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MICHIYASU SHIMA

KYOTO UNIVERSITY, KYOTO, JAPAN

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Abstract

The two dimensional problem of the diffraction of plane elastic P and S pulses of a rectangular type by the crack of a half plane is treated by D. S. Jones' method in the diffraction of a scalar wave. In this method the shorter pulse width is comparing with the distance of observing point from the edge of the crack, the easer the calculation. The results to be noted are as follows ;

a) the phase of some diffracted pulse is reversed at the shadow boundary,

b) while the forms of the incident and reflected pulses are a rectangular type, those of the diffracted P and S pulses are smooth.

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§1. The problems of the diffraction of a sound wave and an electromagnetic wave by a half plane and a slit have been investigated frequently by various methods since A. Sommerfeld[1]. But the diffraction of an elastic wave of simple harmonic type by a crack has been treated approximately only a few times on account of the complexity of method of solution[1),2)] and the diffraction of a elastic pulse has not been solved exactly as far as the writer knows.

In this paper the two dimensional problem of the diffraction of plane elastic P and S pulses by the crack of a half plane is treated by D. S. Jones's method in the diffraction of a scalar wave[4]. That is, first, the formal solutions for the harmonic wave are obtained by his method, and using the principle of superposition, the solutions are calculated for the incidence of the plane P and S pulses of a rectangular type. In this method, the shorter pulse width is comparing with the distance of observing point from the edge of the crack, the easer the calculation.

While the interpretation of the reflected wave from a plane surface is a simple problem, the seismogram is considerably complex for the case of the edge of the crack in the elastic medium. Therefore, the theoretical results of diffraction of the elastic wave are useful to interprete the seismogram.

Notation

a, b, c_R : propagation velocities of P, S, and Rayleigh waves, respectively
k, K : wave numbers of P and S waves, respectively
p_{yy}, p_{xy} : components of stress
u, v : components of displacements in x, y, θ, and r-directions, re-
u_θ, u_r spectively
ρ : density

§2. In this section, we derive the formal solutions for the two dimensional problem of the diffraction of the plane P and S waves of the simple harmonic type vertically incident to the edge of the crack of the half plane in the uniform isotropic elastic medium.

The total displacement $\boldsymbol{u}^t$ can be written in the form of a sum of two quantities

$$\boldsymbol{u}^t = -\operatorname{grad}\phi^t + \operatorname{rot}\boldsymbol{\psi}^t, \tag{1}$$

where ϕ^t is the total scalar potential corresponding to longitudinal wave and $\boldsymbol{\psi}^t$ is the total vector potential corresponding to transverse wave, where the quantities ϕ^t and $\boldsymbol{\psi}^t$ satisfy the equations

$$\begin{aligned} \Delta\phi^t + k^2\phi^t &= 0, \\ \Delta\boldsymbol{\psi}^t + K^2\boldsymbol{\psi}^t &= 0. \end{aligned} \tag{2}$$

Since we treat the two dimensional problem in the xy-plane, ψ^t has only the z-component and can be taken as scalar. Thus, the total stresses $p_{yy}{}^t$, $p_{yx}{}^t$ can be expressed in the following

$$\left.\begin{aligned} p_{yy}{}^t &= 2\rho b^2\left[\left(\frac{\partial^2}{\partial x^2} + \frac{K^2}{2}\right)\phi^t - \frac{\partial^2\psi^t}{\partial x \partial y}\right], \\ p_{xy}{}^t &= -2\rho b^2\left\{\frac{\partial^2\phi^t}{\partial x \partial y} + \left(\frac{\partial^2}{\partial x^2} + \frac{K^2}{2}\right)\psi^t\right\}. \end{aligned}\right\} \tag{3}$$

Let us choose the orthogonal coordinate system, the origin of which coincides with the edge of the crack of the half plane, as shown in Fig. 1.

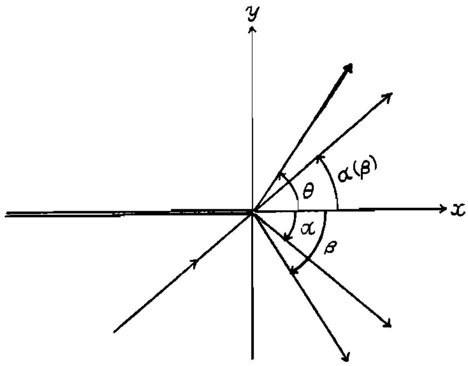


Fig. 1.

Since the plane of the crack may be taken as a free surface, the boundary conditions on this surface for the stress will be

$$p_{yy}{}^t = p_{xy}{}^t = 0 \qquad \text{on } y=0,\ x<0, \tag{i}$$

and

$$p_{yy}{}^t,\ p_{xy}{}^t \text{ and } \boldsymbol{u}^t \text{ are continuous on } y=0,\ x>0. \tag{ii}$$

The boundary condition at the edge is such that the stress on the elastic medium, div $\boldsymbol{u}^t$ and rot $\boldsymbol{u}^t$ at the edge may be zero. Thus, for stresses[5]

$$p_{yy}{}^t,\ p_{xy}{}^t \sim r^{-1/2} \quad \text{for} \quad r \to 0 \tag{iii}$$

And finally, ϕ, ψ must satisfy the radiation condition.

Now split the total potentials ϕ^t and ψ^t into the incident part and the scattered part

$$\phi^t = \phi^0 + \phi, \qquad \psi^t = \psi^0 + \psi. \tag{4}$$

The potential ϕ, ψ as the superposition of plane wave can be written in the form

$$\left.\begin{aligned} \phi_{1,2} &= \int_{-\infty}^{\infty} M_{1,2}(\lambda)\, e^{i(\lambda x + \sqrt{k^2-\lambda^2}|y|}d\lambda, \\ \psi_{1,2} &= \int_{-\infty}^{\infty} N_{1,2}(\lambda)\, e^{i(\lambda x + \sqrt{K^2-\lambda^2}|y|}d\lambda, \end{aligned}\right\} \tag{5}$$

where 1 for $y>0$, 2 for $y<0$. In order to satisfy the radiation condition at $y \to \infty$, we must interprete the integrals in (5) as the limits of those along the path L in the complex plane shown in Fig. 2, when semicircles around four singular points $\pm k$, $\pm K$, $\pm \lambda_R$ are made vanishingly small.

