DISASTER PREVENTION RESEARCH INSTITUTE
BULLETIN No. 7. MARCH, 1954

VIBRATION PROBLEMS OF SKYSCRAPER

DESTRUCTIVE ELEMENTS OF SEISMIC WAVES FOR STRUCTURES

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Preface:

The destructive element of an earthquake for structures that has been recognized, is an effect of the acceleration value. But, from our view point, the value of the maximum acceleration gained from its seismogram is not the only important element usually for the structure. We can show the existence of most undesirable seismic waves having a small acceleration value. As a matter of course, in final aseismic design of structures, it is necessary to determine the value of the lateral acceleration from some past earthquakes, but how much acceleration value we adopt for each structure must be determined according to the sort of factor in the seismic waves causing the destructive power. The seismic waves, as shown by the seismograms, are very complicated and mingled with many various and irregular waves, and we can observe in the seismograms, that the period of the waves from the acceleration record is shorter than that from the displacement record. Thus, as we can never foresee any seismic waves in the future, we must, in designing structures, select the seismic waves which are likely to happen and affect the structures most adversely from the seismogram records in the past. With respect to the present aseismic design considering the acceleration value only, it seems to us that there is no clear consideration about the seismic waves, on which the aseismic

design must be based. So that, we are going to consider our hypothetical seismic waves, which involve all sorts of irregularity, and which are apt to happen and give the most dangerous damage to the structures.

1. Unstationary Vibrations of the Skyscraper.

The absolute value of the seismic acceleration has not the most destructive effect on the structures. We consider now, remarking the irregularity of the seismic waves, the general vibrating state of the framed structures suffering a constant acceleration in a short time interval, by mathematical transaction.

As the fundamental equation necessary to analyse the vibration of the framed structures, we use the equation of the shearing vibration. And as the authors have often pointed out[1)2)3)], an important subject on the aseismic engineering is the consideration of a vibrating system in the unstationary state. So we give as a general fundamental equation the shearing vibrations.[3)]

$$\frac{\partial}{\partial x}\left\{S(x)\frac{\partial y}{\partial x}\right\}-\frac{\partial}{\partial x}\left\{P(x)\frac{\partial y}{\partial x}\right\}-D(x)\frac{\partial y}{\partial t}-\rho A(x)\frac{\partial^2 y}{\partial t^2}=-F(x,t) \tag{1.1}$$

then $S(x)=G(x)\ A(x)$ ..... shearing stiffness.

$P(x)$ ..... axial forces of framed structures by dead and live loads.

$D(x)$ ..... damping coefficient distributing on structures.

ρ ..... unit mass of structures.

$A(x)$ ..... cross-sectional area of structures.

$F(x,\ t)$ ..... generalized external force involving unstationary forces.

By using the Laplace transformation we obtain the solution (1.2) of Eq. (1.1).

$$\begin{aligned} y(\xi,\ t)=&e^{-\varepsilon t}\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu\mu^2\{a(\xi)\}^{\frac{1}{2}}}\times\int_0^t\int_0^1\bar{F}(s,\zeta)\frac{\varphi_\nu(s)}{\{a(s)\}^{\frac{1}{2}}}ds\cdot e^{\varepsilon\zeta}\sin\omega_\nu(t-\zeta)d\zeta \\ &+e^{-\varepsilon t}\sum_{\nu\ 1}^{\infty}\frac{\varphi_\nu(\xi)}{\{a(\xi)\}^{\frac{1}{2}}}\cos\omega_\nu t\int_0^1\{a(s)\}^{\frac{1}{2}}\ M(s)\varphi_\nu(s)ds \\ &+e^{-\varepsilon t}\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu\{a(\xi)\}^{\frac{1}{2}}}\sin\omega_\nu t\int_0^1\{a(s)\}^{\frac{1}{2}}\ [N(s)+\varepsilon\ M(s)]\ \varphi_\nu(s)ds \end{aligned} \tag{1.2}$$

where $\xi=x/h,\quad S(x)=S_0\vartheta(\xi),\quad P(x)=P_0q(\xi),\quad A(x)=A_0a(\xi),$
$D(x)=D_0a(\xi)$

φ_ν, ω_ν^2 are Eigen function and Eigen value of the next differential equation.

$$\frac{d}{d\xi}\left\{p(\xi)\frac{d\varphi}{d\xi}\right\}+\frac{\delta^2}{2}\,\varepsilon\, a(\xi)+\omega^2\,\mu^2\, a(\xi)\varphi=0$$

and

$$p(\xi)=\vartheta(\xi)-\lambda^2\, q(\xi),\quad \lambda^2=P_0/S_0,\quad \mu^2=\frac{\rho A_0}{S_0/h^2}$$

$$\delta^2=\frac{D_0}{S_0/h^2},\quad F=\frac{F}{S_0/h^2}\quad \varepsilon=\frac{\delta^2}{2\mu^2}=\frac{D_0}{2\rho A_0}.$$

$$|y(\xi,\ t)|_{t=0}=M(\xi),\quad \left|\frac{\partial y}{\partial t}\right|_{t=0}=N(\xi) \tag{1.3}$$

Now we consider the case, that the seismic acceleration acts on the structures in an infinitesimal time interval constantly. This is convenient to study the aseismic estimate of the present vibrating system, for the reason that the seismic waves are not steady waves but the waves with few same period[4)5)], and another reason is that we want to know either the seismic acceleration value affects fatally to the vibrating state of structures or not.

When $\Phi(t)$ is the displacement of the ground motion, and α is the value of seismic acceleration, regarding the initial condition $M(\xi)=0$, $N(\xi)=0$ in (1.3), we conclude that we may consider the first term only.

To simplify the discussion, $a(\xi)=1$ say. (the cross-sectional area is constant).

$$\overline{F}=-\mu^2\frac{d^2\Phi}{dt^2} \tag{1.4}$$

So that from Eq. (1.2)

$$y(\xi,\ t)=-e^{-\varepsilon t}\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu}\int_0^1\varphi\ (s)ds\times\int_0^1\frac{d^2\Phi}{dt^2}e^{\varepsilon\zeta}\sin\omega_\nu(t-\zeta)d\zeta \tag{1.5}$$

$$\left.\begin{aligned}\frac{d^2\Phi}{dt^2}&=\alpha\ :\ 0\leqq t\leqq\tau\\ \frac{d^2\Phi}{dt^2}&=0\ :\ t<0,\ t>\tau\end{aligned}\right\} \tag{1.6}$$

The acceleration acts as Fig. 1, and τ is its time interval.

We may regard that α is great and τ is infinitesimal usually. So in Eq. (1.5),

$$\omega_\nu=\frac{k_\nu}{\mu}\left[1-\frac{\varepsilon}{4}\frac{\delta^2}{k_\nu^2}\right]$$

$$\varphi_\nu(\xi)=\sqrt{2}\sin k_\nu\xi \tag{1.7}$$

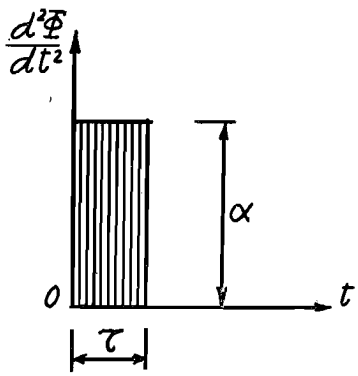


Fig. 1.

