DISASTER PREVENTION RESEARCH INSTITUTE
BULLETIN No. 20 MARCH, 1958

ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES IN THE ULTIMATE STATE

BY

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BULLETINS

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Presented in Japanese originally and published in "Kensetsu-Kogaku" July 1947.

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Introduction

The ultimate strength of a statically indeterminate structure is no longer determined by analysing it as an elasticity problem when the structure consists of plastic ductile materials, and in that case the so-called self-helping effect in the structure have been noticed hitherto. The plastic feature, or plasticity, of materials plays an important role in providing high resistance of structures to violent earthquake shocks.[1]. At the present time, the structural design of buildings is based upon the ultimate strength of structures against such unusual loads.[2] Hence, it is of great importance to evaluate in advance the ultimate resistance of structures of unusual loads, and to estimate the plastic deformation produced in each structural member and remaining strains in the whole structure, as well as to analyse the stresses and strains of the statically indeterminate structures whose stresses are in an elastic range. Nevertheless, very little investigations of this kind has been done. However, the author has made such studies in the past. The author has previously made a general investigation of a step-by-step decrease in the degree of freedom of statically indeterminate frames during the stresses in some partial members of the frames exceed the yield point,[3] and attempted to extend the method so as to apply to the case where the structures consist of elasto-plastic materials.[4] In the study, the states of equilibrium of frames at every step under the action of increasing loads were treated from the fundamental equations of the statically indeterminate frames, assuming that at first the structures had elastic characteristics. Insomuch as the materials have ductile characteristics, the ultimate state of structures must be realized as the state of equilibrium in statically determinate frames, therefore. as far as ultimate states of frames are concerned,[3] it is not always necessary to refer to the fundamental equations of statically indeterminate frames. In this paper, to determine cross-sectional area for every member of the structure realizing the state of equilibrium, the author

intends to assume arbitrarily the ultimate state of equilibrium of a structure, so that it is statically determinate, and also to pay an attention mainly on how much plastic deformation would develop in every member of the structure and how much strain would remain after the loads are removed.

1. Does a Statically Indeterminate Truss fall into any specific ultimate state?

For certain structural materials such as steel, the stress-strain diagram is deemed to be ideally elasto-plastic as shown in Fig. 1. This assumption would be appropriate from the engineer's point of view, and as a matter of fact, the DINE 4114, the German codes for buckling of steel shapes. are also based upon the assumption.

A statically indeterminate truss of n-degrees, consisting of such elasto-plastic materials, will turn into a statically determinate truss when its n redundant members have yielded, and the truss will collapse if one more member begins to yield. Assuming some state of equilibrium in the truss and providing cross-sections for every member so as to make the stresses in the n members equal to those of elastic limit of material, the truss is considered as a statically determinate frame. Hook's law is still valid so far as other members whose stresses are less than the yield stress are concerned. Thus the displacement diagram of the truss is obtained from Williot-Mohr diagram or other procedures. From the displacement diagram, the deformation of members in which plastic stresses have developed can be found.

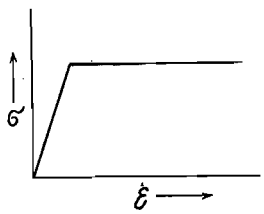

Fig. 1.

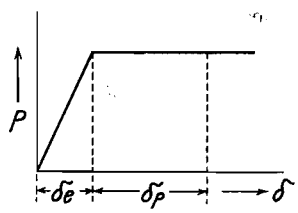

Fig. 2.

Subtracting an elastic deformation, Δe, of a member whose stress exceeds the elastic limit from the total elasto-plastic deformation of the member, $\Delta(e+p)$, we find the plastic deformation of the member, Δp (Fig. 2). As far as the assumed equilibrium is adequate, the quantity, Δp, would be suitable. However, if Δp is too large, the assumed equilibrium is considered as in-

adequate in respect to the plastic deformation limit of materials. Otherwise, if Δp has a negative value, the assumed equilibrium is not appropriate. It means that in spite of the assumption there is no plastic deformation developed actually in the member, so it is shown that another state of equilibrium must exist.

In general, a n-degree indeterminate structure falls into its ultimate state when the stresses of its $(n+1)$ redundant members successively exceed the elastic limit. The member in which the plastic deformation has developed last can be considered as fully elastically deformed because the stress of the member has just exceeded the elastic range. Thus every plastic deformation of n redundant members is obtained from a complementary system where the n yielded members have been removed from the structure. However, it is difficult at first to find the member which has just exceeded the elastc range among $(n+1)$ members.

Another complementary system where the already yielded $(n+1)$ virtual redundant members have been removed is an unstable structure of one degree of freedom. Therefore, the elastic deformation of every part forming a rigidly closed frame will be obtained. Assuming the length of one of these members A be constant, namely $\delta^0{}_a=0$, we get the deformation of other removed members, $\delta^0{}_b$, $\delta^0{}_c$, $\delta^0{}_d$,..... Secondly, when we neglect the elastic deformation of the frame and assume that a plastic deformation, $\Delta a=1$, develops in the member A, the deformation of other deformed members, δ_b, δ_c, δ_d,..... are obtained from the normal velocity diagram. From above considerations, deformations in every member at yield stress state, Δa, Δb, Δc..... are expressed as follows,

$$\Delta a=x,\quad \Delta b=\delta^0{}_b+\delta_b x,\quad \Delta c=\delta^0{}_c+\delta_c x,\ldots\ldots$$

If the limits of elastic deformation of members, A, B,..... are defined as ε_a, ε_b, ε_c,....., the following relations will be obtained.

$$\begin{aligned}
&\varepsilon_a\leq\Delta a=x, && \varepsilon_a/l\leq x,\\
&\varepsilon_b\leq\delta^0{}_b+\delta_b x, && (\varepsilon_b-\delta^0{}_b)/\delta_b\leq x,\\
&\varepsilon_c\leq\delta^0{}_c+\delta_c x, && (\varepsilon_c-\delta^0{}_c)/\delta_c\leq x,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The maximum value of x satisfying above relations will determine every deformation of members, so that the deformation of frames in elasto-plastic stress range will be known.

The case where all members of the frame are in yielding stress range under

an ultimate loading is seldom in practice but is still possible ideally. Even it is reasonable to adopt such a procedure as expressed above, since all the members do not always fall in the yield stress state but only some members (for instance, n members if the frame is of n-degrees of indeterminate nature) do fall in this state prior to the rest of the members. It is not easy to find what members are exceeding beyond the elastic limit, and what is the most appropriate complementary system in that case.

Such a problem, however, comes under a very particular case of the problems of finding the state in equilibrium of a certain framed structure under the action of ultimate loads. Insomuch as we deal with such problems from the practical point of view, the difficulty mentioned above would not appear even seldom, because our problems are almost always related to the designing of a structure which would be subjected to a determined equilibrium under both usual and unusual loads.

2. Residual Deformation in Statically Indeterminate Trusses

From the preceding section, it has been made clear how much plastic deformation is produced in every member of a structure under the action of ultimate loads. Then, the next problem of importance would be how much residual deformation is in the structure after the loads are removed, strictly speaking, after the unusual loads only are taken away.

When the loads are removed after every member had deformed elasto-plastically due to the stresses beyond the yield point, there are considered two patterns of the restoration of the deformation as shown in Fig. 3(a) and (b). Fig. 3(a) is usually obtained from the tension test of metals, and in this figure the specified set at zero stress, or offset, is considered equal to the plastic deformation Δp. The case of Fig. 3(b) is seen in the elastic buckling phenomena, and the path of restoration of the deformation coincides with that under

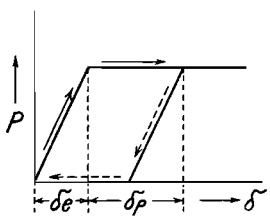

Fig. 3(a).

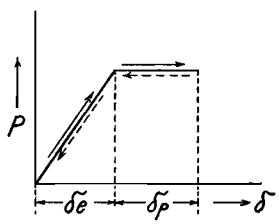

Fig. 3(b).

