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Non-stationary Response of the Linear System to Random Excitation

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Abstract

In relation to the statistical design method of anti-earthquake structures for moderately intense excitations, the basic studies on the statistical quantities such as the co-variance and spectral density in the non-stationary stochastic process are described and the input and output relations of such quantities in the case of a multi-input and -output, linear discrete system having time-variant coefficients are presented.

As the basic statistical quantities in the time and frequency domain, the local co-variance matrix and the local spectral density matrices are considered in this paper. At first, the local co-variance matrix is defined as the product of a two-dimensional cutoff operator and the co-variance matrix in a non-stationary stochastic process. And then, the two-dimensional local spectral density matrix and the several kinds of one-dimensional local spectral density matrices are introduced by defining them as the double and single Fourier transform of the local co-variance matrix, respectively. It is found that the appropriately defined one-dimensional spectral density matrices containing a time variable have the meaning of the power spectral density matrix in the non-stationary stochastic process in the sense that the integral of these quantities over the finite time domain results in the local energy spectral density matrix defined in the square time domain. And also, it is shown that as a limiting case, the one-dimensional local Hermitian spectral density matrix presented in this paper is reduced to the spectral density matrix introduced by D. G. Lampard. Moreover it is shown that the local spectral density matrices are expressed as the weighted averages of the corresponding total spectral density matrices associated with the full time domain.

For the general case of a multi-input and -output linear discrete system having time-variant, complex-valued coefficients, the input and output relations of the local co-variance matrix and the one- or two-dimensional local spectral density matrices are presented. And it is shown that as a special case of a linear discrete system having time-invariant, real-valued coefficients, the input and output relation of the two-dimensional total spectral density matrix is reduced to the relation presented by J. S. Bendat. As an example of the non-stationary input process most applicable to earthquake engineering, the quasi-stationary random process introduced by V. V. Bolotin as well as the locally stationary random process presented by R. A. Silverman are considered, and the basic statistical quantities of the output of a linear system subjected to these random inputs are estimated. And also, it is shown that as a special case of a time-invariant, linear discrete system subjected to a stationary input, the input and output relations of the co-variance matrix and the total spectral density matrices are reduced to the well-known results in the stationary stochastic process.

Finally, in the appendix, it is shown that the ensemble averages of the short-time correlation and power spectral density matrix which are introduced by R. M. Fano, are expressed as the weighted time averages of the local co-variance matrix and the local spectral density matrices, respectively.

1. Introduction

In order to determine the reasonable dynamic characteristics of an elasto-plastic anti-earthquake structure, the structure should be designed according to the following two kinds of aseismic design method depending upon the intensity and the frequency of occurrence of earthquake excitations[1]. For very intense earthquake excitations, the dynamic characteristics concering the elasto-plastic behaviour of the structures during earthquakes should be determined according to the elasto-plastic aseismic design method in which a comparatively small safety factor with respect to the structural response and an appropriate safety factor for the earthquake excitations are introduced, and the aseismic safety of the elasto-plastic structure must be guaranteed in the ultimate state[2]. For moderately intense earthquake excitations having large frequency of occurrence, on the other hand, the elastic aseismic design method should be applied with a comparatively large safety factor in respect to the response, and the aseismic safety of the structure is to be examined in the allowable elastic range. Particularly in the latter case, a statistical approach to establishing the aseismic design method is more plausible because the elastic responses of structures with slight damping subjected to random earthquake excitations are sensitively affected by the spectral characteristics of the excitations which statistically differ from each other according to the seismicity and soil conditions of the construction site of the structure. In this case, it is better to consider that both structural response and earthquake excitations belong to the non-stationary stochastic process since the statistical properties of earthquake excitations essentially vary with time. Even if the earthquake excitations could be equivalently approximated by the finite duration of a stationary random time function, the transient responses due to a suddenly applied excitation are apt to be predominant because little damping effect can be anticipated in the elastic structure designed with a comparatively large safety factor.