$$k_\nu^2=\omega_\nu^2\mu^2-\gamma_0 \qquad \gamma_0=\delta^2\varepsilon/2$$
$$k_\nu=(2\nu-1)\pi/2 \quad [\nu=1,\ 2,\ 3\cdots\cdots] \tag{1.8}$$

and regarding

$$T_\nu=2\pi/\omega_\nu=2\pi/\frac{k_\nu}{\mu}\left(1-\frac{\gamma_0}{4k_\nu^2}\right) \tag{1.9}$$

we get the next equation,

$$y(\xi,\ t)=-2e^{-\varepsilon t}\sum_{\nu=1}^{\infty}\frac{\sin k_\nu\xi}{k_\nu\xi}\times\begin{cases}\alpha\, I_t\cdots\cdots\cdots\cdots\cdots 0\leqq t\leqq\tau\\ \alpha\, I_\tau\cdots\cdots\cdots\cdots\cdots t>\tau\end{cases} \tag{1.10}$$

where

$$\alpha I_t=\int_0^t\frac{d^2\Phi}{dt^2}e^{\varepsilon\zeta}\sin\omega_\nu(t-\zeta)d\zeta$$
$$=\frac{\alpha/\omega_\nu}{(\varepsilon/\omega_\nu)^2+1}\left[e^{\varepsilon\zeta}-\frac{\varepsilon}{\omega_\nu}\sin\omega_\nu t-\cos\omega_\nu t\right] \tag{1.11}$$

$$\alpha I_\tau=\int_0^\tau\frac{d^2\Phi}{dt^2}e^{\varepsilon\zeta}\sin\omega_\nu(t-\zeta)d\zeta$$
$$=\frac{\alpha/\omega_\nu\sin\omega_\nu t}{(\varepsilon/\omega_\nu)^2+1}\left[e^{\varepsilon\zeta}\left\{\left(\frac{\varepsilon}{\omega_\nu}\right)\cos\omega_\nu\tau+\sin\omega_\nu\tau\right\}-\frac{\varepsilon}{\omega_\nu}\right]$$
$$-\frac{\alpha/\omega_\nu\cos\omega_\nu t}{(\varepsilon/\omega_\nu)^2+1}\left[e^{\varepsilon\zeta}\left\{\left(\frac{\varepsilon}{\omega_\nu}\right)\sin\omega_\nu\tau-\cos\omega_\nu\tau\right\}+1\right] \tag{1.12}$$

To make the velocity of the ground motion is constant when the action of the acceleration finished, we take the value of α, τ as follows.

1. $\alpha=0.2g$, $\tau=0.5$ sec.
2. $\alpha=0.4g$, $\tau=0.25$ sec.
3. $\alpha=0.8g$, $\tau=0.125$ sec.

g: acceleration value of gravity.

Then $\alpha\tau=$ const. And when the natural period of the structures in the fundamental mode is 1 sec. and 2 sec., using the Eqs. (1.10)—(1.12), we get Fig. 2 by the numerical computation.

In Fig. 2, y_1 shows the displacement of the fundamental mode of vibration and y_2, y_3,……each shows the displacement of the first and the second……mode of vibration respectively. From this, when the impluse $\overline{\alpha\tau}$ is constant, it is recognized that the vibrating state of the structures scarecely depends upon the time interval τ, so that it depends upon the acceleration value α indirectly. And the vibrating modes of higher orders are, of course, apparent but they vanish earlier than the fundamental mode does, due to the damping.

Comparing the higher modes with the fundamental we can see also that the maximum displacements of higher modes diminish in the ratio

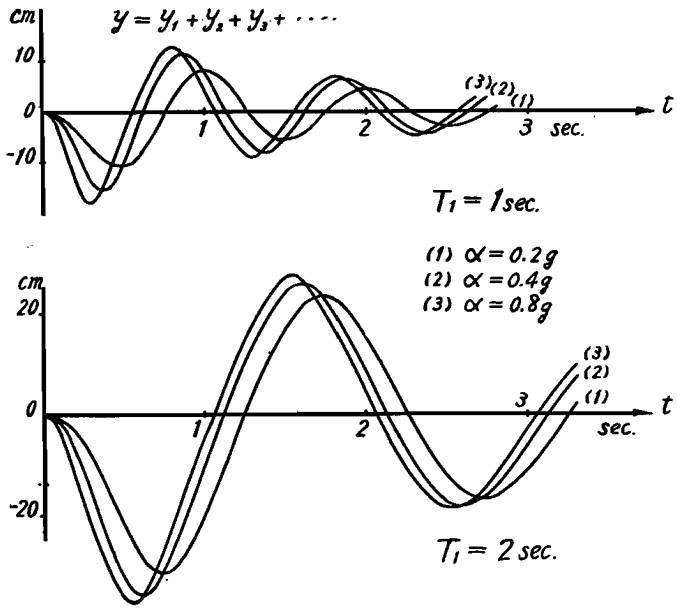


Fig. 2. Deformation at the top of the structure.

of 1 to $\frac{1}{3^3}$, $\frac{1}{5^3}$, $\frac{1}{7^3}$,.....

In the next, comparing with the vibration of the structures, we consider the ground motion, because we want to know the relation between the structural deformation and the ground motion.

Using Eq. (1.6), the ground displacement Φ is

$$\Phi = \frac{1}{2}\alpha t^2 + Ct + D$$

C, D: integral constant.

Substituting the initial condition,

$$\Phi = 0, \quad \frac{d\Phi}{dt} = 0 \qquad \text{when } t = 0, \text{ we get}$$

$$\Phi = \frac{1}{2}\alpha t^2 \qquad 0 \leqq t \leqq \tau \tag{1.13}$$

and when $t > \tau$, $\frac{d^2\Phi}{dt^2} = 0$, $\frac{d\Phi}{dt} = \text{const.} = \alpha\tau$

so

$$\Phi = \alpha\tau\left(t - \frac{1}{2}\tau\right) \tag{1.14}$$

From Fig. 2, looking for the time interval t at the first maximum

displacement of the structures, and substituting this value to Eq. (1.14), we find the displacement of the ground motion.

when $T_1=1$ sec.

1. $\alpha=0.2g$, $\tau=0.5$ sec. $t=1/2$ sec.
2. $\alpha=0.4g$, $\tau=0.25$ sec. $t=3/8$ sec.
3. $\alpha=0.8g$, $\tau=0.125$ sec. $t=5/16$ sec.
1. $\Phi=\alpha\tau(t-1/2\tau)=0.1g(0.5-0.25)=24.5$ cm.
2. $\Phi=$ 〃 $=0.1g(0.375-0.125)=24.5$ cm.
3. $\Phi=$ 〃 $=0.1g(0.3125-0.0625)=24.5$ cm.

When $T_1=2$ sec., we find the same tendency as before. That is, we may say as follows, in these cases, if we take the several values of α, the time at the maximum displacement of the structure is not equal, but an equivalent displacement of the ground motion is equal. Since, we may conclude also that, seeing three curves in Fig. 2, the maximum displacement increases when α becomes greater and τ smaller, but a value of the displacement of the ground motion is quite equal at the corresponding maximum displacement and zero displacement of the structures. (Fig. 3).

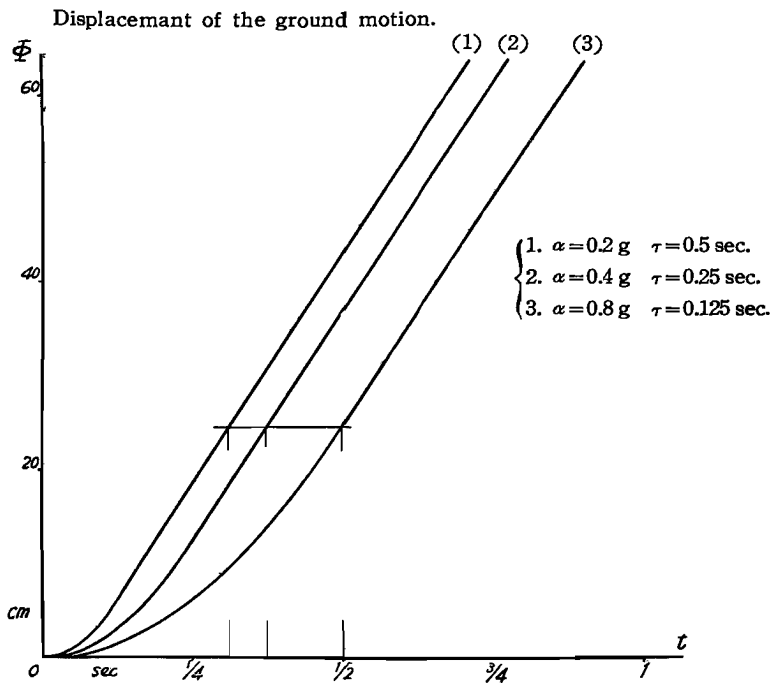


Fig. 3. Ground motion under the constant impulse.