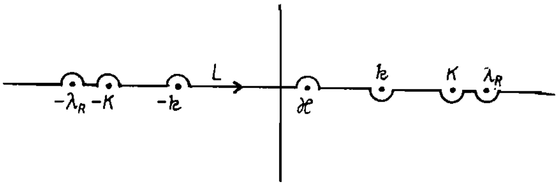


Fig. 2.

Thus, the stresses are

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
p_{yy}=2\rho b^2\displaystyle\int_{-\infty}^{\infty}\Big\{\Big(\frac{K^2}{2}-\lambda^2\Big)M_1(\lambda)e^{-\sqrt{\lambda^2-k^2}y} \\
\quad +\lambda\sqrt{K^2-\lambda^2}N_1(\lambda)e^{-\sqrt{\lambda^2-K^2}y}\Big\}e^{i\lambda x}d\lambda\equiv\displaystyle\int_{-\infty}^{\infty}P_{yy}e^{i\lambda x}\cdot d\lambda \\
p_{xy}=-2\rho b^2\displaystyle\int_{-\infty}^{\infty}\Big\{-\lambda\sqrt{k^2-\lambda^2}M_1(\lambda)e^{-\sqrt{\lambda^2-k^2}y} \\
\quad +\Big(\frac{K^2}{2}-\lambda^2\Big)N_1(\lambda)e^{-\sqrt{\lambda^2-K^2}y}\Big\}e^{i\lambda x}d\lambda\equiv\displaystyle\int_{-\infty}^{\infty}P_{xy}e^{i\lambda x}\cdot d\lambda
\end{array}\right\}\text{ for } y\geqslant 0, \\
\left.\begin{array}{l}
p_{yy}=2\rho b^2\displaystyle\int_{-\infty}^{\infty}\Big\{\Big(\frac{K^2}{2}-\lambda^2\Big)M_2(\lambda)e^{\sqrt{\lambda^2-k^2}y} \\
\quad -\lambda\sqrt{K^2-\lambda^2}N_2(\lambda)e^{\sqrt{\lambda^2-K^2}y}\Big\}e^{i\lambda x}d\lambda\equiv\displaystyle\int_{-\infty}^{\infty}P_{yy}e^{i\lambda x}d\lambda \\
p_{xy}=-2\rho b^2\displaystyle\int_{-\infty}^{\infty}\Big\{\lambda\sqrt{k^2-\lambda^2}M_2(\lambda)e^{\sqrt{\lambda^2-k^2}y} \\
\quad +\Big(\frac{K^2}{2}-\lambda^2\Big)N_2(\lambda)e^{\sqrt{\lambda^2-K^2}y}\Big\}e^{i\lambda x}d\lambda\equiv\displaystyle\int_{-\infty}^{\infty}P_{xy}e^{i\lambda x}d\lambda
\end{array}\right\}\text{ for } y\leqslant 0
\end{array}\right\}\quad (6)
$$

Insert (6) into the boundary conditions (i), (ii) and $M_{1,2}$, $N_{1,2}$ can be expressed in terms of two unknown functions,

$$
\left.\begin{array}{l}
M_1(\lambda)=\Big(\dfrac{K^2}{2}-\lambda^2\Big)R_1(\lambda)-\lambda\sqrt{K^2-\lambda^2}R_2(\lambda), \\
M_2(\lambda)=\Big(\dfrac{K^2}{2}-\lambda^2\Big)R_1(\lambda)+\lambda\sqrt{K^2-\lambda^2}R_2(\lambda), \\
N_1(\lambda)=\lambda\sqrt{k^2-\lambda^2}R_1(\lambda)+\Big(\dfrac{K^2}{2}-\lambda^2\Big)R_2(\lambda), \\
N_2(\lambda)=\Big(\dfrac{K^2}{2}-\lambda^2\Big)R_2(\lambda)-\lambda\sqrt{k^2-\lambda^2}R_1(\lambda).
\end{array}\right\}\quad (7)
$$

Next, split the transformed stresses $P_{yy}(\lambda, y)$, $P_{xy}(\lambda, y)$ into the following two parts

$$
\left.\begin{array}{l}
P_{yy}(\lambda, y)=P_{yy}^{+}(\lambda, y)+P_{yy}^{-}(\lambda, y) \\
P_{yy}^{+}(\lambda,y)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{0}p_{yy}e^{-i\lambda x}dx: \quad P_{yy}^{-}(\lambda,y)=\dfrac{1}{2\pi}\displaystyle\int_{0}^{\infty}p_{yy}e^{-i\lambda x}dx
\end{array}\right\}\quad (8)
$$

For brevity, we shall sometimes write $f(\lambda)$ or $f(y)$ instead of $f(\lambda, y)$

when there is no risk of confusion. An expression like $f(\pm 0)$ will always refer to the value of $f(\lambda, y)$ for $y=0$, where $+0$ means the limit as y tends to zero approached from positive values of y, etc.. Now we define the λ-plane not lower than the real axis as limits of the line L shown in Fig. 2 to be the upper plane and the plane not above the real axis to be the lower plane. When the solutions satisfy the radiation condition, P_{yy}^+ is regular in the upper plane, and P_{yy}^- regular in the lower. We also split the displacements into two parts in the same way

$$\left.\begin{aligned} &U(\lambda)=U^+(\lambda)+U^-(\lambda), \\ &U^+(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{0} ue^{-i\lambda x}dx : \qquad U^-(\lambda)=\frac{1}{2\pi}\int_{0}^{\infty} ue^{-i\lambda x}dx, \end{aligned}\right\} \tag{9}$$

On applying the above definitions to (8), (9) we find

$$P_{yy}^+(0)+P_{yy}^-(0)=2\rho b^2 G(\lambda)R_1(\lambda) \tag{10a}$$

$$P_{xy}^+(0)+P_{xy}^-(0)=-2\rho b^2 G(\lambda)R_2(\lambda) \tag{10b}$$

$$U^+(0)+U^-(0)=i(-\lambda M_1(\lambda)+\sqrt{K^2-\lambda^2}N_1(\lambda)) \tag{11a}$$

$$U^+(-0)+U^-(-0)=i(-\lambda M_2(\lambda)-\sqrt{K^2-\lambda^2}N_2(\lambda)) \tag{11b}$$

$$V^+(0)+V^-(0)=i(-\sqrt{k^2-\lambda^2}M_1(\lambda)-\lambda N_1(\lambda)) \tag{11c}$$

$$V^+(-0)+V^-(-0)=i(\sqrt{k^2-\lambda^2}M_2(\lambda)-\lambda N_2(\lambda)), \tag{11d}$$

where

$$G(\lambda)=\left(\frac{K^2}{2}-\lambda^2\right)+\lambda^2\sqrt{(k^2-\lambda^2)(K^2-\lambda^2)}.$$

