_s-2} \\
&S_{C(N_s-1)}^i K_{N_s-1} & S_{D(N_s-1)}^i K_{N_s-1} & & & S_{C(N_s)}^i K_{N_s}
\end{pmatrix}
\]

(3.84)

\[
R_{3i}^r K = 
\begin{pmatrix}
S_{G1}^i K_1 \\
S_{F1}^i K_2 & S_{G1}^i K_2 \\
& & & \ddots & \ddots \\
& & & & S_{G(N_s-1)}^i K_{N_s-1} \\
&S_{F(N_s-1)}^i K_{N_s-1} & S_{G(N_s-1)}^i K_{N_s-1} & & & S_{F(N_s)}^i K_{N_s}
\end{pmatrix}
\]

(3.85)
3.6.3 Stiffness matrix of total rings

From the preceding section, modify Eq. (3.82) by combining each together to form the overall equilibrium equation of total rings. These equilibrium equations are called the system of linear equations. The unknown vector \( \mathbf{x} \) will be solved by means of this system linear equation (Eq. (3.87)). \( N_r \) is total numbers of ring.

\[
\begin{bmatrix}
R_1^2 & R_1^3 & 0 & 0 & \cdots & 0 \\
R_2^2 & R_2^3 & 0 & 0 & \cdots & 0 \\
R_3^2 & R_3^3 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
R_{N_r-2}^2 & R_{N_r-2}^3 & 0 & 0 & \cdots & 0 \\
R_{N_r-1}^1 & R_{N_r-1}^2 & R_{N_r-1}^3 & 0 & \cdots & 0 \\
R_{N_r}^1 & R_{N_r}^2 & R_{N_r}^3 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
R_1^x \\
R_2^x \\
\vdots \\
\vdots \\
\vdots \\
R_{N_r-1}^x \\
R_{N_r}^x \\
\end{bmatrix}
= \begin{bmatrix}
S_1^p \\
S_2^p \\
\vdots \\
\vdots \\
\vdots \\
S_{N_r-1}^p \\
S_{N_r}^p \\
\end{bmatrix}
\]

(3.87)

3.7 Infinite ring analysis

In general, joint arrangement between rings of shield tunnel is staggered joint arrangement. According to Fig. 3.2, ring 1 and ring 2 are combined together to create one cycle and the combination of all cycles can build overall shield tunnel. Therefore, in order to analyze the mechanical behaviors of infinitely long shield tunnel, it can be supposed that only one cycle
analysis is enough to be a representative of total cycles or overall shield tunnel. To carry out the infinite ring analysis, there are two conditions must be adjusted: boundary condition and loading condition.

3.7.1 Cyclic boundary and loading conditions

The boundary conditions of one end of one cycle are set to connected to the another end. As illustrated in Fig. 3.20, End 1 is set to connect to End 2 and also End 2 is set to connect to End 1. For the loading conditions, the external loads acting on these two rings, are supposed to be a representative of every loading condition of cycle. With this concept of setting of cyclic boundary condition, the infinite ring analysis can be carried out.

3.7.2 Modified equations for infinite ring analysis

After setting cyclic boundary conditions as discussed in Sec. (3.7.1), the stiffness matrix discussed in Sec. (3.6.1) and Sec. (3.6.2) will be subsequently modified as following

According to Eqs. (3.77) and (3.79) of Section 3.6.1, \( i-1x_j \) equals \( i+1x_{j-1} \) and \( i-1x_{j+1} \) equals \( i+1x_j \). Thus for ring 1,

\[
\begin{align*}
S_1K_j = \left( SC_1K_j, SD_1K_j, SE_1K_j, SA_1K_j + SF_1K_j, SB_1K_j + SG_1K_j \right)
\end{align*}
\] (3.88)
for ring 2,

\[
S_2 K_j = \left( \frac{1}{2} S_2^C K_j, \frac{1}{2} S_2^D K_j, \frac{1}{2} S_2^E K_j, \frac{1}{2} S_2^A K_j + \frac{1}{2} S_2^B K_j + \frac{1}{2} S_2^G K_j \right)
\]  

are obtained. Suppose \(S_2^{AF} K_j = S_2^A K_j + S_2^F K_j\) and \(S_2^{BG} K_j = S_2^B K_j + S_2^G K_j\). From Section 3.6.2, Eq. (3.83), Eq. (3.84) and Eq. (3.85) are modified as following

\[
R_1^i K = \begin{pmatrix}
S_1^{AF} K_1 & S_1^{BG} K_1 \\
S_1^{AF} K_2 & S_1^{BG} K_2 \\
& & \ddots \\
S_1^{AF} K_{N_s} & S_1^{BG} K_{N_s - 1} & S_1^{AF} K_{N_s - 1} & S_1^{BG} K_{N_s} \\
S_1^{BG} K_{N_s}
\end{pmatrix}
\]  

(3.92)

\[
R_2^i K = \begin{pmatrix}
S_1^D K_1 & S_1^E K_1 \\
S_1^C K_2 & S_1^D K_2 & S_1^E K_2 \\
& & \ddots \\
S_1^C K_{N_s} & S_1^D K_{N_s - 1} & S_1^E K_{N_s - 1} & S_1^C K_{N_s} \\
S_1^E K_{N_s}
\end{pmatrix}
\]  

(3.93)
3.8 Linear and nonlinear joint-spring models

One of the main objectives of this study is to apply this rigid segment model to the actual shield tunnel construction; therefore, for more reality, the nonlinear joint-spring model is essential to introduce into analysis.

3.8.1 Linear joint-spring model

Fig. 3.21 shows the characteristic of linear joint-spring model. The spring constants at any joint are considered as same values both the compression side and the tension side. With this model, it is not considered whether the adjacent segments are overlapped mutually.
3.8.2 Nonlinear joint-spring model

Generally, segment body can resist large compression and little tension, as illustrated in Fig 3.24. On the other hand, the tensile strength and the compressive strengths of bolt, as illustrated in Fig. 3.23 strength are the same. The nonlinear joint-spring model is based on the fact that segment body carries most of compression and bolt carries most of tension. Nonlinear joint-spring model can be represented by Fig. 3.22. To get rid of the overlapped joints between the adjacent segments, nonlinear joint-spring model must be introduced into consideration and the analysis is carried out until there is no overlapped joint. The flow chart for considering nonlinear joint-spring model is shown in Fig. 3.25.
4.1 Introduction

In this chapter, the application of the proposed model, i.e., rigid segment spring model, to the actual shield tunnel segments is carried out. The construction project namely Kyoto Subway Tozai Line Rokujizo-Kita Construction Section is selected to be simulated. The cross section used in this project is rectangular shape section. The method of analysis of rectangular ring by using rigid segment spring model is discussed in previous chapter (chapter 3). The simulation is based on *nonlinear joint-spring model*. Parametric study is carried out first by simulating the full-scale loading tests of the composite segment (crossover section). The optimal spring constants are determined by comparing the calculated deformation with the observed deformation. Finally, the application of the proposed model to the actual shield tunnel segments in the construction site is carried out using the obtained optimal spring constants.

4.2 Outline of construction project

Kyoto Subway Tozai line Rokujizo-Kita Construction Section is the extension project of Tozai line approximately 2.4 km connecting Rokujizo station in Uji City with Daigo station which has already opened, via Ishida station, as illustrated in Fig. 4.1. Construction work is carried out by the cut and cover method for station parts and by the shield method for running track...
At Rokujizo-Kita Construction Section between Rokujizo Station and Ishida Station the rectangular cross section shield method for complex lines section which includes both crossover section and running track section is adopted for the first time in the world. A launching shaft is constructed at the north side of Rokujizo station, and construction work from Rokujizo station to Ishida station with the distance 759.59m is carried out by rectangular cross section shield method of high-density slurry type. The lining of this shield driven tunnel has an outside width of 9900mm and an outside height of 6500mm. The distance of cross over section is 56m and the distance of running track section is 697m. In this project, there are three types of rectangular cross sections; crossover section, transition section and running track section (see Fig. 4.2). All types of section are described as follows:
4.3. Preparation analysis

In order to check the applicability of the proposed model, the numerical analysis of the longitudinal direction of rectangular cross section tunnel is carried out basically through cantilever beam.

The numerical results are taken to compare to the fundamental beam theory formula i.e., Euler-Bernoulli beam formula. A fundamental cantilever beam applied by point load at the right end is used as the representative model, as shown in Fig. 4.3. Table 4.1 shows the parameters used in this preparation analysis.

For segment spring constants, normal and shear segment spring constants are denoted, respectively, by $k_{sn}$ and $k_{ss}$. For ring spring constants, normal and shear ring spring constants are denoted by $k_{rn}$ and $k_{rs}$, respectively. And for ground spring constants, $k_{en}$ is denoted for spring constant of the radial direction of cross section of tunnel. Spring constants of the tangential
Table 4.1: Parameters used in the preparation analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point load ( P ) (kN)</td>
<td>200</td>
</tr>
<tr>
<td>Span length ( l ) (m)</td>
<td>30</td>
</tr>
<tr>
<td>Number of rings ( N_r )</td>
<td>30 rings</td>
</tr>
<tr>
<td>( k_{en}, k_{es} ) at left end</td>
<td>Large</td>
</tr>
<tr>
<td>( k_{en}, k_{es} ) at elsewhere</td>
<td>0</td>
</tr>
<tr>
<td>( k_{sn} ) (kN/m)</td>
<td>( 1.0 \times 10^4 )</td>
</tr>
<tr>
<td>( k_{ss} ) (kN/m)</td>
<td>( 1.0 \times 10^3 )</td>
</tr>
<tr>
<td>( k_{rn} ) (kN/m)</td>
<td>( 1.0 \times 10^3 )</td>
</tr>
<tr>
<td>( k_{rs} ) (kN/m)</td>
<td>( 1.0 \times 10^3 )</td>
</tr>
</tbody>
</table>

Figure 4.4: Positions of ring springs at ring joints of rectangular ring

The purpose of this study is only to check the accuracy of the program, so the parameter study is not carried out. Because the ratio \( k_{rs}/k_{sn} \) is equal to 1, so effect of shear deformation is very small. Therefore, the numerical deflection curve would be satisfied by Euler-Bernoulli beam formula. According to Fig. 4.5 showing the deflection curves obtained from numerical analysis and Euler-Bernoulli beam formula, it can be concluded that the deflection curve obtained from the analysis is agreed well with Euler-Bernoulli beam formula. Fig. 4.6 shows the bending moment diagrams obtained from numerical analysis and analytical solution, it also can be
4.4. Steps of study

As shown in Fig. 4.7, firstly the application to the full-scale segment loading test is carried out in order to obtain the appropriate spring constants. Secondly, by using the same set of the spring constants, the application to the actual segmental rings in construction site is carried out by taking two conditions of loading, short and long-term conditions, into account. The comparison between the observed data and numerical results can indicate the applicability of the rigid segment spring model in the design of shield tunnel lining.

Note that the simulation of the full-scale segment loading test (first part) is carried out through 3 rings model and the simulation of actual segmental rings in site (second part) is carried out through infinite ring analysis.
Application to the full-scale segment loading test

Compare analysis results with experimental results

Obtain appropriate spring parameters

Application to the actual tunnel ring (in-situ)

Short-term loading condition

Long-term loading condition

With subgrade reaction

Without subgrade reaction

Compare analysis results with experiment results

Evaluate model

**Figure 4.7:** Steps of study on the application of the rigid segment spring model to the actual segments

## 4.5 Application to full-scale segment loading test

### 4.5.1 Outline of the loading test

In the loading test, the safety of the linings and the adequacy of the design method were evaluated by directly applying to single segments and full segmental rings loads that were near to the design loads. Actually, loading tests of both composite segmental rings and DC segmental rings were carried out but in this section only loading test of composite segmental rings will be discussed. Loading test on composite segmental rings is shown in Fig. 4.8 and consisted of three rings assembled and bolted together. The specification of composite segmental rings is illustrated in Table 4.2.

Loads were applied to the composite segmental rings from eight directions. (see Figs. 4.8 and 4.9) First, 10 kN was applied by each load jack. After that supports were removed. Then, for each step segmental rings were applied by the incremental 10% of maximum load of each load jack. The number of loading steps were 10 steps. Each maximum load of each load jack is shown in Fig. 4.9. This maximum load is nearly equivalent to the design loads. After each load increment, the amounts of lining displacement, generated stresses of segment body and joints, joint openings and offset length between adjacent rings were measured.
4.5. Application to full-scale segment loading test

Table 4.2: Specification of composite segments (crossover section) (Nakamura et al., 2001)

<table>
<thead>
<tr>
<th>Description</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height, width, length (mm)</td>
<td>$6500 \times 9900 \times 1000$</td>
</tr>
<tr>
<td>Girder depth (mm)</td>
<td>500</td>
</tr>
<tr>
<td>Ring division</td>
<td>7 segments</td>
</tr>
<tr>
<td>Material (steel)</td>
<td>SM490</td>
</tr>
<tr>
<td>Total weight / ring</td>
<td>375.2 kN (38.26t)</td>
</tr>
<tr>
<td>Steel shell weight / ring</td>
<td>113.7 kN (11.60t)</td>
</tr>
<tr>
<td>Concrete weight / ring</td>
<td>261.4 kN (26.66t)</td>
</tr>
<tr>
<td>Segment weight (max.)</td>
<td>69.9 kN (7.13t)</td>
</tr>
<tr>
<td>Joints between rings</td>
<td>M33 $\times$ 34 bolts / ring</td>
</tr>
<tr>
<td>Joints between segments</td>
<td>M36 $\times$ 8 bolts / joint</td>
</tr>
</tbody>
</table>

Figure 4.8: Schematic diagram of full-scale composite segmental rings loading test
4.5.2 Joint behavior and spring constants

In this construction project, the beam-spring model was mainly used to design the shield tunnel segments. The beam-spring model uses the rotational spring $K_\theta$ at the segment joint and the shear spring $K_s$ at the ring joint. To obtain the spring constants, the segment joint-bending test and the ring joint-shear test were carried out to obtain the rotational spring constants and shear spring constants, respectively (Kyoto Subway Tozai Line Rokujizo-Kita Construction Section, 2000). The behavior of each type of joints are explained as below:

**Segment joint**

According to Fig. 4.10, the segment joint behavior is not linear relationship between rotational angle and bending moment. At the early state of loading, large increment of bending moment for rotating segment joint is required due to the bearing of the adjacent segments. When this early state is beyonded, instead of the bearing of adjacent ring, bolts carry load due to lower increment of bending moment than the early state.

**Ring joint**

The shear force resistance of ring joint is shown in Fig. 4.11. The ring joint behavior is not a linear relationship between displacement and shear force. At the early state of loading, shear force is carried by the friction due to bolt tightening. The displacement of this early state is very small. After the early state, the friction disappears and the space between bolt and hole
4.5. Application to full-scale segment loading test

Bending moment (kN·m).

Rotational angle θ

Kθ1+

Kθ2+

Kθ1−

Kθ2−

450

-150

Figure 4.10: Bending moment vs. rotational angle.

Shear force (kN)

δ (mm)

Ks1

Ks2

Ks3

85

115

Figure 4.11: Shear force vs. displacement.

Table 4.3: Spring constants used in the beam-spring model

<table>
<thead>
<tr>
<th>Segment joint (apply axial force)</th>
<th>Ring joint (not apply axial force)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotational spring constant $K_\theta$ (kN·m/ rad)</td>
<td>Shear spring constant $K_s$ (kN/m)</td>
</tr>
<tr>
<td>$K_{\theta_1}$</td>
<td>$K_{\theta_2}$</td>
</tr>
<tr>
<td>Positive $7.5 \times 10^5$</td>
<td>Positive $3.7 \times 10^5$</td>
</tr>
<tr>
<td>Negative $3.7 \times 10^5$</td>
<td>Negative $1.0 \times 10^9$</td>
</tr>
</tbody>
</table>

causes the additional displacement. Finally, shear force is resisted by the bolts themselves.

Table 4.3 illustrates the spring constants obtained from joint-bending test and ring joint-shear test. Figs. 4.10 and 4.11 shows bending moment-rotational angle and shear force-displacement curves, respectively.

In this study, the numerical analysis by using rigid segment spring model is carried out in order to evaluate the applicability of the rigid segment spring model. The spring constants used in analysis based on the spring constants used in the beam-spring model. However, positions and number of each types of springs of rigid segment spring model are not the same as the beam-spring model. In order to apply spring constants to rigid segment spring model, these spring constants must be modified approximately as follows:

**Normal segment spring constant** $k_{sn}$: According to Fig. 4.12, the segment springs of rigid segment model locate at the edges of segment joints. Because there are two springs locating at one edge of segment joint, total spring force is $F = 2k_{sn}\Delta x$ and bending moment acting on segment joint is $M = Ft = 2k\Delta x$. By using equation $M = k_\theta\theta$, therefore we can obtain

$$2k_{sn}\Delta x t = K_\theta\theta = K_\theta \frac{\Delta x}{t} \quad (4.2)$$
Chapter 4. Simulation of the shield tunnel segments

Section a-a

\[ F = 2k \]

\[ M = Ft \]

Figure 4.12: Comparative diagram of normal segment spring and rotational spring

Thus,

\[ k_{sn} = \frac{K_\theta}{2t^2} \]  \hspace{1cm} (4.3)

Shear ring spring constant \( k_{rs} \): The number of ring springs of each segment is the same as that of other segments. But actually the number of bolts connecting the adjacent rings together is not the same. Therefore, according to Fig. 4.13, a coefficient \( C_i \) is supposed to modify the shear ring spring constant \( k_{rs} \). \( C_i \) is defined as the ratio of the number of bolts (or ring springs in beam-spring model) in part \( i \) to the number of ring springs in rigid segment spring model.

\[ k_{rs}^{mod} = C_i K_s = C_i k_{rs} \]  \hspace{1cm} (4.4)

where, \( k_{rs} \) and \( k_{rs}^{mod} \) are shear ring spring constants before and after modification in rigid segment spring model, respectively, and \( K_s \) is shear ring spring constant in beam-spring model.

Note that, at ring joint, there are only bolts at internal part of segment connecting two adjacent rings together, the ring spring constants at external part of segment are initially set to be zero. For example, seeing Fig. 4.13, at part 2, there are 2 springs at bottom for model and 4 bolts at bottom for the actual ring, thus \( C_2 = 4/2 \) and \( k_{rs}^{mod} \) of each spring in part 2 is \( 2K_s \) or \( 2k_{rs} \).

Shear segment spring constant \( k_{ss} \): Shear segment spring constant should be assumed to be approximately 2.25 times (Kyoto Subway Tozai Line Rokujizo-Kita Construction Section, 2002a) of shear ring spring constant because at segment joint there is axial force acting on it. However, actually there are 8 bolts connecting segment to the adjacent segment and there are 4 springs at segment joint in this study model. Therefore the shear segment spring constant of each spring in this study model should be approximately 2 times. That means \( k_{ss} \) is supposed to be 4 times of \( k_{rs} \).

Normal ring spring constant \( k_{rn} \): \( k_{rn} \) is assumed to be constant and less than \( k_{sn} \) suitably.
4.5. Application to full-scale segment loading test

**Figure** 4.13: Ring spring positions and bolt positions

**Figure** 4.14: Bending moment vs. displacement: segment normal spring constant $k_{sn}$

**Figure** 4.15: Bending moment vs. displacement: ring normal spring constant $k_{rn}$

**Figure** 4.16: Shear force vs. Displacement: segment shear spring constant $k_{ss}$

**Figure** 4.17: Shear force vs. Displacement: ring shear spring constant $k_{rs}$
4.5.3 Condition and parameter settings

According to Fig. 4.13, the adjacent rings are connected by bolts which located at the internal part of segments, thus the spring constants at external part of segments are supposed initially to be zero (node \( k = 21, 23, 25, \ldots, 35 \) in Figs. 3.18 and 3.19). Moreover, at segment joint, the normal segment spring constants of top springs are modified from the negative bending moment rotational spring constants and spring constants of bottom springs are modified from the positive bending rotational spring constants. The initial parameters modified by the method described above, are summarized in Table 4.4.