2. Consideration of the destructive seismic waves.

In order to get a consideration about the more actual ground motion of earthquakes, we adopt the action of waves having an acceleration as shown in Fig. 4(a). This corresponds with the velocity diagram Fig. 4(b), and with the displacement diagram Fig. 4 (c). Fig. 5 represents our hypothetical seismic wave and the seismic motion in simple harmonic manner. So irregular seismic waves may be substituted approximately for these hypothetical waves. Therefore, the period of the seismic motion corresponds to the time interval $2T$ in Fig. 4(d). We consider now the wave, which finishes a definite ground displacement within a constant time interval T, and then study the vibrating state of structures, varying

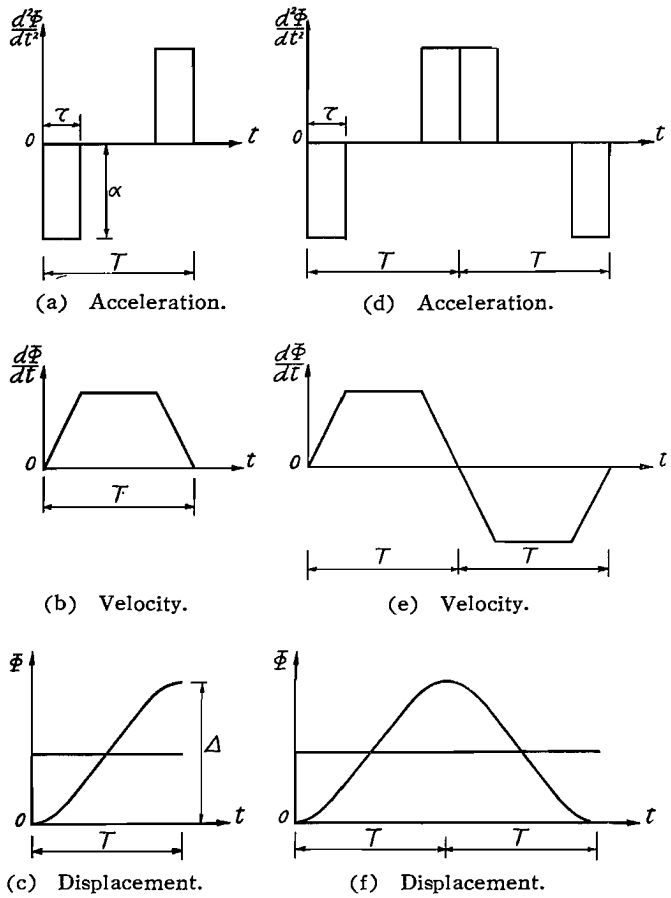


Fig. 4. Our hypothetical seismic waves.

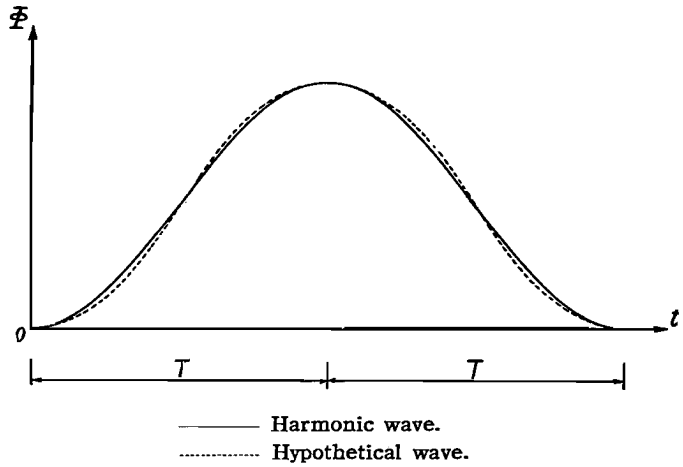
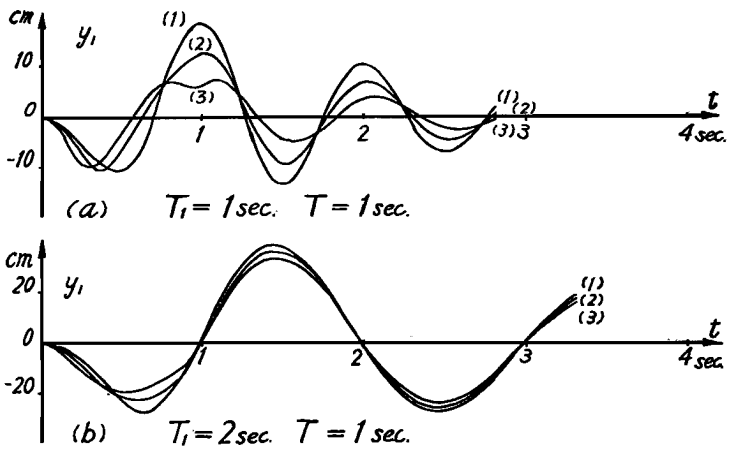


Fig. 5. Comparison of our hypothetical seismic wave with the seismic motion in simple harmonic manner.

in acting acceleration value. That is to say, giving the absolute value of the acceleration α and its acting time interval τ as Table 1. in order to make the displacement of the ground motion in Fig. 4(c) is constant, we get the displacement curves in Fig. 6 using Eqs. (1.10)—(1.12). In Fig. 6, T is the semi-period of the ground motion, T_1 means the natural period of the structure in the fundamental mode, and the number of the curve corresponds to the Table 1.



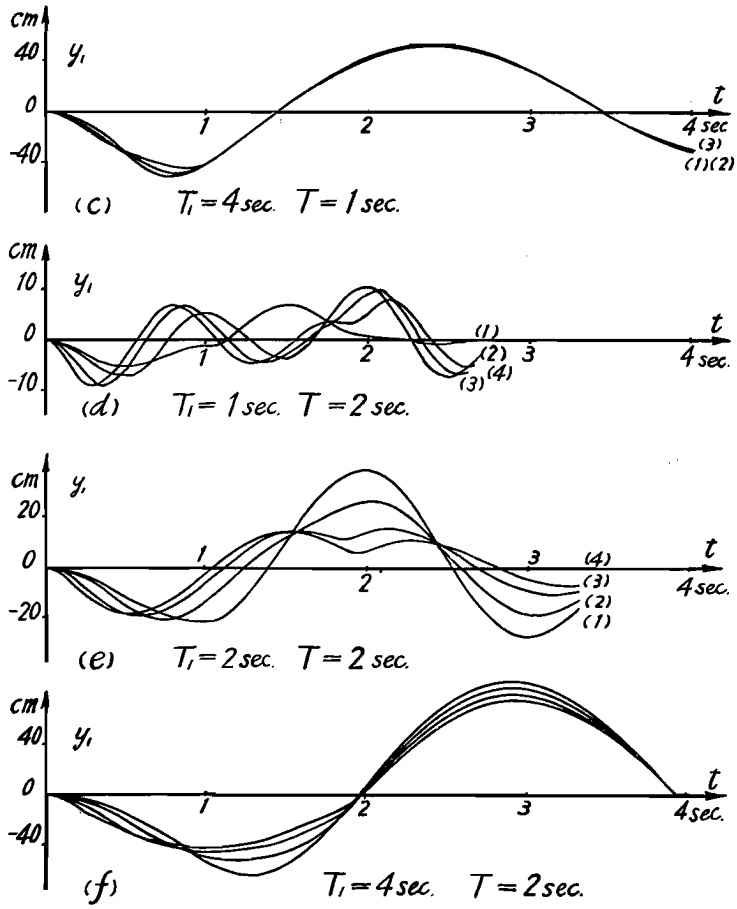


Fig. 6. Structural deformations under the constant mean velocity of the ground motion.

1. T=1 sec. $\Delta=\frac{0.1}{2}$g

No.	α	τ sec.
1	0.2g	0.5
2	0.4g×2/3	0.2
3	0.8g×4/7	0.125

2. T=2 sec. Δ=0.1g

No.	α	τ sec.
1	0.1g	1
2	0.2g× 2/3	0.5
3	0.4g× 4/7	0.25
4	0.8g× 8/15	0.125

Table. 1.

From Fig. 6, which is made in order to make the mean velocity of the ground motion Δ/T costant, we can see the relation between the value of the acceleration α and the maximum displacement of the structures, and show this relation in Fig. 7.