Next subtract (11b) from (11a) and (11d) from (11c). From the boundary conditions (i), (ii), then

$$\left.\begin{aligned} &D^+\equiv U^+(0)-U^+(-0)=iK^2\sqrt{K^2-\lambda^2}R_2(\lambda), \\ &E^+\equiv V^+(0)-V^+(-0)=-iK^2\sqrt{k^2-\lambda^2}R_1(\lambda). \end{aligned}\right\} \tag{12}$$

Eliminate R_1, R_2 between (10) and (12). Then

$$P_{yy}^+(0)+P_{yy}^-(0)=\frac{i\rho b^2(K^2-k^2)\sqrt{K^2-\lambda^2}E^+(\lambda)F(\lambda)}{K^2}, \tag{13a}$$

$$P_{xy}^+(0)+P_{xy}^-(0)=\frac{i\rho b^2(K^2-k^2)\sqrt{k^2-\lambda^2}D^+(\lambda)F(\lambda)}{K^2}, \tag{13b}$$

where

$$F(\lambda)\equiv\frac{2G(\lambda)}{(K^2-k^2)\sqrt{(k^2-\lambda^2)(K^2-\lambda^2)}}=F^+(\lambda)\cdot F^-(\lambda). \tag{14}$$

First, we specify the incident P wave in the form

$$\left.\begin{aligned} &\phi_0 = e^{i(\kappa x+\sqrt{k^2-\kappa^2}y)}, \qquad \psi_0=0, \\ &-k<\kappa<k. \end{aligned}\right\} \tag{15}$$

and seek for the function $R_1(\lambda)$. In the equation (13a) $P_{yy}^+(0)$ is known from the boundary condition (i). In fact

$$P_{yy}^+(0) = \frac{1}{2\pi}\int_{-\infty}^{0}(-p_{yy}^0)e^{-i\lambda x}dx = \frac{\rho b^2}{\pi}\int_{-\infty}^{0} A_1 e^{i(\kappa-\lambda)x}dx = \frac{\rho b^2 A_1}{\pi i(\kappa-\lambda)}, \tag{16}$$

where

$$A_1 = -\left(\frac{K^2}{2}-\kappa^2\right).$$

The equation (13a) now becomes

$$\frac{P^-_{yy}(0)}{2\rho b^2\sqrt{K-\lambda}F^-(\lambda)} + \frac{A_1}{2\pi i(\kappa-\lambda)\sqrt{K-\lambda}F^-(\lambda)}$$
$$= \frac{i(K^2-k^2)\sqrt{K+\lambda}E^+(\lambda)F^+(\lambda)}{2K^2}. \tag{17}$$

Next consider the decomposition of the function $F(\lambda)$ in the form of a product $F^+(\lambda)\cdot F^-(\lambda)$ by means of splitting of $\log F(\lambda)$ into the form of a sum, where $F^+(\lambda)$ and $F^-(\lambda)$ are regular in the upper and lower half planes, respectively. The singular points of $\log F(\lambda)$ are $\pm k$, $\pm K$ and $\pm\lambda_R$, zeros of $F(\lambda)$, and

$$F(\lambda) \sim 1 + \frac{\text{const.}}{\lambda^2} \qquad \text{for} \qquad \lambda\to\infty. \tag{18}$$

$f(\lambda)$ can be written in the form by Cauchy's theorem

$$f(\lambda) = \log F(\lambda) = \frac{1}{2\pi i}\int_{\Gamma^+}\frac{\log F(z)}{z-\lambda}dz - \frac{1}{2\pi i}\int_{\Gamma^-}\frac{\log F(z)}{z-\lambda}dz, \tag{19}$$

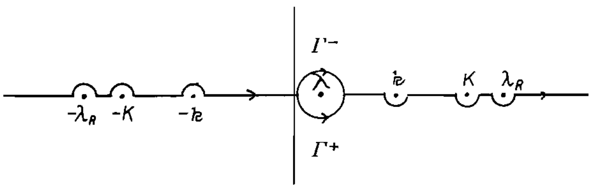


Fig. 3.

where contour Γ^+, Γ^- is shown in Fig. 3, λ is contained in the domain enclosed by Γ^+ and Γ^-. Taking the singular points $\pm k$, $\pm K$, $\pm\lambda_R$ into the consideration, we can write (19) in the following way

$$f^{+,-}(\lambda)=\frac{1}{2\pi i}\int_{\Gamma^{+,-}}\log\left(1+\frac{\left(\frac{K^2}{2}-z^2\right)^2}{z^2\sqrt{(k^2-z^2)(K^2-z^2)}}\right)\frac{dz}{z-\lambda}$$

$$=\log\frac{\lambda_R\pm\lambda}{K\pm\lambda}+\frac{1}{\pi}\int_{\mp k}^{\mp K}\tan^{-1}\left\{\frac{\left(\frac{K^2}{2}-z^2\right)^2}{z^2\sqrt{(z^2-k^2)(K^2-z^2)}}\right\}\frac{dz}{z-\lambda}. \tag{20}$$

Split $g(\lambda)$ in two parts in the same way as (19)

$$\left.\begin{aligned}
g(\lambda)&=g^+(\lambda)+g^-(\lambda)\equiv\frac{A_1}{2\pi i(\kappa-\lambda)\sqrt{K-\lambda}F^-(\lambda)},\\
g^+(\lambda)&=\frac{A_1}{2\pi i\sqrt{K-\kappa}(\kappa-\lambda)F^-(\kappa)},\\
g^-(\lambda)&=\frac{A_1}{2\pi i(\kappa-\lambda)}\left\{\frac{1}{\sqrt{K-\lambda}F^-(\lambda)}-\frac{1}{\sqrt{K-\kappa}F^-(\kappa)}\right\},
\end{aligned}\right\} \tag{21}$$

where $g^+(\lambda)$ is regular in the upper, $g^-(\lambda)$ is regular in the lower. Insert (21) in (17) and rearange

$$\frac{P^-_{yy}(0)}{2\rho b^2\sqrt{K-\lambda}F^-(\lambda)}+g^-(\lambda)=\frac{i(K^2-k^2)\sqrt{K+\lambda}F^+(\lambda)E^+(\lambda)}{2K^2}-g^+(\lambda)\equiv I(\lambda). \tag{22}$$

In this form a function $I(\lambda)$ is regular in the upper plane and also regular in the lower, i.e., in the whole of plane, since these two half planes overlap. And we proceed to examine the behaviour of the functions $I(\lambda)$ as λ tends to infinity.