As a problem of earthquake response analysis, the output-responses of the structure subjected to non-stationary random excitations should be defined by the statistical measures of aseismic safety of each part of the structure which may be, for example, the expected number in excess of the allowable response level, the probability of peak amplitude over the allowable response level and so on. These statistical earthquake responses, however, can be analytically expressible in the non-stationary stochastic process by using the first and second moment of the response and its derivatives with respect to time, at least in the case where the Gausian process is concerned[1][3]. The purpose of the present paper is not to deal directly with such statistical earthquake responses but to study the basic statistical quantities such as the co-variance and the corresponding spectral density in the non-stationary stochastic process and to present the input and output relationship of those basic quantities expressed in matrix from for a multi-input and -output, linear discrete system. For the generality of analytical results, it is assumed that the linear system considered is a time-variant discrete system having complex-valued coefficients and that the non-stationary stochastic processes have at least the first and the second moments. Therefore, the analytical results of the present paper contain those of a time-invariant system having real-valued coefficients and of the so-called quasi-stationary excitation[1)4)5)] as special cases and also tend to the well-known results in a stationary process when a limiting case is considered. As regards the spectral density matrix in a non-stationary process, the one- or two-dimensional local spectral density matrix is defined in this paper as the single or double Fourier transform of the local co-variance matrix which is equal to the co-variance matrix in the prescribed finite two-dimensional time domain and zero outside this domain. And it will be found that this one- or two-dimensional local spectral density matrix can be related to the spectral density defined by D. G. Lampard[6)] and that given by J. S. Bendat[7)], respectively, as the limiting cases. And, it will also be shown that the input and output relationship of the local spectral density matrices is reducible to J. S. Bendat's formula in a non-stationary stochastic process where a time-invariant linear discrete system and the two-dimensional spectral density matrix defined as the double Fourier transform of the co-variance matrix in R^2_∞ are concerned and also reducible to the well-known relationship between the input and output spectral density matrix in the case where a time-invariant linear discrete system and the stationary process are concerned.

2. The local co-variance matrix and local spectral density matrix in the non-stationary stochastic process

The co-variance matrix $[K(\tau_1, \tau_2)]$ of a complex-valued non-stationary random vector, $\{\xi(\tau)\}$ which is defined in the one-dimensional infinite domain, R^1_∞ of a real variable τ, is given as a complex-valued matrix defined in the two-dimensional infinite domain, R^2_∞ of real variables $\tau_1, \tau_2 \in R^1_\infty$ by the following well-known foumula:

$$[K(\tau_1\ \tau_2)] = [K(\tau_1, \tau_2\,;\, R^2_\infty)] = E(\{\xi_d(\tau_1)\}\{\xi_d(\tau_2)\}^*)$$
$$\{\xi_d(\tau_i)\} = \{\xi(\tau_i)\} - E\{\xi(\tau_i)\} \qquad \cdots\cdots\cdots\cdots(1)$$

in which the symbol E and the superscript $*$ denote the ensemble average and the transposed conjugate, respectively.

By making use of the one- and two-dimensional cutoff operator, $D(\tau_i\,;\, R^1_{\tau_i})$ and $D(\tau_1, \tau_2\,;\, R^2_{\tau_1\tau_2})$ which are unit in the inside of any finite closed domain, $R^1_{\tau_i}$ or $R^2_{\tau_1\tau_2}$, half in the boundary domain, $C^1_{\tau_i}$ or $C^2_{\tau_1\tau_2}$ and zero outside the domain, the local co-variance matrix, $[K(\tau_1, \tau_2\,;\, R^2_{\tau_1\tau_2})]$ is defined by the following equation:

$$[K(\tau_1, \tau_2\,;\, R^2_{\tau_1\tau_2})] = E(\{D(\tau_1\,;\, R^1_{\tau_1})\xi_d(\tau_1)\}\{D(\tau_2\,;\, R^1_{\tau_2})\xi_d(\tau_2)\}^*)$$
$$= D(\tau_1, \tau_2\,;\, R^2_{\tau_1\tau_2})[K(\tau_1, \tau_2\,;\, R^2_\infty)] = D(\tau_1, \tau_2\,;\, R^2_{\tau_1\tau_2})[K(\tau_1, \tau_2)] \quad \cdots\cdots\cdots\cdots(2)$$