The numerical analysis is carried out based on incremental analysis because spring constants are not constant through loading as shown in Figs 4.14, 4.15, 4.16 and 4.17.

Table 4.4: Initial parameters for three segmental rings loading test analysis (Unit : kN/m)

<table>
<thead>
<tr>
<th>Spring Joint</th>
<th>Normal spring constant</th>
<th>Shear spring constant</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( k_{sn1} )</td>
<td>( k_{sn2} )</td>
</tr>
<tr>
<td>Segment joint</td>
<td>450</td>
<td>( 1.5 \times 10^6 )</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>( 7.4 \times 10^5 )</td>
</tr>
<tr>
<td>Ring joint</td>
<td>+ none</td>
<td>( 1.0 \times 10^5 )</td>
</tr>
<tr>
<td></td>
<td>- none</td>
<td>( 1.0 \times 10^5 )</td>
</tr>
</tbody>
</table>

1. \( ip \) : force or bending moment value at inflection point. Unit for normal spring is kN \( \cdot \) m and for shear spring is kN.
2. Positive bending moment
3. Negative bending moment
4. \( k_{rs} \) and \( k_{rn} \) at the top part of ring joints are set to be zero.
5. This \( k_{rs} \) is the value before modification (see Eq. (4.4))

4.5.4 Steps of incremental analysis

In order to simulate this full-scale loading test by using rigid segment-spring model, the incremental analysis is carried out step by step as follows:

1. loads are applied increasingly from 0 kN/load jack up to 10 kN/load jack. The incremental analysis is repeated totally 100 times; then
4.5. Application to full-scale segment loading test

Table 4.5: Parameters for three segmental rings loading test analysis for case 1 (Unit: kN/m)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Spring</th>
<th>Normal spring constant</th>
<th>Shear spring constant</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$k_{sn1}$</td>
<td>$k_{sn2}$</td>
</tr>
<tr>
<td>Segment joint</td>
<td>+</td>
<td>450</td>
<td>$7.4 \times 10^6$</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>150</td>
<td>$2.0 \times 10^6$</td>
</tr>
<tr>
<td>Ring joint</td>
<td>+</td>
<td>none</td>
<td>$1.0 \times 10^5$</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>none</td>
<td>$1.0 \times 10^5$</td>
</tr>
</tbody>
</table>

2. loads are applied increasingly from 10 kN/load jack up to the maximum load of each load jack. The incremental analysis of this step is repeated totally 100 times.

Note that the maximum loads of all load jacks are shown in Fig. 4.9.

4.5.5 Parametric studies

The parametric studies are carried out here by considering into two cases:

**Case 1**: the shear spring constants ($k_{rs}$, $k_{ss}$) at ring joints are set to parameter while the normal spring constants ($k_{rn}$, $k_{sn}$) at ring joints and segment joints, are set to be fixed.

**Case 2**: the normal spring constants ($k_{sn}$) of segment springs are set to be parameter while the shear spring constants($k_{rs}$, $k_{ss}$) and normal spring constants ($k_{rn}$) of ring springs are set to be fixed.

Note that the experimental results are referred to technical report no.7 of Kyoto Subway Tozai Line Rokujizo-Kita Construction Section (Kyoto Subway Tozai Line Rokujizo-Kita Construction Section, 2002b).

**Case 1**

Shear spring constants($k_{rs}$, $k_{ss}$), at ring joints and segment joints, are set to be parameter and normal spring constants($k_{rn}$, $k_{sn}$), at ring joints and segment joints, are set to be fixed. The spring parameters used for case 1 are illustrated in Table 4.5. Note that the values at the inflection points of the shear springs are supposed to be 50% of the values shown in Table 4.4.

According to Figs 4.18, 4.19 and 4.20, the numerical result is not satisfied well when we consider the deflection of tunnel. That is the lateral deflections (the deflections of left and right sides of tunnel) obtained from numerical simulation are relative small to the observed data.
In order to obtain more lateral deflection, segment joint should set to be more flexible to rotate. This means normal spring constants would be set to smaller. This is the basic concept of case 2.

**Case 2**

In order to consider how segmental rings deform when segment joints allowed to rotate easier, normal spring constants \( k_{sn} \) of segment springs are set to be parameter while the shear spring constants \( k_{rs}, k_{ss} \) and the normal spring constants \( k_{rn} \) of ring springs are set to be fixed. The spring parameters are shown in Table 4.6.

According to Figs 4.21 and 4.22, when ratio \( \frac{k_{rs1}}{k_{sn1}} = 6.66 \) we obtained the good result. Both of the vertical deflection of tunnel crown and the lateral deflection of tunnel springline obtained from analysis are very close to the observed data. Fig 4.23 shows the deformation of segmental rings at optical spring constants. Not only deflection but also bending moment of segmental rings are satisfied with the observed data as illustrated in Fig. 4.24. Fig 4.25 shows the axial
4.5. Application to full-scale segment loading test

Table 4.6: Parameters for three segmental rings loading test analysis for case 2 (Unit: kN/m)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Spring</th>
<th>Normal spring constant</th>
<th>Shear spring constant</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>ip $k_{sn1}$</td>
<td>$k_{sn2}$</td>
</tr>
<tr>
<td>Segment joint</td>
<td>+ 450</td>
<td>$4.5 \times 10^4$</td>
<td>$2.22 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>– 150</td>
<td>$2.22 \times 10^4$</td>
<td>$6.0 \times 10^4$</td>
</tr>
<tr>
<td>Ring joint</td>
<td>+ none</td>
<td>$1.0 \times 10^5$</td>
<td>$1.0 \times 10^5$</td>
</tr>
<tr>
<td></td>
<td>– none</td>
<td>$1.0 \times 10^5$</td>
<td>$1.0 \times 10^5$</td>
</tr>
</tbody>
</table>

Figure 4.21: Lateral deflection vs. $k_{rs1}/k_{sn1}$ (case 2: $k_{rs1}/k_{sn1} = 6.66$)

Figure 4.22: Crown deflection vs. $k_{rs1}/k_{sn1}$ (case 2: $k_{rs1}/k_{sn1} = 6.66$)

force (hoop force) of segmental rings and it can be seen that the observed data are too varied but there are some points which satisfied by the numerical results. Table 4.7 shows the parameters used in the optimal case.

Table 4.7: Parameters for three segmental rings loading test analysis for optimal case (Unit: kN/m)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Spring</th>
<th>Normal spring constant</th>
<th>Shear spring constant</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>ip $k_{sn1}$</td>
<td>$k_{sn2}$</td>
</tr>
<tr>
<td>Segment joint</td>
<td>+ 450</td>
<td>$4.5 \times 10^4$</td>
<td>$2.22 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>– 150</td>
<td>$2.22 \times 10^4$</td>
<td>$6.0 \times 10^4$</td>
</tr>
<tr>
<td>Ring joint</td>
<td>+ none</td>
<td>$1.0 \times 10^5$</td>
<td>$1.0 \times 10^5$</td>
</tr>
<tr>
<td></td>
<td>– none</td>
<td>$1.0 \times 10^5$</td>
<td>$1.0 \times 10^5$</td>
</tr>
</tbody>
</table>
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Figure 4.23: Deformation of segmental rings when $k_{rs1}/k_{sn1} = 6.66$ (case 2)

Figure 4.24: Bending moment diagram when $k_{rs1}/k_{sn1} = 6.66$ (case 2)

Figure 4.25: Hoop force diagram when $k_{rs1}/k_{sn1} = 6.66$ (case 2)
4.6 Application to in-situ segmental rings

Applications of the rigid segment spring model to the in-situ segment ring are carried out, using the optimal spring parameters obtained from the previous section (Section 4.5). Both short-term loading condition and long-term loading condition are considered to evaluate the applicability of this model for design of the shield tunnel lining.

4.6.1 Short-term loading

Loading condition for short-term condition is shown in Fig. 4.26(Kyoto Subway Tozai Line Rokujizo-Kita Construction Section, 2003). This load condition was modified from the in-situ earth pressure measurement. The reaction forces, acting on the lining, induced by two support columns removal, will be added in the analysis while ground reaction will be considered also. We considered the case that there are only compressive reaction from ground i.e., consider only the reaction springs located on the both lateral sides of segmental ring.

Parameter setting

The tunnel passes through gravel of the upper diluvial beds(including sand and clay partially). N-value ranges from 20 to over 50. According to page 46 of literature (Tunnel engineering
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committee, 2003), for N-value over than 30 soil, the ground reaction spring constant is approximately in the range of 30-50 MN/m$^3$. The tension ground reaction spring is ignored and only compression ground reaction springs are considered. According to Fig. 4.27, then ground reaction springs are placed along the circumferential length of each lateral side of segmental ring. By modifying these ground reaction spring constants to the analysis model where there are 10 reaction springs along the circumferential length of the each lateral side of segmental ring i.e., each side length=8.39 m. Thus, for this analysis, three different ground spring constants ($k_{es}$ and $k_{en}$) i.e., 25000, 35000 and 42000 kN/m for each spring, are used.

Results and discussions

Ring deformation

According to Figs. 4.28, 4.29 and 4.30, these figures shows the deformation of rings 1 and 2 for each ground reaction spring constant. It is found that the deformation obtained from numerical analysis agrees with the deformation measured in construction site. As ground reaction spring constant is higher, the deformation of ring becomes smaller.

Bending moment and hoop force

Figs. 4.31, 4.32 and 4.33 illustrate the bending moment diagram for each case of ground reaction spring constant. It is found that there is no significant difference in bending moment among these three cases. The bending moment along the both sides of segment are fairly satisfied with the the measured data in construction site. At the crown and bottom of ring, however, the bending moment obtained from the numerical analysis is higher than the measured data in construction site.
4.6. Application to in-situ segmental rings

For the hoop force, Figs. 4.34, 4.35 and 4.36, the measured data are very varied and the hoop force obtained from the numerical analysis are quite lower than the the measured data. However, there are good agreements at some positions i.e., at the bottom of ring 1 and the left side of both rings, where the average hoop force is approximately equal to 1000 kN.

Offset length between adjacent segments

Table 4.8 shows the offset length between the adjacent segments. It is noted that the ground reaction spring constant 25000, 35000 and 42000 kN/m are used for case 1, case 2 and case 3, respectively. The number of segment joint is shown in Fig. 4.37. From table 4.8, we can see that the offset length obtained from numerical analysis is larger than the measured data. This result can be explained as follows:

The rigid segment spring model assumed the segment as the rigid body where there is no strain within the body, but the actual segment may be elastic body. According to Fig. 4.38, for both elastic and rigid segments, the reference point P on segment 2 displaces to point P' with the displacement \( v_p \). For rigid segment (Fig. 4.38(a)), the displacement \( v_p \) is equal to the summation of the displacement \( v_1 \) and the offset length \( l_r \) of segment joint. However, for elastic segment (Fig. 4.38(b)), \( v_p \) is equal to the summation of the displacement \( v_1 \), offset length \( l_e \) of segment joint and the elastic deformation \( u_e \) of segment 2. This explanation can be written mathematically as

\[
\begin{align*}
\text{Rigid segment} & : v_p = v_1 + l_r \\
\text{Elastic segment} & : v_p = v_1 + l_e + u_e
\end{align*}
\]

If the elastic deformation \( u_e \) is assumed positively, then

\[
l_r > l_e
\]

which indicates that the offset length of rigid segment case is larger than that of elastic segment.

4.6.2 Long-term loading

The earth pressure distribution acting on the lining will be changed due to the lining deformation. The long-term loading condition, as shown in Fig. 4.39, was obtained by averaging the measured earth pressure on site for two months after support removal (Kyoto Subway Tozai
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Figure 4.28: Cross-sectional deformation for \( k_{en} = k_{es} = 25000 \text{ kN/m} \) (short term)

Figure 4.29: Cross-sectional deformation for \( k_{en} = k_{es} = 35000 \text{ kN/m} \) (short term)

Figure 4.30: Cross-sectional deformation for \( k_{en} = k_{es} = 42000 \text{ kN/m} \) (short term)
4.6. Application to in-situ segmental rings

**Figure 4.31:** Bending moment diagram for $k_{en} = k_{es} = 25000$ kN/m (short term)

**Figure 4.32:** Bending moment diagram for $k_{en} = k_{es} = 35000$ kN/m (short term)

**Figure 4.33:** Bending moment diagram for $k_{en} = k_{es} = 42000$ kN/m (short term)
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Figure 4.34: Hoop force diagram for $k_{en} = k_{es} = 25000$ kN/m (short term)

Figure 4.35: Hoop force diagram for $k_{en} = k_{es} = 35000$ kN/m (short term)

Figure 4.36: Hoop force diagram for $k_{en} = k_{es} = 42000$ kN/m (short term)
4.6. Application to in-situ segmental rings

Table 4.8: Offset length between adjacent segments of Ring 1 (unit:mm)

<table>
<thead>
<tr>
<th>Position</th>
<th>observed</th>
<th>numerical result</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>case 1</td>
<td>case 2</td>
</tr>
<tr>
<td>①</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>②</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>③</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>④</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>⑤</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>⑥</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>⑦</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

Figure 4.37: Numbering of positions of segment joints

Figure 4.38: Comparison between rigid and elastic segments
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Figure 4.39: Long-term loading condition (Kyoto Subway Tozai Line Rokujizo-Kita Construction Section, 2002a)

Line Rokujizo-Kita Construction Section, 2002a). The same spring parameters as those of short-term loading case are used without ground reaction springs.

Fig. 4.40 shows the deformation of segmental rings 1 and 2. It can be seen that the deformation of segmental rings are very small compared to the measured data on site and deformation obtained from short-term loading condition. The long-term loads were measured at the stable state of segmental ring, hence segmental ring will not be deformed much. With this reason, the small deformation obtained from this analysis should be reasonable.

For bending moment and hoop force diagrams (see Fig. 4.41 and 4.42), the numerical results are also small compared to those of the short-term loading condition.

4.7 Conclusions

The rigid segment spring model for analyzing the actual shield tunnel lining by considering the fact that the adjacent segments are not overlapped on each other was developed. Moreover, the methodology for analyzing the infinite segmental rings by considering only two representative rings was also proposed.

The parametric studies in order to determine the appropriate spring constants were carried out first by simulating the large-scale loading test on three segmental rings. The first case in which the shear spring constants at ring joints were varied and normal spring constants are fixed, was carried out. No any set of spring constants was found to make the good agreement between the computed deformation and observed deformation i.e., the computed vertical deformation
4.7. Conclusions

Figure 4.40: Cross-section deflection (long term)

Figure 4.41: Bending moment diagram (long term)

Figure 4.42: Hoop force diagram (long term)
of the crown and bottom of rings and computed horizontal deformation of lateral sides of rings are not satisfied the observed deformation when the same set of spring constants is used. The second case was subsequently carried out in order to allow the segment joint more easily to rotate. The normal spring constants are varied while the shear spring constants are fixed. The appropriate set of spring constants was found when the ratio $k_{rs1}/k_{sn1}$ is approximately 6.66. By using this appropriate set of spring constants in the application of the rigid segment spring model to the actual segmental rings, the two cases of loading conditions, short-term and long-term conditions, were carried out. From comparison between numerical results and observed values from construction site, the deformation obtained from the short-term loading condition are close to the observed valued. For the long-term loading condition, the deformation is almost zero. This comparative results are reasonable because in the actual situation, the segmental ring deformed due to the short-term loads and no further deformation occur when it reaches the stable stage i.e., long-term condition. It can be concluded, therefore, that the rigid segment spring model can be expected to apply in the design of shield tunnel lining if the appropriate spring parameters can be obtained.
Part II

New Austrian Tunneling Method (NATM)
5.1 Introduction

A rigid-plastic finite element method, namely RPFEM, is recently considered as the effective tool to determine the critical strength of soil or rock structures such as dams, tunnels and the foundations for the large structures, without the increase of the numerical errors which is a problem in the conventional finite element method. This method is interpreted as an extension of the classical limit analysis of a frame structure to continuum, being based on the upper-bound theorem in plasticity (Tamura, 1990). The critical state is obtained by minimizing of the rate of internal plastic energy dissipation with respect to the kinematically admissible velocity field under several linear constraint conditions (Tamura et al., 1984).

In this chapter, the theoretical basis for plasticity will be described followed by the stress-strain rate relationships and the fundamental equations and boundary conditions for rigid-plastic analysis. The formulation of the finite element method will be also explained. Furthermore, the methodology to include the effect of groundwater in the rigid-plastic finite element method will be proposed.
5.2 Theoretical basis for plasticity

5.2.1 Fundamental concept of plasticity

The fundamental concept of the elasto-plastic constitutive model is described first (Potts and Zdrvkovic, 1999). Fig. 5.1a) shows the stress-strain relation for the elasto-plastic material. On first material is loaded and deforms elastically and its stress-strain response travels along the line AB. If the loading process is stopped before the stress in material reaches point B and material is unloaded, the stress-strain response moves back down the line BA. Consequently, as long as the loading does not cause the stress to reach point B, the material behaves in an elastic manner and, when unloaded, returns to its original undeformed state, with no permanent strains. For simplicity, the perfectly plastic material is described here. If the material is strained beyond $\varepsilon^e$, to point $C_0$, the stress-strain curve passes through point B. Now the overall strain is equal to $\varepsilon$. At B, the yield stress is reached and the material becomes plastic. The stress remains constant and equal to the yield stress. If the material is unloaded now it becomes elastic and the stress-strain response moves along the line $C_0D$, which is parallel to the line BA. At point D, the stress is zero but there is still strain in the material. This remaining strain is equivalent to the strain experienced along the path $BC_0$ and called the plastic strain which is given by

$$
\varepsilon^p = \varepsilon - \varepsilon^e
$$

(5.1)

where, $\varepsilon^p$ and $\varepsilon^e$ denote, respectively, plastic strain and elastic strain. Therefore, the material does not return to its original shape. If the material is now reloaded, the stress-strain response...
5.2. Theoretical basis for plasticity

travels along the elastic path DC₀ until point C₀ is reached and the material becomes plastic again. Conclusively, the elastic paths AB and DC₀ are reversible but the path BC₀ is not reversible.

In addition, if the stress still increases after the yield stress is passed, this material is called the strain-hardening material as shown by the path ABC₊ in Fig. 5.1. For the strain-softening material, as shown by the path ABC₋ in Fig. 5.1, the stress decreases when the yield stress is reached.

In this dissertation, the elasto-plastic material is not considered but the rigid-perfectly plastic material is mainly considered. As shown in Fig. 5.1b), the elastic component of strain is zero in the limit and the material element is therefore rigid when stress is below the yield point. If the limit state is reached, the material starts to flow with the constant stress.

5.2.2 Yield surface

The material exhibits the plastic deformations if and only if the stresses satisfy the yield condition. This yield condition can be expressed by the yield function which is written as

\[ f(\sigma_{ij}) = k \]  \hspace{1cm} (5.2)

where, \( k \) is the non-negative material constant. This yield function is called the yield surface which separates the elastic and plastic behavior in the stress space as shown in Fig. 5.2. Plastic deformations can occur only if \( f(\sigma_{ij}) = k \). If \( f(\sigma_{ij}) < k \), there are no plastic deformations but only the elastic deformations occur.
5.2.3 Plastic potential

Generally, the plastic deformation can be described by means of the plastic strain rate $\dot{\varepsilon}_{ij}^p$. It is now postulated that the plastic strain rate can be derived from a plastic potential $g$, that depends on the stresses only i.e., $g = g(\sigma_{ij})$, in such a way that the plastic strain rate can be obtained by

$$
\dot{\varepsilon}_{ij}^p = \lambda \frac{\partial g}{\partial \sigma_{ij}} \quad \text{if} \quad f(\sigma_{ij}) = k
$$

(5.3)

where, $\lambda$ is an undetermined constant.