From the edge condition

$$I(\lambda)\sim|\lambda|^{-1} \quad \text{as} \quad \lambda\to\infty. \tag{23}$$

$I(\lambda)$ tends to zero as λ tends to infinity in any direction. Hence, from the Liouville's theorem $I(\lambda)$ must be identically zero, i.e.

$$E^+(\lambda)=-\frac{iK^2\sqrt{k^2-\lambda^2}\sqrt{K-\lambda}g^+(\lambda)\cdot F^-(\lambda)}{G(\lambda)}, \tag{24}$$

$$R_1(\lambda)=-\frac{A_1}{2\pi i}\frac{\sqrt{K-\lambda}F^-(\lambda)}{\sqrt{K-\kappa}F^-(\kappa)G(\lambda)(\lambda-\kappa)}. \tag{25}$$

From (13b) the unknown function $R_2(\lambda)$ also can be found by the same procedure.

$$R_2(\lambda)=-\frac{A_2}{2\pi i}\frac{\sqrt{k-\lambda}F^-(\lambda)}{\sqrt{k-\kappa}F^-(\kappa)G(\lambda)(\lambda-\kappa)}, \tag{26}$$

where

$$A_2=\kappa\sqrt{k^2-\kappa^2}.$$

Finally, the scattered part for the harmonic P plane wave is

$$
\left.\begin{aligned}
\phi=&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sqrt{\frac{K-\lambda}{K-\kappa}}\frac{F^{-}(\lambda)}{F^{-}(\kappa)}\\
&\times\frac{\left(\frac{K^2}{2}-\kappa^2\right)\left(\frac{K^2}{2}-\lambda^2\right)\pm\kappa\lambda\sqrt{(k+\kappa)(k-\lambda)(K+\lambda)(K-\kappa)}}{G(\lambda)}\\
&\times e^{i(\lambda x+\sqrt{k^2-\lambda^2}|y|)}\cdot\frac{d\lambda}{\lambda-\kappa},\\
\psi=&-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sqrt{\frac{k-\lambda}{K-\kappa}}\frac{F^{-}(\lambda)}{F^{-}(\kappa)}\\
&\times\frac{\mp\left(\frac{K^2}{2}-\kappa^2\right)\lambda\sqrt{(k+\lambda)(K-\lambda)}+\left(\frac{K^2}{2}-\lambda^2\right)\kappa\sqrt{(k+\kappa)(K-\kappa)}}{G(\lambda)}\\
&\times e^{i(\lambda x+\sqrt{K^2-\lambda^2}|y|)}\cdot\frac{d\lambda}{\lambda-\kappa},
\end{aligned}\right\}\qquad(27)
$$

Specifying the incident S wave in the form

$$
\psi_0=e^{i(\kappa x+\sqrt{K^2-\kappa^2}y)},\qquad \phi_0=0,\qquad -K<\kappa<K,\qquad(28)
$$

we find also in the similar way

$$
\left.\begin{aligned}
\phi=&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sqrt{\frac{K-\lambda}{k-\kappa}}\frac{F^{-}(\lambda)}{F^{-}(\kappa)}\\
&\times\frac{\left(\frac{K^2}{2}-\lambda^2\right)\kappa\sqrt{(k-\kappa)(K+\kappa)}\mp\left(\frac{K^2}{2}-\kappa^2\right)\lambda\sqrt{(k-\lambda)(K+\lambda)}}{G(\lambda)}\\
&\times e^{i(\lambda x+\sqrt{k^2-\lambda^2}|y|)}\frac{d\lambda}{\lambda-\kappa},\\
\psi=&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sqrt{\frac{k-\lambda}{k-\kappa}}\frac{F^{-}(\lambda)}{F^{-}(\kappa)}\\
&\times\frac{\pm\kappa\lambda\sqrt{(k+\lambda)(k-\kappa)(K+\kappa)(K-\lambda)}+\left(\frac{K^2}{2}-\kappa^2\right)\left(\frac{K^2}{2}-\lambda^2\right)}{G(\lambda)}\\
&\times e^{i(\lambda x+\sqrt{K^2-\lambda^2}|y|)}\cdot\frac{d\lambda}{\lambda-\kappa}.
\end{aligned}\right\}\qquad(29)
$$

The upper signs refer to $y\geqslant 0$, the lower to $y\leqslant 0$ in (27), (29).

§3. In this section we produce the solution of a pulse diffraction with the aid of the known solution of harmonic wave by the principle of superposition.

Assume that the incident P plane pulse has the form

$$u_r{}^0=\cos(\theta-\alpha)D\left(t-\frac{r}{a}\cos(\theta-\alpha)\right)=\begin{cases}0 & : |t-r\cos(\theta-\alpha)/a|>\varepsilon \\ \cos(\theta-\alpha) & : |t-r\cos(\theta-\alpha)/a|<\varepsilon\end{cases} \tag{30}$$

The integral representation of the function is

$$u_r{}^0=\frac{1}{\pi}\int_{-\infty}^{\infty}\cos(\theta-\alpha)\frac{\sin(\varepsilon ka)}{k}e^{-ika\left(t-\frac{r}{a}\cos(\theta-\alpha)\right)}dk \tag{31}$$