where

$$\begin{aligned} D(\tau_i\,;\, R^1_{\tau_i}) &= 1 && \tau_i \in R^1_{\tau_i} - C^1_{\tau_i} \\ &= \frac{1}{2} && \tau_i \in C^1_{\tau_i} \\ &= 0 && \tau_i \in R^1_\infty - R^1_{\tau_i}, \quad i = 1, 2 \qquad \cdots\cdots\cdots\cdots(3) \\ D(\tau_1, \tau_2\,;\, R^2_{\tau_1\tau_2}) &= 1 && (\tau_1, \tau_2) \in R^2_{\tau_1\tau_2} - C^2_{\tau_1\tau_2} \\ &= \frac{1}{2} && (\tau_1, \tau_2) \in C^2_{\tau_1\tau_2} \end{aligned}$$

$$=0 \qquad (\tau_1, \tau_2) \in R^2_\infty - R^2_{\tau_1\tau_2}$$

Therefore, the local co-variance matrix is equal to the co-variance matrix in the inside of an arbitrarily prescribed finite domain, $R^2_{\tau_1\tau_2}$ and zero outside the domain.

The two-dimensional local spectral density matrix associated with the finite domain, $R^2_{\tau_1\tau_2}$ is defined as the double Fourier transform of the local co-variance matrix given by eq. (2).

$$[S(\omega_1, \omega_2 ; R^2_{\tau_1\tau_2})] = \int_{R^2_\infty} [K(\tau_1, \tau_2 ; R^2_{\tau_1\tau_2})] e^{-j(\omega_1\tau_1 - \omega_2\tau_2)} d\tau_1 d\tau_2$$
$$= \int_{R^2_{\tau_1\tau_2}} [K(\tau_1, \tau_2)] e^{-j(\omega_1\tau_1 - \omega_2\tau_2)} d\tau_1 d\tau_2 \qquad \cdots\cdots\cdots\cdots(4)$$

In particular, for the rectangular domain, ${}_rR^2_{\tau_1\tau_2}$, eq. (4) can be expressed by

$$[S(\omega_1, \omega_2 ; {}_rR^2_{\tau_1\tau_2})] = E(\{F_{\xi_d}(\omega_1 ; R^1_{\tau_1})\}\{F_{\xi_d}(\omega_2 ; R^1_{\tau_2})\}^*)$$
$$\{F_{\xi_d}(\omega_i ; R^1_{\tau_i})\} = \int_{R^1_\infty} D(\tau_i ; R^1_{\tau_i})\{\xi_d(\tau_i)\} e^{-j\omega_i\tau_i} d\tau_i$$
$$= \int_{R^1_{\tau_i}} \{\xi_d(\tau_i)\} e^{-j\omega_i\tau_i} d\tau_i, \qquad i=1, 2. \qquad \cdots\cdots\cdots\cdots(5)$$

Inversely transforming eq. (4), the local co-variance matrix can be expressed as the inverse double Fourier transform of the two-dimensional local spectral density matrix except a set of points of zero measure.

$$[K(\tau_1, \tau_2 ; R^2_{\tau_1\tau_2})] = \frac{1}{(2\pi)^2} \int_{R^2_\infty} [S(\omega_1, \omega_2 ; R^2_{\tau_1\tau_2})] e^{j(\omega_1\tau_1 - \omega_2\tau_2)} d\omega_1 d\omega_2 \qquad \cdots\cdots(6)$$

where R^2_∞ denotes the two-dimensional infinite domain of two real frequency variables, ω_1 and ω_2.

