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On the Resonance Effect in a Storm Surge (Part II)

By Hikoji YAMADA and Jun-ichi OKABE[1)]

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Abstract

In Part II of this paper it is shown that, provided some errors be overlooked, effects of meteorological factors can be expressed solely in terms of an atmospheric pressure gradient when a storm surge computation is performed. It is also remarked at the same time that the theoretical equations derived here, taking account of eddy viscosity, coincide approximately with the empirical formulas which have come into general use very recently. By means of our computational scheme, influence of eddy viscosity upon the resonant intensification of waves is studied, and it is concluded, after some investigations and calculations, that generally it is small and in some cases small enough to be safely neglected. Next, the disturbance which has been assumed ever since Part I is too small in its spatial extent to stand for a larger example such as a typhoon. Therefore a resonant high water caused by a depression whose dimension is large compared with the linear scale of a sloping bottom is computed under similar conditions, but no sensible modification is observed provided the present fundamental assumptions are employed. Finally, since it is formidable in a practical computation of a storm surge to renew meteorological data at every step of the calculation, intermittent supply of the data is performed tentatively. With a view to obtaining a general idea of the situation, the high water treated in our problem is computed again by furnishing data at intervals of certain steps. The fact that no marked difference is resulted gives us not only an understanding of the significance of resonance phenomenon itself but also a suggestion that sufficient reliability may be obtained unless too long an interval is chosen.

6. General considerations on equations of motion

Since our problem is confined to one-dimensional water surface, according to the wave theory of shallow water the equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{33}$$

$$p=\rho g(P+\eta-y), \tag{33'}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{33''}$$

are employed as the fundamental laws governing the motion of sea water. But, since it is impossible to treat these equations as they are, we shall take the usual way of simplification by substituting their means over y, the coordinate measured in the upward direction, for their original forms. Meanwhile some considerations become necessary with the boundary layers on the bottom and on the surface of water. In the case, however, when the main stream varies with time or vertical distance from the surface, to distinguish the boundary layer from the main flow would be difficult, sometimes even illegi-

1) Professor, Research Institute for Applied Mechanics, Kyushu University.

timate. But, now that we cannot proceed further without taking some step for it, we shall assume for convenience's sake the following distinction : supposing that the main stream is given by a function, $u(x, y ; t)$ say, changing gradually with respect to time and space, the boundary layer will be defined as the region in the flow where the function mentioned above deviates noticeably from the existing velocity as we approach the bottom or the surface. If the boundary layer may be regarded as sufficiently thin and negligible in comparison to the depth of water, then consequently the slip of the main stream is necessarily needed on the bottom as well as on the surface of the water. Let us begin our computations with this flow picture as the basis.

Turning now to T_s, the traction exerted by wind on the surface of water, we may assume it is determined solely by V_a, the wind velocity at a standard height, this supposition being supported by the fact that the wind velocity is large compared with the water velocity in a higher order and that therefore there is no need to take into account the boundary layers of air and of water formed on their interface. Thus we may put

$$\mu\left(\frac{\partial u}{\partial y}\right)_{y=\eta} \equiv T_s = k\rho_a V_a^2 \quad (k \simeq 0.0024), \tag{34}$$

where ρ_a means the density of air. The real problem lies in estimation of T_b, the frictional force on the bottom of the water. Adopting the picture of the bottom slip stated above, together with the law of squared velocity for the turbulent friction, we may write

$$\mu\left(\frac{\partial u}{\partial y}\right)_{y=-h} \equiv T_b = k'\rho|u_b|u_b \; ; \; u_b \equiv u\,(x, -h \,; t). \tag{35}$$

For the sake of easiness of subsequent treatments, however, we shall resort to an expedient of substituting temporalily $|u_b|$ with v_0, a kind of its mean value with regard to both space and time :

$$T_b = k'\rho v_0 u_b. \tag{35'}$$

Concerning $u(x, y ; t)$, on the other hand, we shall assume it in the form of a polynomial of the second degree, a parabolic distribution in other words, against y, the vertical distance (**9**). Evidently we should be able to grasp the main point of things by means of this simple approximation even if there be a backward stream at the bottom.[1] If we express three indeterminate coefficients with the aid of conditions (34) and (35) together with the average velocity defined by

$$\bar{u} = \frac{1}{h+\eta}\int_{-h}^{\eta} u\,dy,^{2)} \tag{36}$$

then the distribution of velocity may be written

1) We shall have another chance to discuss this problem in more detail.

2) In what follows, a bar placed above a letter indicates that the mean value with regard to y is to be taken.

$$\left.\begin{aligned} u&=\bar{u}+\frac{h'T_s}{2\mu}-\frac{h'(T_s-T_b)}{6\mu}+\frac{h'T_s}{\mu}\left(\frac{y'}{h'}\right)+\frac{h'(T_s-T_b)}{2\mu}\left(\frac{y'}{h'}\right)^2, \\ \text{where}\quad & h'=h+\eta,\ \text{and}\ y'=y-\eta. \end{aligned}\right\} \tag{37}$$

Substituting this value of u at the bottom $y=-h$ for u_b in (35′), we obtain

$$T_b=k'\rho v_0\left(\bar{u}-\frac{h'T_s}{6\mu}-\frac{h'T_b}{3\mu}\right),$$

or

$$T_b=\frac{\lambda}{1+\lambda}k'\rho v_0\bar{u}-\frac{1}{2+2\lambda}T_s, \tag{38}$$

where

$$\lambda=\frac{3\nu}{k'v_0h'}. \tag{39}$$

Further, introducing here the third constant k'', we put

$$k'\rho v_0=k'\rho|u_b|\equiv k''\rho|\bar{u}|\ ; \tag{40}$$

validity of this approximation might be assured only for the case when the flow is almost always in one direction throughout its depth. Consequently, the bottom friction has the form