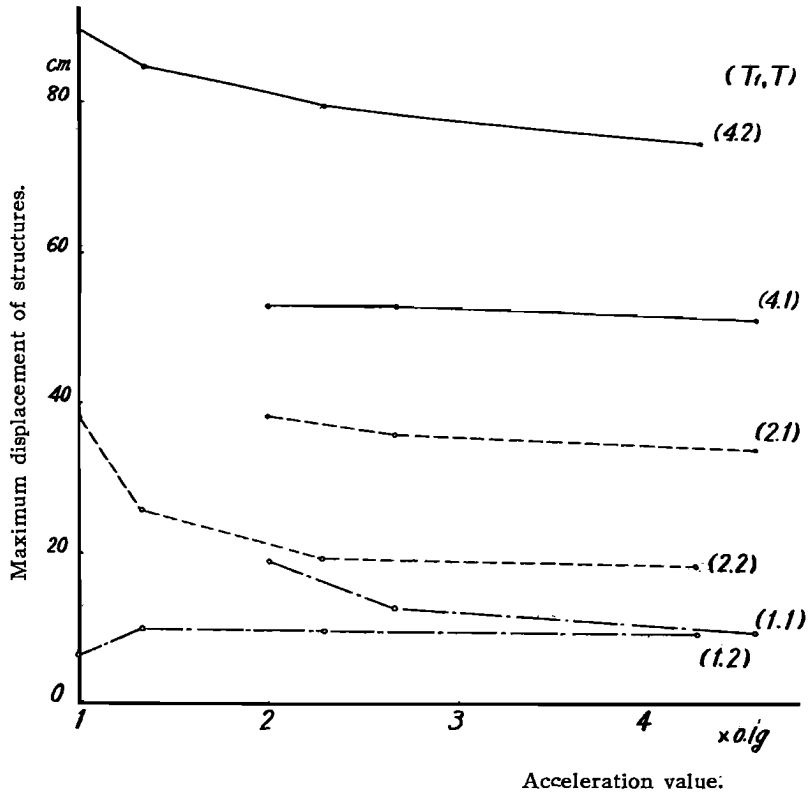
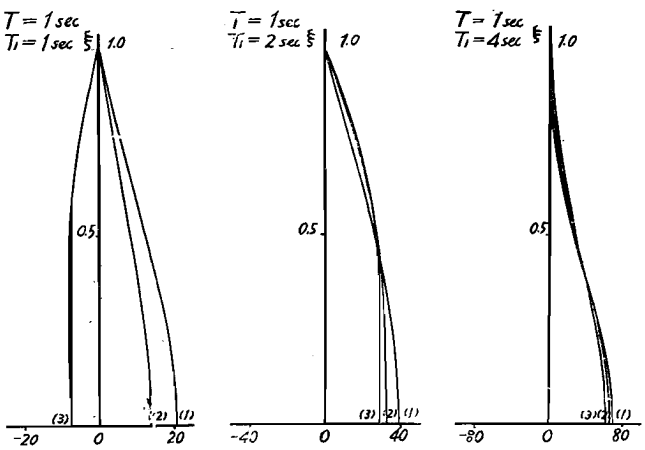


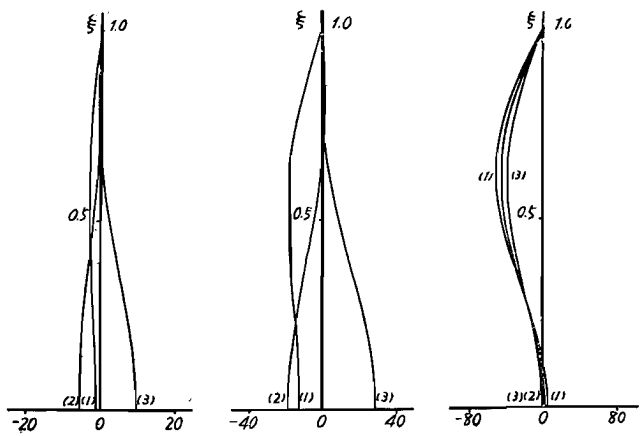
Fig. 7. Relation between the acceleration value and the maximum displacement of structures.

Then putting the above consequences together, we may conclude as follows: The deformation of the structures is greatest when the period of the ground motion approaches to the natural period of the structures in the fundamental mode, and the value of the acceleration has few effects upon the value of the structural deformation. Our hypothetical seismic motion coming into question should have a period coinciding with the natural period of the structures in the fundamental mode, and it is not important how great peak of acceleration the seismic waves have, but the displacement of the ground motion Δ, which has finished within the interval of the semi-period T. In other words, the mean velocity of the ground motion Δ/T prescribes the displacement of the structural deformation. According to this conclusion, we can prescribe the undesirable waves

among the seismic waves, and furthermore if we study precisely into the waves having the same mean velocity of the ground motion, the distribution of the shearing stress in the structures shows some difference, although the quantity of the structural deformation is scarecely affected by the variation of α and τ. (Fig. 8). And the wave, in the case when the value of τ is a quarter of the natural period of the structure in the fundamental mode, gives the most undesirable stress distribution. So, according to the

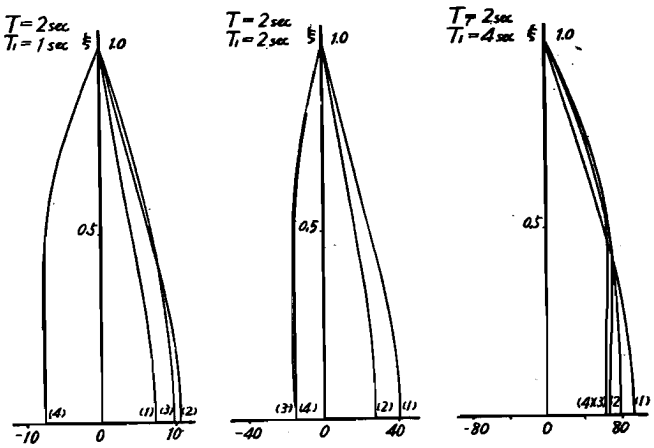


When the fundamental mode is maximum.

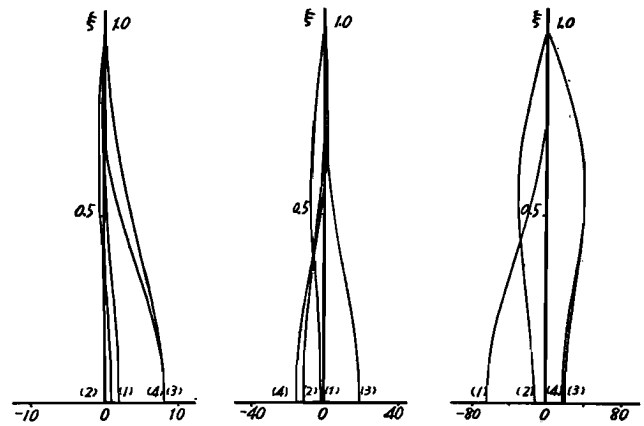


When the 1st higher mode is maximum.

Fig. 8 (a). Shearing stress distributions when $T=1$ sec.



When the fundamental mode is maximum.



When the 1st higher mode is maximum.

Fig. 8 (b). Shearing stress distribution when $T=2$ sec.

above consideration, the most undesirable wave that affects the worst damage to the framed strtuctures has the period nearest to the natural period of the structures in the fundamental mode, and has the greatest displacement (Amplitude). This wave is once prescribed by the mean velocity of the ground motion, but we can say anew, among such prescribed wave by the equal mean velocity the wave having a maximum velocity affects the most undesirable damage to the structures (Fig. 9). Therefore, we conclude, as one of the authors, Ryo Tanabashi, maintained already in

1937, in the report[7], "On the Resistance of Structures to Earthquake Shocks," we must lay great emphasis on the velocity of the earthquake as a measure of the destructive power of the seismic wave. In an aseismic design of structures, we must adopt the smaller acceleration value for the structure having the longer natural period.

