# Part II Singular Hydrologic Amount and Rejection Test

## 1. Introduction

It is needless to say that the statistical treatment of the hydrologic data should be rational in the sense of stochastics to be significant as a useful tool for the hydraulic works. Usually, the very large or small data which seem to be singular in an ordinary sense are to be contained in a family of hydrologic observations. In hydrologic frequency analysis, the rejection test of such data is essential in the sense of stochastics, while evaluation of the singular value defined later is important in the sense of engineering. It seems, however, that the course of treatment for such problems is not always successfully planned, in the field of hydrologic statistics. In this part, a theoretical and practical method of treatment for such problems is developed[10,20].

## 2. Definition of singnlar hydrologic amount

In a family of hydrologic data of size $N$, the very large variate $x_{\varepsilon u}$ or the very small variate $x_{\varepsilon l}$ is often contained, whose population probability of exceedance $1-F(x_{\varepsilon u})$ or of nonexceedance $F(x_{\varepsilon l})$ is very small.

According to the help of the sample theory, the probability $\mathcal{E}\equiv P_N(x\geqq x_{\varepsilon u})$ or $P_N(x\leqq x_{\varepsilon l})$ with which such a variate is contained in a sample of size $N$ is greater than the population probability $1-F(x_{\varepsilon u})$ or $F(x_{\varepsilon l})$, under the condition of finite sample size $N$.

$$\left.\begin{aligned} \mathcal{E} &\equiv P_N(x\geqq x_{\varepsilon u}) \geqq 1-F(x_{\varepsilon u}) \\ \mathcal{E} &\equiv P_N(x\leqq x_{\varepsilon l}) \geqq F(x_{\varepsilon l}) \end{aligned}\right\} \tag{2.1}$$

Or, for a fixed value $x_\varepsilon$, the smaller the sample size $N$ is, the larger the probability $\mathcal{E}$ becomes in spite of the fixed population probability.

In hydrologic frequency analysis, the above facts should always be considered so far as the analysis is based on a sample of small size. And it is desired in the hydraulic planning and design to utilize the reasonable information obtained by the frequency analysis of the hydrologic data as for as possible.

That is, as the expected value for the desired return period $T$, the value of $x_{\varepsilon u}$ or $x_{\varepsilon l}$ for $\varepsilon=1/T$ rather than that for $1-F(x)=1/T$ or $F(x)=1/T$ should be considered for discussion. In this paper, the former estimate is named the expected singular hydrologic amount in a stochastic sense, or simply the (expected) singular value, and the latter estimate is called the expected hydrologic amount in an ordinary sense, or simply the (expected) ordinary value. Moreover, the probability $\varepsilon$ is simply called the singular level for the singular value $x_\varepsilon$.

To find the theoretical relation between the singular value $x_\varepsilon$ and its singular level $\varepsilon$, the sample theory should, strictly speaking, be extended on extremes, but it seems to be difficult since a satisfactory sample theory has not yet been developed. Therefore, in this paper, the theory of rejection limit in the sense of Thompson, which is obtained as a special case of the two-sample theory on normals, will be utilized as a convenient approach. This idea is all but analogous in an essential sense to that adopted by Ogawara[21)], by which he studied the stochastic limit for the maximum possible amount of precipitation.

Now, let $\bar{\eta}$ and $s_\eta{}^2$ be the sample values of mean and variance, respectively, in a sample of size $N$ from a normal population $N(m, \sigma^2)$, defined as

$$\left.\begin{aligned} \bar{\eta}&=\frac{1}{N}\sum_i \eta_i \\ s_\eta{}^2&=\frac{1}{N}\sum_i(\eta_i-\bar{\eta})^2 \end{aligned}\right\} \tag{2.2}$$

If $\eta_\varepsilon$ is another sample of size 1 which is obtained from the same population independent of the adove-state sample, the following equations are satisfactory clearly,

$$E(\eta_\varepsilon-\bar{\eta})=0$$

$$E(\eta_\varepsilon-\bar{\eta})^2=\frac{N}{N+1}\sigma^2$$

The statistics

$$\chi_1{}^2=\frac{Ns^2{}_\eta}{\sigma^2} \quad \text{and} \quad \chi_2{}^2=\frac{N}{N+1}\frac{(\eta_\varepsilon-\bar{\eta})^2}{\sigma^2}$$

are independent of each other and they follow the $\chi^2$-distribution of freedom $N$-1 and 1, respectively, Therefore, the statistics

$$F=\frac{N-1}{N+1}\frac{(\eta_\varepsilon-\bar{\eta})^2}{s_\eta{}^2} \tag{2.3}$$

must follow the $F$-distribution of freedom 1 and $N$-1.