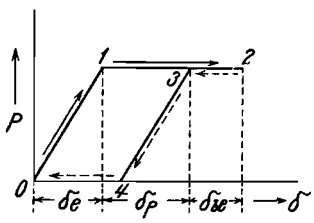

Fig. 4.

the increasing load, so that no offset is found after the load is taken away. Fig. 4 may be assumed as a load-deformation diagram corresponding to buckling of a non-purely elastic column with small slenderness. That is to say, the elastic deformation of the column is shown in this figure by the inclined line 01. If we continue to increase the load, the relation between the load P and the deformation δ is shown by the line 12, and the column will deform plastically due to buckling under the action of the constant load. If we begin unloading, the elastic part of the deformation caused by buckling of the column will be restored until point 3 is rearched, then the column will behave elastically and the relation between the removed load and the restored axial displacement will be given by the line 34, and some deformation 40 will remain. This must be the permanent set which corresponds to the plastic deformation produced by buckling of the column. If, after unloading, a tensile load acts on the column, the permanent set will vanish. Thus it may be considered that the diagram will rearch point 0 at unloading. Generally speaking, even if the unusual load is removed, the members are still stressed due to usual loads. In this sense, members will be treated by assuming that in general the permanent set will be produced. Practically, however, it may be somewhat difficult to derive the quantity of plastic deformation from the total buckling deformation.

In such structures, if those members still have plastic deformations after unloading, it would be in the state of "Selbstspannung" (self-stressed). It is therefore required to treat the problem mathematically.

Now, let us pay attention to a statically indeterminate structure in which some of the members have deformed plastically under the action of given loads. Consider a complementary system of the structure, with removed redundant members, A, B, C,....., which have plastic deformations, δ_a, δ_b, δ_c,....., respectively. Letting X_a, X_b, X_c..... be the axial forces of the self-stressed members, A, B, C,...., the elastic deformations of those members, ε_a, ε_b, ε_c,...., are obtained from

$$\varepsilon_a=\frac{X_aS_a}{EF_a}.\quad \varepsilon_b=\frac{X_bS_b}{EF_b}.\quad \varepsilon_c=\frac{X_cS_c}{EF_c},\ldots\ldots$$

where S_a, S_b, S_c,..... are length of members,

F_a, F_b, F_c,..... are cross-sectional area of members.

If we define here that δ^0_{aa}, δ^0_{ba}, δ^0_{ca},..... are the changes in the distance in the complementary system, due to a force $X_a=1$, corresponding to the position of members, A, B, C,.....; δ^0_{ab}, δ^0_{bb}, δ_{cb},....., are the changes in the distance

due to the force $X_b=1$; and δ^0_{ac}, δ^0_{bc}, δ^0_{cc},..... are the changes in the distance due to the force $X_c=1$, we have

$$\left.\begin{array}{l}\delta^0_{aa}X_a+\delta^0_{ab}X_b+\delta^0_{ac}X_c+\cdots\cdots=\delta_a+K_aX_a\\ \delta^0_{ba}X_a+\delta^0_{bb}X_b+\delta^0_{bc}X_c+\cdots\cdots=\delta_b+K_bX_b\\ \delta^0_{ca}X_a+\delta^0_{cb}X_b+\delta^0_{cc}X_c+\cdots\cdots=\delta_c+K_cX_c\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

in which $K_a=\dfrac{S_a}{EF_a}$. $K_b=\dfrac{S_b}{EF_b}$. $K_c=\dfrac{S_c}{EF_c}$,.....

Using the conventional notations of fundamental simultaneous equations of statically indeterminate structures, Eq. (1) can be expressed as follows;

$$\left.\begin{array}{l}\delta_{aa}X_a+\delta_{ab}X_b+\delta_{ac}X_c+\cdots\cdots=\delta_a\\ \delta_{ba}X_a+\delta_{bb}X_b+\delta_{bc}X_c+\cdots\cdots=\delta_b\\ \delta_{ca}X_a+\delta_{cb}X_b+\delta_{cc}X_c+\cdots\cdots=\delta_c\end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

From the simultaneous equations expressed above, the self-stressed state of the structure must be figured out, so that the permanent set of the statically indeterminate structure will be obtained from the complementary system by using Williot-Mohr diagram or other procedures.

3. Statically Indeterminate Rigid Frames

Insofar as structural materials have an idealized visco-elastic property, the relation between the bending moment and the curvature of a beam can be assumed to have an idealized yield point and to deform ideally-plastically as shown in Fig. 5. Therefore, the procedure explained in the preceding section will be valid for the analysis of a general type of statically indeterminate rigid frames.

Let us consider, for example, a rigid frame as shown in Fig. 6(a). Since the structure is statically indeterminate with three redundant constraints, it will be reduced into a statically determinate structure if it has three plastic hinges A, B,

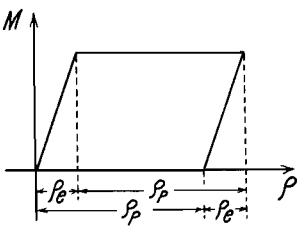

Fig. 5.

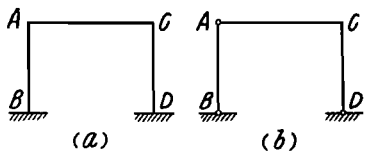

Fig. 6.

and D, as shown in Fig. 6(b). Whenever the elastic deformation of the complementary system is found, we can get the plastic deformation δ_a, δ_b, and δ_d. Thus, if we obtain the values of X_a, X_b, and X_d from the simultaneous equations

$$\left.\begin{aligned} \delta_{aa}X_a+\delta_{ab}X_b+\delta_{ad}X_d&=\delta_a \\ \delta_{ba}X_a+\delta_{bb}X_b+\delta_{bd}X_d&=\delta_b \\ \delta_{da}X_a+\delta_{db}X_b+\delta_{dd}X_d&=\delta_d, \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(3)$$

the self-stressed state of the structure will be known, so that we can find the permanent set of the structure from the elastic deformation in this case.

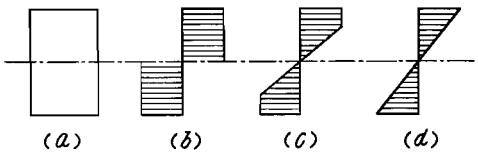

Fig. 7.

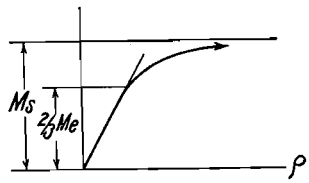

Fig. 8.

However, even if the mechanical property of the structural materials can be assumed as ideally plastic or elasto-plastic, the idealized relation between the bending moment and the rotation angle at a certain point of a member carrying a pure bending moment as shown in Fig. 5 is appropriate only in such a special case where the cross-section of the member concentrates in the centers of gravity of both flanges. In case of a prismatic member whose cross-section is as shown in Fig. 7(a), there are three kinds of stress distribution diagram in accordance with the magnitude of the bending moment. Between two cases shown in Figs. 7(b) and (d) where the strain at every point of the cross-section exceeds the elastic limit and where no parts of the cross-section are plastic, we have another state of stress distribution as shown in Fig. 7(c). The bending moment corresponding to the stress distribution 7(d) is 2/3 times that corresponding to Fig. 7(b). Hence, the relation between the bending moment M and the curvature ρ is not ideally-plastic but must be represented by a curve as shown in Fig. 8. For the values of bending moment larger than a certain limit, therefore, the problem should be treated by taking the nonlinear-elastic property of materials into consideration.

4. Statically Indeterminate Structures of Nonlinear-elastic Members

Let us assume the mechanical property of materials as nonlinear-elastic as

shown in Fig. 9. During loading, the non-linearly elastic deformation, δ_e, (concerning any small part of the materials the property is regarded as ideally plastic) is supposed to be followed by the ideally plastic deformation, δ_p. For the structural members of such materials, similar relations are assumed between the axial force P and the axial deformation Δ, or the bending moment M and the curvature ρ. From the author's experimental studies,[5)6)] it has also been confirmed that such an assumption is appropriate enough for the reinforced concrete construction.