$$T_b=-mT_s+s\rho|\bar{u}|\bar{u}, \tag{41}$$

where

$$m=(2+2\lambda)^{-1},\ \text{and}\ s=\lambda(1+\lambda)^{-1}k''. \tag{41′}$$

Expression (41) for T_b has just the same form as given recently by GROEN and GROVES (10). The coefficients m and s can be evaluated in the following way when the stream is in one direction without backward flow at the bottom : by making use of BOWDEN's formula (11) we have

$$\nu=0.0025|u_b|_{max}h\gtrsim 0.0025v_0h.$$

So that from (39) we get $\lambda\gtrsim 3.0$, since $k'=0.0025$. It follows that

$$m\lesssim 0.125,\ \text{and}\ s\gtrsim 0.75k''\simeq 0.002.$$

These values are found compatible exactly with those mentioned by GROEN and GROVES as appropriate for the North Sea :

$$m\lesssim 0.1,\ \text{and}\ s|\bar{u}|\simeq 0.2\text{cm/sec},$$

if we note $|\bar{u}|$ is of the order of 1m/sec there. Legitimacy of formula (41) for the bottom friction, therefore, has been established.

Now, to take the averages of the equations in regard to the depth, first integrate (33″) from $-h$ to η. If we consider the relations

$$v_b=-u_b\frac{\partial h}{\partial x}$$

on the bottom, and

$$v_s=\frac{\partial\eta}{\partial t}+u_s\frac{\partial\eta}{\partial x}$$

on the surface respectively, where

$$v_b=v(x,-h\ ;\ t),$$

$$u_s=u(x,\ \eta\ ;\ t),\quad \text{and}\quad v_s=v(x,\ \eta\ ;\ t),$$

then we are led to

$$\frac{\partial\eta}{\partial t}=-\frac{\partial}{\partial x}\{\bar{u}(h+\eta)\}, \tag{42}$$

the so-called equation of continuity. Next, after substituting for p in equation (33) the expression given by (33′), integrate the both-hand sides with respect to y, then the left-hand side (LHS) is found to be

$$\text{LHS}=\frac{\partial}{\partial t}\{\bar{u}(h+\eta)\}-u_s\frac{\partial\eta}{\partial t}+u_s\left(\frac{\partial\eta}{\partial t}+u_s\frac{\partial\eta}{\partial x}\right)$$

$$-u_bv_b+\frac{\partial}{\partial x}\{\overline{u^2}\ (h+\eta)\}-u_s{}^2\frac{\partial\eta}{\partial x}+u_bv_b,$$

or

$$\text{LHS}=\frac{\partial}{\partial t}\{\bar{u}(h+\eta)\}+\frac{\partial}{\partial x}\{\overline{u^2}(h+\eta)\}\ ;$$

in carrying out the above computation the following relations were employed :

$$\int\frac{\partial u}{\partial t}\mathrm{d}y=\frac{\partial}{\partial t}\{\bar{u}(h+\eta)\}-u_s\frac{\partial\eta}{\partial t},$$

$$\int u\frac{\partial u}{\partial x}\mathrm{d}y=\frac{1}{2}\left[\frac{\partial}{\partial x}\{\overline{u^2}(h+\eta)\}-u_s{}^2\frac{\partial\eta}{\partial x}+u_bv_b\right],$$

and

$$\int v\frac{\partial u}{\partial y}\mathrm{d}y=u_sv_s-u_bv_b+\int u\frac{\partial u}{\partial x}\mathrm{d}y.$$

On the other hand, making use of (34) and (41), the right-hand side (RHS) becomes

$$\text{RHS}=-g(h+\eta)\left(\frac{\partial P}{\partial x}+\frac{\partial\eta}{\partial x}\right)+(1+m)\frac{T_s}{\rho}-s|\bar{u}|\bar{u}.$$

Expressing the equation (33) in its 'averaged' form instead of the 'integrated' form, we obtain, therefore,

$$\frac{1}{h+\eta}\left[\frac{\partial}{\partial t}\{\bar{u}(h+\eta)\}+\frac{\partial}{\partial x}\{\overline{u^2}(h+\eta)\}\right]$$

$$=-g\frac{\partial P}{\partial x}-g\frac{\partial\eta}{\partial x}+\frac{(1+m)T_s}{\rho(h+\eta)}-\frac{s|\bar{u}|\bar{u}}{h+\eta}. \tag{43}$$

With a view to reducing equation (43) into a form more familiar to us, let us introduce a function $q\ (x, t)$ such as

$$\overline{u^2}=q\bar{u}. \tag{44}$$

Then it follows from (43) that

$$\frac{\partial\bar{u}}{\partial t}+q\frac{\partial\bar{u}}{\partial x}+\frac{\bar{u}}{h+\eta}\left[\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial x}\{q(h+\eta)\}\right]$$

$$=-g\frac{\partial P}{\partial x}-g\frac{\partial\eta}{\partial x}+\frac{(1+m)T_s}{\rho(h+\eta)}-\frac{s|\bar{u}|\bar{u}}{h+\eta}, \tag{43'}$$

and if ignoring some errors we might put q as equal to $\bar{u}$ in the third term on the left-hand side, then this is found to vanish by virtue of (42). We have namely

$$\frac{\partial \bar{u}}{\partial t}+q\frac{\partial \bar{u}}{\partial x}=-g\frac{\partial P}{\partial x}-g\frac{\partial \eta}{\partial x}+\frac{(1+m)T_s}{\rho(h+\eta)}-\frac{s|\bar{u}|\bar{u}}{h+\eta}. \tag{45}$$

Equations (42) and (45) constitute the system of our basic equations. Furthermore considering that what is derived out of (45) by replacing q with $\bar{u}$ is nothing but the 'fundamental equation' usually adopted (**10**), we should say that this can be valid only if $\overline{u^2}$ might be approximated by $\bar{u}^2$, or in other words, if the stream be predominantly one-directional throughout the depth at a fixed value of x. On the basis of this assumption, we also rewrite the fundamental equation (45) in the form