Next, let $F^1{}_{N-1}(2\mathcal{E})$ be value of $F$ for a given singular level $\mathcal{E}$ which corresponds to the level of significance of one tail in the case of the rejection test on a normal sample. Then Eq. (2.3) becomes,

$$\left.\begin{matrix}\eta_{\varepsilon u}\\ \eta_{\varepsilon l}\end{matrix}\right\} = \bar{\eta} \pm S_\eta \sqrt{\frac{N+1}{N-1} F'_{N-1}(2\mathcal{E})} \qquad (2.4)$$

where $\eta_{\varepsilon u}$ and $\eta_{\varepsilon l}$ are the values corresponding to the singular level $\mathcal{E}$, provided on the normals.

Generally, the following relation is satisfactory for the upper tail.

$$\Phi(\eta_{\varepsilon u}) = \frac{1}{\sqrt{2\pi}\, S_\eta} \int_{\eta_{\varepsilon u}}^{\infty} \exp\left\{-\frac{(\eta-\bar{\eta})^2}{2S_\eta{}^2}\right\} d\eta \leqq \Phi_0(\eta_{\varepsilon u}) =$$

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{\eta_{\varepsilon u}}^{\infty} \exp\left\{-\frac{(\eta-m)^2}{2\sigma^2}\right\} dy \leqq \mathcal{E} \qquad (2.5)$$

And the similar relation to it for the lower tail can be presented. It can be easily considered that there is a simple relation between the singular level $\mathcal{E}$ and the value of $\Phi(\eta_{\varepsilon u})$ or $\Phi_0(\eta_{\varepsilon u})$, which will be defined as a function of sample size $N$ if a relation as Eq. (2.4) can be made available.

The above discussions provided for the normals can be easily extended to the extremes.

## 3. Estimation of singnlar hydrologic amount based on extremes

If all parameters included in the asymptotes for the largest value distribution which are expressed as

$$\left.\begin{array}{ll} F(x) = \exp(-e^{-y}), & \\ \text{for the first asymptote;} & y = a(x-u), \\ \text{for the second asymptote;} & y = a\log(x+b)/(u+b), \end{array}\right\} \qquad (2.6)$$

are known, the expected hydrologic amount can be obtained by the following equations,

$$\left.\begin{array}{ll} \text{for the first asymptote;} & x = u + (1/a)y \\ \text{for the second asymptote;} & \log(x+b) = \log(u+b) + (1/a)y \end{array}\right\} \qquad (2.7)$$

And as is well known, in estimating the amount for the desired return period $T$ in an ordinary sense, the value of $y$ defined by the following equation must be adopted in Eq. (2.7)

$$y = -\lg\{\lg T/(T-1)\} \qquad (2.8)$$

Table 2.1 Reduced extremes $y$ for given return period.

| $T$ | $1-F(x)$ | $y$ |
|---|---|---|
| 2000 | 0.00050 | 7.60065 |
| 1000 | 100 | 6.90725 |
| 500 | 200 | 6.21361 |
| 400 | 250 | 5.99021 |
| 300 | 0.00333 | 5.70212 |
| 250 | 400 | 5.51946 |
| 200 | 500 | 5.29581 |
| 150 | 667 | 5.00730 |
| 100 | 0.01000 | 4.60015 |
| 80 | 1250 | 4.37574 |
| 60 | 1667 | 4.08596 |
| 50 | 2000 | 3.90194 |
| 40 | 0.02500 | 3.67625 |
| 30 | 3333 | 3.38429 |
| 25 | 4000 | 3.19853 |
| 20 | 5000 | 2.97020 |
| 10 | 0.10000 | 2.25037 |
| 8 | 12500 | 2.01342 |
| 7 | 14286 | 1.86982 |
| 6 | 16667 | 1.70199 |
| 5 | 0.20000 | 1.49994 |
| 4 | 25000 | 1.24590 |
| 3 | 33333 | 0.90273 |
| 2 | 50000 | 0.36651 |
| | 63212 | 0 |

These values are tabulated in Table 2.1.