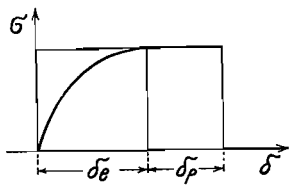

Fig. 9.

Even though it has not been impossible, it has still been very difficult, at a range of nonlinear-elastic stress, to find the state of equilibrium of a statically indeterminate structure of the nonlinear-elastic members. However, in the ultimate state of the structure, namely in the state of statically determinate equilibrium, the nonlinear-elastic deformation up to the ultimate state makes the problem no longer difficult, but we can find the state of equilibrium of the structure in the same manner as mentioned above.

It is simple to find the deflection of a statically determinate structure which contains menbers deformed plastically by the amount prescribed in that manner. Although the value of the modulus of elasticity E is not constant, it is known as it is determined from forces acting on the cross-section, and the forces are known. Thus, whenever the deformations of the statically determinate system of the structure on the whole are known, every deformation of plastically deformed members will be determined.

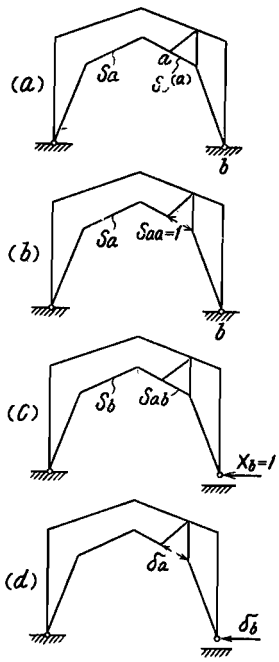

Fig. 10

5. Determination of Total Deformations by means of calculation

We have already seen that the total deformation of members deformed plastically, together with the parts corresponding to the purely plastic deformation, will be found whenever the defor-

mations of the complementary system are known with these members removed from the structure. These values of deformation can, of course, be found also by means of calculation.

Let us consider, for example, a two-hinged frame work with one redundant constraint and assume that member a shown in Fig. 10(a) is deformed plastically. The complementary system will be a three hinged frame with member a removed from the original structure. We let S_0 be axial forces ating on each member in the ultimate state in which the structure has the limit force at member a under the action of loads. Also, as shown in Fig. 10(b) letting S_a be axial forces of each member in the case where a unit force, $S^{\bullet}_{aa}=1$, is acting on the position of member a then the displacement corresponding to the axial deformation of the member a, Δ_a, can be obtained from the equation,

$$\Delta_a = \sum S_0 S_a \rho$$

In above equation if S_0 contains the axial force of member a, namely, the limitting value of the axial force of member a, $S_0^{(a)}$, Δ_a presents the plastic deformation of member a. If there contains no $S_0^{(a)}$, Δ_a is the total deformation of member a. When we calculate the above equation, S_a should be known. However, since the complementary system is not the same as the primary system of the structure adopted in solving statically indeterminate structure, we have a disadvantage finding the value of S_a. By the method described below, however we can make use of the solution of the primary system and save the trouble.

Let us consider now a primary system of the structure having a horizontally movable roller at its right support b as shown in Fig. 10(c). When the redundant force, $X_b=1$, acts at this point, the axial forces in each member, S_b, is obtained. Let us consider a system with one degree of freedom in which the plastically deformed member a is removed from the primary system having roller end b. Then, change of length δ_a produced in member a when the point b has a displacement δ_b is easily found by using the normal velocity diagram. Letting S_{ab} be the axial force in the member a when a unit load, $S_b=1$, is applied at the point b, we have

$$\delta_a S_{ab} + 1 \cdot \delta_b = 0$$

from which

$$S_{ab} = -\delta_b / \delta_a.$$

Also, from the relation, $S_a : S_b = 1 : S_{ab}$, we have

$$S_a = S_b / S_{ab} = -\frac{\delta_a}{\delta_b} S_b.$$

Thus, the deflection Δ_a can be easily obtained from

$$\Delta_a = -\frac{\delta_a}{\delta_b} \sum S_0 S_b \rho$$

Alternatively without refering to Fig. 10 (d), we have the relation

$$\Delta_a = \sum S_0 S_a \rho = \sum \frac{1}{S_{ab}} S_0 S_b \rho$$

when $X_b = 1/S_{ab}$ is applied, because the expression $S_{ab} X_b = 1$ will correspond to the stress state of S_a in the structure.

The method described so far can be applied to the solution of statically indeterminate structures of higher degree. Let us consider, for example, a structure with fixed ends which is statically indeterminate of third degree as shown in Fig. 11. Now, let S_0 be the forces in members due to the ultimate loading when the members, a, b, and c, undergo plastic deformation beyond the elastic limit of axial forces, so that the structure is statically determinate and in the ultimate state of equilibrium. We let S_a be axial forces in each member of the statically indeterminate structure where the members, a, b and c are removed, under the action of a unit force $S_{aa} = 1$ applied at the member a as shown in Fig. 11 (b) ; also S_b and S_c, the axial forces when the structure is acted upon by $S_{bb} = 1$ and $S_{cc} = 1$, respectively as shown in Fig. 11(c) and (d). Then, the plastic deformations, Δ_a, Δ_b, and Δ_c, of the members, a, b, and c, are obtained from the following equations ;

$$\Delta_a = \sum S_0 S_a \rho, \ \Delta_b = \sum S_0 S_b \rho, \text{ and } \Delta_c = \sum S_0 S_c \rho.$$

The trouble in using the above equations is, as well as the preceding example, that it is necessary to know the values of S_a, S_b, and S_c, of the complementary structure. Nevertbeless if we know the stresses of the primary structure, the values are obtained as combinations of the stresses as described below.

Now, in order to obtain the forces in the statically indeterminate structure, we consider a primary system being supported freely at the right hand, and we let X_1, X_2, and X_3, be the redundant forces of the structure as shown in Fig. 11 (e), (f) and (g). While the axial forces of the primary system S_1, S_2, and S_3, corresponding to the redundant forces are obtained, we let S_{a1},

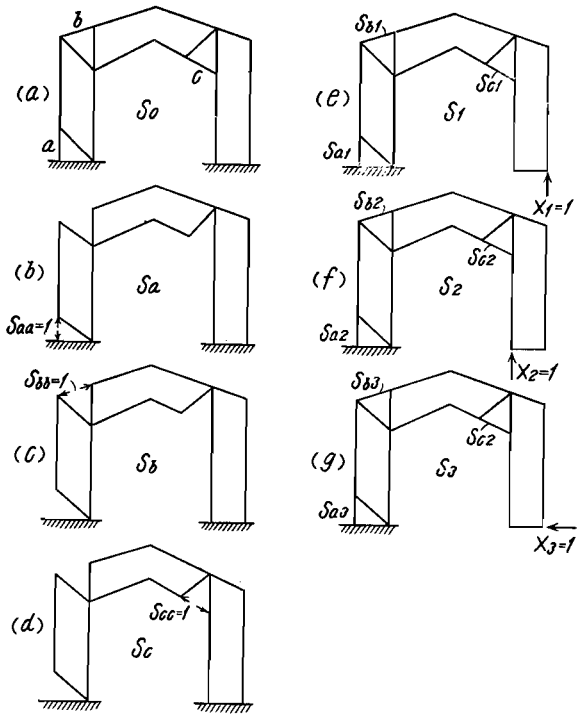

Fig. 11

S_{b1}, and S_{c1}, be the axial forces S_1 in members, a, b, and c: similarly, S_{a2}, S_{b2}, and S_{c2} ; S_{a3}, S_{b3}, and S_{c3} the axial forces S_2 and S_3 in the members respectively.