The scattered pulse for the incidence of the pulse of form (30) can be obtained by superposition of the solution (27) for the incidence of harmonic wave. Namely, the P part of the scattered pulse is for $\theta>0$

$$u_r=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{P(\lambda,\kappa)\{\lambda\cos\theta+\sqrt{k^2-\lambda^2}\sin\theta\}}{k(\lambda-\kappa)}$$

$$\times\frac{\sin\varepsilon ka}{k}e^{-ika\left(t-\frac{r}{ka}(\lambda\cos\theta+\sqrt{k^2-\lambda^2}\sin\theta)\right)}dk\cdot d\lambda \tag{32}$$

$$P(\lambda,\kappa)=\frac{\sqrt{K-\lambda}F^-(\lambda)H(\lambda,\kappa)}{\sqrt{K-\kappa}F^-(\kappa)G(\lambda)}\frac{1}{2\pi i} \tag{33}$$

$$H(\lambda,\kappa)=\left(\frac{K^2}{2}-\kappa^2\right)\left(\frac{K^2}{2}-\lambda^2\right)\pm\kappa\lambda\sqrt{(k+\kappa)(K-\kappa)(k-\lambda)(K+\lambda)} \tag{34}$$

For $\theta<0$, $-\theta$ takes the places of θ.

Exchange the order of integration, change the variable λ to γ defined by $\lambda=k\cos\gamma=k\cos(\theta+is)$ and deform the contour C to the line $\theta=$const., $-\infty<s<\infty$ as in Fig. 4.

Integrate with respect to k, to find

$$u_r=-\cos(\alpha-\theta)\frac{H(\alpha)}{G(\alpha)}D\left(t-\frac{r\cos(\alpha-\theta)}{a}\right)-\frac{i}{\pi}\int_{s_1}^{s_2}\frac{P(\gamma,\alpha)\sin(\theta+is)\cosh s}{\cos(\theta+is)-\cos\alpha}ds$$

$$-\frac{i}{\pi}\int_{-s_2}^{-s_1}\frac{P(\gamma,\alpha)\sin(\theta+is)\cosh s}{\cos(\theta+is)-\cos\alpha}ds \tag{35}$$

$$\left.\begin{aligned}&s_1=\cosh^{-1}\frac{a(t-\varepsilon)}{r}, \qquad s_2=\cosh^{-1}\frac{a(t+\varepsilon)}{r},\\ &\kappa=k\cos\alpha=K\cos\beta,\end{aligned}\right\} \tag{36}$$

where the first term vanishes for $|\theta|<\alpha$ and the third term is the complex conjugate of the second.

When the ratio $2\varepsilon/r$ of the pulse width 2ε to the distance r from the edge of the crack is small and $P(\alpha,\gamma)$ varies slowly, we write approximately for $t\leqslant r/a+\varepsilon$

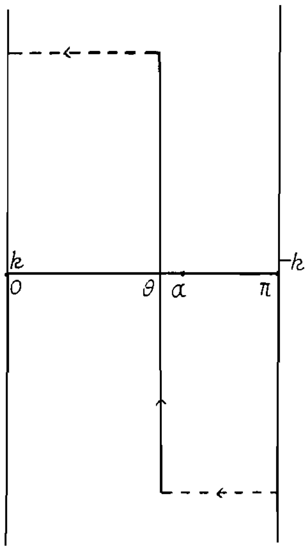


Fig. 4.

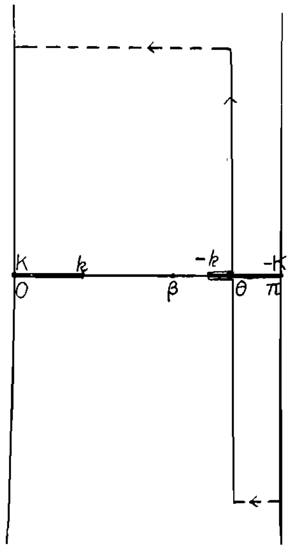


Fig. 5.

$$u_r^p = -\cos(\alpha-\theta)\frac{H(\alpha)}{G(\alpha)}D\left(t-\frac{r}{a}\cos(\alpha-\theta)\right)$$

$$-\frac{\sqrt{2k}\sqrt{K-k\cos\theta}}{\pi(\lambda_R-k\cos\alpha)}\frac{H(\theta,\alpha)F^-(\theta)\sin\dfrac{\theta}{2}}{G(\theta)F_1^-(\alpha)}$$

$$\times\left(\int_{-s_2}^{s_2}\left\{\frac{1}{\sin\{(\theta+\alpha+is)/2\}}-\frac{1}{\sin\{(\theta-\alpha+is)/2\}}\right\}ds\right). \tag{37}$$

Integrate with respect to s, to find

$$u_r^p = -\cos(\alpha-\theta)\frac{H(\alpha)}{G(\alpha)}D\left(t-\frac{r}{a}\cos(\alpha-\theta)\right)$$

$$-\frac{\sqrt{2k}\sqrt{K-k\cos\theta}}{\pi(\lambda_R-k\cos\alpha)}\frac{F^-(\theta)H(\theta,\alpha)\sin\dfrac{\theta}{2}}{F_1^-(\alpha)G(\theta)}\left\{\tan^{-1}\left(\frac{\sinh\dfrac{s_2}{2}}{\sin\dfrac{\theta+\alpha}{2}}\right)-\tan^{-1}\left(\frac{\sinh\dfrac{s_2}{2}}{\sin\dfrac{\theta-\alpha}{2}}\right)\right\}, \tag{38}$$

$$F_1^-(\lambda)=\frac{\sqrt{K-\lambda}\sqrt{k-\lambda}}{\lambda_1-\lambda}F^-(\lambda), \qquad \sinh\frac{s_2}{2}=\sqrt{\frac{a(t+\varepsilon)-r}{2r}},$$

where the first term expresses the reflected pulse and the second term expresses the diffracted pulse.