Now, let $y$ in Eq. (2.7) be denoted as $y_\varepsilon$, which should be especially used for evaluation of the expected singular hydrologic amount for the desired return period $T=1/\varepsilon$ in a stochastic sense as discussed in the preceeding section. Then, the value of $y_\varepsilon$ can be obtained from Eqs. (2.4) and (2.5) by the transformation of variables as follows:

$$\left.\begin{aligned} &\text{For upper tail;} \quad \exp(-e^{-y_{\varepsilon u}}) = F(y_{\varepsilon u}) \equiv 1-\Phi(\eta_{\varepsilon u}) \\ &\text{For lower tail;} \quad \exp(-e^{-y_{\varepsilon l}}) = F(y_{\varepsilon l}) \equiv \Phi(\eta_{\varepsilon l}) \end{aligned}\right\} \qquad (2.9)$$

Table 2.2 Reduced singular extremes $y_\varepsilon$ for given upper singular levels $\varepsilon$.

| $N$-1 | upper singular level $\varepsilon$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 25% | 12.5% | 5% | 2.5% | 1.25% | 0.5% | 0.25% | 0.05% |
| 20 | 1.314 | 2.179 | 3.328 | 4.237 | 5.200 | 6.557 | 7.666 | 10.5 |
| 22 | 308 | 163 | 294 | 182 | 113 | 425 | 483 | 10.2 |
| 24 | 303 | 150 | 265 | 137 | 047 | 319 | 342 | 9.94 |
| 26 | 298 | 140 | 241 | 098 | 4.991 | 230 | 224 | 72 |
| 28 | 294 | 130 | 220 | 067 | 943 | 155 | 124 | 56 |
| 30 | 291 | 122 | 203 | 039 | 903 | 091 | 039 | 40 |
| 32 | 1.289 | 2.116 | 3.188 | 4.015 | 4.867 | 6.038 | 6.969 | 9.27 |
| 34 | 286 | 110 | 175 | 3.994 | 836 | 5.991 | 904 | 16 |
| 36 | 284 | 104 | 163 | 976 | 808 | 950 | 849 | 08 |
| 38 | 282 | 099 | 153 | 959 | 785 | 913 | 799 | 01 |
| 40 | 280 | 095 | 143 | 944 | 763 | 880 | 755 | 8.94 |
| 42 | 1.278 | 2.091 | 3.135 | 3.931 | 4.742 | 5.850 | 6.715 | 8.86 |
| 44 | 277 | 087 | 127 | 919 | 726 | 824 | 680 | 80 |
| 46 | 275 | 084 | 120 | 908 | 711 | 801 | 650 | 74 |
| 48 | 274 | 081 | 113 | 898 | 695 | 778 | 620 | 68 |
| 50 | 273 | 079 | 108 | 889 | 682 | 756 | 593 | 63 |
| 52 | 1.272 | 2.076 | 3.103 | 3.879 | 4.670 | 5.740 | 6.568 | 8.58 |
| 54 | 271 | 073 | 098 | 873 | 658 | 721 | 544 | 54 |
| 56 | 270 | 071 | 093 | 865 | 647 | 706 | 523 | 50 |
| 58 | 269 | 069 | 088 | 859 | 638 | 690 | 503 | 46 |
| 60 | 268 | 066 | 084 | 852 | 628 | 676 | 486 | 43 |
| 65 | 1.267 | 2.063 | 3.076 | 3.839 | 4.610 | 5.645 | 6.443 | 8.36 |
| 70 | 265 | 060 | 068 | 827 | 592 | 620 | 410 | 30 |
| 75 | 264 | 057 | 061 | 817 | 577 | 598 | 382 | 25 |
| 80 | 263 | 054 | 056 | 807 | 565 | 579 | 359 | 20 |

Table 2.3 Reduced singular extremes $y_\varepsilon$ for given lower singular levels $\varepsilon$.