Eq.(5.3) indicates that the strain rate $\dot{\varepsilon}_{ij}$ is perpendicular to the plastic potential $g = \bar{k}$, as shown in Fig. 5.3, and the stresses satisfy the yield surface $f = k$.

5.2.4 Drucker’s stability postulate

Let us consider Fig. 5.4(a) showing the stress-strain relation of the stable material. It can be seen that an additional loading $\Delta \sigma > 0$ gives rise to an additional strain $\Delta \varepsilon > 0$, with the product $\Delta \sigma \Delta \varepsilon > 0$. This indicates that the additional stress $\Delta \sigma$ does positive work. This kind of material is called stable. On the other hand, according to Fig. 5.4(b) (Drucker, 1959), the additional stress of some parts of curve 1 and curve 3 does not do positive work and therefore material which has the stress-strain behavior represented by curves 1 and 3 is unstable material.

According to the closed loading path shown in Fig. 5.5(Kobayashi, 2003), the path starts from the initial stress state $\sigma_{ij}^0$ and crosses the initial yield surface at $\sigma_{ij}^1$ and then reaches the...
5.2. Theoretical basis for plasticity

subsequent yield surface at $\sigma_{ij}^1 + \delta \sigma_{ij}$. After that the path returns back to the initial stress state. The plastic deformations occurs between $\sigma_{ij}^1$ and $\sigma_{ij}^1 + \delta \sigma_{ij}$, the permanent strain remains in a body if stress is removed to the initial state. Drucker (Drucker, 1951) gave a proposition that the material is called stable if and only if the net work $W_p$ done by stresses during this cycle must be non-negative. This proposition is called Drucker’s stability postulate. Net work $W_p$ is determined by subtracting the work done $W_0$ by the initial stress from the total work $W_T$ during the loading cycle, illustrated as the area surrounded by the thick arrow lines in Fig. 5.6. Therefore, the net work $W_p$ can be calculated as

$$W_P = W_T - W_0$$

$$= \int_{t_0}^{t_3} (\sigma_{ij}(t) - \sigma_{ij}^0) \dot{\varepsilon}_{ij} dt$$

$$= \int_{t_0}^{t_3} \left( \sigma_{ij}(t) - \sigma_{ij}^0 \right) \left( \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \right) dt$$

$$= \int_{t_1}^{t_2} \left( \sigma_{ij}(t) - \sigma_{ij}^0 \right) \dot{\varepsilon}_{ij}^p dt \quad \text{since} \quad \int \dot{\varepsilon}_{ij}^e dt = 0.$$  \hspace{1cm} (5.4)

According to Drucker’s postulate, Eq.(5.4) can be written as

$$W_P = \int_{t_1}^{t_2} \left( \sigma_{ij}(t) - \sigma_{ij}^0 \right) \dot{\varepsilon}_{ij}^p dt \geq 0.$$  \hspace{1cm} (5.5)

Since plastic strains materialise only at $t = t_1$, $W_P$ can be expanded as a Taylor series about the neighborhood of $\sigma_{ij}(t_1)$, as follows:

$$W_P(t) = (t - t_1) \left( \sigma_{ij} - \sigma_{ij}^0 \right) \dot{\varepsilon}_{ij}^p + \frac{(t - t_1)^2}{2!} \left[ \sigma_{ij} \dot{\varepsilon}_{ij}^e + (\sigma_{ij} - \sigma_{ij}^0) \ddot{\varepsilon}_{ij}^p \right] + \cdots.$$  \hspace{1cm} (5.6)

By assuming the value of $(t_2 - t_1)$ is small enough, only the first term of the right-hand side of Eq.(5.6) is considered, then we have

$$\left( \sigma_{ij} - \sigma_{ij}^0 \right) \dot{\varepsilon}_{ij}^p \geq 0 \quad \text{if} \quad \sigma_{ij} \neq \sigma_{ij}^0$$

$$\sigma_{ij} \dot{\varepsilon}_{ij}^p \geq 0 \quad \text{if} \quad \sigma_{ij} = \sigma_{ij}^0.$$  \hspace{1cm} (5.7)

The inequality (5.7) is equivalent to the so-called Principle of maximum plastic work proposed by Hill (Hill, 1950). Consequently, the inequality (5.7) leads to the two important properties:
**Figure 5.5**: A closed loading path in relation to the yield surfaces.

**Figure 5.6**: Schematic representation of a stress cycle for stable material.

- Convexity of a yield surface: Yield surface must be convex in a stress space. The definition of convexity is described in appendix B.

- Normality rule: If yield surface is smooth, direction of plastic strain rate coincides with the direction of the outward normal vector of a yield surface if a yield surface is smooth. This suggests that yield surface \( f \) is equal to the plastic potential \( g \) and according to Eq.(5.3), we obtain

\[
\dot{\varepsilon}_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}}
\]  

(5.8)

This rule can be called in another word, the associated flow rule.

Generally, it has been found, by comparing theoretical results with experimental data, that for metals very good agreements is obtained if the associated flow rule is considered. However, many soils do not satisfy the associated flow rule because of the dilatancy, therefore, the non-associated flow rule is introduced as the plastic potential \( g \) does not coincide with the yield function \( f \). The relationship between the strain rate and the plastic potential is written as Eq.(5.3).

### 5.2.5 Rate of plastic energy dissipation

For a given plastic strain rate \( \dot{\varepsilon}_{ij}^p \), the rate of specific plastic energy dissipation \( D(\dot{\varepsilon}_{ij}^p) \) can be defined as:
5.2. Theoretical basis for plasticity

\[ D(\dot{\varepsilon}_{ij}^p) = \sigma_{ij} \dot{\varepsilon}_{ij}^p \]  \hspace{1cm} (5.9)

in which \( \sigma_{ij} \) denotes a stress on the yield surface associated with the plastic strain rate \( \dot{\varepsilon}_{ij}^p \) through the normality rule. It should be noted that all of plastic strain rates are not always allowed to define the rate of energy dissipation. For example, in the case of the von-Mises yield criterion, \( \dot{\varepsilon}_{ij}^p \) with nonvanishing volumetric component never satisfies the associated flow rule for any stress \( \sigma_{ij} \) on the yield surface. Therefore some kind of constraints depending on the adopted yield criterion must be imposed on choosing the plastic strain rate for \( D(\dot{\varepsilon}_{ij}^p) \).

On the other hand, more than one stress states, say \( \sigma_{ij}^{(1)} \) and \( \sigma_{ij}^{(2)} \), may correspond to a given \( \dot{\varepsilon}_{ij}^p \) when the yield surface contains a flat face or a line as some portion of it. But \( D(\dot{\varepsilon}_{ij}^p) \) can be regarded as a single-valued function of \( \dot{\varepsilon}_{ij}^p \) since the difference \( (\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}) \) is always perpendicular to \( \dot{\varepsilon}_{ij}^p \), as illustrated in Fig. 5.7.

It is easy to show the following properties of \( D(\dot{\varepsilon}_{ij}^p) \).

a) \( D(\dot{\varepsilon}_{ij}^p) = \sigma_{ij} \dot{\varepsilon}_{ij}^p \) is homogeneous of degree-one of \( \dot{\varepsilon}_{ij}^p \) since \( \sigma_{ij} \) is independent of the magnitude of \( \dot{\varepsilon}_{ij}^p \).

b) The variation of \( D \), denoted by \( \delta D \), is calculated as

\[ \delta D = \sigma_{ij} \delta \dot{\varepsilon}_{ij}^p \]  \hspace{1cm} (5.10)

, by showing \( \delta \dot{\varepsilon}_{ij}^p = 0 \) which results from the normality rule (see Fig. 5.8).

c) \( D(\dot{\varepsilon}_{ij}^p) \) is convex in \( \dot{\varepsilon}_{ij}^p \) if it is continuously differentiable.

This can be proved as follows:
Chapter 5. Rigid-plastic finite element method

Let us consider the two states of stress and strain rate corresponding to each state of stress; \((\sigma_{ij}^{(1)}, \dot{\varepsilon}_{ij}^{p(1)})\) and \((\sigma_{ij}^{(2)}, \dot{\varepsilon}_{ij}^{p(2)})\), as shown in Fig. 5.9. According to the normality rule or the principle of maximum plastic work, we obtain

\[
\sigma_{ij}^{(1)} \dot{\varepsilon}_{ij}^{p(1)} \geq \sigma_{ij}^{(2)} \dot{\varepsilon}_{ij}^{p(1)}.
\] (5.11)

Subtract both sides of inequality (5.11) by \(\sigma_{ij}^{(2)} \dot{\varepsilon}_{ij}^{p(2)}\), we have

\[
\sigma_{ij}^{(1)} \dot{\varepsilon}_{ij}^{p(1)} - \sigma_{ij}^{(2)} \dot{\varepsilon}_{ij}^{p(2)} \geq \sigma_{ij}^{(2)} (\dot{\varepsilon}_{ij}^{p(1)} - \dot{\varepsilon}_{ij}^{p(2)}).
\] (5.12)

Then,

\[
D(\dot{\varepsilon}_{ij}^{p(1)}) - D(\dot{\varepsilon}_{ij}^{p(2)}) \geq \nabla D(\dot{\varepsilon}_{ij}^{p(2)})(\dot{\varepsilon}_{ij}^{p(1)} - \dot{\varepsilon}_{ij}^{p(2)})
\] (5.13)

which satisfies the characterizations of differentiable convex functions shown by inequality (B.4) in appendix B.

5.3 Rigid-plastic material

5.3.1 Stress-strain rate relations (associated flow rule)

A rigid-plastic material is considered to obey the Drucker-Prager yield criterion:

\[
f(\sigma_{ij}) = -\alpha I_1 + \sqrt{J_2} = k
\] (5.14)
5.3. Rigid-plastic material

where $\alpha$ and $k$ are the material constants. $\sigma_{ij}$ is the stress tensor. $I_1$ and $J_2$ denote respectively the first invariant of stress $\sigma_{ij}$ and the second invariant of the deviatoric stress component of $\sigma_{ij}$. $I_1$ and $J_2$ are defined as shown in Eq.(5.15).

$$I_1 = \sigma_{kk}, \quad J_2 = \frac{1}{2}s_{ij}s_{ij} \quad \text{where,} \quad s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}. \quad (5.15)$$

Note that $\delta_{ij}$ in Eq.(5.15) denotes the Kronecker’s delta symbol.

The yield surfaces are open and composed of an infinite number of linear generators as illustrated in Fig. 5.10. If $\alpha$ is more than zero, the yield surface becomes the cylindrical cone which is the Drucker-Prager yield surface. On the other hand, if $\alpha$ equals zero, the yield surface becomes cylinder which is the yield surface for the von-Mises material, as shown by the dotted-line cylinder in Fig. 5.10.

The stress-strain rate relations will be derived from the Drucker-Prager yield criterion(Eq.(5.14)) under the assumption of the associated flow rule(or normality rule) as shown in the following equation.

$$\dot{\varepsilon}_{ij} = \Lambda \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} \quad (5.16)$$

where, $\Lambda$ is a positive unknown variable. Note that, for rigid-plastic material, $\dot{\varepsilon}_{ij}$ denotes the plastic strain rate. From Eqs.(5.14) and (5.16), the following equation is obtained.

$$\dot{\varepsilon}_{ij} = \Lambda \left( -\alpha \delta_{ij} + \frac{s_{ij}}{2\sqrt{J_2}} \right) \quad (5.17)$$
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The magnitude of strain rate $\bar{\dot{e}}$ is defined as

$$\bar{\dot{e}} = \sqrt{\dot{e}_{ij} \dot{e}_{ij}}.$$  \hfill (5.18)

Hence, using Eq.(5.17),

$$\bar{\dot{e}}^2 = \dot{e}_{ij} \dot{e}_{ij} = \Lambda^2 \left(3\alpha^2 + \frac{1}{2}\right)$$  \hfill (5.19)

and

$$\Lambda = \frac{\sqrt{2}}{\sqrt{6}\alpha^2 + 1} \dot{e}$$  \hfill (5.20)

are obtained. Then, Eq.(5.17) can be written as

$$\dot{e}_{ij} = \frac{\sqrt{2}}{\sqrt{6}\alpha^2 + 1} \left(-\alpha \delta_{ij} + \frac{s_{ij}}{2\sqrt{J_2}}\right).$$  \hfill (5.21)

Multiplying $\delta_{ij}$ into both sides of Eq.(5.21), we have

$$\dot{e}_{kk} + \frac{3\sqrt{2}\alpha}{\sqrt{6}\alpha^2 + 1} \bar{\dot{e}} = 0,$$  \hfill (5.22)

where $\dot{e}_{kk}$ is defined as the volume change rate. Consequently, pulling the term $\dot{e}_{ij}$ out of the parenthesis in Eq.(5.21), we obtain

$$\dot{e}_{ij} \left(\delta_{ij} + \frac{3\sqrt{2}\alpha}{\sqrt{6}\alpha^2 + 1} \frac{\dot{e}_{ij}}{\bar{\dot{e}}}\right) = 0.$$  \hfill (5.23)

This equation indicates that the strain rate $\dot{e}_{ij}$ is perpendicular to $\left(\delta_{ij} + \frac{3\sqrt{2}\alpha}{\sqrt{6}\alpha^2 + 1} \frac{\dot{e}_{ij}}{\bar{\dot{e}}}\right)$. According to Fig. 5.11 and the normality rule, the strain rate $\dot{e}_{ij}$ must be perpendicular to the generator of the yield surface. Therefore, in case of the Drucker-Prager material, the direction of the generator can be represented by $\left(\delta_{ij} + \frac{3\sqrt{2}\alpha}{\sqrt{6}\alpha^2 + 1} \frac{\dot{e}_{ij}}{\bar{\dot{e}}}\right)$. To derive the stress-strain rate relation, the stress $\sigma_{ij}$ is divided into the principal component $\sigma_{ij}^{(1)}$ and the indeterminate component $\sigma_{ij}^{(2)}$, i.e,

$$\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}.$$  \hfill (5.24)

$\sigma_{ij}^{(1)}$ locates in the dual cone and has the same direction as that of $\dot{e}_{ij}$, while $\sigma_{ij}^{(2)}$ locates in the
5.3. Rigid-plastic material

Figure 5.11: Stress decomposition for the associated flow rule

Conical yield surface and points to one of the generators of the yield surface. More precisely, these stress components can be expressed by

\[
\sigma_{ij}^{(1)} = G_1 \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}} \quad \text{and,} \quad \sigma_{ij}^{(2)} = \lambda \left( \delta_{ij} + \frac{3\sqrt{2}\alpha}{\sqrt{6\alpha^2 + 1}} \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}} \right)
\]  

(5.25)  

(5.26)

where, \(G_1\) is a positive coefficient which can be determined by the fact that \(\sigma_{ij}^{(1)}\) fulfills the yield condition

\[
f(\sigma_{ij}^{(1)}) = k.
\]  

(5.27)

For the Drucker-Prager yield condition (Eq.(5.14)), it follows that

\[
G_1 = \frac{\sqrt{2}k}{\sqrt{6\alpha^2 + 1}}.
\]  

(5.28)

The indeterminate stress parameter \(\lambda\) is the unknown variable and will be determined after the whole problem is solved.

Conclusively, the stress \(\sigma_{ij}\) can be expressed as follows:

\[
\sigma_{ij} = \frac{\sqrt{2}k}{\sqrt{6\alpha^2 + 1}} \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}} + \lambda \left( \delta_{ij} + \frac{3\sqrt{2}\alpha}{\sqrt{6\alpha^2 + 1}} \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}} \right)
\]  

(5.29)

\[
= (G_1 + G_2\lambda) \frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}} + \lambda \delta_{ij}
\]
Chapter 5. Rigid-plastic finite element method

in which

\[ G_2 = \frac{3\sqrt{2}\alpha}{\sqrt{6\alpha^2 + 1}} \]  \tag{5.30}

### 5.3.2 Stress-strain rate relations (non-associated flow rule)

The stress-strain rate relation based upon the associated flow rule often produces too large volume expansion due to the dilatancy, when the large angle of internal friction is considered. In order to decrease the effect of the dilatancy without changing the internal friction angle, the non-associated flow rule was adopted. According to Fig. 5.12, stress is satisfied by the yield criterion but the plastic potential is not the same as the yield surface, i.e., not parallel to the generator of the Drucker-Prager yield surface. The plastic potential function for the non-associated flow rule, given by

\[ g(\sigma_{ij}) = -\bar{\alpha}I_1 + \sqrt{J_2} = \bar{k} \]  \tag{5.31}

is assumed to be the conical shape and any strain rate \( \dot{\varepsilon}_{ij} \) must be normal to this plastic potential. Note that the plastic potential given by Eq.(5.31) is similar to Eq.(5.14) but the material constants, i.e., \( \bar{\alpha} \) and \( \bar{k} \), are different. The condition for the normality to the potential is

\[ \dot{\varepsilon}_{ij} = \bar{\Lambda} \frac{\partial g(\sigma_{ij})}{\partial \sigma_{ij}} \]  \tag{5.32}

where, \( \bar{\Lambda} \) is a positive unknown variable. By using similar procedure as the associated flow rule, we have

\[ \frac{\dot{\varepsilon}_{ij}}{\bar{\varepsilon}} = \frac{\sqrt{2}}{\sqrt{6\bar{\alpha}^2 + 1}} \left( -\bar{\alpha}\delta_{ij} + \frac{s_{ij}}{2\sqrt{J_2}} \right) \]  \text{and,} \tag{5.33}

\[ \dot{\varepsilon}_{kk} + \frac{3\sqrt{2}\bar{\alpha}}{\sqrt{6\bar{\alpha}^2 + 1}} \bar{\varepsilon} = 0. \]  \tag{5.34}

The stress tensor is divided again into two components and the same forms are assumed as in Eqs.(5.25) and (5.26):

\[ \sigma_{ij}^{(1)} = \bar{G}_1 \frac{\dot{\varepsilon}_{ij}}{\bar{\varepsilon}} \]  \tag{5.35}

\[ \sigma_{ij}^{(2)} = \bar{\Lambda} \left( \delta_{ij} + \frac{3\sqrt{2}\bar{\alpha}}{\sqrt{6\bar{\alpha}^2 + 1}} \frac{\dot{\varepsilon}_{ij}}{\bar{\varepsilon}} \right). \]  \tag{5.36}
5.3. Rigid-plastic material

5.3.1. Stress decomposition for the non-associated flow rule

The indeterminate parameter \( \lambda \) plays the same role as \( \lambda \) in the case of the associated flow rule. It is unknown and will be determined when the whole problem is solved.

Since, the stress \( \sigma_{ij} \) must satisfy the yield condition (Eq. 5.14):

\[
f(\sigma_{ij}) = f(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) = k.
\] (5.37)

Then, the parameter \( G_1 \) can be obtained as

\[
G_1 = \sqrt{2} \frac{3(\alpha - \overline{\alpha}) \lambda + (1 + 6\alpha^2)k}{(1 + 6\alpha^2) \sqrt{6\alpha^2 + 1}}
\] (5.38)

Substituting \( G_1 \) into Eqs.(5.35) and (5.36), the stress-strain rate relation can be written as

\[
\sigma_{ij} = \sqrt{2} \frac{3(\alpha - \overline{\alpha}) \lambda + (1 + 6\alpha^2)k}{(1 + 6\alpha^2) \sqrt{6\alpha^2 + 1}} \dot{\varepsilon}_{ij} + \lambda \left( \delta_{ij} + \frac{3\sqrt{2}\alpha}{\sqrt{6\alpha^2 + 1}} \frac{\dot{\varepsilon}_{ij}}{\dot{c}} \right)
\] (5.39)

where

\[
G_2 = \frac{3\sqrt{2}\alpha}{\sqrt{6\alpha^2 + 1}}.
\] (5.40)

There is almost no difference between the cases for the associated and non-associated flow rules from the viewpoint of the numerical analysis. It is possible to derive the material constants for the Drucker-Prager yield criterion for the non-associated flow rule corresponding to Mohr-
Coulomb’s yield criterion in plane strain condition:

\[ k = \begin{cases} 
\frac{3c\alpha}{\tan \phi} & \text{if } \phi > 0 \\
c & \text{if } \phi = 0 
\end{cases} \]  
(5.41)

\[ \alpha = \begin{cases} 
\frac{(3 + 2r \sin^2 \phi) - \sqrt{9 + 12r(1-r)\sin^2 \phi}}{6r^2(3 + \sin^2 \phi)} & \text{if } r \neq 0 \\
\frac{\sin \phi}{3} & \text{if } r = 0 
\end{cases} \]  
(5.42)

where

\[ \bar{\sigma} = r\alpha \quad (0 \leq r \leq 1) \]  
(5.43)

and \( c \) and \( \phi \) denote respectively the cohesion and the angle of internal friction of the Mohr-Coulomb material.