Thus, the redundant forces X_1, X_2 and X_3, which will make $S_{aa}=1$, and $S_{bb}=S_{cc}=0$, are determined from the following simultaneous equations,

$$\left.\begin{aligned} S_{a1}X_1+S_{a2}X_2+S_{a3}X_3&=1 \\ S_{b1}X_1+S_{b2}X_2+S_{b3}X_3&=0 \\ S_{c1}X_1+S_{c2}X_2+S_{c3}X_3&=0 \end{aligned}\right\}$$

If we denote the solutions of the simultaneous equations as $X_1{}^a$, $X_2{}^a$, and $X_3{}^a$, we have

$$S_a=S_1X_1{}^a+S_2X_2{}^a+S_3X_3{}^a$$

The redundant forces corresponding to Fig. 11 (c) will be found from

$$\left.\begin{aligned} S_{a1}X_1+S_{a2}X_2+S_{a3}X_3&=0 \\ S_{b1}X_1+S_{b2}X_2+S_{b3}X_3&=1 \\ S_{c1}X_1+S_{c2}X_2+S_{c3}X_3&=0 \end{aligned}\right\}$$

Denoting them as $X_1{}^b$, $X_2{}^b$, and $X_3{}^b$, we have

$$S_b = S_1X_1{}^b + S_2X_2{}^b + S_3X_3{}^b$$

In the same manner, we have

$$S_c = S_1X_1{}^c + S_2X_2{}^c + S_3X_3{}^c$$

For this computation, the redundant forces, X_1, X_2, and X_3 for various combinations on the right hand side of the simultaneous equations are obtained with relative case, if we know the general solutions of the left hand side of the simultaneous equations with respect to the general values of the right hand side.

Hence, we have the values of Δ_a, Δ_b, and Δ_c, from the following expressions, including

$$\Delta_a = \sum S_0S_a\rho = \sum S_0(S_1X_1{}^a + S_2X_2{}^a + S_3X_3{}^a)\rho,$$
$$\Delta_b = \sum S_0S_b\rho = \sum S_0(S_1X_1{}^b + S_2X_2{}^b + S_3X_3{}^b)\rho,$$
$$\Delta_c = \sum S_0S_c\rho = \sum S_0(S_1X_1{}^c + S_2X_2{}^c + S_3X_3{}^c)\rho,$$

which is obtained by elimtneting S_a, S_b, and S_c, in the equations described previously.

6. Process of Finding the Intermediate States from the Ultimate State Equilibrium.

A procedure of finding the process in a structure undergoing a change in stress in its members from elastic state to elasto-plastic has been investigated by the author (ref. 3) basing upon the linear equations of equilibrium when the structure was in elastic equilibrium. In that case, unknown plastic deformations of the structure were considered as agents in finding the equilibrium for a statically indeterminate structure of lower degree, which was changed from the original statically indeterminate structure of n-degrees.

On the contrary, since the final state of the structures is found in this procedure, the finding of the intermediate elasto-plastic state of equilibrium from the final state of structure requires another procedure.

The problem is simple so far as a statically indeterminate structure of one degree is concerned. Denoting dead and live loads usually acting upon the structure as W_g, and a unit of the load which will act in the event of earthquakes or wind as W_h, we assume that the structure is in its final state when

a load $W_g+\alpha W_h$ is applied to it. In this case, if we let δ_{pa} be the quantity of plastic deformation, and δ_{ea} be the elastic deformation of the member A of the structure, which is deformed elasto-plastically, the quantity $\delta_{pa}+\delta_{ea}$ will be obtained from the displacement diagram for a complementary structure under the action of the ultimate load, or from computation. Now, let us consider a complementary system in which the member A has no axial force under a unit unusual load W_h. Letting δ_{au} be the relative displacement between the both ends of the member A which corresponds the deformation of it and is obtained from the displacement diagram, and denoting the unusual load corresponding to the elastic limit of stress in the member A as $(\alpha-\beta)W_h$, we have the value for β as

$$\delta_{pa}=\beta\delta_{au}, \text{ so that } \beta=\frac{\delta_{pa}}{\delta_{au}},$$

For a statically indeterminate structure of higher degree, the problem is not so simple. Assuming now that members A, B, C, ... have their plastic deformations δ_{pa}, δ_{pb}, δ_{pc}, ..., and that their elastic deformations as δ_{ea}, δ_{eb} .....under the action of the loads $W_g+\alpha W_h$, it is apparent, that the quantities $(\delta_{pa}+\delta_{ea})$, $(\delta_{pb}+\delta_{eb})$,....., or δ_{pa}, δ_{pb},..... can be obtained from the displacement diagram with respect to the complementary system of the structure or from computation. Whereas, in the final state, a statically indeterminate structure of n degrees must have n members in which the stresses have exceeded the elastic limit, unusual load $(\alpha-\beta)W_h$ to make the stress in the last one of the n members reach the elastic limit can be obtained as follows. By applying a unit unusual load on the complementary structure in which no forces act upon the positions of redundant members being removed, we fined from the displacement diagram or computations the change in length δ_{au}, δ_{bu}, δ_{cu}, ... corresponding to the positions of members, A, B, C,..... in the original structure. Thus, the value of β is determined as the minimum value among those which are obtained from the following expressions.

$$\beta=\frac{\delta_{pa}}{\delta_{au}}, \quad \beta=\frac{\delta_{pb}}{\delta_{bu}}, \quad \beta=\frac{\delta_{pc}}{\delta_{cu}}, \ldots\ldots\ldots$$

When the value of $\beta=\frac{\delta_{pa}}{\delta_{au}}$ is the least among those values of β expressed above, it means that the stress of member A exceeds the elastic limit. It might be best to consider the problem to be statically indeterminate of one degree with the redundant member A while other members, B, C,.....are deformed

plastically as far as the load is not more than $(\alpha-\beta)W_h$.

In order to find the load $W_g+(\alpha-\beta-\beta')W_h$, under whose action one more member is deformed beyond the elastic limit, the least value among $\beta'=\frac{\delta_{pb}-\beta\delta_{bu}}{\delta'_{bu}}$, $\beta'=\frac{\delta_{pc}-\beta\delta_{cu}}{\delta'_{cu}}$, ... should be known. Although the denominators of these expressions already known, the numerators have to be obtained from the solution of an statically indeterminate structure of one degree. The method of finding them is explained below.

While δ_{au} is the change in length corresponding to the deformation of the member A in the complementary structure due to a unit unusual load W_h, if we let δ_{aa} be the deformation corresponding to the axial force in the member A in which a force $X_a=1$ is applied, the redundant force X_a, when the unusual load is W_h, can be found from the following expression.

$$X_a=-\frac{\delta_{au}}{\delta_{aa}}$$

If the value of X_a is once obtained, the values, δ'_{bu}, δ'_{cu}, ... can be calculated from

$$\delta'_{bu}=\delta_{bu}+X_a\delta_{ba},\quad \delta'_{cu}=\delta_{cu}+X_a\delta_{ca},\ \ldots\ldots$$

where δ_{ba}, δ_{ca}, ... are the deformations due to axial forces in members, B, C, ... of the complementary structure under the action of $X_a=1$.

Hence, if β' is the least when $\beta'=(\delta_{pb}-\beta\delta_{bu})/\delta'_{bu}$, the stress in member B reaches the elastic limit under the action of the load $(\alpha-\beta-\beta')W_h$, and then the axial force in member A is $N_a-\beta'X_a$, in which N_a is the axial force in member A corresponding to the elastic limit.