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}=-g\frac{\partial P}{\partial x}-g\frac{\partial \eta}{\partial x}+\frac{(1+m)T_s}{\rho(h+\eta)}-\frac{s|\bar{u}|\bar{u}}{h+\eta}. \tag{45'}$$

In the third term on the right-hand side of this equation, if we neglect η in the denominator against h, then the term becomes a known function of x and t, and accordingly by introducing a known function defined as

$$Q(x,t)\equiv P-\frac{1+m}{\rho g}\int^x\frac{T_s}{h(x)}\mathrm{d}x, \tag{46}$$

(45′) can be transformed into

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}=-g\frac{\partial \eta}{\partial x}-g\frac{\partial Q}{\partial x}-\frac{s|\bar{u}|\bar{u}}{h+\eta}\ ; \tag{47}$$

this would reduce to equation (1) of Part I, provided we substituted Q by P and neglected the last term corresponding to the bottom friction. From this we know that the effect of wind traction can be evaluated in terms of that of pressure gradient. Only, in the case when a stationary disturbance advances with a uniform speed, does inconvenience in treatment arise from the fact that Q remains still non-stationary owing to $h(x)$ involved in the denominator of T_s. However, when the disturbance itself is not stationary, this property does not constitute an additional difficulty, and the formulation (47) is expected very useful for calculations of storm surges. When the variation of h is small enough, or merely qualitative result is aimed at, we may approximate h with h_0, the average value over the region in question; thus representing the disturbance by the function

$$Q_0(x,t)\equiv P-\frac{1+m}{\rho g h_0}\int^x T_s\mathrm{d}x, \tag{46'}$$

the problem is simplified to some extent.

7. Effect of bottom friction (1)

Let us first investigate the influence of bottom friction upon the stationary attendant tidal wave in the area of uniform depth h_1. Reducing concerning quantities into non-dimensional forms in the same manner as in Section 2, we put

$$\frac{\bar{u}}{V}=u_0,\ \frac{x}{L_1}=x_0,\ \frac{Vt}{L_1}=t_0,\ \frac{\eta}{h_1}=\eta_1,\ \text{and}\ \frac{Q}{h_1}=q_1, \tag{48}$$

then (47) and (42) become

$$\left.\begin{aligned}\frac{\partial u_0}{\partial t_0}+u_0\frac{\partial u_0}{\partial x_0}&=-\frac{1}{m^2}\frac{\partial}{\partial x_0}(\eta_1+q_1)-\frac{L_1 s}{h_1}\frac{|u_0|u_0}{1+\eta_1},\\ \text{and}\qquad \frac{\partial \eta_1}{\partial t_0}&=-\frac{\partial}{\partial x_0}\{u_0(1+\eta_1)\}\end{aligned}\right\} \tag{49}$$

respectively, where

$$m=V/\sqrt{gh_1}=\sqrt{h_0/h_1}$$

is the parameter defined in equation (12) of Part I. Now, if we assume that

$$Q=Q\left(\frac{x-Vt}{L_1}\right),$$

namely

$$q_1=q_1(x_0-t_0),$$

then the wave in question will be found as a solution of

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{u_0-\frac{1}{2}u_0{}^2-\frac{1}{m^2}(\eta_1+q_1)\right\}&=\frac{L_1 s}{h_1}\frac{|u_0|u_0}{1+\eta_1},\\ \text{and}\qquad \eta_1&=u_0(1+\eta_1),\end{aligned}\right\} \tag{50}$$

the non-linear ordinary differential equation of the first order, where we write $x_0-t_0\equiv\xi$ for short.

In order to solve this equation, we need to know the boundary conditions. Evidently they are $u_0=0$, accordingly $\eta_1=0$, as $\xi\to\pm\infty$ (the region where $q_1\equiv 0$ holds true). To begin with, from (50) the equation governing the tidal level is known to be

$$\{m^2-(1+\eta_1)^3\}\frac{\mathrm{d}\eta_1}{\mathrm{d}\xi}=(1+\eta_1)^3\frac{\mathrm{d}q_1}{\mathrm{d}\xi}+\frac{L_1 s}{h_1}m^2|\eta_1|\eta_1, \tag{51}$$

and putting $q_1\equiv 0$ we have

$$\{m^2-(1+\eta_1)^3\}\frac{\mathrm{d}\eta_1}{\mathrm{d}\xi}=\frac{L_1 s}{h_1}m^2|\eta_1|\eta_1. \tag{51'}$$

Let us scrutinize the property of this equation in the two cases which follow. First, when $\eta_1>0$, (51′) is readily integrated into the form

$$\frac{L_1 s}{h_1}m^2\xi=\frac{1-m^2}{\eta_1}-3\log|\eta_1|-3\eta_1-\frac{1}{2}\eta_1{}^2+\text{const.}, \tag{52}$$

and so $\eta_1\to+0$,

as $\xi\to+\infty$ when $1\geqq m^2$,

and as $\xi\to-\infty$ when $m^2>1$.

Secondly, since the integrated form for the case $\eta_1<0$ is obtained simply by changing the sign of the left-hand side of (52), we know that $\eta_1\to-0$

as $\xi \to +\infty$ when $1 > m^2$,

and as $\xi \to -\infty$ when $m^2 \geqq 1$.

Combining these two cases, we may conclude :— It is the case when $1 \gtrapprox m^2$ that there can exist a tide extending forward to an infinite distance and vanishing there (a so-called *preceding* tide), and on the contrary when $m^2 \gtrapprox 1$ that there can exist a tide extending backward to an infinite distance and vanishing there (a so-called *receding* tide).