It should be noted that when \( r = 1 \), the non-associated flow rule becomes the same as the associated flow rule. \( \alpha \) in Eq.(5.42) can be reduced in the form:

\[ \alpha = \frac{\sin \phi}{\sqrt{3(3 + \sin^2 \phi)}}. \]  
(5.44)
5.4 Fundamental equations and boundary conditions for rigid-plastic analysis

Region $V$ of a rigid-plastic material is divided into two disjointed subsets $S_{\sigma}$ and $S_u$ as shown in Fig. 5.13. The material is subjected to a body force $\overrightarrow{f_i}$ and a surface traction $\overrightarrow{T_i}$ on the stress boundary $S_{\sigma}$. The smooth velocity field $\dot{u}_i$ is defined at each point of the velocity boundary $S_u$. The external forces such as $\overrightarrow{T_i}$, $\overrightarrow{f_i}$ are multiplied by the load factor $\mu$ under which the whole region of the body begins to yield and flow with a constant rate of deformation. This steady flow is governed by the equations of equilibrium:

$$\sigma_{ij,j} + \mu f_i = 0 \quad \text{in } V$$  \hspace{1cm} (5.45)

the stress boundary conditions:

$$-\sigma_{ij} n_j = \mu T_i \quad \text{on } S_{\sigma}$$  \hspace{1cm} (5.46)

and the velocity boundary conditions:

$$\begin{cases} \text{Homogeneous velocity boundary conditions:} & \dot{u}_i = 0 \quad \text{on } S_u \\ \text{Non-homogeneous velocity boundary conditions:} & \dot{u}_i = \overline{u}_i \quad \text{on } S_u \end{cases}$$  \hspace{1cm} (5.47)

where $n_i$ denotes the outward unit normal on $S_{\sigma}$. By adopting the small deformation theory, the compatibility conditions can be written as

$$\dot{\epsilon}_{ij} = -\frac{1}{2} (\dot{u}_{i,j} + \dot{u}_{j,i}) .$$  \hspace{1cm} (5.48)

5.5 Upper-bound theorem for rigid-plastic analysis

The concept of the upper-bound theorem is described below. The loads, determined by equating the external rate of work to the internal rate of dissipation in an assumed deformation mode(or velocity field) that satisfies: (a) velocity boundary condition (Eq.(5.47)); and (b) strain and velocity compatibility conditions (Eq.(5.48)), are not less than
the actual collapse load (Chen, 1975). A velocity field satisfying the above conditions has been termed a *kinematically admissible velocity* field. The dissipation of energy in plastic flow $D(\dot{\varepsilon}^p_{ij})$ associated with such a field can be computed from the stress-strain rate relation.

According to the above-mentioned concept of the upper-bound theorem, assume that the external forces $(f_i, T_i)$ be given and introduce the subset $K_p (\subset K)$:

$$K_p = \left\{ (\ddot{u}_i, \dot{\varepsilon}_{ij}) \in K \mid \int_V f_i \ddot{u}_i dV + \int_{S_p} T_i \dot{u}_i dS > 0 \right\} \quad (5.49)$$

For $(\ddot{u}_i, \dot{\varepsilon}_{ij}) \in K_p$, define the load factor $\mu$ by

$$\mu \left\{ \int_V f_i \ddot{u}_i dV + \int_{S_p} T_i \dot{u}_i dS \right\} = \int_V D(\dot{\varepsilon}_{ij}) dV = \int_V \sigma_{ij} \dot{\varepsilon}_{ij} dV \quad (5.50)$$

The upper-bound theorem states that

$$\mu^* \leq \mu \quad (5.51)$$

where $\mu^*$ is the load factor at the limit state. To prove Eq.(5.51), we first compare Eq.(5.50) with

$$\mu^* \left\{ \int_V f_i \ddot{u}_i dV + \int_{S_p} T_i \dot{u}_i dS \right\} = \int_V \sigma_{ij}^* \dot{\varepsilon}_{ij} dV \quad (5.52)$$

which represents the principle of virtual work for a kinematical field $(\ddot{u}_i, \dot{\varepsilon}_{ij})$ and a equilibrium system $\mu^*(f_i, T_i), \sigma_{ij}^*$. Subtracting Eq.(5.50) by Eq.(5.52), we obtain

$$(\mu - \mu^*) \left\{ \int_V f_i \ddot{u}_i dV + \int_{S_p} T_i \dot{u}_i dS \right\} = \int_V (\sigma_{ij} - \sigma_{ij}^*) \dot{\varepsilon}_{ij} dV \quad (5.53)$$

Considering the definition of $K_p$ and the principle of maximum plastic work(Eq.(5.7)), we obtain Eq.(5.51).

### 5.6 Formulation of rigid-plastic finite element method (RPFEM)

#### 5.6.1 Description of the problem

The formulation of rigid-plastic finite element method starts from defining the main problem by considering the upper-bound theorem and some constraint conditions. According to section
5.6. Formulation of rigid-plastic finite element method (RPFEM)

5.5, it is understood that $\mu^*$ is searched as the minimum value of $\mu$ defined as

$$
\mu = \frac{\int_V D(\dot{\varepsilon}_{ij})dV}{\int_V f_i \dot{u}_i dV + \int_{S_p} T_i \dot{u}_i dS} \quad (5.54)
$$

by surveying all of the elements in $K_p$.

Since a rigid-plastic material is rate-insensitive, it is possible to specify the magnitude of velocity without loss of generality. Hence

$$
\int_V f_i \dot{u}_i dV + \int_{S_p} T_i \dot{u}_i dS = 1 \quad (5.55)
$$

can be assumed. Furthermore, the constraint condition on strain rate $\dot{\varepsilon}_{ij}$ i.e., Eq.(5.22) for associated flow rule or Eq.(5.34) for non-associated flow rule, must be taken into account. Also, if there is additional constraint condition on velocity field, it is added in the consideration also.

As mentioned above, it can be concluded that the problem becomes equivalent to

---

**Problem 1**

**Minimize** \[\int_V D(\dot{\varepsilon}_{ij})dV\]

**subject to** \[\int_V f_i \dot{u}_i dV + \int_{S_p} T_i \dot{u}_i dS = 1\]

\[\Rightarrow\] **Constraint condition on strain rate $\dot{\varepsilon}_{ij}$:**

- Eq.(5.22) for associated flow rule, or
- Eq.(5.34) for non-associated flow rule

\[\Rightarrow\] **Additional constraint conditions:**

\[a_1 \dot{u}_i = 0\]
\[a_2 \dot{u}_i = 0\]
\[a_3 \dot{u}_i = 0\]
\[\vdots\]

where $a_1$, $a_2$, $a_3$, ... are the coefficients for each constraint condition.
5.6.2 Numerical procedure

According to the description on the rigid-plastic material mentioned in section 5.3 and the concept of upper-bound theorem mentioned section 5.5, the unknown variables of the overall problem are assigned differently as follows:

- velocities at limit state $\dot{u}_i$ are assigned for each nodal point,
- the indeterminate magnitude of the isotropic stress component $\lambda$ or $\lambda$ is assigned for each element,
- the load factor $\mu$ which indicates the limit state and,
- other parameters associated to the additional constraint conditions.

Now, the above-mentioned problem 1 is investigated by the finite element technique. In this dissertation, the isoparametric quadrilateral elements composed of 4 nodes and hexahedral elements composed of 8 nodes are used in 2D and 3D analysis, respectively. The notation necessary to formulate the rigid-plastic finite element analysis will be introduced as follows:

$\dot{u}_e$: vector of nodal velocities in each element

$\dot{u}$: vector of all nodal velocities

$\sigma_e$: vector of stresses of an element defined as

$$\sigma_e = \begin{cases} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}^T & \text{for 2D analysis} \\ \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}\}^T & \text{for 3D analysis} \end{cases} \quad (5.56)$$

$\sigma$: vector of stresses of all elements

$\dot{\epsilon}_e$: vector of engineering strain rate of an element, defined as

$$\dot{\epsilon}_e = \begin{cases} \{\dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}, \dot{\gamma}_{xy}\}^T & \text{for 2D analysis} \\ \{\dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}, \dot{\epsilon}_{zz}, \dot{\gamma}_{xy}, \dot{\gamma}_{yz}, \dot{\gamma}_{zx}\}^T & \text{for 3D analysis} \end{cases} \quad (5.57)$$

$\dot{\epsilon}$: vector of engineering strain rate of all elements
\( \hat{\mathbf{E}}_e \): vector of tensorial strain rate \( \hat{\varepsilon}_{ij} \) of an element, defined as

\[
\hat{\mathbf{E}}_e = \begin{cases} 
\{\hat{\varepsilon}_{xx}, \hat{\varepsilon}_{yy}, \hat{\varepsilon}_{xy}\}^T & \text{for 2D analysis} \\
\{\hat{\varepsilon}_{xx}, \hat{\varepsilon}_{yy}, \hat{\varepsilon}_{zz}, \hat{\varepsilon}_{xy}, \hat{\varepsilon}_{yz}, \hat{\varepsilon}_{zx}\}^T & \text{for 3D analysis}
\end{cases}
\] (5.58)

\( \mathbf{F} \): vector of all nodal forces

\( \mathbf{B}_e \): strain rate-velocity matrix for an element, defined as shown in the compatibility condition as

\[
\hat{\varepsilon}_e = \mathbf{B}_e \hat{\mathbf{u}}_e
\] (5.59)

\( \mathbf{B} \): strain rate-velocity matrix defined as

\[
\hat{\varepsilon} = \mathbf{B} \hat{\mathbf{u}}
\] (5.60)

Therefore, according to Eq.(5.59), the relationship between vector of engineering strain rate for each element \( \hat{\varepsilon}_e \) and the vector of tensorial strain rate \( \hat{\mathbf{E}}_e \) is written as

\[
\hat{\mathbf{E}}_e = \mathbf{Q} \hat{\varepsilon}_e \\
= \mathbf{Q} \mathbf{B}_e \hat{\mathbf{u}}_e
\] (5.61)

in which, for 2D analysis

\[
\mathbf{Q} = \begin{pmatrix} 
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\] (5.62)

and for 3D analysis,

\[
\mathbf{Q} = \begin{pmatrix} 
1 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 1 & 0 & \ldots & \ldots & 0 \\
\vdots & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & 0 & \frac{1}{2} & 0 & 0 \\
\vdots & \vdots & \vdots & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}
\] (5.63)
According to Eq.(5.18), the magnitude of strain rate $\tilde{\varepsilon}$ can be written as

$$\tilde{\varepsilon} = \sqrt{\dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}}$$

$$= \sqrt{\dot{\varepsilon}_e^T B_e^T Q B_e \dot{\varepsilon}_e}$$

(5.64)

The volume change rate $\dot{\varepsilon}_{kk}$ is also written as

$$\dot{\varepsilon}_{kk} = \int_V \delta^T B_e \dot{\varepsilon}_e \, dV$$

$$= L_e \dot{\varepsilon}_e$$

(5.65)

where the matrix $L_e$ is defined as

$$L_e = \int_V \delta^T B_e \, dV$$

(5.66)

and the Kronecker’s delta vector $\delta$ is written as

$$\delta = \begin{cases} 
1, 1, 0^T & \text{for 2D analysis} \\
1, 1, 1, 0, 0^T & \text{for 3D analysis} 
\end{cases}$$

(5.67)

Furthermore, the vector of rates of volume change $\dot{v}$ of all elements is defined as

$$\dot{v} = L \dot{u}$$

(5.68)

in which $L$ consists of $L_e$ of each element.

Therefore the problem 1 as mentioned above can be written in the matrix forms as follows:
Minimize \[ \int_V D(\dot{\varepsilon}_{ij})dV \]
subject to \[ F^T \dot{u} = 1 \]
\[ L \dot{u} + G_2 \int_V \varepsilon dV = 0 \] for associated flow rule
\[ L \dot{u} + \overline{G}_2 \int_V \varepsilon dV = 0 \] for non-associated flow rule
\[ A_i \dot{u} = 0 \] for each additional constraint condition.

Note that the constraint conditions for strain rate \( \dot{\varepsilon}_{ij} \) are applied to each element (not to all elements).

Suppose \( \dot{u} \) be a solution of problem 2. It means
\[ \int_V D(\dot{u} + \Delta \dot{u}) \geq \int_V D(\dot{u})dV \] (5.69)
for an arbitrary \( \Delta \dot{u} \) which satisfies
\[ F^T \Delta \dot{u} = 0 \] (5.70)
\[ \left( L^T + \overline{G}_2 \int_V \frac{B^T Q B}{\varepsilon} \dot{u} \right)^T \Delta \dot{u} = 0 \] (5.71)
\[ A_i \Delta \dot{u} = 0 \] (5.72)

Remember Eq.(5.10) and we have
\[ \Delta \int_V D(\dot{u}) dV = \int_V \sigma^T \Delta \dot{\varepsilon} dV = \left( \int_V \sigma^T B dV \right) \Delta \dot{u} \]
\[ = \left( \int_V B^T \sigma^{(1)} dV \right)^T \Delta \dot{u} = 0 \] (5.73)
in which the last equation is justified apparently since the isotropic component of stress does
no work for \(\Delta \mathbf{u}\) which satisfies \(\mathbf{L} \dot{\mathbf{u}} + \bar{G}_2 \int_V \mathbf{e} dV = 0\).

Now, a basic theorem in algebra that a system of linear equation \(\mathbf{A} \mathbf{x} = \mathbf{b}\) has at least one
solution \(\mathbf{x}\) if and only if \(\mathbf{b}^T \mathbf{y} = 0\) for an arbitrary \(\mathbf{y}\) such that \(\mathbf{A}^T \mathbf{y} = 0\), is taken into account.
The system of equations composed of Eqs.(5.70), (5.71) and (5.72) can be written as
\[
\begin{pmatrix}
\mathbf{F}^T \\
\left( \mathbf{L}^T + \bar{G}_2 \int_V \frac{\mathbf{B}^T \mathbf{Q} \mathbf{B}}{\mathbf{e}} \dot{\mathbf{u}} dV \right)^T \\
\mathbf{A}_i
\end{pmatrix}
\begin{pmatrix}
\Delta \dot{\mathbf{u}} \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\] (5.74)
where matrix \(\mathbf{A}\) and \(\Delta \dot{\mathbf{u}}\) are supposed by \(\left( \mathbf{F}^T, \left( \mathbf{L}^T + \bar{G}_2 \int_V \frac{\mathbf{B}^T \mathbf{Q} \mathbf{B}}{\mathbf{e}} \dot{\mathbf{u}} dV \right)^T, \mathbf{A}_i^T \right)\) and vector \(\mathbf{y}\), respectively. Also, according to Eq.(5.73), \(\int_V \mathbf{B}^T \sigma^{(1)} dV\) is supposed by \(\mathbf{b}\). Hence,
by considering the above-mentioned theorem on algebra, we can conclude that \(\mathbf{A} \mathbf{x} = \mathbf{b}\), i.e.,
\[
\begin{pmatrix}
\mathbf{F}, \left( \mathbf{L}^T + \bar{G}_2 \int_V \frac{\mathbf{B}^T \mathbf{Q} \mathbf{B}}{\mathbf{e}} \dot{\mathbf{u}} dV \right), \mathbf{A}_i^T
\end{pmatrix}
\begin{pmatrix}
\mu \\
\lambda \\
\nu
\end{pmatrix}
= 
\int_V \mathbf{B}^T \sigma^{(1)} dV
\] (5.75)
in which \(\bar{\lambda}\) is composed of the indeterminate stress parameters \(\bar{\lambda}\) of all elements.
Finally, for non-associated flow rule, the system of equations of the overall problem can be
written as
\[
\left\{ \int_V \left( \mathbf{G}_1 + \lambda \mathbf{G}_2 \right) \frac{\mathbf{B}^T \mathbf{Q} \mathbf{B}}{\mathbf{e}} dV \right\} \mathbf{\dot{u}} + \mathbf{L}^T \bar{\lambda} - \mathbf{A}_i^T \nu - \mu \mathbf{F} = 0
\] (5.76)
\[
\mathbf{F}^T \mathbf{\dot{u}} - 1 = 0
\] (5.77)
\[
\mathbf{L} \mathbf{\dot{u}} + \bar{G}_2 \int_V \mathbf{\dot{e}} dV = 0
\] (5.78)
\[
\mathbf{A}_i \mathbf{\dot{u}} = 0
\] (5.79)
where Eq.(5.76) is called the equations of equilibrium. Eqs.(5.77), (5.78) and (5.79) are the constraint conditions. And for associated flow rule,

\[
\left\{ \int_V \left( G_1 + \lambda G_2 \right) \frac{B^T Q B}{\mathbf{\varepsilon}} \, dV \right\} \dot{\mathbf{u}} + \mathbf{L}^T \lambda - \mathbf{A}_i^T \nu - \mu \mathbf{F} = 0 \quad (5.80)
\]

\[
\mathbf{F}^T \dot{\mathbf{u}} - 1 = 0 \quad (5.81)
\]

\[
\mathbf{L} \dot{\mathbf{u}} + G_2 \int_V \mathbf{\varepsilon} \, dV = 0 \quad (5.82)
\]

\[
\mathbf{A}_i \dot{\mathbf{u}} = 0 \quad (5.83)
\]

are obtained.

### 5.6.3 Newton-Raphson method

According to the above-mentioned system of equations, \( \mathbf{\varepsilon} \) is a nonlinear function of the velocities \( \dot{\mathbf{u}} \), therefore, the system of equations will be solved iteratively by using the Newton-Raphson method.

Let us consider a non-linear function

\[
f(x) = 0 \quad (5.84)
\]

To solve the above equation, the Newton-Raphson method requires the evaluation of both the function \( f(x) \) and the derivative \( f'(x) \), at arbitrary points \( x \) (Press et al., 1992). The function \( f(x) \) is approximately derived from Taylor series expansion of a function in the neighborhood of point \( x \),

\[
f(x + \Delta x) = f(x) + \Delta x f'(x) + \Delta x^2 \frac{f''(x)}{2} + \ldots = 0 \quad (5.85)
\]

If \( \Delta x \) is small enough, the terms beyond linear are unimportant, hence \( f(x + \Delta x) = 0 \) implies

\[
\Delta x = - \frac{f(x)}{f'(x)} \quad \text{or} \quad f'(x) \Delta x = -f(x) \quad (5.86)
\]

To obtain the solution of \( f(x) = 0 \), the iterative analysis is required. Let us start from the initial \( x_0 \). The next step \( x_1 \) is illustrated by

\[
x_1 = x_0 + \Delta x = x_0 - \frac{f(x_0)}{f'(x_0)} \quad (5.87)
\]
More generally,

\[ x_{i+1} = x_i + \Delta x = x_i - \frac{f(x_i)}{f'(x_i)} \]  \hspace{1cm} (5.88)

can be written for step \( i \).