When the unusual load is not more than $(\alpha-\beta-\beta')W_h$, the structure is statically indeterminate of two degrees, and then we have to find the value of an unusual load $(\alpha-\beta-\beta'-\beta'')W_h$ which make another member reach the elastic limit. The value of β'' will be found as the least value among

$$\beta''=\frac{\delta_{pc}-\beta\delta_{cu}-\beta'\delta'_{cu}}{\delta''_{cu}}.\quad \beta''=\frac{\delta_{pd}-\beta\delta_{du}-\beta'\delta'_{du}}{\delta''_{du}}$$

Whereas the denominators of the right hand side of the above equations are known, the numerators δ''_{cu}, δ''_{du},.....have to be calculated as follows. Namely, letting δ_{aa}, δ_{ba}, δ_{ca},..... (which are described again) and δ_{ab}, δ_{bb}, δ_{cb}, ... be the deformations in the directions of members, A, B, C,....., when $X_a=1$ and $X_b=1$ are applied to complementary system respectively, we have the following relations among δ_{au}, δ_{bu}, δ_{cu}, δ_{du},..... corresponding to the load W_h.

$$\left.\begin{aligned}\delta_{aa}X_a+\delta_{ab}X_b+\delta_{au}=0,\\ \delta_{ba}X_a+\delta_{bb}X_b+\delta_{bu}=0.\end{aligned}\right\}$$

Denoting the solutions of the simultaneous equations as X'_a, and X'_b, we have

$$\delta_{cu}''=\delta_{cu}+X_a'\delta_{ca}+X_b'\delta_{cb},$$
$$\delta_{du}''=\delta_{du}+X_a'\delta_{da}+X_b'\delta_{db},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus, the member whose stress has just reached the elastic limit will be known when the least value of β'' is determined. For the time being, we let C be the member. The axial forces of members, A, and B, are

$$N_a-\beta'X_a-\beta''X_a',\quad N_b-\beta''X_b'$$

Also, the plastic deformation in members, D, ... are obtained from

$$\delta_{pd}-\beta\delta_{du}-\beta'\delta_{da}'-\beta''\delta_{du}''$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

After all, if we find the the changes in length in the complementary structure at the positions of removed members, A, B, C,..... under the action of a unit unusual load W_h, or of a unit load applied respectively at the positions of removed members, A, B, C, ..., we have the comprehensive features of equilibrium of the structure from its final state back to every successive intermediate state. This case is, however, different from the case where we find the successive intermediate state of axial forces in the member of the structure from the initial state, because as much labor will be required as we trace back further to the successive intermediate state of axial forces. Therefore, it is apparent that finding the intermediate state close to the final state is simple but it is not if we want to know the intermediate state close to the initial state.

7. Change in Equilibrium due to the Partial Reinforcement of Structure.

From the practical point of view, we will sometimes encounter such case as described below. As will be seen in the following example, there are cases where the plastic deformation of members, or the lateral displacement at the mid-span of columns especially due to buckling, are so large in the final state that they are required to be reinforced further. Actually such reinforcement would result in strengthening not only of the members deformed plastically but

also the members disposed symmetrically with those members, in which the stress remains within the elastic limit even when it is in the final state of structure.

In those cases, the quantity of elastic deformation of the complementary structure is associated with the decrease in deformation, due to the increase of crosssectional area of reinforced members, and the quantity of the decrease would effect the displacement diagram of the complementary structure by the amount of the members. The ultimate resistance of reinforced members increases when the members are further deformed up to the range of plastic deformation, therefore, the axial forces in members and the displacement diagram of the complementary structure due to the increase of the ultimate resistance of members must be obtained. Using this displacement diagram, the displacement and acting

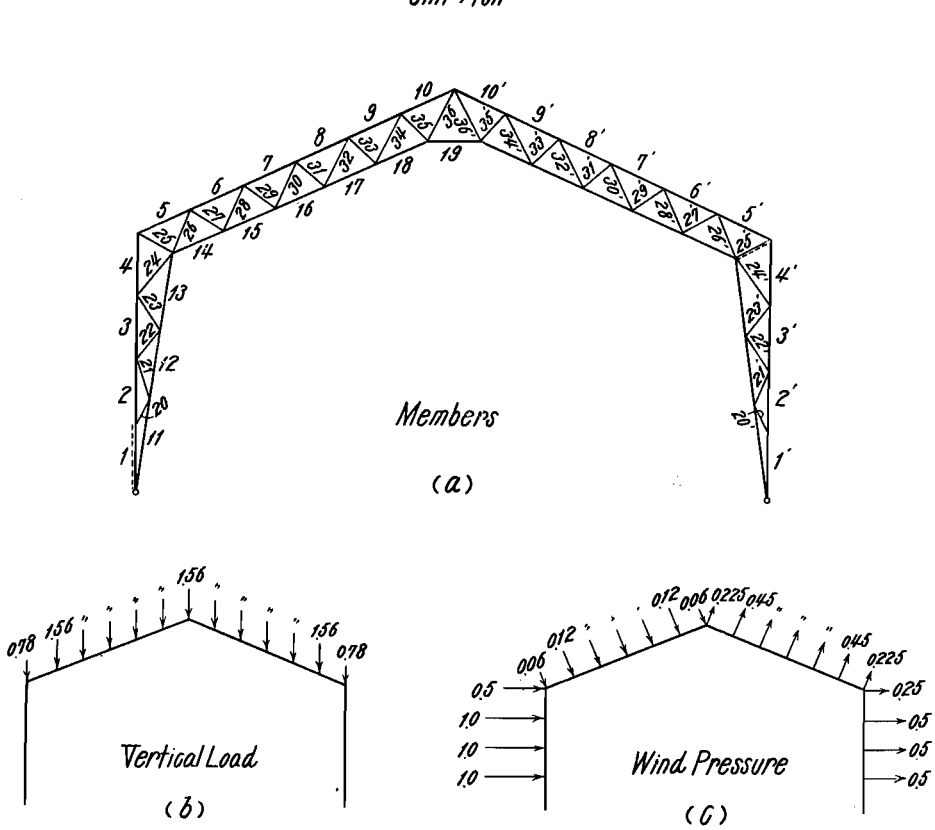

Fig. 12

forces of the reinforced structure can be obtained. However, if it is seen that, as the result of reinforcement, the reinforced members are not deformed plastically but, on the contrary, other members of the complementary structure are deformed plastically, we have to analize the structure in another way.

8. An Example : Two Hinged Statically Indeterminate Truss of One Degree

(1) Shape and size of the truss :

The shape and lengths of steel mambers are shown in Fig. 12 and Table 1, respectively.

(2) Loads :

A dead load for the truss is assumed as shown in Fig. 12(b). A wind pressure is considered in Fig. 12(c) as to act upon the hinges of the truss.

(3) Assumptions :

Steel used here is considered to have an ideal elasto-plastic features between stress and strain. Young's modulus and the yield stress of steel are assumed as

$$E=2{,}100{,}000\ \mathrm{kg/cm^2},\quad \sigma_y=2{,}400\ \mathrm{kg/cm^2}$$

The allowable buckling stress is determined by the German Building Codes, the DIN 4114, of Chwalla.

(4) Stresses in members calculated from the prevalent design method :

Prior to treating an example in this section, let us find the forces, N_s and N_w, in members of the structure under the action of loads as shown in Fig. 12(b) and (c), by applying the conventional method of linear simultaneous equations for this problem.

As a redundant force, we choose the horizontal reaction at a support as shown in Fig. 13, and let $\bar{N}$ be the stresses of the structure due to the unit redundant force; N_{s0} and N_{w0}, respectively be the stresses in members when the dead load and the wind pressure act

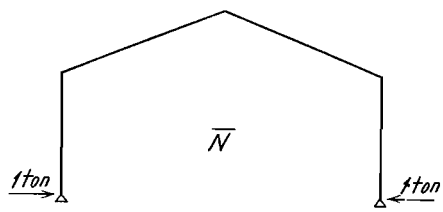

Fig. 13

upon the primary structure. The cross-sectional area of every member is assumed as shown in Table 1. The axial forces N_s and N_w in members calculated from the conventional methods are shown in Tables 1 and 2.

(5) **The following design method is based upon the ultimate state of structures along the reasoning explained in this paper.**

(a) As the load which will cause the structure to be in the final state, we assume 3.167 times the wind pressure as shown in Fig. 12(c). It means that a member of the structure has already been deformed plastically and another member has the stress epual to the elastic limit when the structure is under the action of the dead load and of 3.167 times the wind pressure adopted by the conventional design codes.