Then, in order to have a tidal level vanishing at an infinite distance in front and in the rear of the disturbance for the cases $m^2 \geqq 1$ and $1 \geqq m^2$ respectively, the tide should be necessarily null ($\eta_1 = 0$, accordingly $u_0 = 0$) in those regions; this is in fact a particular solution of equation (51′). In other words, when $1 \gtrapprox m^2$, at the rear end of the distribution of disturbance η_1 becomes zero and the water is calm in the background area, while on the other hand when $m^2 \gtrapprox 1$ η_1 vanishes at the forward end of the q_1-distribution and in front of it a domain is left free from all influences of disturbance (12). In this way we can clarify the pertinent boundary conditions when integrating equation (51) for the purpose of computing the tidal wave throughout the whole region. To sum up, we have to put $\eta_1 = 0$ at the rearmost and the foremost ends of q_1 for the cases $1 \gtrapprox m^2$ and $m^2 \gtrapprox 1$ respectively.

It is just the same as we saw in Section 3 of Part I that equation (51) has two solutions when $m^2 = 1$, and there is no question about it. At present, however, a difficulty arises in its neighborhood. Namely as is evident from the equation transformed from (51), namely

$$\frac{d\eta_1}{d\xi} = \left\{(1+\eta_1)^3 \frac{dq_1}{d\xi} + m^2 \frac{L_1 s}{h_1} |\eta_1|\eta_1\right\} \Big/ \{m^2 - (1+\eta_1)^3\} \equiv N/D, \tag{53}$$

D, the denominator of the right-hand side, vanishes at a point when m^2 approaches unity beyond a certain limit, and the position is found where the gradient of surface elevation increases or decreases indefinitely. Since the sign of N, the numerator, is kept generally invariable in a small domain including that point, $d\eta_1/d\xi$ jumps abruptly from $-\infty$ up to $+\infty$ (or else from $+\infty$ down to $-\infty$) and we cannot elongate the water surface continuously in the forward (or backward) direction. With a view to studying the state of affairs in more detail, for the cases $m \lessapprox 1$ various phases of the integral curves $\eta_1(\xi)$ are sketched in FIGURES 5 taking into consideration the sign of $\frac{d\eta_1}{d\xi}$ (or N/D) in the neighborhood of the forward end of the q_1-distribution, which is assumed in the form

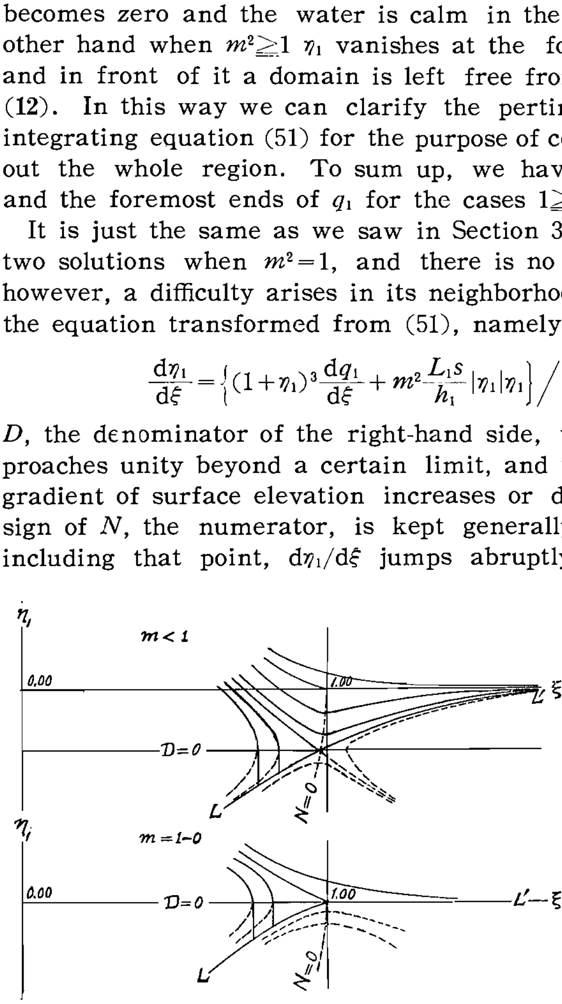

Fig. 5. Possible forms of attendant water surface at the forward end of the disturbance ; $m \lessapprox 1$.

$$q_1=\frac{Q}{h_1}=\begin{cases}-\dfrac{P_0}{h_1}\{1+\cos(\pi\xi)\}, & (|\xi|<1),\\ 0, & \text{otherwise},\end{cases} \tag{54}$$

the same expression as P in equation (15) of Part I.

What is most important for us is the position, $(\xi_0,\ \eta_1{}^0)$ say, where not only the denominator D but also the numerator N vanishes at the same time. In order to know the orientation of the curve $\eta_1(\xi)$, the limiting value of the right-hand side of (53) at that point is worked out. After some simple manipulations we obtain

$$\left(\frac{d\eta_1}{d\xi}\right)_0^2+2\left\{\frac{1}{2}\left(\frac{dq_1}{d\xi}\right)_0-\frac{1}{3}\frac{L_1s}{h_1}(1+\eta_1{}^0)\eta_1{}^0\right\}\left(\frac{d\eta_1}{d\xi}\right)_0+(1+\eta_1{}^0)\left(\frac{d^2q_1}{d\xi^2}\right)_0=0.$$

Thus with the aid of the relations

$$(dq_1/d\xi)_0>0, \qquad (d^2q_1/d\xi^2)_0<0,$$
$$1+\eta_1{}^0>0, \quad \text{and} \quad \eta_1{}^0<0,$$

two branches of $\eta_1(\xi)$ passing through the point $(\xi_0,\ \eta_1{}^0)$ are found : for one of them we have $(d\eta_1/d\xi)_0<0$, and for the other, $(d\eta_1/d\xi)_0>0$. The water surface generated by extending the latter forwards and backwards indefinitely (the curve LL′ in the figure), we shall call it the limiting surface, for it is the lowest water surface possible. Some of the typical surfaces realized in practice are shown in full lines. Of these, concerning the surface which intersects vertically the horizontal line denoted as $D=0$, it is assumed, in order to ensure continuity of that surface in the forward region, that it is jointed with the limiting surface by means of a vertical surface as shown in the figure. Since, as stated before, vertical acceleration is ignored in our equations of motion, a surface having a vertical tangent at a point must be mentioned as lying beyond the scope of our approximation, and accordingly, all we can do is to connect through that point more or less arbitrarily, surfaces situated in front of it with those in the rear. Behaviors of surfaces in the neighborhood of that point have to be examined in detail by more elaborate treatment ; at any rate we may expect that a bore-like surface is generated near the point.