In the general case of the complex-valued non-stationary process, the following relations are valid:

$$[K(\tau_1, \tau_2 ; R^2_{\tau_1\tau_2})]^* = [K(\tau_2, \tau_1 ; R^2_{\tau_2\tau_1})] \qquad \cdots\cdots\cdots\cdots(7)$$

$$[S(\omega_1, \omega_2 ; R^2_{\tau_1\tau_2})]^* = [S(\omega_2, \omega_1 ; R^2_{\tau_2\tau_1})] \qquad \cdots\cdots\cdots\cdots(8)$$

In particular, for the real-valued non-stationary process, eqs. (7) and (8) are reduced to the following expressions, respectively:

$$[K(\tau_1, \tau_2 ; R^2_{\tau_1\tau_2})]^T = [K(\tau_2, \tau_1 ; R^2_{\tau_2\tau_1})] \qquad \cdots\cdots\cdots\cdots(9)$$

$$[S(\omega_1, \omega_2 ; R^2_{\tau_1\tau_2})]^T = [S(-\omega_2, -\omega_1 ; R^2_{\tau_2\tau_1})] \qquad \cdots\cdots\cdots\cdots(10)$$

where the superscript, T denotes the transposed matrix. Also, in the case of a real process, the real and imaginary parts of the inverse double Fourier transform given by eq. (6) are expressed by

$$\frac{1}{(2\pi)^2} \int_{R^2_\infty} \boldsymbol{R}([S(\omega_1, \omega_2 ; R^2_{\tau_1\tau_2})] e^{j(\omega_1\tau_1 - \omega_2\tau_2)}) d\omega_1 d\omega_2 = [K(\tau_1, \tau_2 ; R^2_{\tau_1\tau_2})]$$
$$\frac{1}{(2\pi)^2} \int_{R^2_\infty} \boldsymbol{I}([S(\omega_1, \omega_2 ; R^2_{\tau_1\tau_2})] e^{j(\omega_1\tau_1 - \omega_2\tau_2)}) d\omega_1 d\omega_2 = [0] \qquad \cdots\cdots\cdots\cdots(11)$$

in which the symbols $\boldsymbol{R}$ and $\boldsymbol{I}$ represent the real and imaginary parts, respectively.

Particularly considering the square domain, $_sR^2_{\tau_1\tau_2}=R^2_{\tau_L\tau_U}$, in which τ_L and τ_U denote the common lower and upper limits of the variables τ_1 and τ_2, and substituting $\omega_1=\omega_2=\omega$ in the two-dimensional local spectral density matrix defined by eq. (4) or (5), we can define the local energy spectral density matrix by the equation,

$$[S(\omega, \omega\,;R^2_{\tau_L\tau_U})]=E(\{F\xi_d(\omega\,;R^1_{\tau_L\tau_U})\}\{F\xi_d(\omega\,;R^1_{\tau_L\tau_U})\}^*)$$
$$=\int_{R^2_\infty}[K(\tau_1, \tau_2\,;R^2_{\tau_L\tau_U})]e^{-j(\tau_1-\tau_2)\omega}d\tau_1 d\tau_2 \quad \cdots\cdots\cdots\cdots(12)$$

Integrating eq. (12) over the infinite domain R^1_∞ with respect to ω and deviding by 2π, we obtain

$$\frac{1}{2\pi}\int_{R^1_\infty}[S(\omega, \omega\,;R^2_{\tau_L\tau_U})]d\omega=\int_{\tau_L}^{\tau_U}[K(\tau, \tau\,;R^2_{\tau_L\tau_U})]d\tau \quad \cdots\cdots\cdots\cdots(13)$$

From the definition given by eq. (12), it is clear that the local energy spectral density matrix is a Hermitian matrix, that is

$$[S(\omega, \omega\,;R^2_{\tau_L\tau_U})]^*=[S(\omega, \omega\,;R^2_{\tau_L\tau_U})] \quad \cdots\cdots\cdots\cdots(14)$$

Then, the diagonal elements of the local energy spectral density matrix are real numbers and each of them corresponds to the integral over the interval $[\tau_L, \tau_U]$ of the ensemble average of the square absolute value of each element of complex-vector.