3. Adoption of K as the ratio of the lateral acceleration to gravity.

According to the consequence of the above discussion, we study further on the records of the Kwantō (S. E. Japan) earthquake on Sept. 1, 1923, which was the greatest scale in the last half century, and select our hypothetical seismic waves from the record[6][7]. Calculating the shearing stress distribution of structures suffered from this seismic wave, we can adopt the quantitative coefficient K from this great Kwantō earthquake. At the time of this earthquake the seismologists got the displacement record at Tokyo Imperial University, Hongo, Tokyo, and succeeded in recording a wave with 8.86cm displacement and 1.35 sec. period in horizontal direction. Our old records of earthquake were all got from the displacement seismograph, and so we culculated the maximum acceleration of the seismic motion assuming it as a simple harmonic motion. Thus the calculated max. acceleration value is 959.62

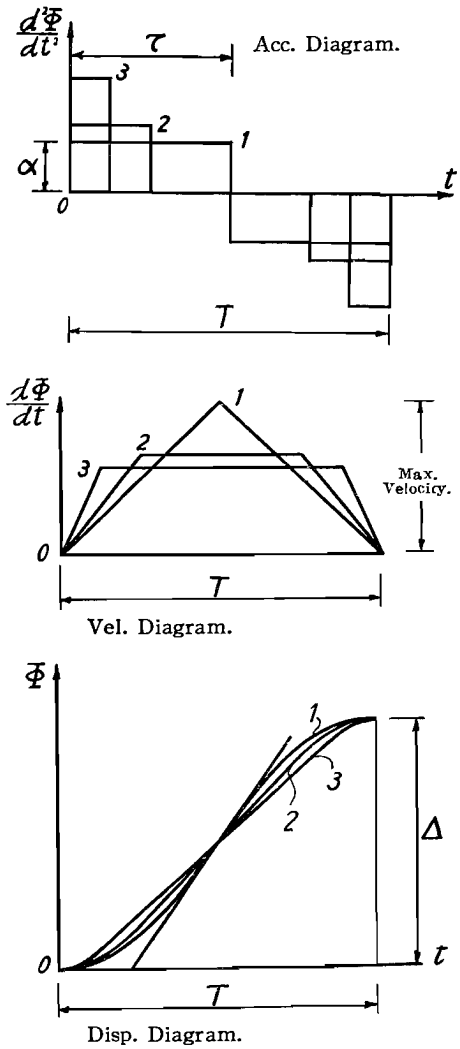


Fig. 9. Hypothetical seismic wave having the max. velocity.

mm/sec². The ratio of amplitude to period is 66.0, and the quantity proportional to the "maximum velocity square" is 4350. Among the seismogram records from the year 1890 to 1930, this calculated max. acceleration is not greatest but the damage of structures and the value proportional to the "max. velocity square" are maximum respectively.

Now $\Delta=8.86$ cm as the displacement of the ground motion and

$2T=1.35$ sec. say, we get the mean velocity of the ground motion

$V_m=\Delta/T=13.125$ cm/sec.

Our hypothetical ground motion on the structures having T_1 sec. period in the fundamental mode is the wave which has the mean velocity V_m, and its period coincides to the time T_1. And among the waves which have the same mean velocity of the ground motion the condition of the wave having a max. velocity is $\tau=\frac{T}{2}=\frac{T_1}{4}$

So the amplitude of our hypothetical wave Δ_1 is

$$\Delta_1=2\times\frac{1}{2}\alpha\tau^2=V_m\times T \tag{3.1}$$

$$\therefore \quad \alpha\left(\frac{T_1}{4}\right)=V_m\times\frac{T_1}{2}$$

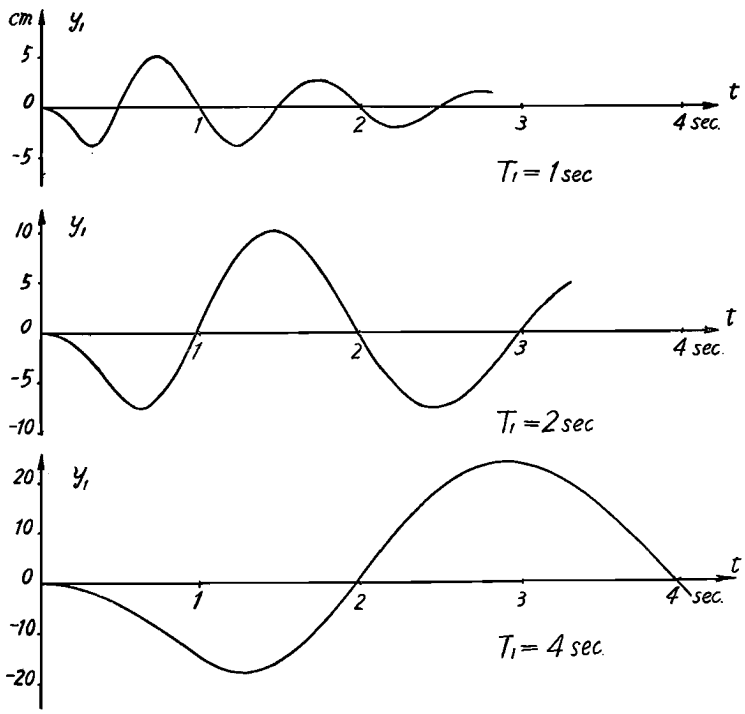
$$\therefore \quad \alpha=\frac{8V_m}{T_1} \tag{3.2}$$

Eq. (3.2) shows $\alpha T_1=$ const. That is to say, in final asismic design of structures, if we fix the standard of the acceleration for unit period, we may take half an acceleration value for the structures having twice period. Therefore from Eq. (3.2), we can determine the acceleration value of our hypothetical waves for structures having each natural period in the fundamental mode 1 sec, 2 sec or 4 sec. This result is as follows.

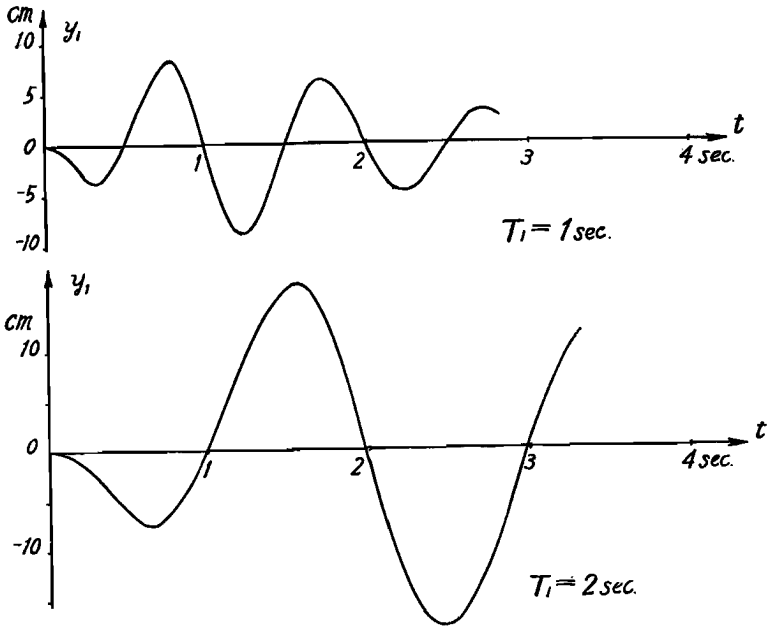
$$\left.\begin{aligned}
&T_1=1 \text{ sec.} \quad \alpha=\frac{8\times13.125}{1}=105 \text{ cm/sec}^2 \\
&\qquad\qquad\qquad\qquad\qquad\quad =0.107g \\
&T_1=2 \text{ sec.} \quad \alpha=\frac{8\times13.125}{2}=52.5 \text{ cm/sec}^2 \\
&\qquad\qquad\qquad\qquad\qquad\quad =0.0535g \\
&T_1=4 \text{ sec.} \quad \alpha=\frac{8\times13.125}{4}=26.25 \text{ cm/sec}^2 \\
&\qquad\qquad\qquad\qquad\qquad\quad =0.02676g
\end{aligned}\right\} \tag{3.3}$$

Calculating the vibrating state of structures suffered the actions of our hypothetical waves with the acceleration values α gained from Eq. (3.3),

we get Fig. 10(a), (b). Fig. 10(a) shows the case the hypothetical waves act on the structures at the interval of semi-period T, and Fig. 10(b) shows another case the waves act at one period $2T$. It is a matter of course to increase the structural deformation when our hypothetical wave acts continueously, because we adopt the wave resonated with the period of the structure. However, studying the seismogram records, we can notice that the seismic waves are unstationary and mingled with the waves having various amplitude and period usually, and the wave with maximum amplitude and same period appears only once, so that we need not to fear for the so-called resonance phenomena. In accordance with the curves of Fig. 10, we can calculated the shearing stress distribution of the structures when the structural deformation is maximum. Comparing this result with the shearing stress distribution calculated by the static lateral load which is determined in the present aseismic design code adopting $K=0.2$ as the ratio