Next, we find the S part of scattered pulse in the similar way for $\theta>0$

$$u_\theta^s = -\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Q(\lambda,\kappa)\{\lambda\cos\theta+\sqrt{K^2-\lambda^2}\sin\theta\}}{k(\lambda-\kappa)}$$

$$\times \frac{\sin \varepsilon Kb}{K} e^{-iKb\left(t-\frac{r}{Kb}(\lambda\cos\theta+\sqrt{K^2-\lambda^2}\sin\theta)\right)} dKd\lambda, \tag{39}$$

$$Q(\lambda,\kappa)=\frac{1}{2\pi i}\sqrt{\frac{k-\lambda}{K-\kappa}}\frac{F^-(\lambda)}{F^-(\kappa)}\frac{I(\lambda,\kappa)}{G(\lambda)}, \tag{40}$$

$$I(\lambda,\kappa)=\mp\left(\frac{K^2}{2}-\kappa^2\right)\lambda\sqrt{(k+\lambda)(K-\lambda)}+\left(\frac{K^2}{2}-\lambda^2\right)\kappa\sqrt{(k+\kappa)(K-\kappa)}. \tag{41}$$

Exchange the order of integration and transform the contour in λ plane to the path in δ plane defined by $\lambda=K\cos\delta=K\cos(\theta+is)$ as shown in Fig. 5.

When the ratio $2\varepsilon/r$ is small, we find for $t\leqslant r/b+\varepsilon$

$$u_\theta{}^s=c\sqrt{K/k}\cos(\beta-\theta)\frac{I(\beta)}{G(\beta)}D\left(t-\frac{r}{b}\cos(\beta-\theta)\right)$$

$$+\frac{C\sqrt{2K}}{\pi\sqrt{k}}\frac{\sqrt{k-K\cos\theta}F^-(\theta)I(\theta,\beta)\sin\frac{\theta}{2}}{(\lambda_R-K\cos\beta)F_1{}^-(\beta)G(\theta)}$$

$$\times\left\{\tan^{-1}\left(\frac{\sinh\frac{s_2'}{2}}{\sin\frac{\theta+\beta}{2}}\right)-\tan^{-1}\left(\frac{\sinh\frac{s_2'}{2}}{\sin\frac{\theta-\beta}{2}}\right)\right\},$$

$$c=\sin\frac{\alpha}{2}\Big/\sin\frac{\beta}{2},\qquad \sinh\frac{s_2'}{2}=\sqrt{\frac{b(t+\varepsilon)-r}{2r}}. \tag{42}$$

In this case the part $-k\theta$ of the contour does not contribute to the integral and for $\theta>-\beta$, the first term vanishes.

In the similar way as the above, we can obtain the displacements due to the incidence of a plane S pulse of rectangular form.

Assume that the incident pulse has the form

$$u_\theta{}^{s0}=\cos(\beta-\theta)D\left(t-\frac{r}{b}\cos(\beta+\theta)\right) \tag{43}$$

The scattered pulse for the incidence of the pulse of form (43) can be obtained by superposition. The P part is for $\theta>0$

$$u_r{}^p=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{P(\lambda,\kappa)\{\lambda\cos\theta+\sqrt{k^2-\lambda^2}\sin\theta\}}{K(\lambda-\kappa)}$$

$$\times\frac{\sin\varepsilon ka}{k}e^{-ika\left\{t-\frac{r}{ka}(\lambda\cos\theta+\sqrt{k^2-\lambda^2}\sin\theta)\right\}}dkd\lambda, \tag{44}$$

$$P(\lambda,\kappa)=\frac{1}{2\pi i}\sqrt{\frac{K-\lambda}{k-\kappa}}\frac{F^-(\lambda)}{F^-(\kappa)}\frac{H(\lambda,\kappa)}{G(\lambda)}, \tag{45}$$

$$H(\lambda,\kappa)=\left(\frac{K^2}{2}-\lambda^2\right)\kappa\sqrt{(k-\kappa)(K+\kappa)}\mp\left(\frac{K^2}{2}-\kappa^2\right)\lambda\sqrt{(k-\lambda)(K+\lambda)}. \tag{46}$$

We have the same transformation as the case of P part for the incidence of P pulse and integrate with respect to k to find for $t \leqslant r/a+\varepsilon$

$$u_r^p = -c'\sqrt{\frac{k}{K}}\frac{\cos(\theta-a)H(a)}{G(a)}D\left(t-\frac{r}{a}\cos(a-\theta)\right)$$

$$-\frac{c'\sqrt{2k}}{\pi\sqrt{K}}\frac{\sqrt{K-k\cos\theta}}{\lambda_R-k\cos a}\frac{F^-(\theta)H(\theta,a)\sin\dfrac{\theta}{2}}{G(\theta)}\int_{-s_2}^{s_2}$$

$$\times\left\{\frac{1}{\sin\{(\theta+a+is)/2\}}-\frac{1}{\sin\{(\theta-a+is)/2\}}\right\}ds. \tag{47}$$

where for $\theta>-a$ the first term vanishes. When the ratio $2\varepsilon/r$ is small, the approximate solution can be obtained

$$u_r^p = -c'\sqrt{\frac{k}{K}}\frac{\cos(\theta-a)H(a)}{G(a)}D\left(t-\frac{r}{a}\cos(a-\theta)\right)$$

$$-\frac{c'\sqrt{2k}}{\pi\sqrt{K}}\frac{\sqrt{K-k\cos\theta}}{\lambda_R-k\cos a}\frac{F^-(\theta)}{F_1^-(a)}\frac{H(\theta,a)\sin\dfrac{\theta}{a}}{G(\theta)}$$

$$\times\left\{\tan^{-1}\left(\frac{\sinh\dfrac{s_2}{2}}{\sin\dfrac{\theta+a}{2}}\right)-\tan^{-1}\left(\frac{\sinh\dfrac{s_2}{2}}{\sin\dfrac{\theta-a}{2}}\right)\right\}, \tag{48}$$

$$c'=\sin\frac{\beta}{2}\Big/\sin\frac{a}{2},$$

Transform in the same way as the case of S part for the incidence of P pulse, then the S part is for $\theta>0$

$$u_\theta^s = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Q(\lambda,\kappa)\{\lambda\cos\theta+\sqrt{K^2-\lambda^2}\sin\theta\}}{K(\lambda-\kappa)}\frac{\sin\varepsilon Kb}{K}$$

$$\times e^{-iKb\left\{t-\frac{r}{Kb}(\lambda\cos\theta+\sqrt{K^2-\lambda^2}\sin\theta)\right\}}dKd\lambda, \tag{49}$$

$$Q(\lambda,\kappa)=\frac{1}{2\pi i}\sqrt{\frac{k-\lambda}{k-\kappa}}\frac{F^-(\lambda)I(\lambda,\kappa)}{F^-(\kappa)G(\lambda)}, \tag{50}$$

$$I(\lambda,\kappa)=\pm\kappa\lambda\sqrt{(k-\kappa)(K+\kappa)(k+\lambda)(K-\lambda)}+\left(\frac{K^2}{2}-\kappa^2\right)\left(\frac{K^2}{2}-\lambda^2\right). \tag{51}$$