In particular, if the two-dimensional local spectral density matrix exists in the case of $R^2_{\tau_1\tau_2}\to R^2_\infty$, we define it as the two-dimensional total spectral density matrix.

$$[S_{\tau_1\tau_2}(\omega_1, \omega_2)]=[S(\omega_1, \omega_2\,;R^2_{\tau_1\tau_2}\to R^2_\infty]=[S(\omega_1, \omega_2\,;R^2_\infty)] \quad \cdots\cdots\cdots\cdots(15)$$

Since the local co-variance matrix and the two-dimensional spectral density matrix are additive as the set functions of the two-dimensional domain $R^2_{\tau_1\tau_2}$, that is,

$$[K(\tau_1, \tau_2\,;\bigcup_i R^2_{\tau_1\tau_2 i})]=\sum_i[K(\tau_1, \tau_2\,;R^2_{\tau_1\tau_2 i})]$$
$$[S(\omega_1, \omega_2\,;\bigcup_i R^2_{\tau_1\tau_2 i})]=\sum_i[S(\omega_1, \omega_2\,;R^2_{\tau_1\tau_2 i})] \quad \cdots\cdots\cdots\cdots(16)$$

where

$$R^2_{\tau_1\tau_2\lambda}-C^2_{\tau_1\tau_2\lambda}\cap R^2_{\tau_1\tau_2\mu}-C^2_{\tau_1\tau_2\mu}=0 \qquad \text{for} \quad \lambda\neq\mu \quad \cdots\cdots\cdots\cdots(17)$$

the co-variance matrix and the corresponding two-dimensional total spectral density matrix can be expressed by the sum of the relevant quantities defined in finite or denumerably infinite disjunctive domains, $R^2_{\tau_1\tau_1 i}$'s.

$$[K(\tau_1, \tau_2)]=\sum_i[K(\tau_1, \tau_2\,;R^2_{\tau_1\tau_2 i})]$$
$$[S_{\tau_1\tau_2}(\omega_1, \omega_2)]=\frac{1}{(2\pi)^2}\int_{R^2_\infty}[K(\tau_1, \tau_2)]e^{-j(\omega_1\tau_1-\omega_2\tau_2)}d\tau_1 d\tau_2$$
$$=\sum_i[S(\omega_1, \omega_2\,;R^2_{\tau_1\tau_2 i})] \quad \cdots\cdots\cdots\cdots(18)$$

where

$$\bigcup_i R^2_{\tau_1\tau_2 i}=R^2_\infty, \quad R^2_{\tau_1\tau_2\lambda}-C^2_{\tau_1\tau_2\lambda}\cap R^2_{\tau_1\tau_2\mu}-C^2_{\tau_1\tau_2\mu}=0 \qquad \text{for} \quad \lambda\neq\mu$$

Inversely, as far as the rectangular domain ${}_rR^2{}_{\tau_1\tau_2}$ is concerned, the two-dimensional spectral density matrix can be expressed by using the total spectral density matrix as follows :

$$[S(\omega_1, \omega_2 ; {}_rR^2{}_{\tau_1\tau_2})] = \frac{1}{(2\pi)^2} Q(\omega_1 ; R^1{}_{\tau_1}) \underset{\omega_1}{*} [S_{\tau_1\tau_2}(\omega_1, \omega_2)] \underset{\omega_2}{*} Q^*(\omega_2 ; R^1{}_{\tau_2}) \quad \cdots\cdots\cdots(19)$$

where

$$Q(\omega_i ; R^1{}_{\tau_i}) = \int_{R^1{}_\infty} D(\tau_i ; R^1{}_{\tau_i}) e^{-j\omega_i\tau_i} d\tau_i$$
$$= ((\sin \omega_i\tau_{iU} - \sin \omega_i\tau_{iL}) + j(\cos\omega_i\tau_{iU} - \cos\omega_i\tau_{iL}))/\omega_i \quad \cdots\cdots(20)$$

$$\frac{1}{2\pi}\int_{R^1{}_\infty} Q(\omega_i ; R^1{}_{\tau_i}) e^{j\omega_i\tau_i} d\omega_i = D(\tau_i ; R^1{}_{\tau_i}) \quad \cdots\cdots\cdots\cdots(21)$$

and the symbol $\underset{\omega_i}{*}$ denotes the convolution with respect to ω_i, and in eq. (20) τ_{iL}, τ_{iU} represent the lower and the upper limits of the domain $R^1{}_{\tau_i}$, respectively.