(b) Assume that the plastically deformed member is member 1 at the left support, which is buckled in the final state of the structure, and that the member consists of 2-L $100\times100\times10$, which gives the cross-sectional area $F=38$ cm^2, radius of gyration $i=3.03$ cm, length of the member $L=250$ cm, so that, $L/i=82.6$. From the DINE 4114, the actual buckling stress for the member is obtained as

$$\sigma_k=1.383 \text{ ton/cm}^2,$$

thus, we have the actual buckling force (load) P_k as

$$P_k=1.383\times38=52.57 \text{ ton.}$$

Under the assumption as mentioned above, the state of epuilibrium in the structure is considered as just the same as that of a statically determinate three hinged truss, in which the member 1 is removed from the structure, under the action of the ultimate load and a compressive force of 52.57 ton at the position of member 1. Hence, from a method of solving statically determinate trusses, we obtain stresses N_{fs} of the members in the final state of the structure. The results are shown in Table 3.

(c) If we suppose that member 25′ at the right hand eaves is about to collapse due to the axial force N_{fs}, then an appropriate cross-sectional area should be given to the member 25′ so as to make $P_k=N_{fs}=60.2$ ton. From Table 1, the length of the member is given as $L=175$ cm. So that, if we use 2-L $90\times90\times10$, we have $F=34$ cm^2, $i=2.71$ cm, and $L/i=64.5$. From the German design code, the DINE, we have $\sigma=1.769$ ton/cm^2, therefore, $P_k=1.769\times34=60.2$ ton. Thus, it is seen that this cross-section is suitable.

Table 1.

Member	L cm	F cm²	L/F 1/cm	$\bar{N}_{s0}$ ton	$\bar{N}$ ton	$N_{s0}\times\bar{N}L/F$	$\bar{N}^2L/F$	$\bar{N}X_1$ ton	$N_{s0}+\bar{N}X_1=N_s$ ton
1	250	38	6.58	−9.4	+6.0	−371	237	+18.1	+8.7
2	〃	〃	〃	〃	〃	〃	〃	〃	〃
3	〃	〃	〃	〃	〃	〃	〃	〃	〃
4	〃	〃	〃	〃	〃	〃	〃	〃	〃
5	225	38	5.92	− 9.4	+6.4	− 356	242	+19.3	+ 9.6
6	〃	34	6.62	−19.9	+6.8	− 896	306	+20.5	+ 0.6
7	〃	〃	〃	−28.1	+7.3	−1358	353	+22.1	− 6.0
8	〃	〃	〃	−34.2	+7.7	−1743	393	+23.3	−10.9
9	〃	〃	〃	−38.1	+8.1	−2042	434	+24.5	−13.6
10	〃	〃	〃	−39.9	+8.5	−2245	479	+25.7	−14.2
11	375	63	5.95	0	−6.1	0	221	−18.4	−18.4
12	250	〃	3.97	0	〃	0	148	〃	〃
13	300	〃	4.76	0	〃	0	177	〃	〃
14	214	38	5.63	+13.0	−7.6	− 556	325	−22.9	− 9.9
15	〃	34	6.30	+23.2	−8.0	−1169	403	−24.2	− 1.0
16	〃	〃	〃	+31.0	−8.0	−1660	455	−25.7	+ 5.3
17	〃	〃	〃	+36.3	−8.5	−2035	499	−26.9	+ 9.4
18	〃	〃	〃	+39.3	−8.9	−2303	545	−28.1	+11.2
19	200	〃	5.88	+30.1	−7.3	−1292	314	−22.1	+ 8.0
20	135	25	5.40	0	0	0	0	0	0
21	143	〃	5.72	0	0	0	0	0	0
22	157	〃	6.28	0	0	0	0	0	0
23	167	〃	6.68	0	0	0	0	0	0
24	225	〃	9.00	0	0	0	0	0	0
25	175	34	5.15	+10.1	−7.0	− 364	252	−21.0	−11.0
26	〃	25	7.00	−10.4	+0.4	− 29	1	+ 1.2	− 9.2
27	170	〃	6.80	+ 7.8	−0.3	− 16	1	− 0.9	+ 6.9
28	185	〃	7.40	− 8.6	+0.4	− 25	1	+ 1.2	− 7.4
29	167	〃	6.68	+ 5.7	−0.3	− 11	1	− 0.9	+ 4.8
30	195	〃	7.80	− 6.9	+0.4	− 22	1	+ 1.2	− 5.7
31	167	〃	6.68	+ 3.9	−0.3	− 8	1	− 0.9	+ 3.0
32	205	〃	8.20	− 4.9	+0.4	− 16	1	+ 1.2	− 3.7
33	163	〃	6.52	+ 2.2	−0.3	− 4	1	− 0.9	+ 1.3
34	215	〃	8.60	− 2.9	+0.4	− 10	1	+ 1.2	− 1.7
35	163	〃	6.52	+ 0.6	−0.6	− 1	1	− 0.9	− 0.3
36	225	〃	9.00	+15.4	−3.5	− 485	110	−10.6	+ 4.8

$2\times(-18838)$ $\quad 6300\times 2$

-1292 $\quad +314$

$=-38968$ $\quad =12914$

$X_1=38968/12914=3.02\text{ton}$

Table. 2

Member	N_{w0} ton	$N_{w0}\bar{N}L/F$	$\bar{N}X_2$ ton	$N_{w0}+\bar{N}X_2=N_w$ ton	Member	N_{w0} ton	$N_{w0}\bar{N}L/F$	$\bar{N}X_2$ ton	$N_{w0}+\bar{N}X_2=N_w$ ton
1	−36.9	−1456	+15.3	−21.6	1′	− 0.1	− 4	+15.3	+15.2
2	−35.0	−1381	〃	−19.7	2′	− 1.1	− 43	〃	+14.2
3	−32.1	−1266	〃	−16.8	3′	− 2.4	− 95	〃	+12.9
4	−28.7	−1132	〃	−13.4	4′	− 3.9	− 154	〃	+11.4
5	−30.8	−1166	+16.3	−14.5	5′	− 4.1	− 155	+16.3	+12.2
6	−29.0	−1305	+17.3	−11.7	6′	− 5.8	− 261	+17.3	+11.5
7	−27.2	−1313	+18.6	− 8.6	7′	− 8.1	− 391	+18.6	+10.5
8	−25.1	−1279	+19.6	− 5.5	8′	−11.0	− 560	+19.6	+ 8.6
9	−23.1	−1238	+20.6	− 2.5	9′	−14.4	− 772	+20.6	+ 6.2
10	−20.9	−1175	+21.6	+ 0.7	10′	−17.8	−1001	+21.6	+ 3.8
11	+37.5	−1360	−15.5	+22.0	11′	0	0	−15.5	−15.5
12	+36.3	− 879	〃	+20.8	12′	+ 1.4	− 34	〃	−14.1
13	+33.1	− 961	〃	+17.6	13′	+ 2.9	− 84	〃	−12.6
14	+33.3	−1424	−19.3	+14.0	14′	+ 6.5	− 278	−19.3	−12.8
15	+31.4	−1581	−20.4	+11.0	15′	+ 8.6	− 433	−20.4	−11.8
16	+29.4	−1573	−21.6	+ 7.8	16′	+11.4	− 610	−21.6	−10.2
17	+27.2	−1524	−22.6	+ 4.6	17′	+14.8	− 830	−22.6	− 7.8
18	+25.0	−1464	−23.7	+ 1.3	18′	+18.3	−1071	−23.7	− 5.4
19	+17.1	− 734	−18.6	− 1.5					
20	− 2.2	0	0	− 2.2	20′	+ 1.0	0	0	+ 1.0
21	+ 1.2	0	0	+ 1.2	21′	− 0.6	0	0	− 0.6
22	− 2.4	0	0	− 2.4	22′	+ 1.1	0	0	+ 1.1
23	+ 1.7	0	0	+ 1.7	23′	− 0.8	0	0	− 0.8
24	− 2.9	0	0	− 2.9	24′	+ 1.4	0	0	+ 1.4
25	+33.0	−1188	−17.8	+15.2	25′	+ 4.9	− 176	−17.8	−12.9
26	+ 1.4	+ 4	+ 1.0	+ 2.4	26′	− 1.3	− 4	+ 1.0	− 0.3
27	− 1.5	+ 3	− 0.8	− 2.3	27′	+ 1.7	− 3	− 0.8	+ 0.9
28	− 1.6	+ 5	+ 1.0	+ 2.6	28′	− 1.8	− 5	+ 1.0	− 0.8
29	− 1.5	+ 3	− 0.8	− 2.3	29′	+ 2.1	− 4	− 0.8	+ 1.3
30	+ 1.8	+ 6	+ 1.0	+ 2.8	30′	− 2.5	− 8	+ 1.0	− 1.5
31	− 1.6	+ 3	− 0.8	− 2.4	31′	+ 2.6	− 5	− 0.8	+ 1.8
32	+ 2.0	+ 7	+ 1.0	+ 3.0	32′	− 3.1	− 10	+ 1.0	− 2.1
33	− 1.6	+ 3	− 0.8	− 2.4	33′	+ 2.9	− 6	− 1.8	+ 2.1
34	+ 2.2	+ 8	+ 1.0	+ 3.2	34′	− 3.6	− 12	+ 1.0	− 2.6
35	− 1.7	+ 3	− 0.8	− 2.5	35′	+ 3.3	− 6	− 0.8	+ 2.5
36	+11.5	− 362	− 8.9	+ 2.6	36′	+ 4.6	− 145	− 8.9	− 4.3
		−25716					−7160		
							−25716		
							=−32876		