| $N$-1 | lower singular level $\varepsilon$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 25% | 12.5% | 5% | 2.5% | 1.25% | 0.5% | 0.25% | 0.05% |
| 20 | −0.3688 | −0.8042 | −1.208 | −1.446 | −1.649 | −1.881 | −2.036 | −2.350 |
| 22 | 647 | −0.7971 | 197 | 433 | 632 | 860 | 012 | 320 |
| 24 | 615 | 921 | 189 | 422 | 619 | 844 | −1.993 | 298 |
| 26 | 589 | 875 | 182 | 413 | 608 | 830 | 977 | 277 |
| 28 | 567 | 835 | 176 | 405 | 599 | 817 | 964 | 257 |
| 30 | 547 | 801 | 171 | 398 | 590 | 807 | 952 | 240 |
| 32 | −0.3529 | −0.7772 | −1.166 | −1.392 | −1.583 | −1.798 | −1.941 | −2.227 |
| 34 | 513 | 746 | 162 | 387 | 577 | 791 | 932 | 214 |
| 36 | 499 | 722 | 158 | 382 | 571 | 784 | 924 | 204 |
| 38 | 487 | 701 | 155 | 378 | 566 | 778 | 917 | 195 |
| 40 | 476 | 682 | 152 | 375 | 562 | 772 | 910 | 187 |
| 42 | −0.3466 | −0.7665 | −1.149 | −1.371 | −1.557 | −1.767 | −1.905 | −2.178 |
| 44 | 457 | 649 | 147 | 369 | 554 | 762 | 899 | 171 |
| 46 | 449 | 633 | 145 | 366 | 551 | 758 | 895 | 165 |
| 48 | 441 | 621 | 143 | 363 | 548 | 754 | 890 | 159 |
| 50 | 434 | 610 | 141 | 361 | 545 | 751 | 886 | 153 |
| 52 | −0.3428 | −0.7598 | −1.139 | −1.358 | −1.542 | −1.747 | −1.882 | −2.148 |
| 54 | 422 | 587 | 138 | 357 | 540 | 744 | 879 | 144 |
| 56 | 416 | 577 | 136 | 355 | 537 | 741 | 876 | 139 |
| 58 | 410 | 568 | 135 | 353 | 535 | 739 | 873 | 136 |
| 60 | 406 | 560 | 134 | 351 | 533 | 736 | 870 | 132 |
| 65 | −0.3396 | −0.7542 | −1.131 | −1.348 | −1.529 | −1.731 | −1.863 | −2.124 |
| 70 | 387 | 527 | 129 | 345 | 525 | 727 | 858 | 117 |
| 75 | 379 | 514 | 127 | 342 | 522 | 723 | 854 | 111 |
| 80 | 371 | 502 | 125 | 340 | 519 | 719 | 850 | 105 |

where $$\Phi(\eta_{\varepsilon l}) = \frac{1}{\sqrt{2\pi} s_\eta} \int_{-\infty}^{\eta_{\varepsilon l}} \exp\left\{-\frac{(\eta-\bar{\eta})^2}{2s_\eta^2}\right\} d\eta$$

These values, $y_{\varepsilon u}$ and $y_{\varepsilon l}$, for various singular levels, $\varepsilon$, are tabulated as a function of sample size $N$ in Tables 2.2 and 2.3 for the practical facilities. It may be easily seen by the help of the values in Tables 2.1 and 2.2 that the value of $y_{\varepsilon u}$ for $T=1/\varepsilon$ approaches to the value of $y$ for $T=1/1-F(x)$ with increase of the sample size $N$.

It is needless to say that the expected singular hydrologic amount for the desired return period $T=1/\varepsilon$ is obtained from Eq. (2.7) by using $y_\varepsilon$ instead of $y$, that is,

$$\left.\begin{array}{ll} \text{for the first asymptote ;} & x_\varepsilon = u + (1/a) y_\varepsilon \\ \text{for the second asymptote ;} & \log(x_\varepsilon + b) = \log(u+b) + (1/a) y_\varepsilon \end{array}\right\} \quad (2.10)$$

The necessity for such an idea or the usefulness of such a proposal as discussed above in detail, may be easily understood even from several examples of application presented by Fig. 2.1 which is shown with the proposal in the next section, and by Fig. 1.8 which has already been shown in Part I.