When $m=1-0$, the axis $\eta_1=0$ coincides with the line $D=0$. Change in the pattern is therefore brought about more or less as is shown in the figure. On the other hand, when $m\geqq 1$, a similar problem arises near the rear end of the disturbance $q_1(\xi)$, but we can verify without difficulty that the general behavior of surfaces may be visualized by the figures produced from both of FIGURES 5 by inverting them upside down, interchanging right and left at the same time. In both of these cases, the bore-like jump through which the level ascends steeply takes place at a point where the condition

$$m^2=(1+\eta_1)^3, \quad \text{or} \quad \frac{V^2}{g(h_1+\eta)}=\left(\frac{h_1+\eta}{h_1}\right)^2 \tag{55}$$

is satisfied.

In order to estimate the effect of friction in practice, let us first calculate an attendant swell appearing in the case $m=1-0$. Substituting (54) for q_1 in (53), and assuming the same constant values as given already by (16) and (16′) of Part I, the surface elevation of water is computed through numerical integration. The result is shown in FIGURE 6 ; for ready comparison the curve is also drawn for the case when viscosity is negligible. In both of them, h_1 is replaced by $h_0=24$m and the following values are employed :

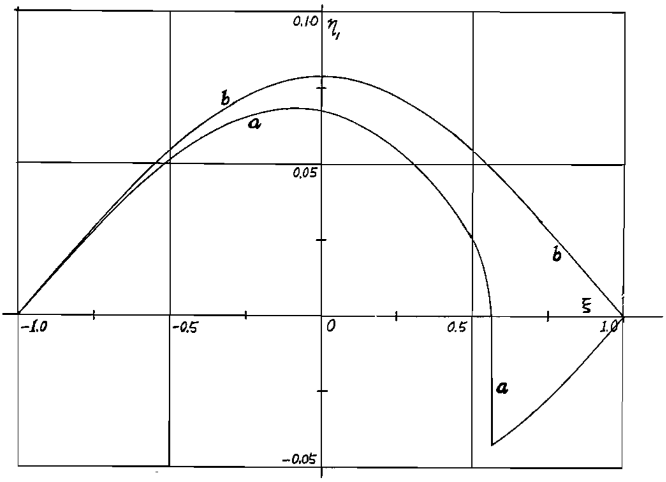

Fig. 6. Resonance of high water (a) with and (b) without friction ; $m=1-0$.

$$\frac{\pi P_0}{h_0}=0.013090, \quad \text{and} \quad \frac{L_1 s}{h_0}=1.0000 \ ;$$

s is assumed to be 0.0024, the value not so far from those accepted in general. From this result we may say that the surface elevation is well reproduced by the curve yielded from the assumption of an inviscid fluid except the neighborhood of the bore, where the approximation is found less satisfactory. Besides considering that $m=1$ is one of those cases in which the effect of friction appears most conspicuously, the above comparison suggests to us that it would be quite small altogether. TABLE 3 shows another example in which

TABLE 3.

η computed from nonlinear equations without (0) and with (x) friction ; $m^2=0{,}6$.

X	0	x	X	0	x
−10km	0.000m	0.000m	1km	0.475m	0.473m
− 9	012	012	2	442	439
− 8	048	048	3	388	385
− 7	102	102	4	322	318
− 6	171	171	5	246	243
− 5	246	246	6	171	167
− 4	322	321	7	102	098
− 3	388	388	8	048	044
− 2	442	440	9	012	+0.008
− 1	475	474	10	0.000	−0.004
− 0	0.487	0.485			

$$h_1=40\text{m}, \quad m^2=0.6, \quad \text{and} \quad m^2L_1s/h_1=0.375.$$

The difference between the cases with and without viscosity is found reasonably small as previously expected. What is more important in the bottom friction features is a low or a high water of small elevation which appears in front ($m<1$) or in the rear ($m>1$) of the disturbance respectively, extending however to a great distance. For example, in the case of $m^2=0.6$ abovementioned, if the boundary value at $\xi=1.0$:

$$\eta_1 \simeq \frac{-0.004}{40} = -0.0001$$

is introduced in (52) in order to determine the constant involved we obtain accordingly the relation

$$-0.375\,\xi \doteqdot \frac{0.4}{\eta_1}+4000\;; \tag{56}$$

from this we know that the distance through which η_1 decreases to one half or -0.00005 is $\xi \simeq 10^4$ or about a hundred thousand kilometers in the forward direction. If larger η_1 be chosen, we obtain a lesser distance, but still it would not be unusual to find a distance of several thousand kilometers. If, therefore, two-dimensional extent of sea is confined by some reason or another against a disturbance whose intensity and scale are large enough at the same time, then we should expect with certainty that the influence of the disturbance may be transferred to a greater distance : this might be significant towards understanding the physics of typhoons.

8. Effect of bottom friction (2)

Since we saw in the preceding section that the influence of friction upon a surge would not be so large, we shall study in a simple way its effect upon the resonance phenomenon by introducing linearized frictional force into the system of our linearized equations. Ignoring the inertia terms and approximating $h+\eta$, the depth, by h, and $s|\bar{u}|$, the friction term, by r, a constant velocity, (42) and (47) become

$$\left.\begin{aligned} \frac{\partial \bar{u}}{\partial t} &= -g\frac{\partial \eta}{\partial x} - g\frac{\partial Q}{\partial x} - \frac{r}{h}\bar{u}, \\ \text{and} \qquad \frac{\partial \eta}{\partial t} &= -\frac{\partial}{\partial x}(h\bar{u}) \end{aligned}\right\} \tag{57}$$

respectively ; this is the system of the fundamental linearized equations governing a surge of a dissipative medium.