$X_2=32876/12914=2.545$ ton

Table 3.

Nember	N_{fs} ton		$\bar{N}'$ ton	$N_{fs}N'L/F$	
	left	right		left	right
1	−52.6	+64.0	+1.0	−346	+421
2	−46.6	+60.8	〃	−307	+400
3	−37.4	+56.7	〃	−246	+373
4	−26.6	+51.9	〃	−175	+342
5	−28.4	+56.1	+1.1	−179	+354
6	−28.4	+45.1	+1.1	−213	+338
7	−24.6	+35.9	+1.2	−198	+289
8	−19.2	+25.4	+1.3	−165	+218
9	−11.9	+15.6	+1.4	−106	+139
10	− 1.9	+ 7.9	+1.4	− 18	+ 74
11	+44.1	−74.7	−1.0	−267	+452
12	+40.3	−70.3	〃	−163	+284
13	+30.2	−65.5	〃	−146	+317
14	+25.4	−59.4	−1.3	−181	+423
15	+24.3	−47.9	−1.3	−204	+402
16	+19.9	−37.1	−1.4	−177	+331
17	+13.5	−25.8	−1.5	−126	+241
18	+ 4.3	−16.9	−1.6	− 42	+165
19	− 5.4		−1.2	+ 39	
20	− 7.0	+ 3.2	0	0	0
21	+ 3.8	− 1.9	0	0	0
22	− 7.6	+ 3.5	0	0	0
23	+ 5.4	− 2.5	0	0	0
24	− 9.2	+ 4.4	0	0	0
25	+28.9	−60.2	−1.2	−174	+362
26	− 1.1	− 9.7	+0.1	0	− 4
27	− 0.8	+ 9.4	−0.1	0	− 3
28	+ 1.3	− 9.4	+0.1	+ 1	− 5
29	− 2.9	+ 8.5	−0.1	+ 1	− 3
30	+ 3.7	−10.0	+0.1	+ 2	− 5
31	− 5.0	+ 8.3	−0.1	+ 2	− 3
32	+ 6.3	− 9.9	+0.1	+ 3	− 5
33	− 6.7	+ 7.6	−0.1	+ 2	− 3
34	+ 8.9	− 9.4	+0.1	+ 5	− 5
35	− 8.6	+ 7.2	−0.1	+ 3	− 2
36	+ 8.9	−12.9	−0.6	− 47	+ 68
				−3422	+5955
					−3422
					+2533

Table 4.

Member	Type of Steel Member	F cm²	i cm	L/i	σ_k ton/cm²	P_k ton	P_y ton
1	2L100.100.10	38	3.03	82.6	1.383	52.57	91.2
2	〃	〃	〃	〃	〃	〃	〃
3	〃	〃	〃	〃	〃	〃	〃
4	〃	〃	〃	〃	〃	〃	〃
5	2L100.100.10	38	3.03	74.3	1.556	59.13	91.2
6	2L 90. 90.10	34	2.71	83.1	1.372	46.65	81.6
7	〃	〃	〃	〃	〃	〃	〃
8	〃	〃	〃	〃	〃	〃	〃
9	〃	〃	〃	〃	〃	〃	〃
10	〃	〃	〃	〃	〃	〃	〃
11	L150.150.11	63	4.75	79.0	1.457	91.80	151.2
12	〃	〃	〃	52.6	2.008	126.40	〃
13	〃	〃	〃	63.2	1.796	113.20	〃
14	2L100.100.10	38	3.03	70.6	1.638	62.23	91.2
15	2L 90. 90.10	34	2.71	79.0	1.457	49.55	81.6
16	〃	〃	〃	〃	〃	〃	〃
17	〃	〃	〃	〃	〃	〃	〃
18	〃	〃	〃	〃	〃	〃	〃
19	2L 90. 90.10	34	2.71	73.8	1.566	53.25	81.6
20	2L 75. 75. 9	25	2.25	60.0	1.864	46.60	60.0
21	〃	〃	〃	63.6	1.788	44.71	〃
22	〃	〃	〃	69.8	1.655	41.39	〃
23	〃	〃	〃	74.3	1.556	38.90	〃
24	〃	〃	〃	100.0	1.064	26.60	〃
25	2L 90. 90.10	34	2.71	64.6	1.769	60.20	81.0
26	2L 75. 75. 9	25	2.25	77.8	1.482	37.06	60.0
27	〃	〃	〃	75.6	1.528	38.20	〃
28	〃	〃	〃	82.2	1.390	34.75	〃
29	〃	〃	〃	74.2	1.558	38.96	〃
30	〃	〃	〃	86.7	1.302	32.55	〃
31	〃	〃	〃	74.2	1.558	38.96	〃
32	〃	〃	〃	91.2	1.216	30.40	〃
33	〃	〃	〃	72.5	1.595	39.89	〃
34	〃	〃	〃	95.6	1.138	28.46	〃
35	〃	〃	〃	72.5	1.595	39.89	〃
36	〃	〃	〃	100.0	1.064	26.60	〃

(d) For other members except members 1 and 25′, the cross-sections are chosen as to have a reserve of more strength against N_{fs}. The cross-sections