## 4. Method of rejection test

In the frequency analysis of observed hydrologic data, the decision of adoption or rejection of the singular variate contained in a family of the data must be objectively made from a stochastic viewpoint, because it is the duty of the hydrologic statistics to offer the most likely information to hydraulic works.

Since the rejection test proposed in this section is based on the idea of binomial distributin, it is not necessarly new as an idea, but it may be recognized to be useful as a method.

Generally, the probability of an event that $r$ variates which are not smaller (larger) than $x_\varepsilon$ for the upper (lower) singular level $\varepsilon$ are at least contained in a family of observed data of size $N$ is given by

$$P\{np(x \geqq x_{\varepsilon u}) \geqq r \text{ or } np(x \leqq x_{\varepsilon l}) \geqq r\} = 1 - \sum_{j=0}^{r} \frac{\Gamma(N+1)}{\Gamma(i)\Gamma(N-j+1)} (1-\varepsilon)^{N-j} \varepsilon^j \quad (2.11)$$

If this probability is smaller than a certain level $\beta_0$ of significance, the danger of rejecting such variates under the hypothesis that the event is very rare may be said to be 100 $\beta_0$ % at most.

Up to now, if the singular variates in a family of observed data of small size $N$ are two or more in number in one tail, it is not desirable to treat them simply in the usual sense of statistics, because a certain physical reason may exist in this event. Therefore, the singular variate to be made the object of rejection test should be either a set of two extreme data in both tails or an extreme datum in one tail. This concept of rejection test seems to be inevitable from the viewpoint that the treatment or the evaluation of singular value itself is to be of importance in the hydrologic statistics.

Thereupon, putting $r=1$, Eq. (2.11) becomes,

$$\beta=1-(1-\varepsilon)^N \tag{2.12}$$

Table 2.4 Singular levels $\varepsilon_0$ for given levels of significance, $\beta_0$.

| $N$ \ $\beta_0$ | 10% | 5% | 1% |
|---|---|---|---|
| | % | % | % |
| 18 | 0.584 | 0.285 | 0.056 |
| 20 | 525 | 256 | 049 |
| 22 | 478 | 233 | 046 |
| 24 | 438 | 214 | 042 |
| 26 | 404 | 197 | 039 |
| 28 | 376 | 183 | 036 |
| 30 | 351 | 171 | 034 |
| 32 | 0.329 | 0.160 | 0.032 |
| 34 | 309 | 151 | 030 |
| 36 | 292 | 142 | 028 |
| 38 | 276 | 135 | 027 |
| 40 | 263 | 128 | 025 |
| 42 | 0.251 | 0.122 | 0.024 |
| 44 | 239 | 117 | 023 |
| 46 | 229 | 111 | 022 |
| 48 | 219 | 107 | 021 |
| 50 | 210 | 103 | 020 |
| 55 | 0.191 | 0.093 | 0.018 |
| 60 | 175 | 085 | 017 |
| 65 | 162 | 079 | 016 |
| 70 | 150 | 073 | 014 |
| 75 | 140 | 068 | 013 |
| 80 | 132 | 064 | 013 |

Table 2.5 Examples of rejection test.

| Station | Period of Observation | Sample Size $N$ | Observed Value in Question $x_\varepsilon'$ (mm/day) | Equation for Rejection Test | Rejection Test | | | Equation for Estimates |
|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon$ (%) | $\beta$ (%) | Decision | |
| Wakayama | 1911—1955 | 45 | 394 | $x_\varepsilon=96+29.7y_\varepsilon$ | <0.05 | $\leqq 5$ | Rej. | $x_\varepsilon=96+29.7y_\varepsilon$ |
| Ueno | 1901—1953 | 53 | 287 | $\log(x_\varepsilon+513)=\log 590+0.020y_\varepsilon$ | 0.27 | >10 | Adop. | $\log(x_\varepsilon+67)=\log 144+0.0783y_\varepsilon$ |
| Kyoto | 1911—1950 | 40 | 282 | $x_\varepsilon=85+21.4y_\varepsilon$ | 0.055 | $\leqq 5$ | Rej. | $x_\varepsilon=85+21.4y_\varepsilon$ |
| Mineyama | 1911—1955 | 44 | 308 | $\log(x_\varepsilon+635)=\log 704+0.0162y_\varepsilon$ | 0.13 | > 5 | Adop. | $\log(x_\varepsilon-5)=\log 660+0.1303y_\varepsilon$ |
| Maizuru | 1911—1955 | 42 | 324 | $\log x=\log 81+0.0964y_\varepsilon$ | 0.35 | >10 | Adop. | $\log(x_\varepsilon-29)=\log 54+0.1433y_\varepsilon$ |