Representing the domain $R^1{}_{\tau_i}$ in the form,

$$R^1{}_{\tau_i} = [\tau_{iL}, \tau_{iU}] = \left[\tau_{ic} - \frac{\tau_{ib}}{2}, \ \tau_{ic} + \frac{\tau_{ib}}{2}\right] \quad \cdots\cdots\cdots\cdots(22)$$

where

$$\tau_{ic} = (\tau_{iL} + \tau_{iU})/2, \quad \tau_{ib} = \tau_{iU} - \tau_{iL} \quad \cdots\cdots\cdots\cdots(23)$$

and transforming the variable τ_i and the domains $R^1{}_{\tau_i}, {}_rR^2{}_{\tau_1\tau_2}$ to new variable τ_i' and new domains $R^1{}_{\tau_i'}, {}_rR^2{}_{\tau_1'\tau_2'}$ respectively, the two-dimensional local spectral density matrix with respct to ${}_rR^2{}_{\tau_1'\tau_2'}$ is expressed by

$$[S(\omega_1, \omega_2 ; {}_rR^2{}_{\tau_1'\tau_2'})] = e^{j(\omega_1\tau_{1c} - \omega_2\tau_{2c})}[S(\omega_1, \omega_2 ; {}_rR^2{}_{\tau_1\tau_2})]$$
$$= \frac{1}{(2\pi)^2} Q(\omega_1 ; R^1{}_{\tau_1'}) \underset{\omega_1}{*} [S_{\tau_1'\tau_2'}(\omega_1, \omega_2)] \underset{\omega_2}{*} Q^*(\omega_2 ; R^1{}_{\tau_2'}) \quad \cdots\cdots(24)$$

Eqs. (19) and (24) show that the two-dimensional local spectral density can be expressed as a kind of extended weighted averages of the two-dimensional total spectral density if the latter exists. In these equations, the weights have the following property.

$$\frac{1}{2\pi}\int_{R^1{}_\infty} Q(\omega_i ; R^1{}_{\tau_i}) d\omega_i = 1 \qquad \text{if} \quad 0 \in R^1{}_{\tau_i} - C^1{}_{\tau_i}$$
$$= \frac{1}{2} \qquad \text{if} \quad 0 \in C^1{}_{\tau_i}$$
$$= 0 \qquad \text{if} \quad 0 \in R^1{}_\infty - R^1{}_{\tau_i} \quad \cdots\cdots\cdots\cdots(25)$$

$$\frac{1}{2\pi}\int_{R^1{}_\infty} Q(\omega_i ; R^1{}_{\tau_i'}) d\omega_i = 1, \qquad IQ(\omega_i ; R^1{}_{\tau_i'}) = 0 \quad \cdots\cdots\cdots\cdots(26)$$

In particular, if the domain is square and $\omega_i = \omega$, that is, where the local energy spectral density is concerned, the following equation is valid :

$$[S(\omega, \omega ; {}_sR^2{}_{\tau_1\tau_2})] = [S(\omega, \omega ; {}_sR^2{}_{\tau_1'\tau_2'})]$$
$$= \frac{1}{(2\pi)^2}\int_{R^1{}_\infty} d\nu_1 \frac{\sin(\omega - \nu_1)\tau_b/2}{(\omega - \nu_1)/2}\int_{R^1{}_\infty} d\nu_2 \frac{\sin(\omega - \nu_2)\tau_b/2}{(\omega - \nu_2)/2} [S_{\tau_1'\tau_2'}(\nu_1, \nu_2)] \quad \cdots\cdots(27)$$