By putting $Q=Q(X)$, the stationary attendant wave is first computed. Employing the notations defined previously :

$$\frac{\bar{u}}{V} \equiv u_0, \quad \frac{\eta}{h_1} \equiv \eta_1, \quad \frac{Q}{h_1} \equiv q_1\;; \quad x-Vt \equiv X\;; \quad \frac{V^2}{gh_1} \equiv m^2,$$

equations (57) may be written

$$\left.\begin{aligned} &(1-m^2)\frac{d\eta_1}{dX}+\frac{m^2 r}{h_1 V}\eta_1=-\frac{dq_1}{dX}, \\ \text{and}\qquad &\eta_1=u_0, \end{aligned}\right\} \tag{58}$$

which can be integrated without difficulty into the form

$$\left.\begin{aligned} &\eta_1=e^{-\epsilon X}\left(\text{const.}-\frac{1}{1-m^2}\int_{-\infty}^{X}\frac{dq_1}{dX'}e^{\epsilon X'}dX'\right), \\ \text{where}\qquad &\epsilon=\frac{m^2}{1-m^2}\frac{r}{h_1 V}. \end{aligned}\right\} \tag{59}$$

When, therefore, $m<1$ namely $\epsilon>0$, we have

$$\eta_1=-\frac{1}{1-m^2}e^{-\epsilon X}\int_{-\infty}^{X}\frac{dq_1}{dX'}e^{\epsilon X'}dX', \tag{60}$$

and on the other hand when $m>1$ or $\epsilon<0$,

$$\eta_1=\frac{1}{1-m^2}e^{-\epsilon X}\int_{X}^{\infty}\frac{dq_1}{dX'}e^{\epsilon X'}dX'. \tag{61}$$

As $m\rightarrow 1\pm 0$, we are led to a finite solution

$$\eta_1=-\frac{h_1 V}{r}\frac{dq_1}{dX}, \tag{62}$$

but seeing that this is too crude to be looked upon as an approximate value, we should expect that both of (60) and (61) would also cease to be valid when m approaches unity beyond a certain limit.

Now assuming (54) for the expression of q_1 and also

$$P_0=0.10\text{m},\quad L_1=10\text{km},\quad \text{and}\quad V=15.3362\text{m/sec},$$

the same values as used previously, together with the constant $r=0.002$m/sec, we shall compute the high water for the two cases $h_1=40$m and $h_1=8$m. In the former we have $m^2=0.6$, and $\epsilon=4.891\times10^{-5}\text{km}^{-1}$, while in the latter $m^2=3.0$, and $\epsilon=-2.445\times10^{-2}\text{km}^{-1}$; integrations involved in (60) and (61) may be

TABLE 4.

η computed from linear equations without (0) and with (x) friction ; $m^2=0.6$ and 3.0.

X	$m^2=0.6$		$m^2=3.0$	
	0	x	0	x
−10km	0.000m	0.000m	0.0000m	+0.0192m
− 7.5	073	073	− 0146	+ 0055
− 5	250	248	− 0500	− 0306
− 2.5	427	420	− 0854	− 0690
0	500	488	− 1000	− 0886
2.5	427	409	− 0854	− 0792
5	250	228	− 0500	− 0478
7.5	073	050	− 0146	− 0143
10	0.000	−0.023	0.0000	0.0000

carried out in the domain $|X| \leqq L_1$, and we get
when $m^2=0.6$

$$\eta_1 = -0.001989\{a \sin(\pi\xi) - \pi\cos(\pi\xi) - \pi e^{-a(1+\xi)}\},$$

and when $m^2=3.0$

$$\eta_1 = 0.001977\{a \sin(\pi\xi) - \pi\cos(\pi\xi) - \pi e^{a(1-\xi)}\},$$

where $\xi = X/L_1$, and $a = \epsilon L_1$.

η's computed through both of these formulas are shown in TABLE 4, in contrast with the results from linear approximation for an inviscid fluid (see TABLE 2 of Part I). In TABLE 4 the column $m^2=0.6$ gives the linear approximation corresponding to TABLE 3. By comparing both of them with each other, it is noticed that the dissipation term contained in the linear theory is particularly too great. Since this results from the fact that the assumption $r=0.002$ m/sec was in reality too large, we shall estimate the appropriate value of $s|\bar{u}|$ inversely. From (58), $\bar{u}=V\eta/h_1 \leqq 0.2$ m/sec, and using $s \simeq 0.0025$ we obtain $s|\bar{u}| \leqq 0.0005$ m/sec. r defined as a kind of its average should be estimated therefore as of the order of 0.0004 m/sec at the most. Accordingly, we know that the value of r used by us was about five times as large as the correct one. Also for the case $m^2=3.0$, the situation is found to be completely similar. But, as already remarked, in cases of the resonance on a slope between them, the velocity of water can attain 0.8 m/sec at its maximum, so that it gives $s|\bar{u}| \simeq 0.002$ m/sec. Since our object in view is primarily to show that the influence of frictional force is quite small, we shall use this maximum value 0.002 m/sec throughout all cases. Furthermore, it is evident that in the case $m^2=0.6$ we may substitute for the attendant wave the tidal level taking place in the limit of vanishing viscosity and to adopt it as the initial condition for subsequent integrations.