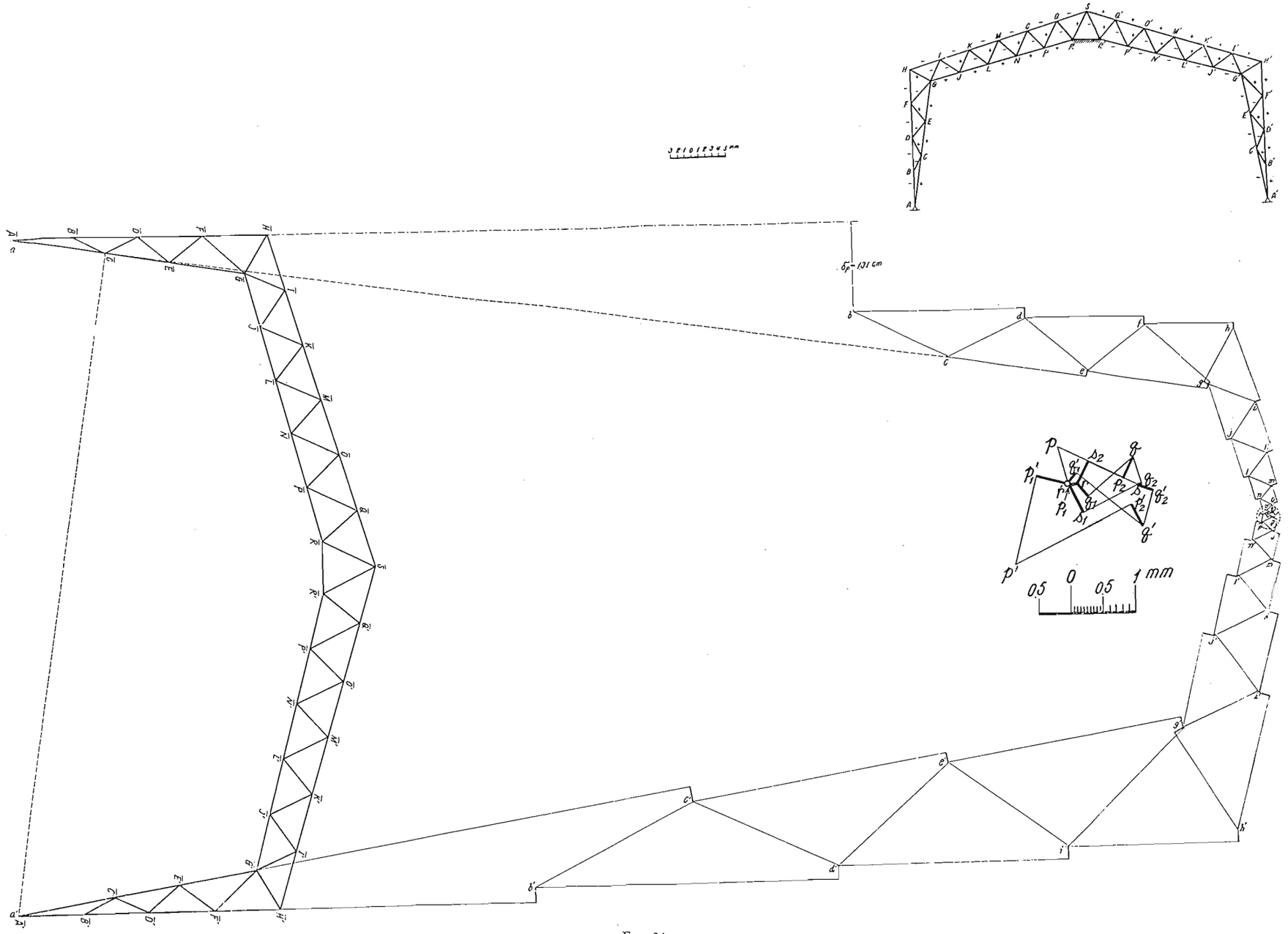

Fig. 14.

are shown in Table 4 where, in addition to the values of L/i for the cross-section and σ_k determined by the DINE, the buckling load $P_k=\sigma_k F$ and the allowable tensile forces $P_y=2{,}400\times F$ (provided $\sigma_{yield}=2{,}400$ kg/cm^2) are shown. Comparing the values of P_k and P_y in Table 4 with N_{rs} in Table 3, we can see that all members fulfil the conditions described above.

(e) In the next, let us find the plastic deformation δ_p in the member 1 (or $\delta_p+\delta_{be}$ as shown in Fig. 4). For this purpose, we let $\bar{N}'$ be axial forces in members of a primary structure, in which the member 1 is removed, due to a unit load applied at the position of member 1. Hence, applying the theorem of virtual work with respect to N_{rs} and $\bar{N}'$, and using the values in Table 3, we have

$$E\delta_p=2533 \text{ ton/cm}$$
$$\delta_p=1.206 \text{ cm}.$$

And graphically solving the plastic deformatin δ_p in the same member by means of Williot's Displacement Diagram, we have also (in Fig. 14)

$$\delta_e+\delta_p=1.310 \text{ cm}$$
$$\delta_p=1.310-\frac{P_k L}{E\bar{F}}=1.293 \text{ cm}$$

So these values of δ_p show the validity of numerical and graphical method respectively.

Thus, the fact that we have obtained the value of δ_p as plastic contraction was nothing but to show what we asumed in advance that the member 1 has compressive force was right,

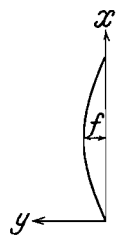

Fig. 15

Now, let us obtain the lateral displacement of member 1 due to buckling. For the time being, we assume that when the compressive force increases from zero to P_k the axial contraction of the member is independent on the sideward deformation of column due to buckling, and that the plastic deformation δ_p is equal to the axial contraction of the member which depends on the sideward deformation of it due to buckling. Computation is done as follows: Assuming the deflection curve of the column as a trigonometric function, we say,

$$y=f\sin\frac{\pi x}{L}$$

the plastic deformation δ_p is, therefore,

$$\delta_p=\frac{1}{2}\int_0^L\left(\frac{dy}{dx}\right)^2 dx=\frac{f^2\pi^2}{4L}$$

from which

$$f=\frac{2}{\pi}\sqrt{\delta_p L}$$

Substituting $\delta_p=1.206$ cm, $L=250$ cm into the above expression, we have

$$f=\frac{2}{\pi}\sqrt{1.206\times 250}=11.05 \text{ cm}$$

(f) Finally, let us check on the stress state and the residual strain in the structure, in which the plastic deformation exists, after the wind pressure is unloaded. Assuming that there are some residual compressive stresses which are in equilibrium with the dead loads, we can calculate the compressive force X in member 1 by using Ep. (1) in Section 2.

We let N_0 be the axial forces in the primary structure, in which the member 1 is removed, under the action of the dead load only, and we show these values in Table 5.

Making use of the values $\overline{N}'$ shown in Table 3, Ep. (1) becomes

$$\frac{1}{E}(3066.3+X\cdot 360.6)=-1.206-X\frac{L_1}{EF_1}$$

in which no positive X is found. This means that the assumption that the member 1 is subjected to a compressive force is no longer valid, but that is applied to the member a tensile force. After all, there is no plastic deformation δ_p. Hence, it is apparent that the state of equilibrium of the structure in this case is just the same as in the elastic range obtained from Table 1, and that the same result for the values of X would be found by letting $\delta_p=0$ in the above equation.

Table. 5

Member	N_0 ton	$-N_0\overline{N}'L/F$	$\overline{N}'^2 L/F$
1	0	0	6.6
2	0	0	〃
3	0	0	〃
4	0	0	〃
5	+ 0.6	− 3.8	6.9
6	− 9.3	+ 69.8	8.3
7	−16.6	+133.7	9.7
8	−22.1	+190.0	11.2
9	−25.3	+226.1	12.5
10	−26.5	+248.5	13.1
11	− 9.5	− 57.5	6.1
12	〃	− 38.4	4.0
13	〃	− 46.0	4.8
14	+ 1.1	+ 7.8	9.3
15	+10.6	+ 89.1	10.9
16	+17.6	+156.9	12.5
17	+22.3	+208.1	14.0
18	+24.7	+213.7	15.6
19	+18.6	+133.1	8.6
20	0	0	0
21	0	0	0
22	0	0	0
23	0	0	0
24	0	0	0
25	− 0.8	− 4.8	7.2
26	− 9.8	+ 4.6	0
27	+ 7.3	+ 2.5	0
28	− 8.0	+ 3.9	0
29	+ 5.2	+ 1.7	0
30	− 6.3	+ 3.3	0.1
31	+ 3.4	+ 1.1	0
32	− 4.3	+ 2.4	0.1
33	+ 1.7	+ 0.6	0
34	− 2.3	+ 1.3	0.1
35	+ 0.1	0	0
36	+ 9.0	+ 52.0	3.2
		1466.6×2 +133.1 =3066.3	176×2 +8.6 =360.6

Acknowledgment :

The author wishes to express his sincerest thanks to the Academic Council of Japan for its financial aid for this study. Acknowledgement is also due to Messrs. Yoshio Baba, assistant professor of Osaka University, Takuji Kobori, assistant professor of Kyoto University, Kiyoshi Kaneta and Ryoichiro Minai, assistants of Kyoto University, for their kind cooperations in preparing this paper.

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Bulletin No. 20 Published March, 1958

昭和 33 年 3 月 20 日 印 刷
昭和 33 年 3 月 31 日 発 行

編輯兼発行者 京都大学防災研究所
印刷者 山 代 多 三 郎
京都市上京区寺之内通小川西入
印刷所 山代印刷株式会社