When the process is stationary, the two-dimensional total spectral density matrix is expressed by

$$[S_{\tau_1\tau_2}(\omega_1, \omega_2)]=[S(\omega_1, \omega_2)]=2\pi\delta(\omega_1-\omega_2)[S(\omega_1)] \quad\cdots\cdots\cdots\cdots(28)$$

where $\delta(\omega)$ denotes the delta-function with respect to ω and $[S(\omega)]$ is the spectral density matrix in the stationary process. Substituting eq. (28) to eq. (24), the two-dimensional local spectral density matrix is given by

$$[S(\omega_1, \omega_2\,;\,{}_rR^2{}_{\tau_1'\tau_2'})]=e^{j(\omega_1\tau_{1c}-\omega_2\tau_{2c})}[S(\omega_1, \omega_2\,;\,{}_rR^2{}_{\tau_1\tau_2})]$$
$$=\frac{1}{2\pi}\int_{R^1_\infty}\frac{\sin(\omega_1-\nu)\tau_{1b}/2}{(\omega_1-\nu)/2}\cdot\frac{\sin(\omega_2-\nu)\tau_{2b}/2}{(\omega_2-\nu)/2}[S(\nu)]e^{j(\tau_{1c}-\tau_{2c})\nu}d\nu \quad\cdots\cdots(29)$$

Moreover, the local energy spectral density per unit time, that is, the local power spectral density matrix can be expressed by a kind of conventional weighted average of the power spectral density as follows:

$$\frac{1}{\tau_b}[S(\omega, \omega\,:\,{}_sR^2{}_{\tau_1\tau_2})]=\frac{\tau_b}{2\pi}\left(\frac{\sin\omega\tau_b/2}{\omega\tau_b/2}\right)^2\underset{\omega}{*}[S(\omega)]$$
$$\overset{\omega}{\underset{\lambda}{\subset}}D_1(\lambda\,;\,\tau_b)[R(\lambda)] \quad\cdots\cdots\cdots\cdots(30)$$

where

$$\tau_b\left(\frac{\sin\omega\tau_b/2}{\omega\tau_b/2}\right)^2\overset{\omega}{\underset{\lambda}{\subset}}D_1(\tau\,;\,\tau_b),\quad [S(\omega)]\overset{\omega}{\underset{\lambda}{\subset}}[R(\lambda)] \quad\cdots\cdots\cdots\cdots(31)$$

In the above equations, the symbol $\overset{\omega}{\underset{\lambda}{\subset}}$ denotes the correspondence between the Fourier transform pair. Then denoting $[R(\lambda)]$ as the co-variance matrix in the stationary process, the second equation in (31) represents the well-known Wiener-Kintchin relation in the stationary process. And, the first equation in (31) shows the so-called Bartlett's pair[8]. When τ_b tends to infinity, eq. (30) is reduced to the following equation:

$$\lim_{\tau_b\to\infty}\frac{1}{\tau_b}[S(\omega, \omega\,;\,{}_sR^2{}_{\tau_1\tau_2})]=[S(\omega)]\subset[R(\lambda)] \quad\cdots\cdots\cdots\cdots(32)$$

Particularly, if the stationary process is ergodic, the above equation tends to the power spectral density conventionally defined on the ergodic stationary process as follows:

$$\lim_{\tau_b\to\infty}\frac{1}{\tau_b}[S(\omega, \omega\,;\,{}_sR^2{}_{\tau_1\tau_2})]=\lim_{\tau_b\to\infty}\frac{1}{\tau_b}[S(\omega, \omega\,;\,{}_sR^2{}_{\tau_1'\tau_2'})]$$
$$=\lim_{\tau_b\to\infty}\frac{1}{\tau_b}E(\{F_{\xi_d}(\omega\,;\,R^1{}_{\tau_1'})\}\{F_{\xi_d}(\omega\,;\,R^1{}_{\tau_2'})\}^*)$$
$$=\lim_{\tau_b\to\infty}\frac{1}{\tau_b}\{F_{\xi_d}(\omega\,;\,R^1{}_{\tau_1'})\}\{F_{\xi_d}(\omega\,;\,R^1{}_{\tau_2'})\}^*=[S(\omega)] \quad\cdots\cdots\cdots\cdots(33)$$