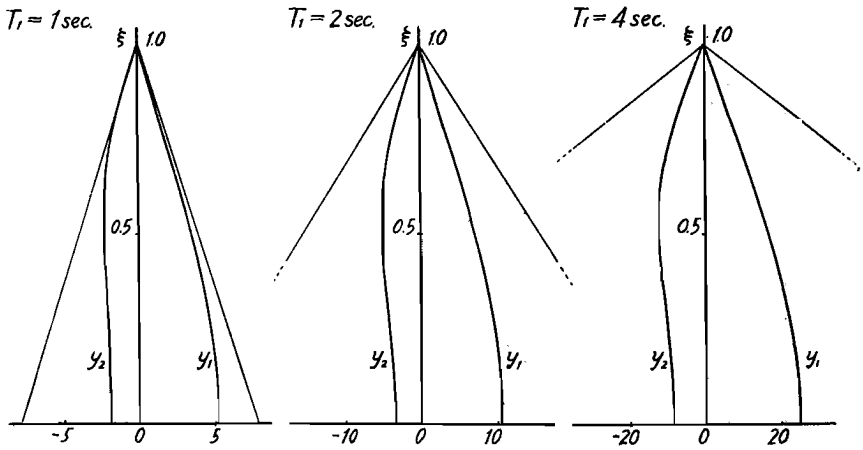
(a) When the hypothetical waves act at the interval of semi-period T.



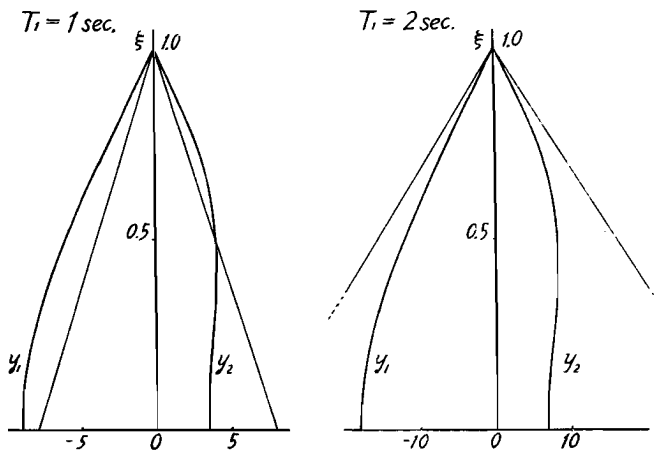
(b) When the hypothetical waves adt at one period $2T$.

Fig. 10. Structural deformations calculated by the mean velocity of the ground motion from the records of the Kwantō earthquake.

of the lateral force to gravity, we get Fig. 11(a), (b). Fig. 11 shows that



(a) When the hypothetical waves act at the interval of semi-period T.

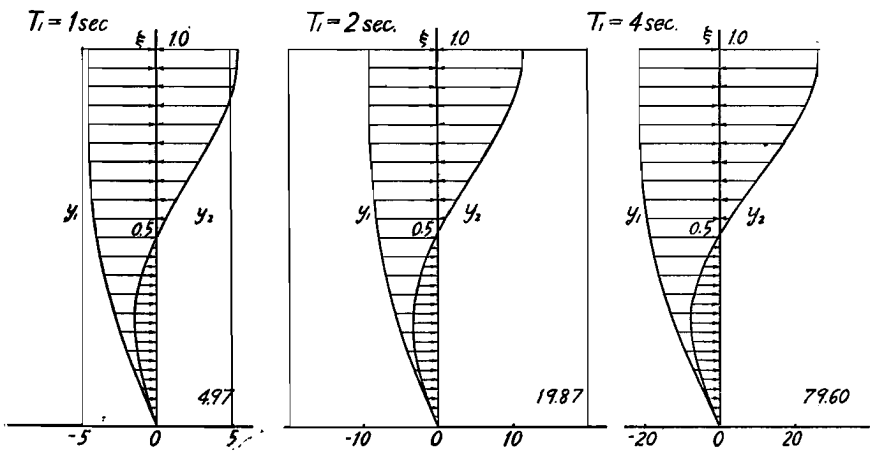


(b) When the hypothetical waves act at one period $2T$.

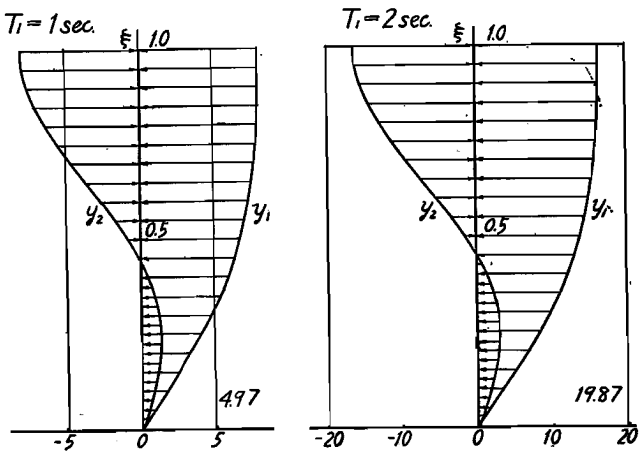
Fig. 11. Shearing stress distributions of the structures when the structural deformation is maximum.

—— shearing stress distribution calculated by the static lateral load.

the base shear in the vibrating state is smaller than that in the statical state. Therefore, in the process of establishing our final aseismic design, we take the lateral load distribution in statically to make the shearing stress distribution as shown in Fig. 11. This result is as Fig. 12, and so it is recognized that the statical load distribution varies according to "sine mode" in the vertical direction.



(a) When the hypothetical waves act at the interval of semi-period T.



(b) When the hypothetical waves act at one period $2T$.

Fig. 12. Lateral load distribution in statically to make the Shearing stress distribution as shown in Fig. 11.

——— Lateral load distribution by the current code of Japan.

Thus, summarizing above all discussions, we may conclude as follows: When we design a structure with the natural period T_1, we should take the value of K——the ratio of the lateral acceleration to gravity——from the records of the great Kwantō earthquake as Eq. (3.4).

$$K=\frac{0.2}{T_1}\sin\frac{\pi}{2}\frac{x}{h}=K_1\sin\frac{\pi}{2h}x \tag{3.4}$$

Eq. (3.4) means that we may give a smaller lateral force distribution for the structures having the longer natural period, but we must adopt very great value of K, on the contrary, for the rigid structures having the short natural period. However, it proves to be apparent from the principle of the acceleration seismograph that the seismic acceleration value acts a leading rôle now for the deformation of the structures having the short period.

In recent years, the Aseismic Property Testing Committee of Japan made an experiment on the ultimate state of a full scale building under the large vibration as in the event of precedented earthquakes, and found a fact that the period of a structure increases about twice under the large vibration compared with the small vibration. We can see that the maximum seismic acceleration in seismogram records reaches nearly 0.4g—0.5g, and so it is unanimous to select $K=0.5\sin\frac{\pi}{2h}x$ as the maximum ratio of lateral acceleration to gravity for buildings. Therefore the relation between

the natural period T_1 of structures and the ratio K_1 in Eq. (3.4) is shown in Fis. 13. And the relation between the natural frequency of structures f and the ratio K_1 is also shown in Fig. 14.