When the ratio $2\varepsilon/r$ is small, for $t\leqslant r/b+\varepsilon$

$$u_\theta^s = -\cos(\beta-\theta)\frac{I(\beta)}{G(\beta)}D\left(t-\frac{r}{b}\cos(\theta-\beta)\right)$$

$$-\frac{\sqrt{2K}}{\pi}\frac{\sqrt{k-K\cos\theta}}{\lambda_R-K\cos\beta}\frac{F^-(\theta)I(\theta,\beta)\sin\dfrac{\theta}{2}}{F_1^-(\beta)G(\theta)}$$

$$\times\left\{\tan^{-1}\left(\frac{\sinh\frac{s_2'}{2}}{\sin\frac{\theta+\beta}{2}}\right)-\tan^{-1}\left(\frac{\sinh\frac{s_2'}{2}}{\sin\frac{\theta-\beta}{2}}\right)\right\}, \tag{52}$$

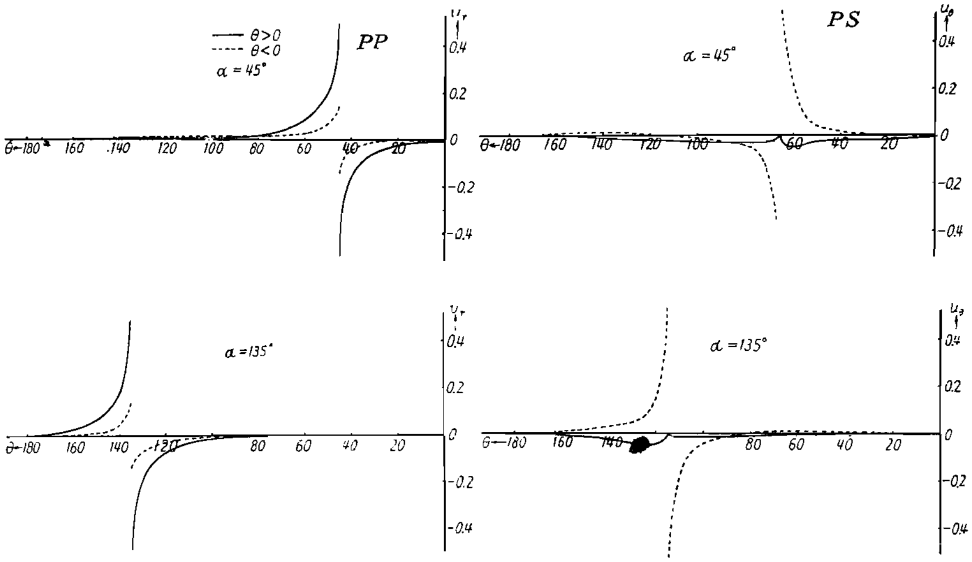
where for $|\theta|<\beta$ the first term vanishes.

§4. In our numerical example we assumed

$K=\sqrt{3}k$

$\alpha=45°,\ 135°$ for the incidence of P pulse

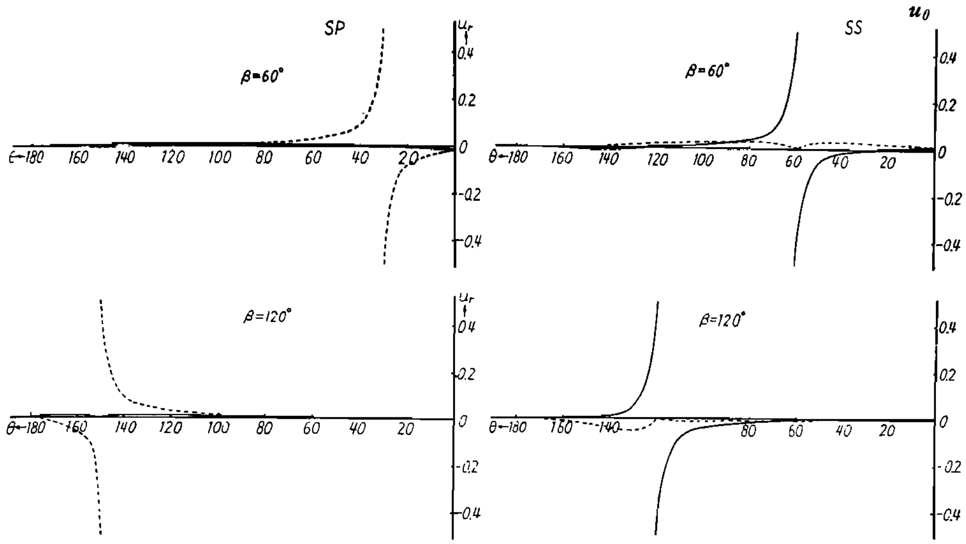
$\beta=60°,\ 120°$ for the incidence of S pulse



a) diffracted P pulse b) diffracted S pulse

Fig. 6. The azimuthal distribution of the amplitude of displacement u_r, u_θ of the diffracted pulse for the incidence of P pulse.

We calculated the azimuthal distribution of the amplitude of displacement of the diffracted pulse which is the second term in the equation (38), (42), (48), (52) at $\sqrt{a\varepsilon/r}=0.025$ and $t=r/a+\varepsilon$ for the incident pulse of unit amplitude. u_r, u_θ thus calculated are shown in Fig. 6, 7. For example, the amplitude of P part of the diffracted pulse for the incident angle $\alpha=45°$ of the P pulse increases with approach of θ to $\theta=45°$ for $\theta>0$ or to $\theta=-45°$ for $\theta<0$ and the phase is reversed at the boundary of $\theta=45°$ or $\theta=-45°$. But the composite amplitude of this pulse with



a) diffracted P pulse b) diffracted S pulse

Fig. 7. The azimuthal distribution of the amplitude of displacement u_r, u_θ of the diffracted pulse for the incidence of S pulse.

the reflected P pulse for $\theta<0$ or the incident pulse for $\theta>0$ is continuous at the boundary. While with respect to the S part the amplitude varies continuously and becomes zero at $\theta=65°54'$ for $\theta>0$, the phase is reversed at the boundary of $\theta=-65°54'$ for $\theta<0$ and the composite amplitude of this pulse with the reflected S pulse is continuous at this angle.