Our next step will be the calculation of the resonance. Computation scheme arranged for the electronic computer is completely the same as already mentioned in Section 5 of Part I when we treated the linearized theory with vanishing viscosity. Using $\lambda = h\bar{u}/V$ in place of $\bar{u}$, and writing $h_0H(x)$ for $h(x)$, the depth, h_0 being 24m, we obtain

$$\left.\begin{aligned} &\frac{\partial\lambda}{\partial t} + \frac{r}{h_0}\frac{\lambda}{H} = -VH\left(\frac{\partial\eta}{\partial x} + G\right), \\ \text{and}\quad &\frac{\partial\eta}{\partial t} = -V\frac{\partial\lambda}{\partial x}, \end{aligned}\right\} \tag{63}$$

where $G = \partial Q/\partial x$. They can be expressed therefore in terms of differences, completely similar as in Section 5, and after all is done we arrive at the system of equations which may be derived simply by adding

$$\frac{0.002}{24}\Delta t\frac{\lambda_M}{H_M} = \frac{0.002}{24}\Delta t\frac{\lambda_L+\lambda_R}{2H_M} = 0.00045281\frac{\lambda_L+\lambda_R}{H_M}$$

to the right-hand side of the first equation of (32) of Part I. Thus, our difference equations are found to be

$$\left.\begin{aligned}\lambda_P &= \left(\frac{1}{2} - \frac{0.00045281}{H_M}\right)(\lambda_L + \lambda_R) \\ &\quad + \frac{1}{3} H_M \left\{\eta_L - \eta_R - 0.015708 < \sin\frac{\pi}{L_1}(x - Vt) >_M\right\}, \\ \text{and} \qquad \eta_P &= \frac{1}{2}(\eta_L + \eta_R) + \frac{1}{3}(\lambda_L - \lambda_R)\,;\end{aligned}\right\} \tag{64}$$

as the inital condition use is invariably made of the value $\eta = \lambda$ for an inviscid fluid (see TABLE 2).

This computation results in a series of curves in FIGURE 7A. Only in the neighborhood of their highest profile, are they compared with the results yielded through the linear theory under the assumption of an inviscid fluid, and the differences between them are observed to be quite small. In fact they are much smaller than those between the theories non-linear = inviscid and linear = inviscid, both of which are shown in the same figure. Out of it we may expect with certainty that by modifying the value of r, the constant for friction, which was assumed too large, into a more appropriate one, the difference can be reduced much further. In this way we may conclude that the maximum level of the resonant high water is not influenced possibly by error contained in estimation of bottom friction nor by its very existence, but this does not necessarily mean that the friction itself might be ignored in all cases. Since the effect of bottom friction is supposed to make its appearance most remarkably for the tide which has entered into the region of shallow water, we should take the mean value placing more importance upon the shallow region if we want to determine the value of r valid throughout the whole area.

9. Effect of scale of disturbance

As stated in Section 2 of Part I, the ratio $L_2/L_1 \equiv \sigma$, where L_1 and L_2 stand respectively for the scales of a disturbance and of a sloping bottom, is one of the factors determining the values of $\eta/h_0 \equiv \eta_0$ and $\bar{u}/V \equiv u_0$. All our computations performed up to now are concerned exclusively with the case $\sigma = 1$, but when dealing with typhoons σ takes a value, we suppose, sensibly less than unity. So in what follows, a calculation will be performed tentatively corresponding to $\sigma = 1/2$. To be more concrete, this is the case when $L_1 = 20$ km and $L_2 = 10$ km, namely preserving the sloping bottom constant we have doubled only L_1, the dimension of a disturbance. For the sake of simplicity, we make use of the linearized theory and introduce at the same time the bottom friction assuming $r = 0.002$ m/sec. The height of the attendant wave is found independent of L_1, and we have only to magnify the transversal dimension. Computation is started from the instant when the foremost end of the disturbance touches the rearmost point of the slope. By naming it -60 th step of our calculation, we shall be able to make the middle point of the wide disturbance we are now interested in coincide with that of the narrow one previously treated.

Computation formulas are completely the same as (63), except that the expression of G becomes