If $\beta$ is smaller than a given significant level $\beta_0$, the variate for the singular level $x_\varepsilon$ may be rejected. This is represented by,

$$\varepsilon<\varepsilon_0=1-(1-\beta_0)^{1/N} \qquad (2.31)$$

That is, Eq. (2.13) indicates that when the singular level $\varepsilon$ of variate is smaller than $\varepsilon_0$, such a variate is rejected at the $\beta_0$ level of significance. The relations between $\varepsilon_0$ and $N$ for several values of $\beta_0$ are shown in Table 2.4 for practical facilities.

In practice, the singular level of variate in question may be evaluated from all of the other data in which such a variate is not included, and as the value of the level of significance $\beta_0$, 5% may be usually adopted.

Several examples of application of those proposed approaches in hydrologic frequency analysis are shown in Fig. 2.1, basing on the data in Table 2.5.

## 5. Conclusion

In this part, several discussions on the singular data included in a actual sample are made from a stochastic viewpoint. After definition of the singular value, a practical method of evaluation of such a value was successfully developed by help of the theory of stochastic limit on normals, and also a course of the rejection test for such a data was defined practically by the use of the binomial distribution.

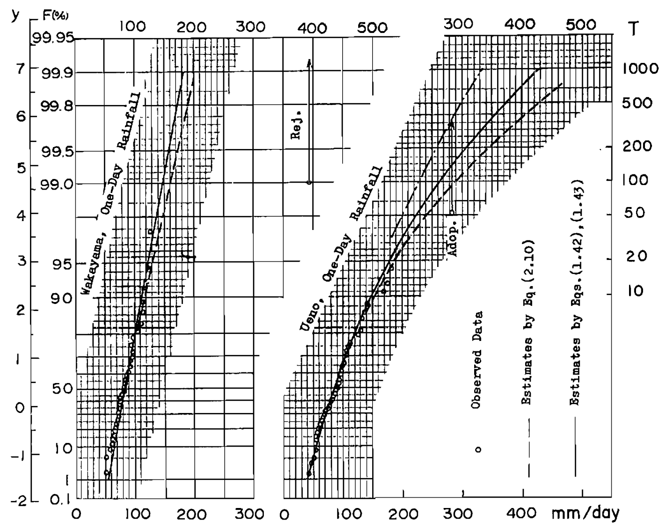


Fig. 2.1-(1) Examples of rejection test and of evaluation of singular value, (1).

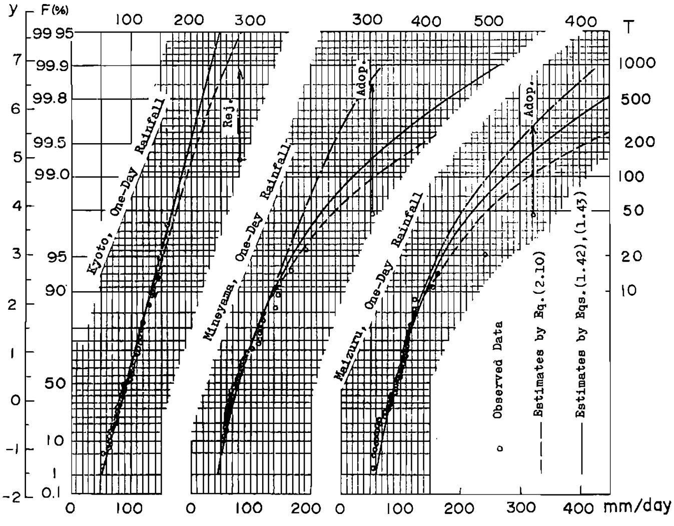


Fig. 2.2-(2) Examples of rejection test and of evaluation of singular value, (2).

The author believes that these treatments of the singular data are strongly desirable, and that the results obtained in this part must provide a useful tool in the field of hydrologic frequency analysis, although these studies were performed in 1959.

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