Newton-Raphson is not restricted to one dimension. The method readily generalizes to multiple dimensions. According to the system of equations mentioned in the preceding section, the unknown variables can be determined by

\[
\begin{align*}
\dot{u}_{i+1} &= \dot{u}_i + \Delta \dot{u} \\
\lambda_{i+1} &= \lambda_i + \Delta \lambda \\
\mu_{i+1} &= \mu_i + \Delta \mu \\
\nu_{i+1} &= \nu_i + \Delta \nu
\end{align*}
\] for non-associated flow rule

(5.89)

Note that, for associated flow rule, \( \lambda \) is considered instead of \( \bar{\lambda} \).

For associated flow rule, determine the derivatives of Eqs.(5.80), (5.82), (5.81) and (5.83) and make a system of equations following the form shown in Eq.(5.86) as

\[
\begin{pmatrix}
(G_1 + G_2\lambda) \int_V \left( \frac{B^TQB}{\varepsilon} - \frac{B^TQB\alpha \sigma^T B^TQB}{\varepsilon} \right) dV - L + G_2 \int_V \frac{B^TQB}{\varepsilon} \dot{u} dV & L^T + G_2 \int_V \frac{B^TQB}{\varepsilon} \dot{u} dV - F - A_i^T \\
\int_V \frac{B^TQB}{\varepsilon} \dot{u} dV & -F^T - A_i \\
-L - G_2 \left( \int_V \frac{B^TQB}{\varepsilon} \dot{u} dV \right)^T & F^T & A_i^T & \frac{\Delta \dot{u}}{} & \frac{\Delta \lambda}{\Delta \mu} & \frac{\Delta \nu}{-1}
\end{pmatrix} \begin{pmatrix}
\Delta \dot{u} \\
\Delta \lambda \\
\Delta \mu \\
\Delta \nu
\end{pmatrix} = \begin{pmatrix}
F_b
\end{pmatrix}
\]  \hspace{1cm} (5.90)
For non-associated flow rule, we obtain

\[
\begin{pmatrix}
    \left( \mathcal{C}_1 + \mathcal{C}_2 \overline{x} \right) f_v \left( \frac{B^T Q_B}{\overline{e}} \right) dV \\
    \mathcal{L} + \mathcal{C}_2 \left( f_v \frac{B^T Q_B}{\overline{e}} \hat{u} dV \right)^2 \\
    -F^T
\end{pmatrix}
\begin{pmatrix}
    \frac{\Delta u}{\Delta \lambda} \\
    \frac{\Delta \lambda}{\Delta \mu} \\
    \frac{\Delta \mu}{\Delta \nu}
\end{pmatrix}
= 
\begin{pmatrix}
    A_i \\
    -F_b
\end{pmatrix}
\]

where \( F_b \) is a loading vector which is constant through the analysis and

\[
\mathcal{G}_2^\otimes = \frac{3\sqrt{2\alpha\sqrt{6\alpha^2 + 1}}}{1 + 6\alpha\alpha}
\]

Eqs. (5.90) and (5.91) are the system of linear equations. The unknown variables \( \Delta \dot{u}, \Delta \lambda (\Delta \overline{x}), \Delta \mu \) and \( \Delta \nu \) of each step are determined by solving the system of linear equations. Furthermore, in order to guarantee the convergence of the iterated solutions, the positive constant \( \eta \) is introduced to reduce the increment \( \Delta \) for each step so that

\[
\begin{align*}
\dot{u}_{i+1} &= \dot{u}_i + \eta \Delta \dot{u} \\
\overline{x}_{i+1} &= \overline{x}_i + \eta \Delta \overline{x} & \text{for non-associated flow rule} \\
\mu_{i+1} &= \mu_i + \eta \Delta \mu \\
\nu_{i+1} &= \nu_i + \eta \Delta \nu
\end{align*}
\]

For associated flow rule, \( \lambda \) is used instead of \( \overline{x} \).

5.7 RPFEM with consideration of groundwater

In this section, the methodology to include the groundwater into the rigid-plastic analysis is described. The critical state of the soil structures is evaluated by determining the load factor \( \mu \) which magnifies the acceleration of gravity \( g \). Therefore the actual collapse occurs under the \( 1g \) condition, if the obtained load factor \( \mu \) is smaller than 1. This evaluation method has a
similar concept to that of the centrifugal modeling test. At the critical state, not only the unit weight of soil $\gamma$ but also the pore water pressure $p$ is multiplied by the load factor $\mu$.

To include the groundwater in the analysis, the following three factors are taken into account.

- Change in the total self-weight due to the change in the groundwater table.
- Change in the effective stress due to the change in the total self-weight and pore water pressure.
- Strength of the ground due to the change in the effective stress.

### 5.7.1 Self-weight of soil

According to Fig. 5.14(a), the soil above the groundwater table is treated as the dry soil and its unit weight is denoted by dry unit weight $\gamma_d$. On the other hand, the soil under the groundwater table is considered as the saturated soil and its total unit weight is denoted by the saturated unit weight $\gamma_s$. The unit weight of the soil element through which the groundwater table passes as shown in Fig. 5.14(b), is estimated as follows:

$$\gamma = (1 - \alpha_\gamma)\gamma_d + \alpha_\gamma\gamma_s.$$  \hspace{1cm} (5.94)

In Eq.(5.94), the coefficient $\alpha_\gamma$ is defined by Eq. 5.95.

$$\alpha_\gamma = \frac{H_c - y_{\min}}{y_{\max} - y_{\min}}$$  \hspace{1cm} (5.95)

where $y_{\max}$ and $y_{\min}$ denote the maximum $y$ coordinate and minimum $y$ coordinate of the element, respectively, while $H_c$ is the water head at the center of the element as shown in Fig. 5.14(b).

### 5.7.2 Pore water pressure

The distribution of the total water head $H$ in the steady seepage state is determined in advance. The pore water pressure $p$ for each element can be calculated by averaging the pore water
pressures of all 4 nodes as shown in the following equation.

\[ p_i = \begin{cases} 
\gamma_w (H_i - y_i) & \text{if } H_i \geq y_i \\
0 & \text{if } H_i < y_i 
\end{cases} \] (5.96)

where \(\gamma_w\) is the unit weight of water. \(y_i\) and \(H_i\) denote, respectively, the \(y\) coordinate and the water head of the nodal point \(i\). The pore water pressure \(p\) of each element is calculated by

\[ p = \frac{\sum_{i=1}^{4} p_i}{4}. \] (5.97)

### 5.7.3 Effective stress

The effective stress \(\sigma'_{ij}\) is equal to the total stress \(\sigma_{ij}\) subtracted by the pore water pressure \(p\), as follows:

\[ \sigma'_{ij} = \sigma_{ij} - \mu \delta_{ij} p \] (5.98)

where \(\delta_{ij}\) is the Kronecker delta.

The rigid plastic finite element method with the consideration of groundwater is carried out by considering the soil as non-homogenous soil i.e., the unit weight is varied by the groundwater table. The important point is the effective stress \(\sigma'_{ij}\) is considered in the stress-strain rate
relation based on the Drucker-Prager yield criterion (Eq.(5.29) or Eq.(5.39)) and the stress
tensor in the equations of equilibrium(Eq.(5.45)) is the *total stress* tensor. We can conclude the
above-mention point as follows:

Equations of equilibrium: \[ \sigma_{ij,j} + \mu \bar{f}_i = 0 \] (5.99)

where \( \sigma_{ij} \) is the *total stress* tensor and \( \bar{f}_i \) is the body force vector. Therefore, combining
Eq.(5.98) with Eq.(5.39), the total stress-strain rate relation for non-associated flow rule can be
written as

\[ \sigma_{ij} = (G_1 + G_2 \lambda) \frac{\dot{\varepsilon}_{ij}}{\dot{e}} + (\lambda + \mu p) \delta_{ij} \] (5.100)

Similarly, combining Eq.(5.98) with Eq.(5.29), the total stress-strain rate relation for associated
flow rule

\[ \sigma_{ij} = (G_1 + G_2 \lambda) \frac{\dot{\varepsilon}_{ij}}{\dot{e}} + (\lambda + \mu p) \delta_{ij} \] (5.101)

is obtained.

Hence, the equations of equilibrium for associated flow rule, can be written as

\[
\left\{ \int_V (G_1 + \lambda G_2) \frac{B^T Q B}{\dot{e}} \, dV \right\} \dot{u} + L^T (\lambda + \mu P) - A_i^T \nu - \mu F = 0
\] (5.102)

and that for non-associated flow rule is written as

\[
\left\{ \int_V (G_1 + \lambda G_2) \frac{B^T Q B}{\dot{e}} \, dV \right\} \dot{u} + L^T \lambda - A_i^T \nu - \mu (F - L^T P) = 0
\] (5.103)

where \( P \) consists of the pore water pressure \( p \) of all elements.

Consequently, *the systems of equations to be solved can be written by substituting \( F \) in Eqs.(5.90)
or (5.91) by \( (F - L^T P) \).
Chapter 6

Numerical analysis of the tunnel stability in soft clay

6.1 Introduction

The safety of the tunnel construction in clay is usually considered into two problems. Firstly, the stability of tunnel heading (area near the face) is necessary to be evaluated for the safety of the construction. Secondly, the ground deformation due to the construction is also required to be determined to prevent damage to existing surface or subsurface structures. This chapter focuses on the first problem which is concerned with the determination of the tunnel pressures $\sigma_T$ that are necessary to maintain the stability of the tunnel heading and of the acceleration factor of the weight of soil mass that causes the collapse of the tunnel. The stability of tunnel heading is evaluated by using the rigid-plastic finite element method and the calculated tunnel pressure $\sigma_T$ and calculated acceleration factor $n$ (or load factor $\mu$ for RPFEM) are compared with the observed values from the centrifugal modeling tests conducted by Mair (Mair, 1979). In the last section of the chapter, the effect of surface load on the stability of tunnel face is also studied through various distance of the surface load apart from the tunnel face.
Chapter 6. Numerical analysis of the tunnel stability in soft clay

6.2 Centrifugal modeling tests by Mair

Mair carried out the experiments to investigate the stability of tunnel both in two-dimensional cross-section tunnel and in three-dimensional tunnel headings (Mair, 1979). These experiments were carried out by using two apparatuses i.e., small and large ones. The small apparatus was used in the two-dimensional cross-section tunnel tests and the three-dimensional tunnel heading test A, while the large apparatus was used in the tunnel heading tests B, B’ and C, which are explained below. The experimental method was centrifugal modeling using the Cambridge Geotechnical Centrifuge.

6.2.1 Two-dimensional cross-section tunnel tests

Two-dimensional test on cross-section tunnels in clay was carried out in order to investigate the overall stability of tunnel, deformations and pore-pressure responses around tunnel. Fig. 6.1 shows the typical dimension of the 2D cross-section tunnel test model using the small apparatus. ➀ in Tables 6.1 and 6.2 show the summary of the 2D cross-section tunnel test. In the test, the tunnel support pressure at equilibrium state (the acceleration factor $n = 75$), which is equal to the overburden pressure at the spring line, was applied to the tunnel and then gradually reduced until the collapse of tunnel occurred. The tunnel pressure at collapse is denoted by $\sigma_{TC}$. 

Figure 6.1: Typical dimensions of clay models (small apparatus) in 2D cross-section test series
6.3. Numerical analysis of 2D cross-section tunnel

![Diagram of a 3D tunnel cross-section with dimensions: 660 mm x 400 mm x 390 mm, labeled P, C, and D, with support lining.]  

**Figure 6.2:** Typical dimensions of clay models in 3D test series (large apparatus)

### 6.2.2 Tunnel heading tests

Further test series modeled the tunnel headings and investigated the influence of the non-supported heading length on the deformation behavior and stability. A summary of tunnel heading tests is given by \( ②, ③, ④, ⑤ \) in Tables 6.1 and 6.2. \( P \) is the unlined length (non-supported heading length). Four categories of tests were undertaken, namely the tunnel heading tests A, B, B' and C. All tests in the tunnel heading test A were carried out in the small apparatus by increasing the acceleration factor \( n \), without applied tunnel pressure at the tunnel face, until the collapse of tunnel occurred. After proof-testing and commissioning the new large apparatus on the centrifuge, the tunnel heading tests B and B' were conducted following a similar procedure to the tunnel heading test A in order to compare the behavior of a plane strain heading with a circular tunnel heading. For the tunnel heading test C, the acceleration factor \( n \) was fixed at 118 and the tunnel pressure when the collapse of tunnel occurred was investigated, which is similar to the 2D cross-section test(①). The typical dimension of clay models in the tunnel heading tests B and C is shown in Fig. 6.2.

### 6.3 Numerical analysis of 2D cross-section tunnel

According to ① in Table 6.1, the eight successful tests, denoted by 2DB to 2DV from Mair’s tests were selected to be simulated using the rigid-plastic finite element method to determine the tunnel pressure at collapse. In the numerical analysis, the size of the finite element mesh...
Chapter 6. Numerical analysis of the tunnel stability in soft clay

region is, as shown in Table 6.2 for 2D cross-section tunnel tests(1), the same as the test model in Fig. 6.1.

**Table 6.1: Summary of all tunnel test series by Mair**

<table>
<thead>
<tr>
<th>Test no.</th>
<th>D (mm)</th>
<th>C/D</th>
<th>P/D</th>
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<tr>
<td>3DD/R</td>
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<td>3.0</td>
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</tr>
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<td>2.0</td>
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<td></td>
</tr>
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<td>3DF/R</td>
<td></td>
<td>60</td>
<td>3.0</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Cross-section
** Semi-circular heading
*** Plane strain heading
6.3. Numerical analysis of 2D cross-section tunnel

Figure 6.3: Undrained shear strength in compression and extension of one-dimensionally normally consolidated and lightly overconsolidated kaolin (Mair(1979))

6.3.1 Parameters

For the rigid-plastic finite element method for the von Mises material, the required parameters are the undrained shear strength($c_u$) and unit weight($\gamma$) of clay alone. In the 2D cross-section tests, the spestone kaolin was used. The spestone kaolin was consolidated under the effective stress $\sigma'_{v0} = 171$ kN/m$^2$ before tests. The undrained shear strength of spestone kaolin is determined by graph shown in Fig. 6.3. As the tunnel pressure is reduced, most of the elements of clay around and above tunnel are subjected to extension stress paths. The undrained shear strength applicable to the 2D cross-section tests is therefore the strength of one-dimensionally lightly overconsolidated kaolin in plane strain extension. As a result, the undrained shear strength used in this case can be calculated by Eq. (6.1).

$$c_u = 0.16\sigma'_{v0} = 0.16 \times 171 \approx 27.4 \text{ kN/m}^2$$ (6.1)

During model preparation, the moisture content of the models increased by 0.5%, causing the reduction of undrained shear strength. According to literature (Mair, 1979), the undrained shear strength used in this analysis is $0.95 \times 27.4 = 26$ kN/m$^2$. For the unit weight of spestone kaolin in 2D cross-section tests, it was not stated in the literature directly. According to the
Table 6.2: Summary of experimental models and finite element meshes

<table>
<thead>
<tr>
<th>Test</th>
<th>Centrifugal model</th>
<th>FEM mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1 2D cross-section test</strong></td>
<td><img src="image1" alt="2D cross-section model" /></td>
<td><img src="image2" alt="2D cross-section mesh" /></td>
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<tr>
<td></td>
<td>$n$: constant</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sigma_T$: decrease</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>small apparatus</em></td>
<td></td>
</tr>
<tr>
<td><strong>2 Tunnel heading test A</strong></td>
<td><img src="image3" alt="Tunnel heading A model" /></td>
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<tr>
<td>(semi-circular heading)</td>
<td>$n$: increase</td>
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</tr>
<tr>
<td></td>
<td>$\sigma_T = 0$</td>
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</tr>
<tr>
<td></td>
<td><em>small apparatus</em></td>
<td></td>
</tr>
<tr>
<td><strong>3 Tunnel heading test B</strong></td>
<td><img src="image5" alt="Tunnel heading B model" /></td>
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</tr>
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<td></td>
<td>$\sigma_T = 0$</td>
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</tr>
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<td></td>
<td><em>large apparatus</em></td>
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</tr>
<tr>
<td><strong>4 Tunnel heading test B’</strong></td>
<td><img src="image7" alt="Tunnel heading B’ model" /></td>
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<td>(plane-strain heading)</td>
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<td></td>
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</tr>
<tr>
<td></td>
<td>$\sigma_T$: decrease</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>large apparatus</em></td>
<td></td>
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</table>
literature (p.100, Mair 1979) in test 2DH, the tunnel pressure at equilibrium state was set to be 157 kN/m² at 75g. The unit weight $\gamma$ of spestone kaolin at acceleration factor 75, therefore, can be calculated by

$$\gamma_{n=75} = \frac{\text{tunnel pressure at equilibrium}}{C + \frac{D}{2}}$$

$$= \frac{157}{(1.8 \times 0.06) + \left(\frac{0.06}{2}\right)} \approx 1137.7 \text{ kN/m}^3.$$  

6.3.2 Results and discussions

➀ in Table 6.3 is the summary of the collapse tunnel pressure by experimental tests and RPFEM for 2D cross-section tunnel tests. First, the right side of the mesh region is set to be smooth. The tunnel pressure $\sigma_{TC}$ obtained from RPFEM (smooth), in Table 6.3, is slightly larger than the observed pressure. As the ratio $C/D$ increases, the difference becomes larger. It can be explained by the friction effect at the interface between the apparatus and clay in the test. Some amount of the vertical stress was lost in the side friction (front, back, left and right). Because the boundary condition of the interface between apparatus and clay is set as vertically free-to-move i.e., no friction at boundary is assumed, the vertical stress should be larger than that with friction at boundary. This is the main reason why $\sigma_{TC}$ by RPFEM (smooth) is larger than the observed ones as $C/D$ increases. Then, for simplicity, the displacement of the right side boundary is set to be fixed.

The tunnel pressure obtained from this case, RPFEM (fixed), is also shown in ➀ in Table 6.3. According to the results, the side friction can decrease the tunnel pressure $\sigma_{TC}$ obtained from RPFEM and improve the agreement between the values of RPFEM and experimental results. The normalized tunnel pressures obtained from experiments, RPFEM and the upper bound solution obtained by Davis et al. (Davis et al., 1980; Mair, 1979) are plotted in Fig. 6.4. This figure indicates that RPFEM (fixed) can estimate the tunnel pressure at collapse well and better than the upper bound solution.
Table 6.3: Summary of the results of tunnel tests and RPFEM

<table>
<thead>
<tr>
<th>Test no.</th>
<th>(D) (mm)</th>
<th>(C/D)</th>
<th>(P/D)</th>
<th>Tunnel pressure (\sigma_{TC}) (kN/m²)</th>
<th>(N_C)</th>
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<td>RPFEM(3D)</td>
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</table>
6.4 Numerical analysis of tunnel heading tests

The tunnel heading test A, B, B′ and C are analyzed by using the rigid-plastic finite element method. The finite element meshes for all tests are summarized in Table 6.2. The finite element mesh for all 3D semi-circular tunnel heading tests is composed of 1246 elements and 1668 nodes. The analysis of the tunnel heading test A uses the finite element mesh for the small apparatus. According to the aforementioned 2D cross-section test, the horizontal distance of the small apparatus is too short to ignore the effect of the side friction so that the displacement of the right side boundary should be fixed while the front, back and left sides are smooth and bottom is fixed. On the other hand, the finite element mesh for the tunnel heading tests B and C is the same size as the large apparatus of which its horizontal distance is long enough compared to the diameter of tunnel to ignore the side friction. The right side of the finite element mesh of the tunnel test B and C is set to be smooth while the left, front and back sides are smooth and the bottom is fixed. For the tunnel tests A, B and C, the boundary condition of tunnel lining is set by assuming that the direction which is perpendicular to the lining plane is not allowed to move.

Figure 6.4: Normalized tunnel pressure at collapse versus C/D for 2D cross-section tunnel tests
Additionally at the same time, all tests of the tunnel tests A, B and C are simulated using the plane strain meshes for 2D headings as shown in Fig. 6.5 to compare with the 3D analysis and test results.