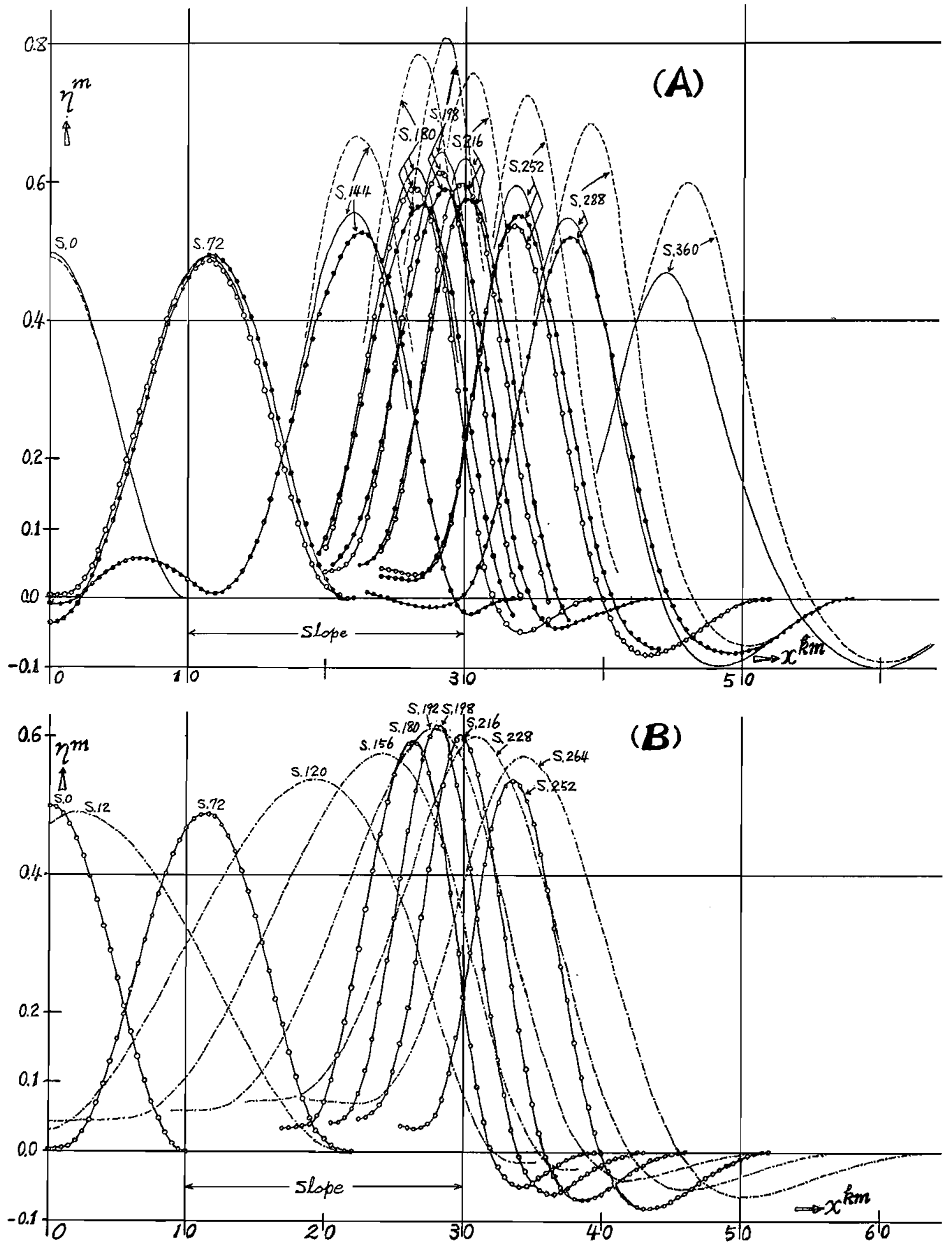

Fig. 7. Changes in surge height according to (A) several approximations adopted, and (B) to assumed scales of disturbance; ······ nonlinear inviscid, —— linear inviscid, O—O—O linear viscous, •—•—• intermittent data supply, ·—·— larger scale distuıbance.

$$G=\frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}\left\{-P_0\left(1+\cos\frac{\pi X}{2L_1}\right)\right\}=\frac{\pi P_0}{2L_1}\sin\frac{\pi X}{2L_1}, \tag{65}$$

where, just as in the case of the narrow disturbance, $L_1=10$ km. Therefore the corresponding difference equations on the whole also remain unchanged from (64), only the last term in the last parentheses of the first equation should read

$$-0.007854<\sin\frac{\pi}{20000}(x-Vt)>_M \quad (x \text{ and } Vt \text{ in m}). \tag{66}$$

Expression (66) shows that at the instant $t=-60\,\Delta t$, the middle point and the forward end of the disturbance pass through the points $x=-10$ km, and 10 km respectively, traversing the distance Δx (=500 m) in time $3\Delta t$. The inviscid values ($\eta=\lambda$, TABLE 2) are used as the initial tide.

Calculations by means of the electronic computer were performed in the same way as in the preceding report, and the results are reproduced in FIGURE 7B together with those of the preceding example. A cursory glance at the figure shows the maximum heights coincident practically with each other. In other words, it is concluded that the resonant wave height seems to be either independent or very slightly dependent upon σ. We are not in a position at present to infer in what circumstance does this interrelation break down. However, judging from the numericals put to use in our examples, we can suppose that generally the tidal level might not possibly be influenced by σ to a great degree. But some difference, more or less, has to be expected in the dissipation stage of the wave which has been raised already. Namely an immense quantity of water sucked up by a depression of a larger extent is certainly inclined to preserve its form as a free tidal wave for a longer time than does the water of less volume. This fact is of some importance in understanding the mechanics of a surge which makes an attack on a coast when a disturbance is landing after passing through the resonance point.

10. Intermittent supply of storm data

In our problems treated in the foregoing sections, data of a disturbance used to be supplied at every mesh point in every step of the computation. It was possible merely because we assumed not only one-dimensional water surface but also an extremely simplified storm pattern. When we deal with data of a typhoon (tangential stress from wind accompanied by gradient of atmospheric pressure) advancing on an ordinary two-dimensional extent of sea surface, renewal of data at every step would be formidable. It has been customary up to this time therefore to put to use, at every step of the computation, data estimated through a crude formula. But we might conceive another idea as a more pertinent measure: that of renewing the data at intervals of certain steps on the understanding that the computation proceeds using the same data until the next replacing station is reached. In other words, we assume that during a certain interval of time, the behavior of a disturbance may as a matter of fact be approximated to those of the middle point of that interval. It is unquestionable that this method would be very accurate in so far as the interval chosen be so short, but also in a general case we shall be able to determine an appropriate interval of time taking account of facility of the computation on the one hand as well as of steepness of variation of data on the other.

One of the good examples of testing validity of this expedient is provided by the resonance phenomenon. In the problem where advancing speed of a disturbance is important, is it possible that procedure yield sufficient approximation to allow the advance to be replaced by a series of intermittent jumps (*a multi-step function*, so to speak) and for resonance to take place only in a sense of a rough mean? In order to answer this question, let us take up the linearized equations for an inviscid fluid, (29) of Part I or the equations derived from (63) by putting $r=0$, and renew tentatively the values of G, the disturbance term, intermittently. Namely, by fixing the value of $<\sin\frac{\pi}{L_1}(x-Vt)>_M$ in equations (32′) of Part I or that of equations (64), where however the coefficient of the first term on the right-hand side of λ_P should be put simply as 1/2, invariably at their values in the middle of the 36 step ones, behaviors of waves are computed throughout the interval consisting of these 36 steps. It corresponds to the distance of 6 km traversed by a disturbance in the time of 6.57 min, or in other words, to a mesh length, about three of which are sufficient to cover 20 km, the range of the sloping bottom.

The result of this computation is shown by a series of curves in FIGURE 7A. Most of these steps are situated at the junctions of the intervals, but the 198 th step lies at the middle of the interval. Variations may be found more or less according to the relative position of a step in an interval, but a better approximation than anticipated is observed in the figure. This fact not only supports the intermittent renewal of data, but also indicates that, in spite of localization feeling suggested by the word 'resonance' the phenomenon is in reality nothing but a synthetic effect over a wider region.

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