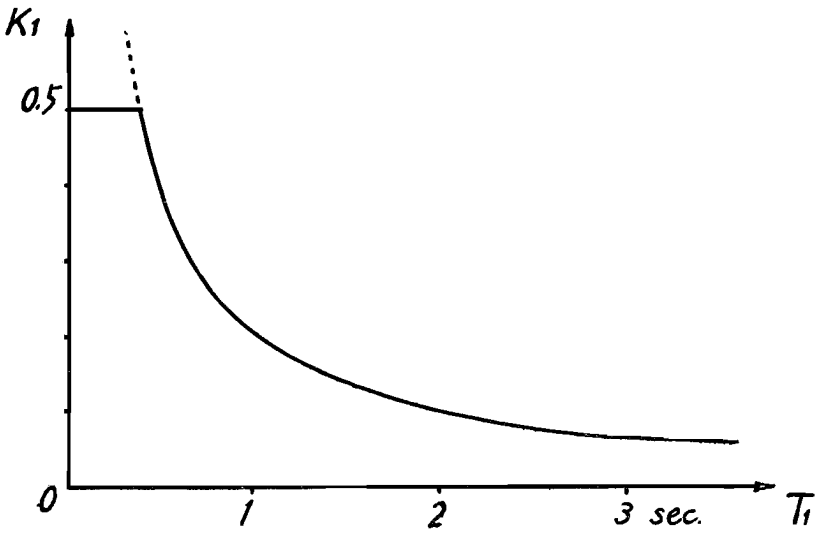


Fig. 13. Relation between the natural period T_1 and the ratio K_1.

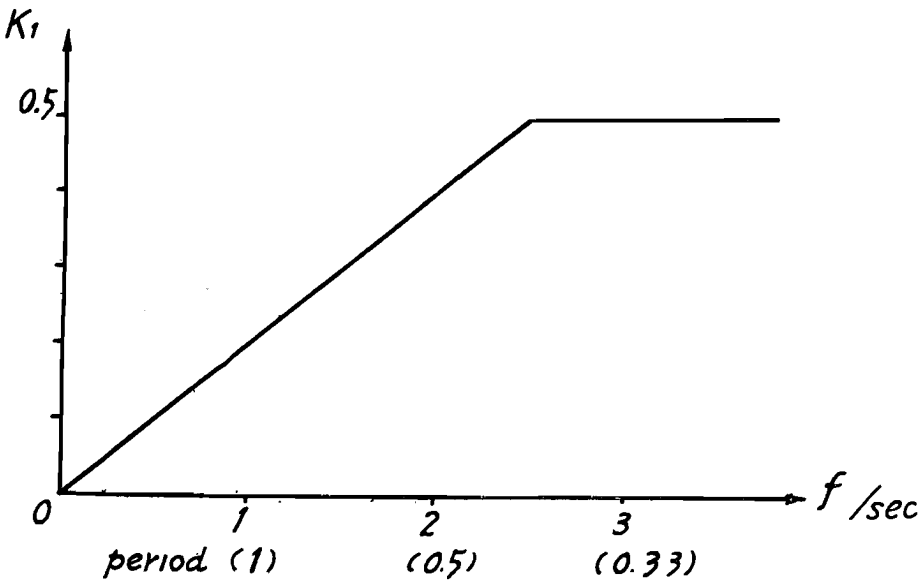


Fig. 14. Relation between the natural frequency of structures f and the ratio K_1.

In the year 1951, the Joint Committee of San Francisco, California Section, ASCE, was unanimous in its selection of $C=0.06$ as the maximum base coefficient for buildings and $C=0.02$ as the minimum base coefficient based on engineering judgement and experience.[8] This base coefficient C corresponds to our ratio K in essentially, but this value of C is so small as compared with our K and lateral force acceleration in the current aseismic design code of Japan. It seems to be brought through the difference between the precedented earthquakes' scale in Japan and that in America.

On the determination of a building period, it is not a clear method to decide merely by the width and the height of the building as the Joint Committee of San Francisco. It is recognized that the period from this method is not coincides with the experimental period of many buildings. Therefore we recognized, for instance, the Lord Rayleigh's approximately calculating method much better, which decides the building period by computing the deformation of the building under the current design code.

In addition of the argument, we must consider the effects of the elasticity of ground etc. Studying these problems would be our new theme.

Appendix:

General Solution of the Equation of Shearing Vibrations

In Section 1 we gave the equation of the shearing vibrations (1.1) in order to analyse a vibrating system in the unstationary state, and obtained the solution Eq. (1.2) by using the Laplace transformation. In this appendix a rigorous proof of the above computation is given as follows.

$$\frac{\partial}{\partial x}\left\{S(x)\frac{\partial y}{\partial x}\right\}-\frac{\partial}{\partial x}\left\{P(x)\frac{\partial y}{\partial x}\right\}-D(x)\frac{\partial y}{\partial t}-\rho A(x)\frac{\partial^2 y}{\partial t^2}=-F(x,\ t) \tag{1.1}$$

In Eq. (1.1) we introduce a new variable ξ, by $\xi=x/h$ to make the equation dimensionless. Where h is the height of the structure. This result is

$$\frac{1}{h}\frac{\partial}{\partial\xi}\left\{S_0\vartheta(\xi)\frac{1}{h}\frac{\partial y}{\partial\xi}\right\}-\frac{1}{h}\frac{\partial}{\partial\xi}\left\{P_0 q(\xi)\frac{1}{h}\frac{\partial y}{\partial\xi}\right\}-D_0 a(\xi)\frac{\partial y}{\partial t}$$
$$-\rho A_0\, a(\xi)\frac{\partial^2 y}{\partial t^2}=-F(\xi,\ t) \tag{2}$$

in which $\quad S(x)=S_0\,\vartheta(\xi),\quad P(x)=P_0\,q(\xi),$

$\quad A(x)=A_0\,a(\xi),\quad D(x)=D_0\,a(\xi).$

By taking new constants and variables $\lambda^2=P_0/S_0$, $\delta^2=\dfrac{D_0}{S_0/h^2}$,

$\mu^2=\dfrac{\rho A_0}{S_0/h^2}$, and $\vartheta(\xi)-\lambda^2 q(\xi)=p(\xi)$, $\bar{F}=\dfrac{F}{S_0/h^2}$.

Eq. (2) is transformed into Eq. (3).

$$\frac{\partial}{\partial\xi}\left\{p(\xi)\frac{\partial y}{\partial\xi}\right\}-\delta^2 a(\xi)\frac{\partial y}{\partial t}-\mu^2 a(\xi)\frac{\partial^2 y}{\partial t^2}=-\bar{F}(\xi,\ t) \tag{3}$$

Considering that the function $y(\xi,\ t)$ converges in exponential mode when the time t increases, we transform the function $y(\xi,\ t)$ into $u(\xi,\ t)$ as

$$y(\xi,\ t)=e^{-\varepsilon t}u(\xi,\ t) \tag{4}$$

in which $\varepsilon=\dfrac{1}{2}\dfrac{\delta^2}{\mu^2}=\dfrac{D_0}{2\rho A_0}$

Then we get Eq. (5)

$$\frac{\partial}{\partial\xi}\left\{p(\xi)\frac{\partial u}{\partial\xi}\right\}+\alpha(\xi)u-\mu^2 a(\xi)\frac{\partial^2 u}{\partial t^2}=-f(\xi,\ t) \tag{5}$$

where $\alpha(\xi)=\dfrac{\delta^2}{2}\varepsilon\, a(\xi)$, $e^{-\varepsilon t}\bar{F}(\xi,\ t)=f(\xi,\ t)$.

Now we apply the Laplace transformation into Eq. (5) and define the relations $|y(\xi,\ t)|_{t=0}=M(\xi)$, $\left|\dfrac{\partial y}{\partial t}\right|_{t=0}=N(\xi)$ as the initial conditions, then the initial conditions about u become

$$|u(\xi,\ t)|_{t=0}=M(\xi),\quad \left|\frac{\partial u}{\partial t}\right|_{t=0}=N(\xi)+\varepsilon M(\xi).$$

Then we get Eq. (6) as the transformed equation of Eq. (5).

$$\frac{d}{d\xi}\left\{p(\xi)\frac{d\eta}{d\xi}\right\}+\alpha(\xi)\eta-\mu^2 a(\xi)s^2\eta=-\Phi(\xi,\ s) \tag{6}$$

where $\eta(\xi,\ s)=\int_0^\infty e^{-st}u(\xi,\ t)dt$

$$\phi(\xi,\ s)=\int_0^\infty e^{-st}f(\xi,\ t)dt$$

$$\Phi(\xi,\ s)=\phi(\xi,\ s)+s\mu^2 a(\xi)M(\xi)+\mu^2 a(\xi)\{N(\xi)+\varepsilon M(\xi)\} \tag{7}$$