The tunnel heading tests A, B and B′ are analyzed to determine the load factor(µ), which is equivalent to the acceleration factor(n) obtained from tests. On the other hand, the tunnel pressures σ_{TC} at collapse are determined in the tunnel heading test C. The summary of tunnel heading tests is given by ② to ⑤ in Table 6.1.

### 6.4.1 Parameters

Spestone kaolin was used in the tunnel heading test A and Speswhite kaolin was used in the tunnel heading tests B, B′ and C. The undrained shear strength of both types of kaolin are almost the same. However, in three-dimensional condition, the appropriate undrained shear strength applicable to the 3D tunnel heading tests is the strength of one-dimensionally consolidated kaolin in triaxial extension. With similar approach to the 2D cross-section tunnel tests, the undrained shear strength is calculated by considering Fig. 6.3 then

\[
e_u = 0.18\sigma_{v0}' = 0.18 \times 171 \approx 31 \text{kN/m}^2.
\]  

(6.3)

The unit weight was estimated from the tunnel pressure at equilibrium of 3DD/B test written in the literature(p.113, Mair 1979). The tunnel pressure at 118g when \(C/D = 3.0\) and \(D = 60\) mm is 394 kN/m². The unit weight \(\gamma\) at 118g is calculated by
6.4. Numerical analysis of tunnel heading tests

\[ \gamma_{n=118} = \frac{\text{tunnel pressure at equilibrium}}{C + \frac{D}{2}} = \frac{394}{(3.0 \times 0.06) + \left(\frac{0.06}{2}\right)} \approx 1876.2 \text{ kN/m}^3. \]  

(6.4)

However, for tests A, B and B' which is done under increasing the acceleration ratio \( n \), the unit weight used for these test is equal to 1876.2/118= 15.9 kN/m\(^3\)

### 6.4.2 Results and discussions

\( \text{②, ③, ④ and ⑤} \) in Table 6.3 summarize the results of the tunnel heading tests and RPFEM. The stability ratio \( N_C \) for each test can be calculated by Eq. (2.1). In this section, \( \sigma_s \) equals to zero and \( \gamma \) is the unit weight of soil at acceleration factor \( n \), which depends on the condition of each test. The results and discussions of each test and its numerical analysis by RPFEM are as follows:

The tunnel heading test A(②): The comparison between 2D and 3D RPFEM will be discussed. The average \( n \) (load factors) as well as the stability ratio \( N_C \) obtained from plane strain RPFEM are smaller than those obtained from 3D RPFEM. This result confirms that the level of stability in the plane strain heading condition is lower than the 3D semi-circular tunnel heading. Additionally, the comparison between 3D RPFEM and the centrifugal modeling tests will be discussed. The stability ratio \( N_C \) obtained from 3D RPFEM are larger than those observed in the centrifugal modeling tests. This indicates that the level of the stability obtained from 3D RPFEM is overestimated when the undrained shear strength \( c_u \) of clay calculated by Eq. (6.3) i.e., 31 kN/m\(^2\) is considered. The further discussion will be explained later.

The tunnel heading tests B(③) and B'(④): The difference between the semi-circular heading test and the plane strain test is investigated here. As the same as the experimental result obtained by Mair, the stability ratio \( N_C \) obtained from the plane strain RPFEM (\( N_C = 6.5 \)) is less than that obtained from 3D RPFEM which simulates the tunnel heading test B(③, \( N_C = 11.1 \)). This again confirms that the 3D semi-circular heading is more stable than the plane strain heading. Next, the comparison between RPFEM and the centrifugal modeling test will be discussed. For the numerical analysis of the tunnel heading test B, the stability ratio \( N_C \) obtained
Chapter 6. Numerical analysis of the tunnel stability in soft clay

from 3D RPFEM ($N_C = 11.1$) is larger than that obtained from the tunnel heading centrifugal modeling test B ($N_C = 9.1$). This suggests again that the predicted average $n$ obtained from 3D RPFEM is overestimated when $c_u = 31$ kN/m$^2$ is considered. For the tunnel heading test $B'$, $N_C$ obtained from the plane strain RPFEM ($N_C = 6.5$) agreed well with the observed one ($N_C = 6.8$). Therefore, we can conclude that RPFEM can effectively estimate the stability of both the plane strain heading and the unlined cross-section tunnel.

The tunnel heading test C: The tunnel pressures $\sigma_{TC}$ at collapse are investigated. The tunnel pressures obtained from plane strain RPFEM are larger than those obtained from 3D RPFEM. This result satisfies the fact that the 3D semi-circular heading is more stable than the plane strain heading. In addition, the tunnel pressures obtained from 3D RPFEM are smaller than the observed tunnel pressure. This indicates, as the same as the previous discussions on the tunnel heading test A and B, that the level of the stability of the tunnel heading predicted by 3D RPFEM is overestimated when $c_u = 31.0$ kN/m$^2$. The improvement of the prediction by 3D RPFEM will be discussed later.

Comparison between 2D and 3D RPFEM: Table 6.4 shows the comparison among the stability ratios of 2D and 3D conditions for the tunnel heading tests A, B and C. The consistency of the difference between 2D RPFEM and 3D RPFEM will be considered here. According to this table, the stability ratios obtained from 2D RPFEM are varied between 0.5 and 0.6 of the stability ratios obtained from 3D RPFEM. This suggests that the 3D RPFEM stability ratio can be approximately calculated by dividing the 2D RPFEM stability ratio by the value between 0.5 and 0.6.

Comparison between 3D RPFEM and experiments: The improvement of the prediction of the stability of the tunnel heading by 3D RPFEM will be discussed now. As mentioned above, the level of stability of the tunnel heading obtained from 3D RPFEM is overestimated when $c_u = 31$ kN/m$^2$ is considered. According to Table 6.4, the stability ratios obtained from 3D RPFEM varied consistently in the small range between 1.15 and 1.25 times of those obtained from the experiments. This result indicates that using the smaller undrained shear strength $c_u$ of clay than that obtained by Eq. (6.3) will improve the agreement between the 3D RPFEM results and the experimental results.

Comparison among 3D RPFEM, experiments and the lower and upper bound solutions: Fig.
6.4. Numerical analysis of tunnel heading tests

Figure 6.6: Stability ratios at collapse for tunnel headings fully lined to the face (i.e., P/D=0)

6.6 shows the comparison between the stability ratios obtained from the tunnel heading tests and other solutions when tunnel is fully supported i.e, P/D = 0. According to the figure, the stability ratios $N_C$ obtained from RPFEM are much closer to those of the tunnel heading test C than those of upper bound solution by Mair (Mair, 1979). Furthermore, both $N_C$ obtained from RPFEM and the tunnel heading test C are bounded well by the upper and lower bound solutions. The upper bound solution and lower bound solution are illustrated in Eqs. (6.5) and (6.6), respectively.

- Upper bound solution by Mair (Mair, 1979):
  \[
  N_C = \frac{3}{2} \pi + 4 \frac{C}{D} \quad (6.5)
  \]

- Lower bound solution by Davis et al. (Davis et al., 1980):
  \[
  N_C = 4 \ln \left(2 \frac{C}{D} + 1\right) \quad (6.6)
  \]

Influence of non-supported heading length on the stability: Fig. 6.7 shows the influence of non-
supported heading length on the stability ratios at collapse. The stability ratios obtained from 3D analysis are larger than those of the tunnel heading test C. As the unlined length increases, the stability of tunnel heading decreases. In order to show that the stability of tunnel heading decreasingly approaches to the stability of 2D cross-section tunnel which is consideringly equivalent to the infinitely unlined length, the further two cases of analysis on cross-section tunnel i.e., \( C/D = 1.5 \) and \( C/D = 3.0 \), are carried out. The finite element mesh for 2D cross-section tunnel is used. As shown in Fig. 6.7, ⑤ and ⑥ in Table 6.3, the stability ratio of tunnel heading approaches to the stability ratio of 2D cross-section tunnel.

Velocity field: Not only the tunnel pressure \( \sigma_{TC} \) or the acceleration factor \( n \) but also the velocity field can be determined by RPFEM. For instance, Fig. 6.8 illustrates the the velocity field for 3DD/R test where unlined heading length \( P \) is three times of the tunnel diameter. This figure indicates that soils at tunnel face and tunnel crown flow inwardly into the tunnel and the magnitudes of velocity at tunnel face and tunnel crown are much larger than those of other parts.
6.4. Numerical analysis of tunnel heading tests

Figure 6.8: Velocity field for 3DD/R test ($P/D = 3.0$)

Table 6.4: Comparison among the stability ratios ($N_C$) of 2D and 3D conditions.

<table>
<thead>
<tr>
<th>Test no.</th>
<th>2D RPFEb</th>
<th>3D RPFEb</th>
<th>3D RPFEb experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C/D = 1.2, P/D = 0.0$</td>
<td>0.59</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>$C/D = 2.3, P/D = 0.0$</td>
<td>0.62</td>
<td>1.22</td>
<td></td>
</tr>
<tr>
<td>$C/D = 3.5, P/D = 0.0$</td>
<td>0.61</td>
<td>1.42</td>
<td></td>
</tr>
<tr>
<td>Test B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C/D = 3.4, P/D = 0.0$</td>
<td>0.59</td>
<td>1.22</td>
<td></td>
</tr>
<tr>
<td>Test C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C/D = 1.5, P/D = 0.0$</td>
<td>0.58</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>$C/D = 1.5, P/D = 0.5$</td>
<td>0.61</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>$C/D = 1.5, P/D = 1.0$</td>
<td>0.61</td>
<td>1.21</td>
<td></td>
</tr>
<tr>
<td>$C/D = 1.5, P/D = 2.0$</td>
<td>0.57</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>$C/D = 3.0, P/D = 0.0$</td>
<td>0.54</td>
<td>1.10</td>
<td></td>
</tr>
<tr>
<td>$C/D = 3.0, P/D = 0.5$</td>
<td>0.56</td>
<td>1.15</td>
<td></td>
</tr>
<tr>
<td>$C/D = 3.0, P/D = 0.92$</td>
<td>0.52</td>
<td>1.19</td>
<td></td>
</tr>
<tr>
<td>$C/D = 3.0, P/D = 2.0$</td>
<td>0.44</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>$C/D = 3.0, P/D = 3.0$</td>
<td>0.42</td>
<td>1.25</td>
<td></td>
</tr>
</tbody>
</table>
The influence of surface load on tunnel stability in soft clay is fundamentally studied. The stability of the plane strain heading is only considered in this study. According to Fig. 6.9(a), the fully supported plane strain heading of height $D$ is shown being constructed with a depth of cover $C$. An uniformly distributed load $q$ of width $B$ is applied on the ground surface and its center places at the horizontal distance $L$ apart from the face. The finite element for this problem and boundary condition are illustrated in Fig. 6.9(b). The critical tunnel pressure $\sigma_{TC}$ acting on the face is determined by using RPFEM. The analysis is carried out under various distance $L$ for each width $B$ of the distributed load. The dimensions of the problem and parameters for this study is summarized in Table 6.5. Specifically, the undrained shear strength $c_u$ of soft clay is equal to 31 kN/m$^2$ and unit weight of soil $\gamma$ is 15.9 kN/m$^3$. For the uniformly distributed load $q$, $q/c_u=2.58$ which is approximately 50% of the bearing capacity of clay with cohesion $c_u=31$ kN/m$^2$ is used.

The objective of this study is to determine the distance $L$ that affects the tunnel stability most as defined by the weak zone as shown in Fig. 6.10. The ground surface boundary of the failure
zone may be obtained in this study. The left and right boundaries of the failure zone at ground surface are also illustrated in Fig. 6.10.

6.5.2 Results and discussions

Weak zone, boundary of the failure zone

Fig. 6.11 shows the influence of the distance $L$ on the tunnel pressure required to maintain the stability of the tunnel heading (plane strain heading) for various cover-to-diameter ratios i.e., $C/D = 0.5, 1.0, 2.0$ and $3.0$. Graphs shown this figure suggests that, for the same $C/D$, the weakest distances $L$ of the applied load $q$ i.e., the distance $L$ that the largest tunnel pressure $\sigma_{TC}$ is obtained(notice the dark marks), for all width $B$ of the surface load are not significant different. This means the weakest distance $L$ is independent of the width $B$ of the surface load if the same $C/D$ is considered.

The weakest distances $L/D$ and the boundaries of the failure zone at ground surface of all cases of $B/D$ are plotted against the cover-to-diameter ratio $C/D$ in Fig. 6.12. All data are fitted by the straight line using least square method. The straight line, which approximately expresses the relationship between the weakest distances $L$ and the cover depth $C$, is written as

$$\frac{L}{D} = \left(\frac{C}{D} + 0.59\right) \approx 0.54 \frac{C}{D} + 0.32. \quad (6.7)$$

The left and right boundaries of the failure zone are almost linearly proportional to the cover depth $C$. Fig. 6.12 indicates that the width of the failure zone at ground surface i.e., the distance between the left and right boundaries, increases with the cover-to-diameter ratio $C/D$. 

Figure 6.11: The ratio $L/D$ plotted against the dimensionless tunnel pressure $\sigma_{TC}$ at critical state for various cover-to-diameter ratio $C/D$.

Also, according to the same figure, the width of the failure zone at ground surface i.e., the length between the left and right boundaries, of the wider surface load is larger than that of the narrower one.

The normalized peak tunnel pressures $\sigma_{TC}/c_u$ are plotted against the cover-to-diameter ratio $C/D$ in Fig. 6.13. The tunnel pressure $\sigma_{TC}$ when there is no load applied on the ground surface, is agreed well, especially for $C/D = 0.5$ and $1.0$, with the upper bound solution proposed by Davis et al. (Davis et al., 1980) which is written as

$$\frac{\sigma_{TC}}{c_u} = \gamma D \left( \frac{C}{D} + \frac{1}{2} \right) - 4 \sqrt{\frac{C}{D} + \frac{1}{4}}. \quad (6.8)$$

Also, this figure suggests that the increment of the tunnel pressure $\sigma_{TC}$ due to the surface load at the weakest distance, decreases with the increasing of the depth of tunnel. This means that the influence of the surface load on the tunnel pressure decreases while the deeper tunnel is constructed.
6.5. Influence of surface load on shallow tunnel stability

**Figure 6.12:** The weak zone, left and right boundaries of the failure zone for each cover-to-diameter ratio $C/D$ and their fitted curves.

**Figure 6.13:** The peak tunnel pressure $\sigma_{TC}$ plotted against the cover-to-diameter ratio $C/D$ (Surface load at weakest distance)
Chapter 6. Numerical analysis of the tunnel stability in soft clay

Velocity field

The velocity fields of all cases of cover depth, when \( B/D = 2/3 \), are shown in Figs. 6.14, 6.15, 6.16 and 6.17 for \( C/D = 0.5, 1.0, 2.0 \) and 3.0, respectively. Fig. 6.14(b), 6.15(b), 6.16(b) and 6.17(b) shows the velocity field when the surface load is applied at the weakest distance for all cases of cover depth. It is found that the weakest distance \( L \) is almost equivalent to the zone where the vertical velocity is found on the ground surface when no surface load is applied (see Figs. 6.14(a), 6.15(a), 6.16(a) and 6.17(a)). This indicates that at the weakest distance \( L \), not only the surface load affects the tunneling most and also the tunneling has the most effect on the surface structures. Furthermore, the velocity under the surface load, as shown in Fig. (b), is almost vertically downward and the failure zone is smaller than the failure zone investigated in the no loading case.

In addition, if the surface load which is approximately equal to the bearing capacity of soft clay i.e., \( q \approx 160 \text{ kN/m}^2 \), is applied at the weakest distance \( L \), the velocity field of this case is drawn in Figs. 6.14(c), 6.15(c), 6.16(c) and 6.17(c) for all cases of cover depth. Particularly, for the shallower tunnel \( (C/D = 0.5 \) and 1.0), the velocity under the surface load is almost equivalent to the failure mechanism i.e., Prandtl mechanism, in the conventional limit analysis considering upper bound theorem for bearing capacity analysis of strip footing (Chen, 1975). This means the failure due to the surface load is more significant than the failure due to the tunneling.

6.6 Conclusions

The rigid-plastic finite element method based on the upper bound theorem is recently a powerful numerical tool in the evaluation of the stability problems in geotechnical engineering. In this chapter, the stability of tunnel in soft clay has been investigated using the rigid-plastic finite element method. The centrifugal modeling tests carried out at Cambridge University by Mair were selected to prove the effectiveness of RPFEM in the evaluation of the stability of tunnel in soft clay. 2D cross-section tunnel (unlined tunnel), 3D circular tunnel heading and plane strain heading were considered. According to the comparison between experimental and numerical results, the following conclusions can be summarized.

1. For 2D cross-section tunnel cases, RPFEM can simulate the the centrifugal modeling
6.6. Conclusions

(a) No load

\[ q = 80 \text{ kN/m}^2 \]

(b) \( \frac{q}{c_u} \approx 2.58 \text{ kN/m}^2 \)

(c) \( \frac{q}{c_u} \approx 5.16 \text{ kN/m}^2 \)

Figure 6.14: Velocity field of the weakest distance \( L \) case for \( C/D = 0.5 \) and \( B/D = 2/3 \)

(a) No load

\[ q = 80 \text{ kN/m}^2 \]

(b) \( \frac{q}{c_u} \approx 2.58 \text{ kN/m}^2 \)

(c) \( \frac{q}{c_u} \approx 5.16 \text{ kN/m}^2 \)

Figure 6.15: Velocity field the weakest distance \( L \) case for \( C/D = 1.0 \) and \( B/D = 2/3 \)
Chapter 6. Numerical analysis of the tunnel stability in soft clay

(a) No load

\[ q = 80 \text{ kN/m}^2 \]

(b) \[ q/c_u \approx 2.58 \text{ kN/m}^2 \]

(c) \[ q/c_u \approx 5.16 \text{ kN/m}^2 \]

\( q = 80 \text{ kN/m}^2 \)

\( q = 160 \text{ kN/m}^2 \)

\( q = 80 \text{ kN/m}^2 \)

\( q = 160 \text{ kN/m}^2 \)

Figure 6.16: Velocity field the weakest distance L case for \( C/D = 2.0 \) and \( B/D = 2/3 \)

Figure 6.17: Velocity field the weakest distance L case for \( C/D = 3.0 \) and \( B/D = 2/3 \)
tests effectively. The difference between numerical and experimental result is larger for the deeper tunnel due to vertical friction at the wall of apparatus. The better results can be obtained when considering the effect of this friction.

2. The 3D semi-circular tunnel heading is more stable than the plane strain heading. The stability ratios obtained from 2D RPFEM are consistently varied between 0.5 and 0.6 of the stability ratios obtained from 3D RPFEM. This suggests that the 3D RPFEM stability ratios can be approximated by dividing the 2D RPFEM stability ratios by the value in range 0.5～0.6.

3. For 3D semi-circular tunnel heading cases, the stability ratios obtained from RPFEM are a bit larger than the experimental results. According to Table 6.4, the stability ratios \( N_C \) obtained from 3D RPFEM are approximately 15%～25% larger than those obtained from experiments. This means that the stability ratio predicted by RPFEM is consistently overestimated and the better prediction can be obtained by using the smaller undrained shear strength \( c_u \) of clay than that obtained by Eq. (6.3).

4. In the case of 2D analysis, RPFEM can predict the stability of both the unlined tunnel and the plane strain heading in soft clay very well.

5. In fully lined tunnel case, the stability ratio increases with the cover. Furthermore, if the non-supported heading length is larger, the stability ratio decreases and approaches to the stability ratio of the unlined tunnel.

The conclusions associated with the study on the influence of the surface load on the tunnel stability are summarized as follows:

1. The location of the surface load that affects the tunnel stability most is independent of the width of the surface load. The weakest distance \( L \) is increasingly and almost linearly proportional to the cover depth.

2. The width of the failure zone at the surface increases with the cover depth of tunnel and also the width of the surface load.