For the incidence of S plane pulse with incident angle $\beta=60°$, the amplitude of the diffracted P pulse is continuous and decreases to zero at $\theta=30°$ for $\theta>0$, while the amplitude increases to maximum at the angle $\theta=-30°$, at which the reflected pulse is inclined to the x-axis, and the phase is reversed at the boundary for $\theta<0$.

The diffracted S pulse increases to maximum with approach of θ to $\theta=-60°$ or $\theta=60°$ and the phase is reversed at these boundaries. The composite amplitude of the diffracted pulse with the incident for $\theta>0$ or the reflected pulse for $\theta<0$ is continuous at these boundaries as in the incidence of the P pulse. The amplitudes of the above all diffracted pulses decrease with the increase of the angular distance measured from the shadow boundaries as known from Fig. 6, 7. Still, such a decrease of the amplitude may be slow with the increase of the pulse width 2ε. In

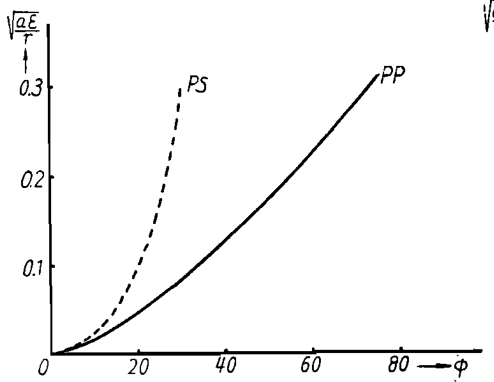


Fig. 8. $\sqrt{a\varepsilon/r}$ versus ϕ for the incidence of P pulse.

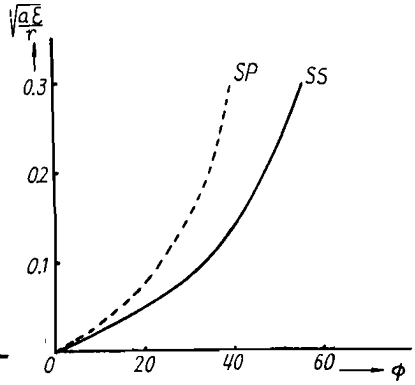
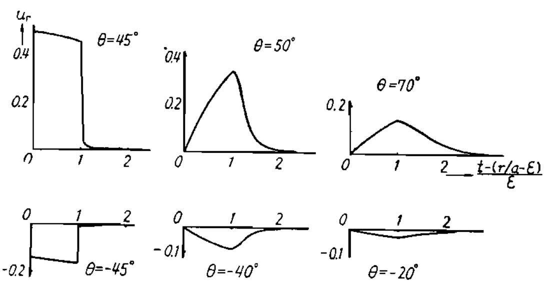


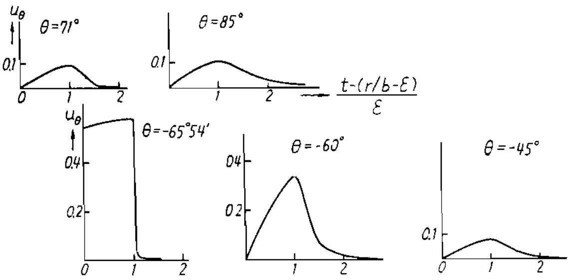
Fig. 9. $\sqrt{a\varepsilon/r}$ versus ϕ for the incidence of S pulse.

order to show such a feature, we plot $\sqrt{a\varepsilon/r}$ versus an angle ϕ in Fig. 8, 9. The angle at which the amplitude of diffracted pulse decreases to 1/10 of that of incidence is measured from the shadow boundary. For the incidence of P pulse, ϕ of PP, the diffracted P pulse, in Fig. 8 is measured from $\theta=45°$ towards the crack plane and ϕ of PS, the diffracted S pulse is measured from $\theta=-60°$ to 0°. For the incidence of S pulse ϕ of SP, the diffracted P pulse, is measured from $\theta=-30°$ to 0° and ϕ of SS, the diffracted S pulse, is measured from $\theta=60°$ to 180°. Generally these extents ϕ are approximately proportional to $\sqrt{a\varepsilon/r}$.

Next, inquire the dependency of the displacement at each point on time. The time variations of the calculated displacements on a circle of $r=100a\varepsilon$ are shown in Fig. 10, 11, when the plane pulse is incident on the crack at the incident angle $\alpha=45°$ or $\beta=60°$.

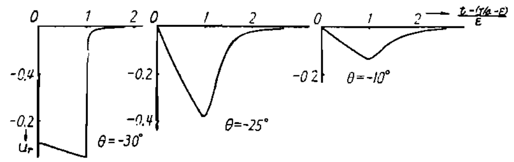


a) diffracted P pulse

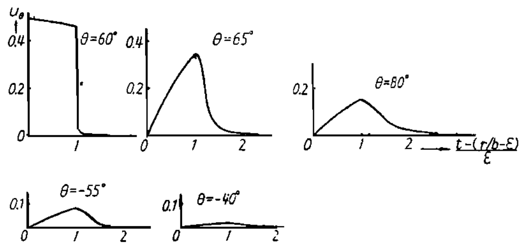


b) diffracted S pulse

Fig. 10. The displacement u_r, u_θ for the incidence of P pulse.



a) diffracted P pulse



b) diffracted S pulse

Fig. 11. The displacement u_r, u_θ for the incidence of S pulse.

It is remarkable that the displacements of diffracted P pulses as well as S pulses vary smoothly with time, while the forms of the incident and reflected pulses are the rectangular type. And the sharpness of the forms of diffracted pulses decreases with the increase of the angular distance measured from the shadow boundary. We see also from the time variation that the phases of diffracted pulses are reversed at the boundaries, as found in the azimuthal distribution of the amplitude. At the shadow boundary, the incident or reflected pulse appears simultaneously with the diffracted one and with the deviation from this direction the diffracted P pulse appears after the reflected or incident P pulse, followed by the diffracted S pulse.

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