In Eq. (6), if we adopt a function $G(\xi,\ z)$ as the Green's function of

$$\frac{d}{d\xi}\left\{p(\xi)\frac{d\eta}{d\xi}\right\}+\alpha(\xi)\eta=0 \tag{8}$$

Eq. (6) is shown by the following integral equation (9).

$$\eta(\xi)=\int_0^1 G(\xi,\ z)\Phi(z,\ s)\,ds-s^2\int_0^1 \mu^2 a(z)\eta(z)G(\xi,\ z)dz \tag{9}$$

Providing $\eta^*(\xi)=\{\mu^2 a(\xi)\}^{\frac{1}{2}}\eta(\xi)$, Eq. (9) is transformed as follows.

$$\eta^*(\xi)=\int_0^1 K(\xi,\ z)\frac{\Phi(z,\ s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}dz-s^2\int_0^1 K(\xi,\ z)\eta^*(z)dz \tag{10}$$

If we take a function $H(\xi, s)$ determined by

$$H(\xi, s)=\int_0^1 K(\xi, z)\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}dz.$$

Eq. (10) is shown as Eq. (11) and $K(\xi, z)$ means a symmetric kernel in the theory of Integral equations. i. e. $K(\xi, z)=\{\mu^2 a(\xi)\}^{\frac{1}{2}}\{\mu^2 a(z)\}^{\frac{1}{2}}G(\xi, z)$

$$\eta^*(\xi)=H(\xi, s)-s^2\int_0^1 K(\xi, z)\eta^*(z)dz \tag{11}$$

Eq. (11) is usually called the Fredholm's integral equation of the 2nd kind, and its solution is shown by E. Schmidt.

$$\eta^*(\xi)=H(\xi, s)-\sum_{\nu=1}^{\infty}\frac{s^2}{\omega_\nu^2+s^2}\varphi_\nu(\xi)H^*(s) \tag{12}$$

where

$$H^*(s)=\int_0^1 H(z, s)\varphi_\nu(z)dz \tag{13}$$

In Eq. (12), $\varphi_\nu, \omega_\nu^2$ means respectively the Eigen function and the Eigen value of the homogeneous integral equation (14) corresponding to Eq. (11).

$$\varphi_\nu(\xi)=\omega^2\int_0^1 K(\xi, z)\varphi_\nu(z)dz \tag{14}$$

Eq. (14) corresponds to its identical differential equation

$$\frac{d}{d\xi}\left\{p(\xi)\frac{d\varphi}{d\xi}\right\}+\frac{\delta^2}{2}\varepsilon a(\xi),\varphi+\omega^2\mu^2 a(\xi)\varphi=0 \tag{15}$$

We have now to transform Eq. (12) by Mellin's inversion theorem

$$u(\xi, t)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\eta(\xi, s)e^{st}ds=L^{-1}\{\eta(\xi, s)\} \tag{16}$$

So that

$$L^{-1}\{\eta^*(\xi, s)\}=L^{-1}[\{\mu^2 a(\xi)\}^{\frac{1}{2}}\eta(\xi, s)]=\{\mu^2 a(\xi)\}^{\frac{1}{2}}u(\xi, t) \tag{17}$$

By using Eq. (12) we can show the next relations

$$L^{-1}\{\eta^*(\xi, s)\}=L^{-1}\{H(\xi, s)\}-L^{-1}\left\{\sum_{\nu=1}^{\infty}\frac{s^2}{\omega_\nu^2+s^2}H^*(s)\right\}\varphi_\nu(\xi)=L_I^{-1}-L_{II}^{-1} \tag{18}$$

$$L_I^{-1}=L^{-1}\left\{\int_0^1 K(\xi, z)\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}dz\right\}=\int_0^1 K(\xi, z)L^{-1}\left[\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\right]dz$$

$$=\sum_{\nu=1}^{\infty}\varphi_\nu(\xi)\int_0^1\int_0^1 K(z_1, z)L^{-1}\left[\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\right]\varphi_\nu(z_1)dz_1\,dz$$

$$=\sum_{\nu=1}^{\infty}\varphi_\nu(\xi)\int_0^1 L^{-1}\left[\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\right]\varphi_\nu(z)dz \tag{19}$$

From Eq. (13)

$$H^*(s)=\int_0^1 H(z, s)\varphi_\nu(z)dz=\int_0^1\int_0^1 K(z, z_1)\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\varphi_\nu(z_1)dz\,dz \tag{20}$$

From Eq. (14)

$$\int_0^1 K(z, z_1)\varphi(z_1)\,dz_1 = \frac{\varphi_\nu(\xi)}{\omega_\nu^2}$$

So

$$H^*(s) = \frac{1}{\omega_\nu^2}\int_0^1 \frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\varphi_\nu(z)dz \tag{21}$$

And from Eq. (18) we get

$$\begin{aligned} L_{II}^{-1} &= L^{-1}\left\{\sum_{\nu=1}^{\infty}\frac{s^2}{\omega_\nu^2+s^2}H^*(s)\right\}\varphi_\nu(\xi) \\ &= \sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu^2}\int_0^1 L^{-1}\left[\frac{s^2}{\omega_\nu^2+s^2}\,\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\right]\varphi_\nu(z)dz \\ &= \sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu^2}\int_0^1 L^{-1}\left[\frac{\Phi_\nu(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\right]\varphi_\nu(z)dz \\ &\quad -\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu^2}\int_0^1 L^{-1}\left[\omega_\nu\frac{\omega_\nu}{\omega_\nu+s^2}\,\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\right]\varphi_\nu(z)dz \end{aligned} \tag{22}$$

Thus, from Eqs. (7), (18), (19) and (22) we get the next equation.

$$\begin{aligned} \{\mu^2 a(\xi)\}^{\frac{1}{2}}u(\xi, t) &= \sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu^2}\int_0^1 L^{-1}\left[\omega_\nu\frac{\omega_\nu}{\omega_\nu^2+s^2}\,\frac{\Phi(z, s)}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\right]\varphi_\nu(z)dz \\ &= \sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu^2}\left[\omega_\nu\int_0^1[\sin\omega_\nu t^*\cdot f(z, t)]\frac{\varphi_\nu(z)dz}{\{\mu^2 a(z)\}^{\frac{1}{2}}}\right. \\ &\quad + \omega_\nu^2\int_0^1 \cos\omega_\nu t\{\mu^2 a(z)\}^{\frac{1}{2}}\,M(z)\varphi_\nu(z)dz \\ &\quad \left. + \omega_\nu\int_0^1 \sin\omega_\nu t\{\mu^2 a(z)\}^{\frac{1}{2}}\,\{N(z)+\varepsilon M(z)\}\varphi_\nu(z)dz\right] \end{aligned} \tag{23}$$

So that, we reach the equation (24)

$$\begin{aligned} y(\xi, t) &= e^{-\varepsilon t}\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu\mu^2\{a(\xi)\}^{\frac{1}{2}}}\int_0^t\int_0^1 \overline{F}(s, \zeta)\frac{\varphi_\nu(s)}{\{a(z)\}^{\frac{1}{2}}}ds\cdot e^{\varepsilon\zeta}\sin(t-\zeta)d\zeta \\ &\quad + e^{-\varepsilon t}\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\{a(\xi)\}^{\frac{1}{2}}}\cos\omega_\nu t\int_0^1\{a(s)\}^{\frac{1}{2}}\,M(s)\varphi_\nu(s)ds \\ &\quad + e^{-\varepsilon t}\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(\xi)}{\omega_\nu\{a(\xi)\}^{\frac{1}{2}}}\sin\omega_\nu t\int_0^1\{a(s)\}^{\frac{1}{2}}\,\{N(s)+\varepsilon M(s)\}\varphi_\nu(s)ds \end{aligned} \tag{24}$$

Eq. (24) is same as Eq. (1.2) in Section 1, and this completes the explanation of the general solution of the equation of shearing vibrations.

This proof was already shown in 1947 by one of the authors, Takuzi Kobori (Suzuki is his former name), in the report[2] "General Solutions and Several Examples on the Unstationary Vibrations of Structures."

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Bulletin No. 7 Published March, 1954

昭和29年3月20日 印刷
昭和29年3月25日 発行
編集兼発行者 京都大学防災研究所
印刷者 山本善治郎
大阪市福島区上福島北1の68
印刷所 旭印刷株式会社