3. The weak zone is equivalent to the zone where the velocity is found vertically downward for the velocity field of the non-loading case. This indicates that this distance is the distance where the tunnel and surface load interact mutually the most.
Chapter 7

Effects of groundwater on the tunnel stability

7.1 Introduction

So far, some tunnels collapsed caused by the groundwater pressure or seepage force. During tunnel construction, the effective overburden pressure is reduced slightly by the arching effect at the tunnel face but the seepage pressure acting on the tunnel face remains the same. Thus, if tunnel is excavated under the groundwater table, the stability of tunnel face should be reduced. In this chapter, fundamental studies on the effect of groundwater on the tunnel stability are carried out by using the rigid-plastic finite element method with consideration of groundwater as described in Section 5.7. The two-dimensional cross-section tunnel embedded in the cohesive-frictional ground is mainly considered. First, the groundwater table will be decided by analyzing various cases of Dupuit’s groundwater equation. Then, the stability of the unlined tunnel is considered at various groundwater table. The influences of earth cover and soil cohesion are also discussed. Furthermore, the stability of the lined tunnel is also studied by varying the groundwater table and the patterns of lining as well.
7.2 Groundwater configuration

Konishi and Tamura (Konishi and Tamura, 2002) used the Dupuit equation to represent the groundwater table as illustrated in Fig. 7.1. The Dupuit equation is shown as

$$H^2(x) = H_f^2 \left( \beta \left( \frac{x}{L} - 1 \right) + 1 \right)$$

(7.1)

where $H(x)$ is the groundwater table at the distance $x$ apart from the center of the tunnel and $H_f$ is the groundwater table at the distance $L$ apart from the center of the tunnel. $\beta$ is the drainage parameter which is shown in the following equation:

$$\beta = \frac{Q}{Q_{\text{max}}}$$

(7.2)

where $Q$ is the rate of inflow and the definition of $Q_{\text{max}}$ is the rate of inflow when the water table draws down to the bottom of the region as shown by line 1 in Fig. 7.1. Line 2 presents the groundwater table if $0 < \beta < 1$.

According to the Dupuit equation (Eq. (7.1)), the curve of the groundwater table depends on three factors i.e., $H_f$, $\beta$ and $L$. This means these three factors have to be assumed to present the curve of groundwater table. Note that when $L$ is large enough the groundwater table is almost horizontal at any $\beta$ value. Therefore, the influence of the parameter $L$ on the tunnel stability analysis will be investigated first by using RPFEM. The parameters $L$ used here are $4x_0$, $10x_0$ and $100x_0$ where $x_0$ is the width of the finite element region. The finite element mesh is shown in $\mathbb{D}$ of Table 7.1 where $D$ denotes by the diameter of the tunnel and the cover-to-diameter ratio $C/D$ is equal to unity. The dimensionless cohesion of soil $c/\gamma_d D$ is 1.33. $\phi = 0^\circ$ and $\phi = 30^\circ$ of the angle of the internal friction are used here.

Fig. 7.2 shows the influence of the groundwater table on the tunnel stability when $L$ is varied. This figure indicates that for the tunnel in clay and sandy soil, the parameter $L$ does not give the significant effect on the load factor $\mu$ at any groundwater table. Therefore, for simplicity, the horizontal groundwater table will be used for other cases instead of the Dupuit groundwater table in this chapter. The horizontal groundwater table and its definition are shown in Fig. 7.3. In this figure, the vertical distance $H$ is measured from the center of the tunnel to the groundwater table.
7.2. Groundwater configuration

Figure 7.1: Dupuit’s groundwater table.

(a) $\phi = 0^\circ$

(b) $\phi = 30^\circ$, $r=0.0$

(c) $\phi = 30^\circ$, $r=0.5$

(d) $\phi = 30^\circ$, $r=1.0$

Figure 7.2: Groundwater table vs. load factor($\mu$) when considering the influence of $L$.

Figure 7.3: Horizontal groundwater table.


7.3 Numerical analysis of unlined tunnel

The fundamental studies on the effect of the groundwater table on the unlined tunnel stability are carried out in this section. The influence of the soil cohesion $c$ and the influence of the earth cover $C$ are also investigated here.

The finite element mesh and parameters used in the unlined case are shown in (2) of Table 7.1 and of Table 7.2, respectively. The groundwater table is assumed to be horizontal and varied from the dry condition up to the fully saturated condition.

Fig. 7.4 shows the influence of the groundwater table on the stability of the unlined tunnel when the dimensionless cohesion $c/(\gamma_d D) = 0.4$.

The load factor $\mu$ or the stability of the unlined tunnel in cohesive soil (i.e., $\phi = 0^\circ$) is not affected by the groundwater table significantly. On the other hand, when $\phi = 30^\circ$ is considered, the groundwater table has the great effect on the stability of the tunnel. When the groundwater table is lower than the invert of tunnel, the groundwater table does not affect the tunnel stability but when the groundwater table is higher than the invert of the tunnel the stability of tunnel starts to decrease and, moreover, when the groundwater table is higher than the springline of the tunnel the stability of tunnel abruptly decreases. In addition, according to the same figure, when $\phi = 30^\circ$ is considered, the load factors obtained from the non-associated ($r = 0.0$ and $0.5$) and associated flow rules ($r = 1$) are not significantly different each other.

7.3.1 Velocity and principal effective stress

Figs. 7.5, 7.6 and 7.7 illustrate the principal effective stress diagrams and the velocity fields of the soil elements surrounding the tunnel for $\phi = 0^\circ$, $\phi = 30^\circ$ with the non-associated flow rule and $\phi = 30^\circ$ with the associated flow rule, respectively.

At any groundwater table, the horizontal tensile stress occurs at the ground surface except the ground surface over the center of the tunnel. Furthermore, when the groundwater table is higher than the springline, the tensile stress occurs around the tunnel especially at the lower half of tunnel. This is due to the existence of the pore water pressure. In addition, as the groundwater table draws down, the compressive effective stress increases and more significant increase of the effective stress can be found for $\phi = 30^\circ$ than that for $\phi = 0^\circ$.

When the collapse of tunnel occurs, the soil surrounding the tunnel tends to move inward to the
### Table 7.1: Summary of the finite element meshes for all cases.

<table>
<thead>
<tr>
<th>Groundwater configuration</th>
<th>Unlined tunnel</th>
<th>Lined tunnel</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Lower lining</td>
</tr>
<tr>
<td>Upper lining</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Incomplete lining</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Complete lining</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Groundwater configuration**
  - C = D
  - D
  - 2C + D
- **Unlined tunnel**
  - C = D
  - D
  - 2C + D
- **Lined tunnel**
  - Upper lining
  - Incomplete lining
  - Complete lining
Table 7.2: Summary of the parameters used in unlined and lined tunnel cases.

<table>
<thead>
<tr>
<th>Analysis case</th>
<th>Analysis case</th>
<th>$D$ (m)</th>
<th>$C/D$</th>
<th>Soil parameters</th>
<th>Lining parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$c$ (kN/m$^2$)</td>
<td>$\phi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\gamma_d$ (kN/m$^3$)</td>
<td>$\gamma_s$ (kN/m$^3$)</td>
</tr>
<tr>
<td>Unlined tunnel</td>
<td>$5$</td>
<td>$0.5 \sim 3.0$</td>
<td>$30 \sim 200$</td>
<td>$0^\circ, 30^\circ$</td>
<td>15</td>
</tr>
<tr>
<td>Lined tunnel</td>
<td>$1.0$</td>
<td>$30$</td>
<td>$3000$</td>
<td>$0^\circ$</td>
<td>$24$</td>
</tr>
</tbody>
</table>

* All cases are considered under both non-associated and associated flow rules ($0.0 \leq r \leq 1.0$).

Figure 7.4: Influence of groundwater table on the load factor $\mu$ when $c/(\gamma_d D) = 0.4$.

tunnel center and the settlement of the ground surface can be found especially in clay ($\phi = 0^\circ$). The horizontal distance of the settlement of the ground surface when $\phi = 0^\circ$ is wider than that for $\phi = 30^\circ$. For $\phi = 30^\circ$ soil with the associated flow rule case (i.e., $r = 1$), the velocity is apparently high at the invert of the tunnel, compared to the velocity at the tunnel crown when the groundwater table is higher than the springline.

Additionally, we would like to introduce an interesting point that, especially for $\phi = 30^\circ$ ground, if the groundwater table is lowering than the springline of the tunnel, the invert of the tunnel tends not to move so much and the tunnel crown tends to settle down more. On the other hand, the different results are obtained when the groundwater is high. Therefore the effect of tunnel lining on the stability will be considered in the later section.
Figure 7.5: Principal effective stress and velocity when unlined tunnel collapses for \( c/(\gamma_d D) = 0.4 \) and \( \phi = 0^\circ \).
Chapter 7. Effects of groundwater on the tunnel stability

(a) Ground surface  
(b) Tunnel crown  
(c) Springline  
(d) Invert  
(e) Dry (No water)

Figure 7.6: Principal effective stress and velocity when unlined tunnel collapses for $c/(\gamma_d D) = 0.4$ and $\phi = 30^\circ$ with $r = 0$. 
7.3. Numerical analysis of unlined tunnel

(a) Ground surface  
(b) Tunnel crown  
(c) Springline  
(d) Invert  
(e) Dry (No water)

Figure 7.7: Principal effective stress and velocity for unlined tunnel collapses for $c/(\gamma_d D) = 0.4$ and $\phi = 30^\circ$ with $r = 1$. 

For stress
- Tensile stress
- Compressive stress
Figure 7.8: Influence of the soil cohesion on the stability of unlined tunnel when $\phi = 30^\circ$ with $r = 0$.

### 7.3.2 Influence of the soil cohesion

The influence of the soil cohesion is described here. The dimensionless soil cohesion $c/(\gamma_d D)$ varying between 0.4 and 2.67 are used in the analysis and the angle of internal friction is fixed to be $30^\circ$. The cover-to-diameter ratio $C/D$ is constantly equal to unity. In Fig. 7.8, the load factors $\mu$ were plotted against various dimensionless soil cohesion $c/(\gamma_d D)$ for $\phi = 30^\circ$ ground with the non-associated flow ($r = 0$). According to this figure, the load factor $\mu$ is equal to zero when the soil cohesion $c$ equals zero and then the load factor $\mu$ almost linearly increases with the soil cohesion at any groundwater table.

### 7.3.3 Influence of the earth cover

The influence of the earth cover on the stability of unlined tunnel is investigated in this section. The cover-to-diameter ratio is varied from 0.5 to 3.0. Both of the purely cohesive soil and the cohesive-frictional soil are considered and $c/(\gamma_d D)$ is fixed to be 0.4.

Fig. 7.9 shows the relationship between the stability ratio $N$, as defined in Eq.(2.1), of tunnel in cohesive ground and the cover-to-diameter ratio $C/D$. Apparently, the stability ratio of tunnel increases with the earth cover depth. The curve of the higher groundwater table locates lower
7.3. Numerical analysis of unlined tunnel

Figure 7.9: Influence of the earth cover on the stability ratio of tunnel in cohesive soil ($\phi = 0^\circ$).

than the lower groundwater table. This indicates that for every earth cover depth, the stability of the unlined tunnel decreases when the higher groundwater table is considered. Furthermore, for $C/D > 1$, the stability ratio $N$ of dry condition agrees well with $N$ obtained from 3D centrifugal modeling tests for unlined tunnel carried out by Cambridge University (Schofield, 1980).

On the other hand, the dimensionless cohesions $c/\left(\mu \gamma d D \right)$ of tunnel in cohesive-frictional soil ($\phi = 30^\circ$) are plotted against the cover-to-diameter ratio $C/D$ in Fig. 7.10. From this figure, it is found that at any groundwater table, the cover depth of tunnel has no significant effect on the stability of the tunnel in frictional soil. Moreover, the dimensionless cohesion $c/\left(\mu \gamma d D \right)$ of the dry condition, approximately 0.18, agrees very well with the tunnel stability range 0.13 $\sim$ 0.17 for $\phi = 30^\circ$ soil, mentioned by Mashimo (Mashimo, 1998).

7.3.4 Required cohesion at critical state

In the practical point of view, there will be the question how high the groundwater table can rise keeping the tunnel still stable. The minimum cohesion $c$ of soil is determined when the tunnel does not collapse or the load factor $\mu$ is not less than unity. The minimum cohesion $c$ can be determined by considering the cohesion $c$ at the load factor $\mu$ is equal to 1.

Fig. 7.11 plotted the required dimensionless cohesion $c/\left(\gamma_d D \right)$ for each earth cover $C$ against
Figure 7.10: Influence of the earth cover on the stability ratio of tunnel in cohesive-frictional soil ($\phi = 30^\circ$, $r = 1$).

various groundwater table. The left and right sides of each curve indicate the unstable zone and the stable zone, respectively. Each curve in this figure shows that the required cohesion is almost constant if the groundwater table is lower than the invert of the tunnel and starts to increase when the groundwater table is higher than the invert. If the groundwater table is higher than the springline of the tunnel, the required cohesion abruptly and almost linearly increases. When the angle of internal friction of soil $\phi$ is equal to $30^\circ$ and the associated flow rule ($r = 1$) is considered, all of the minimum cohesion curves for each earth cover are almost the same. This indicates that for the tunnel embedded in $\phi = 30^\circ$ soil, the minimum required cohesion curves are almost the same for all cover depth. In other words, it can be confirmed again that the load factor $\mu$ or the required cohesion at the critical state for the unlined tunnel is independent of the earth cover.

On the other hand, for the tunnel embedded in the cohesive soil, the required cohesion curve is different if the earth cover is different. All curves for the cohesive soil are almost parallel one another and the required cohesion of the deeper tunnel is larger than that of the shallower tunnel.
7.4 Numerical analysis of lined tunnel

In this section, the lining will be included in the analysis in order to investigate how the lining affects the stability of tunnel when groundwater table varies. Note that the word called 'lining' does not mean only what is called the lining but also other reinforcement methods. Four patterns of lining are considered i.e., upper lining, lower lining, incomplete lining and complete lining (see ➂ of Table 7.1). The comparison between the upper and lower linings is considered first and the comparison between the incomplete and complete linings is considered later. The soil parameters and the lining parameters are shown in ➂ of Table 7.2. Also, in this case the cover-to-diameter ratio $C/D$ is fixed to be unity.

7.4.1 Comparison between upper and lower linings

Fig. 7.12 shows the influence of the groundwater on the stability of the tunnel when the upper and lower linings are considered. For the tunnel in cohesive soil($\phi = 0^\circ$), both of the upper and lower linings do not increase the stability of the tunnel much. On the other hand, for the tunnel embedded in the cohesive-frictional soil($\phi = 30^\circ$), both of these two lining patterns give some interesting effects on the stability of the tunnel. The upper lining strengthens the tunnel much when the groundwater table is lower than the springline. But if the groundwater table is higher...
Chapter 7. Effects of groundwater on the tunnel stability

Figure 7.12: Influence of the groundwater for upper and lower linings.

than the springline, the upper lining does not improve the stability of the tunnel. For the lower lining, the lining improve the stability of the tunnel when the groundwater table is higher than the springline and does not improve the stability of the tunnel much when the groundwater table is lower than the springline.

According to these results, we can conclude that the reinforcement at the invert of the tunnel plays the important role to improve the stability of the tunnel when the groundwater table is high.

7.4.2 Comparison between incomplete and complete linings

The influence of the groundwater on the stability of the tunnel for the incomplete and complete linings is shown in Fig. 7.13. Apparently, the complete lining can strengthen the tunnel much both in cohesive and cohesive-frictional soils at any groundwater table. The incomplete lining does not affect the stability of the tunnel embedded in cohesive soil much. But for the tunnel in the cohesive-frictional soil, the stability of the tunnel is improved much by the incomplete lining when the groundwater table is lower than the springline but if the groundwater table
7.5 Conclusions

The new method to include the effect of groundwater in the rigid-plastic finite element method was proposed. The conclusions are divided into two categories: unlined and lined tunnels.

7.5.1 Unlined tunnel

1. According to the Dupuit groundwater table, the parameter \( L \) in Eq.(7.1) has no significant effect on the obtained load factor \( \mu \). The horizontal groundwater table can be used instead of the Dupuit groundwater table without any significant change in the obtained results.

These results are similar to the results obtained from the case of the upper lining. It can be again concluded that the reinforcement at the invert of the tunnel is vital for the stability of the tunnel.

Figure 7.13: Influence of the groundwater for incomplete and complete linings.
2. The groundwater table has no much effect on the stability of the tunnel embedded in the cohesive soil but gives great effect on the stability of the tunnel embedded in the sandy soil. Especially, if the groundwater table is higher than springline of the tunnel, the stability of the tunnel decreases abruptly. This indicates that it is necessary to consider the existence of the groundwater during the excavation of the tunnel in sandy soil.

3. For cohesive-frictional soil, the load factor $\mu$ obtained from RPFEM is almost linearly proportional to the soil cohesion at any groundwater table.

4. The load factor $\mu$ obtained from RPFEM is independent to the earth cover if the angle of friction $\phi$ of soil is equal to $30^\circ$. On the other hand, for the tunnel in cohesive soil, the stability ratio $N$ of the tunnel increases with the earth cover.

### 7.5.2 Lined tunnel

The conclusions of the lined tunnel case are summarized below:

1. The upper lining and lower lining has not so much effect on the stability of the tunnel embedded in clay at any groundwater level.

2. For the tunnel in sandy soil, the upper lining cannot improve the stability of the tunnel when the groundwater table is higher than the springline but the stability of the tunnel is improved by the upper lining when the groundwater table is lower than the springline.

3. For the tunnel in sandy soil, the lower lining can strengthen the tunnel stability when the groundwater table is higher than the springline. On the other hand, the lower lining cannot improve the stability of the tunnel much when the groundwater table is lower than the springline.

4. The complete lining is apparently improve the stability of the tunnel much for both the tunnel in clay and sandy soil.

5. The incomplete lining cannot improve the stability of the tunnel when the groundwater table is higher than the springline, as the same as the result of the upper lining case.

6. The invert of the tunnel is necessary to be reinforced if the groundwater table is high.
The conclusion of the improvement of the lining patterns in sandy soil is shown in Table 7.3.

**Table 7.3**: Conclusion of the improvement of the stability due to the lining patterns for the sandy soil.

<table>
<thead>
<tr>
<th>Lining pattern</th>
<th>High groundwater table</th>
<th>Low groundwater table</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper lining</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>Lower lining</td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Incomplete lining</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>Complete lining</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

× Not improve the stability
○ Improve the stability
Nowadays, several tunneling methods are utilized to excavate tunnels. Particularly, the most popular methods are shield tunneling method and the New Austrian Tunneling Method (NATM). Hence, the studies on both tunneling methods are necessary to be carried out. The researches in this dissertation focused on both tunneling methods and the objectives of this dissertation are (1) to develop the new structural model, called rigid segment spring model, for the shield tunnel segment design and apply the model to the actual construction; (2) to investigate the stability of tunnel heading in soft clay (purely cohesive soil) considering the ground as green field; (3) to study the influence of the surface load on the tunnel stability; (4) to evaluate the effects of the groundwater on the tunnel stability. According to the results presented in the previous chapters, the following conclusions can be made:

**Simulation of shield tunnel segment**

The rigid segment spring model for analyzing the actual shield tunnel lining by considering the fact that the adjacent segments are not overlapped on each other was developed. Moreover, the methodology for analyzing the infinite segmental rings by considering only two representative rings was also proposed.

The parametric studies in order to determine the appropriate spring constants were carried out first by simulating the large-scale loading test on three segmental rings. The first case in which
the shear spring constants at ring joints were varied and normal spring constants are fixed, was carried out. No any set of spring constants was found to make the good agreement between the computed deformation and observed deformation, i.e. the computed vertical deformation of the crown and bottom of rings and computed horizontal deformation of lateral sides of rings are not satisfied the observed deformation when the same set of spring constants is used. The second case was subsequently carried out in order to allow the segment joint more easily to rotate. The normal spring constants are varied while the shear spring constants are fixed. The appropriate set of spring constants was found when the ratio $k_{rs1}/k_{sn1}$ is approximately 6.66.

By using this appropriate set of spring constants in the application of the rigid segment spring model to the actual segmental rings, the two cases of loading conditions, short-term and long-term conditions, were carried out. From comparison between numerical results and observed values from construction site, the deformation obtained from the short-term loading condition are close to the observed valued. For the long-term loading condition, the deformation is almost zero. This comparative results are reasonable because in the actual situation, the segmental ring deformed due to the short-term loads and no further deformation occur when it reaches the stable stage, i.e. long-term condition. It can be concluded, therefore, that the rigid segment spring model can be expected to apply in the design of shield tunnel lining if the appropriate spring parameters can be obtained.

**Numerical analysis of tunnel stability in soft clay**

Both the stabilities of 2D cross-section unlined tunnel and 3D tunnel heading in soft clay were investigated using the rigid finite element method and the computed results were compared with the observed values from the centrifugal modeling tests done by Mair. The main conclusions can be summarized as follows:

1. For 2D cross-section tunnel cases, RPFEM can simulate the the centrifugal modeling tests effectively. The difference between numerical and experimental result is larger for the deeper tunnel due to vertical friction at the wall of apparatus. The better results can be obtained when considering the effect of this friction.

2. The 3D semi-circular tunnel heading is more stable than the plane strain heading. The stability ratios obtained from 2D RPFEM are consistently varied between 0.5 and 0.6 of
the stability ratios obtained from 3D RPFEM. This suggests that the 3D RPFEM stability ratios can be approximated by dividing the 2D RPFEM stability ratios by the value in range $0.5 \sim 0.6$.

3. For 3D semi-circular tunnel heading cases, the stability ratios obtained from RPFEM are a bit larger than the experimental results. According to Table 6.4, the stability ratios $N_C$ obtained from 3D RPFEM are approximately $15\% \sim 25\%$ larger than those obtained from experiments. This means that the stability ratio predicted by RPFEM is consistently overestimated and the better prediction can be obtained by using the smaller undrained shear strength $c_u$ of clay than that obtained by Eq. (6.3).

4. In the case of 2D analysis, RPFEM can predict the stability of both the unlined tunnel and the plane strain heading in soft clay very well.

5. In fully lined tunnel case, the stability ratio increases with the cover. Furthermore, if the non-supported heading length is larger, the stability ratio decreases and approaches to the stability ratio of the unlined tunnel.

In addition, the study on the influence of the surface load on the tunnel stability was carried out. The conclusions associated with this study are summarized as follows:

1. The location of the surface load that affects the tunnel stability most is independent of the width of the surface load. The weakest distance $L$ is increasingly and almost linearly proportional to the cover depth.

2. The width of the failure zone at the surface increases with the cover depth of tunnel and also the width of the surface load.

3. The weak zone is equivalent to the zone where the velocity is found vertically downward for the velocity field of the non-loading case. This indicates that this distance is the distance where the tunnel and surface load interact mutually the most.

**Study on the effect of groundwater on the tunnel stability**

As discussed in the literature, the groundwater has a great effect on the tunnel stability. If the tunnel is excavated under the groundwater table, the stability of the tunnel is reduced. In
this chapter, the evaluation of the effect of groundwater on the tunnel stability was carried out. Both of the tunnels driven in purely cohesive soil and frictional soil were considered. The conclusions are summarized into two cases, unlined tunnel and lined tunnel as follows:

### Unlined tunnel

1. According to the Dupuit groundwater table, the parameter $L$ in Eq.(7.1) has no significant effect on the obtained load factor $\mu$. The horizontal groundwater table can be used instead of the Dupuit groundwater table without any significant change in the obtained results.

2. The groundwater table has no much effect on the stability of the tunnel embedded in the cohesive soil but gives great effect on the stability of the tunnel embedded in the sandy soil. Especially, if the groundwater table is higher than springline of the tunnel, the stability of the tunnel decreases abruptly. This indicates that it is necessary to consider the existence of the groundwater during the excavation of the tunnel in sandy soil.

3. For cohesive-frictional soil, the load factor $\mu$ obtained from RPFEM almost linearly proportional to the soil cohesion at any groundwater table.

4. The load factor $\mu$ obtained from RPFEM is independent to the earth cover if the angle of friction $\phi$ of soil is equal to $30^\circ$. On the other hand, for the tunnel in cohesive soil, the stability ratio $N$ of the tunnel increases with the earth cover.

### Lined tunnel

The conclusions of the lined tunnel case are summarized below:

1. The upper lining and lower lining has not so much effect on the stability of the tunnel embedded in clay at any groundwater level.

2. For the tunnel in sandy soil, the upper lining cannot improve the stability of the tunnel when the groundwater table is higher than the springline but the stability of the tunnel is improved by the upper lining when the groundwater table is lower than the springline.

3. For the tunnel in sandy soil, the lower lining can strengthen the tunnel stability when the groundwater table is higher than the springline. On the other hand, the lower lining
cannot improve the stability of the tunnel much when the groundwater table is lower than the springline.

4. The complete lining is apparently improve the stability of the tunnel much for both the tunnel in clay and sandy soil.

5. The incomplete lining cannot improve the stability of the tunnel when the groundwater table is higher than the springline, as the same as the result of the upper lining case.

6. The invert of the tunnel is necessary to be reinforced if the groundwater table is high.
Part III

Appendices
Appendix A

Introduction to continuum mechanics

The fundamental theories of continuum mechanics are described here. General motion and deformation are described first and then the motion of rigid body will be represented.

A.1 Motion and deformation of a Continuum

A.1.1 Body and Motion

According to Fig. A.1, let us suppose that a body, at time $t = t_0$, occupies a region of the physical space. The position of a particle at this time can be described by its coordinates $X_i$ with respect to a fixed rectangular Cartesian coordinate system.

![Figure A.1: Motion of body and its reference and present configuration](image-url)
Let the body undergo a motion and point $P$ move to $P'$ whose coordinates with respect to the same fixed axes are $x_i$. Then an equation of the form

$$x_i = x_i(X_1, X_2, X_3, t) \quad \text{or} \quad x = x(X, t) \quad (A.1)$$

describes the path of the particle which at $t = t_0$ is located at $X_i$. In Eq. (A.1) the triplet $(X_1, X_2, X_3)$ serves to identify different particles of the body and is known as reference coordinates. The triplet $(x_1, x_2, x_3)$ gives the present position of the particle which at time $t = t_0$ was at the place $X_i$. Note that for a specific particle Eq. (A.1) defines the path line (or trajectory) of the particle. Of course

$$X_i = x_i(X_1, X_2, X_3, t_0) \quad \text{or} \quad X = x(X, t_0) \quad (A.2)$$

which merely verifies the fact that the particle under consideration occupied the place $X_i$ at $t = t_0$. Furthermore, Eq. (A.1) is the one-to-one function, and its inverse function is given as follow,

$$X = X(x, t) \quad (A.3)$$

Note that Eq. (A.1) is called Lagrangian description and Eq. (A.3) is called Eulerian description.

### A.1.2 Deformations and deformation gradients

From Fig. A.2 and Eq. (A.1), consider two particles in the reference configuration with an elemental distance $dX$ apart. Then in the present configuration the same two particles will be an elemental $dx$ apart, given by

$$dx = x(X + dX, t) - x(X, t) \quad (A.4)$$

As the magnitude $|dX|$ decreases toward zero, we may approximate $x(X + dX, t)$ using Taylor approximation:

$$x(X + dX, t) \approx x(X, t) + \frac{\partial x}{\partial X}dX \quad (A.5)$$

From Eqs. (A.4) and (A.5),

$$dx = \frac{\partial x}{\partial X}dX \quad (A.6)$$
in which the deformation gradient $F$ defined as

$$F = \frac{\partial x}{\partial X} \quad (A.7)$$

We have

$$dx = F dX \quad (A.8)$$

The linear transformation or tensor $F$ maps the neighborhood of the particle $P$ in the reference configuration into the present configuration. This is depicted in Fig. A.2 for a spherical neighborhood. In order to ensure that such a transformation is reversible and one-to-one in a certain region, the Jacobian $J$, defined by

$$J = \det F \quad (A.9)$$

must not vanish at any point of the region. If the Jacobian is positive anywhere, then a right-hand set of coordinates is transformed into another right-hand set, and the transformation is said to be proper. If the Jacobian is negative everywhere, a right-hand set of coordinates is transformed into a left-hand one, and the transformation is said to be improper. Moreover, the local volumetric ratio by deformation is shown by Jacobian in the following equation,

$$dV = J dV \quad (A.10)$$

$dV$ and $dv$ denote the infinitesimal volumes of the object in the reference and the present configurations, respectively.
In term of displacement, the displacement vector \( \mathbf{u} \) and relative displacement vector \( d\mathbf{u} \) are written as follows,

\[
\mathbf{u} = x - X \tag{A.11}
\]

\[
d\mathbf{u} = dx - dX \tag{A.12}
\]

Substituting Eq. (A.8) into Eq. (A.12) and using \( \mathbf{F} \), obtain

\[
d\mathbf{u} = (\mathbf{F} - I) dX \tag{A.13}
\]

The gradient \( \mathbf{W} \) is known as the displacement gradient.

### A.1.3 Polar decomposition

Recall from Section (A.1.2), since \( \mathbf{F} \) is reversible, we may write the polar decomposition of deformation gradient \( \mathbf{F} \) as,

\[
\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{A.14}
\]

in which \( \mathbf{R} \) is a proper orthogonal tensor. \( \mathbf{U} \) and \( \mathbf{V} \) are positive definite symmetric tensors and \( \det \mathbf{F} \) is assumed to be positive. The tensors \( \mathbf{U} \) and \( \mathbf{V} \) are positive tensors, and they have in general three positive principal values and three orthogonal principal directions. By Eq. (A.14) \( \mathbf{U} \) and \( \mathbf{V} \) are also similar to each other, so that they both have the same principal values, which we shall denote by \( \lambda_1, \lambda_2, \lambda_3 \). Furthermore, the principal directions \( u'_i \) of \( \mathbf{V} \) are related to the principal directions of \( u_i \) of \( \mathbf{U} \) by

\[
\mathbf{u} = \mathbf{Ru}' \quad \text{or} \quad u'_i = R_{ij}u_j \tag{A.15}
\]

We may thus interpret the deformation \( \mathbf{F} = \mathbf{RU} \) in the following manner. The tensor \( \mathbf{U} \) is applied first to the neighborhood of \( \mathbf{X} \) and causes elemental distances in the principal directions \( u_i \) to change in length by the ratios \( \lambda_i \), called therefore the principal stretches, but not to change in direction. Thus \( \mathbf{U} \) and \( \mathbf{V} \) are called stretch tensors; \( \mathbf{U} \) is the right stretch tensor and \( \mathbf{V} \) the left stretch tensor. An elemental box having faces initially perpendicular to the \( u_i \) would still have its faces perpendicular to the \( u_i \). The stretch \( \mathbf{U} \) is then followed by the rigid-body
rotational $R$ to complete the local deformation produced by the deformation gradient $F$. The decomposition $F = VR$ represents first the rigid-body rotation $R$, followed by the stretches along the principal directions $u'_i$ of $V$. Both of the decompositions of $F$ are depicted in Fig. A.3, where the dashed lines indicate intermediate mappings. It must be remembered that $F$, $R$, $U$ and $V$ in general are all functions of the particle $P$ and can vary throughout the body.

The square of the right stretch tensor is

$$C = U^2 = F^T F$$

(A.16)

and is called the right Cauchy-Green (strain) tensor of the deformation. And the square of the left stretch tensor is

$$B = V^2 = FF^2$$

(A.17)

is called the left Cauchy-Green (strain) tensor.
Appendix A. Introduction to continuum mechanics

A.2 Rigid body motion

A.2.1 Deformation gradient and displacement gradient

A rigid body is a system of particles in which the distances between any two particles do not vary, i.e. no strain occurs in this body. From (A.16) and (A.17), it is concluded that

\[ C = B = U = V = I \]  \hspace{1cm} (A.18)

Clearly,

\[ F = R \]  \hspace{1cm} (A.19)

So we can see that in the rigid body motion, the rotation is included in it. And then, from Eq. (A.13), we get

\[ W = R - I \]  \hspace{1cm} (A.20)

A.2.2 Infinitesimal rotation

The rotation of rigid body is assumed to be infinitesimal. Since \( R \) is orthogonal tensor, i.e. \( R^T R = I \) and by using Eq. (A.20), we get

\[ (I + W)^T (I + W) = I + W + W^T + W^T W = I \]  \hspace{1cm} (A.21)
A.2. Rigid body motion

Since this rotation is assumed to be infinitesimal, rigid body rotates by small angle. That is \( W^T W \) can be neglected. Then

\[
W^T = -W \tag{A.22}
\]

Clearly, \( W \) is anti-symmetric tensor and its components are supposed as \( W_{ij} \). Thus

\[
W_{ij} = \begin{cases} 
-W_{ji} & \text{if } i \neq j, \\
0 & \text{if } i = j. 
\end{cases} \tag{A.23}
\]

Thus, tensor \( W \) has only three independent components: \( W_{12}, W_{23}, W_{31} \). These three components form a vector as shown,

\[
\omega = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} = -\begin{pmatrix}
W_{12} \\
W_{23} \\
W_{31}
\end{pmatrix} \tag{A.24}
\]

Then, by using the concept of permutation symbol we can rewrite Eq. (A.24) as

\[
\omega_k = -\frac{1}{2} e_{kij} W_{ij} \quad \text{and,} \tag{A.25}
\]

\[
W_{ij} = -e_{kij} \omega_k \tag{A.26}
\]

where, \( e_{kij} \) is the permutation symbol which has definition as shown in Eq. (A.27)

\[
e_{ijk} = \begin{cases} 
1 & \text{if } ijk \text{ is an even permutation of } 1, 2, 3 \\
-1 & \text{if } ijk \text{ is an odd permutation of } 1, 2, 3 \\
0 & \text{if any two of } i, j, k \text{ are same}
\end{cases} \tag{A.27}
\]

And then \( W \) can be written as

\[
W = \begin{pmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{pmatrix} \tag{A.28}
\]

Note that \( W \) and \( \omega \) are called as rotation tensor and rotation vector, respectively.
Appendix A. Introduction to continuum mechanics

For arbitrary vector $a$, the relation between $W$ and $\omega$ is expressed as follow

$$Wa = \omega \times a \quad (A.29)$$

Thus, if we suppose $a = dX$ and substitute it into Eq. (A.13), obtain

$$du = \omega \times dX \quad (A.30)$$

The relative displacement is the vector product of $\omega$ and $d\mathbf{x}$. This is exactly what would have been produced by an infinitesimal rotation $|\omega|$ about an axis through considered point in the direction of $\omega$. Fig. A.4 shows the components of the rotation vector and right-hand side rule of rotation.

A.2.3 General motion of rigid body

When a rigid body is applied by external force, it will move in the rule of translation and rotation. The relation between the position of particle before movement $X$ and after movement $x$ is expressed in following equation.

$$x = RX + \delta \quad (A.31)$$

where, $\delta$ is the translation vector of the origin. Its components are shown as

$$\delta = \begin{cases} 
\delta_1 \\
\delta_2 \\
\delta_3 
\end{cases} \quad (A.32)$$
A.2. Rigid body motion

Therefore, by the assumption of the infinitesimal rotation, the displacement vector is expressed as follows

\[ u = WX + \delta \]  \hspace{1cm} (A.33)
\[ = \omega \times X + \delta \]  \hspace{1cm} (A.34)
\[ = \delta - X \times \omega \]  \hspace{1cm} (A.35)
\[ = \delta - WX \omega \]  \hspace{1cm} (A.36)
\[ = (I, -WX) \begin{bmatrix} \delta \\ \omega \end{bmatrix} \]  \hspace{1cm} (A.37)
\[ = Qx \]  \hspace{1cm} (A.38)

in which,

\[ Q = (I, -WX) \] , \hspace{0.5cm} x = \begin{bmatrix} \delta \\ \omega \end{bmatrix} \]  \hspace{1cm} (A.39)

are defined. \( WX \) is antisymmetric tensor corresponding to \( X \) axis. Note that \( x \) in Eq. (A.39) and \( x \) in Eq. (A.33) are different. Conclusively, from Eq. (A.38) we can obtain the general equation of rigid body motion as

\[
\begin{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & X_3 & -X_2 \\
0 & 1 & 0 & -X_3 & 0 & X_1 \\
0 & 0 & 1 & X_2 & -X_1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\omega_1 \\
\omega_2 \\
\omega_3 \\
\end{bmatrix}
\]  \hspace{1cm} (A.40)

We can see that there are six degree of freedoms which are the components of vector \( x \). If \( x \) is known, the position of considered point after the motion occurs can be determined. Thus, in rigid body motion analysis, the determination of the unknown vector \( x \) must be solved first and subsequently the displacement of any arbitrary point in that rigid body can be determined.
Convexity plays a fundamental role in thermodynamics and in many problems of mechanics, especially in plasticity. The definitions of convex sets and functions are introduced and also the condition for the differentiable convex functions is described here. For more detail, refer to (Bertsekas, 2003) and (Maugin, 1992).

### B.1 Convex sets

A subsets \( C \) of \( \mathbb{R}^n \) is called *convex* if

\[
\alpha x + (1 - \alpha)y \in C, \quad \forall x, y \in C, \forall \alpha \in [0, 1]. \tag{B.1}
\]

In other words, the linear interpolation between any two points of the set must yield points that lie within the set, as illustrated in Fig. B.1.

### B.2 Convex functions

A convex function is defined below and is illustrated in Fig. B.2.

Let \( C \) be a convex subset of \( \mathbb{R}^n \). A function \( f : C \rightarrow \mathbb{R} \) is called *convex* if

\[
f (\alpha x + (1 - \alpha)y) \leq \alpha f (x) + (1 - \alpha)f (y), \quad \forall x, y \in C, \forall \alpha \in [0, 1]. \tag{B.2}
\]
Appendix B. Convexity

Figure B.1: Definition of a convex set

Convex sets
Nonconvex sets

Figure B.2: Definition of a convex function

Figure B.3: Characterization of convexity in terms of first derivatives

Figure B.4: Geometric illustration of the ideas underlying the proof of the proposition (a)

A function is also said to be *strictly convex* if

\[ f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y) \quad \forall \, x, y \in C \text{ with } x \neq y \text{ and all } \alpha \in (0, 1) \]

(B.3)

### B.3 Characterizations of differentiable convex functions

For differentiable functions, a useful alternative characterization of convexity is given below and is illustrated in Fig. B.3.

Let \( C \) be a convex subset of \( \mathbb{R}^n \) and let \( f : \mathbb{R}^n \to \mathbb{R} \) be differentiable over \( \mathbb{R}^n \).
B.3. Characterizations of differentiable convex functions

(a) $f$ is convex over $C$ if and only if

$$f(z) \geq f(x) + (z - x)\nabla f(x), \quad \forall x, z \in C. \quad \text{(B.4)}$$

(b) $f$ is strictly convex over $C$ if and only if the above inequality is strict whenever $x \neq z$.

The proposition (a) can be proved as follows:
Assume that the inequality (B.4) holds. Choose any $x, y \in C$ and $\alpha \in [0, 1]$, and let $z = \alpha x + (1 - \alpha)y$(see Fig. B.4). Using the inequality (B.4) twice, we obtain

$$f(x) \geq f(z) + (x - z)\nabla f(z),$$
$$f(y) \geq f(z) + (y - z)\nabla f(z).$$

We multiply the first inequality by $\alpha$, the second by $(1 - \alpha)$, and add them to obtain

$$\alpha f(x) + (1 - \alpha)f(y) \geq f(z) + (\alpha x + (1 - \alpha)y - z)\nabla f(z) = f(z),$$

which proves that $f$ is convex.