Now we consider the integral of the complex spectrum of $D(\tau_i'\,;\,R^1{}_{\tau_i'})\{\xi_d(\tau_i')\}$ over the frequency domain, $(0, \omega_i]$.

$$\{J(\omega_i\,;\,R^1{}_{\tau_i'})\}=\frac{1}{2\pi}\int_0^{\omega_i}\{F_{\xi_d}(\nu_i\,;\,R^1{}_{\tau_i})\}e^{j\nu_i\tau_{ic}}d\nu_i=\frac{1}{2\pi}\int_0^{\omega_i}\{F_{\xi_d}(\nu_i\,;\,R^1{}_{\tau_i'})\}d\nu_i$$
$$=\frac{1}{2\pi}\int_{R^1{}_{\tau_i}}\frac{e^{-j(\tau_i-\tau_{ic})\omega_i}-1}{-j(\tau_i-\tau_{ic})}\{\xi_d(\tau_i)\}d\tau_i=\frac{1}{2\pi}\int_{R^1{}_{\tau_i'}}\frac{e^{-j\tau_i'\omega_i}-1}{-j\tau_i'}\{\xi_d(\tau_i'+\tau_{ic})\}d\tau_i' \cdots(34)$$

Considering the rectangular domain, ${}_rR^2{}_{\tau_1'\tau_2'}$, the ensemble average of the product of the increments of $\{J(\omega_i\,;R^1{}_{\tau_i'})\}$'s defined above can be expressed by making use of the two-dimensional local spectral density matrix as follows:

$$E(\{J(\omega_{1U}\,;R^1{}_{\tau_1'})-J(\omega_{1L}\,;R^1{}_{\tau_1'})\}\{J(\omega_{2U}\,;R^1{}_{\tau_2'})-J(\omega_{2L}\,;R^1{}_{\tau_2'})\}^*)$$
$$=\frac{1}{(2\pi)^2}\int_{R^2{}_\infty}D(\omega_1,\omega_2\,;{}_rR^2{}_{\omega_1\omega_2})[S(\omega_1,\omega_2\,;{}_rR^2{}_{\tau_1\tau_2})]e^{j(\omega_1\tau_{1c}-\omega_2\tau_{2c})}d\omega_1 d\omega_2$$
$$=\frac{1}{(2\pi)^2}\int_{R^2{}_\infty}D(\omega_1,\omega_2\,;{}_rR^2{}_{\omega_1\omega_2})[S(\omega_1,\omega_2\,;{}_rR^2{}_{\tau_1'\tau_2'})]d\omega_1 d\omega_2 \qquad\cdots\cdots\cdots\cdots(35)$$

or

$$E(d\{J(\omega_1\,;R^1{}_{\tau_1'})\}d\{J(\omega_2\,;R^1{}_{\tau_2'})\}^*)=\frac{1}{(2\pi)^2}[S(\omega_1,\omega_2\,;{}_rR^2{}_{\tau_1'\tau_2'})]d\omega_1 d\omega_2\cdots\cdots(36)$$

Eq. (35) or (36) shows that ensemble average of the product of $\{J(\omega_i\,;R^1{}_{\tau_i'})\}$'s is a set function of ${}_rR^2{}_{\omega_1\omega_2}$ and it is additive with respect to the domain ${}_rR^2{}_{\omega_1\omega_2}$. In the stationary process, putting $R^1{}_{\tau_i'}\to R^1{}_\infty$ and ${}_rR^2{}_{\tau_1'\tau_2'}\to R^2{}_\infty$ and making use of eq. (28), we obtain

$$E(\{J(\omega_{1U}\,;R^1{}_{\tau_1'}\to R^1{}_\infty)-J(\omega_{1L}\,;R^1{}_{\tau_1'}\to R^1{}_\infty)\}\{J(\omega_{2U}\,;R^1{}_{\tau_2'}\to R^1{}_\infty)$$
$$-J(\omega_{2L}\,;R^1{}_{\tau_2'}\to R^1{}_\infty)\}^*)=\frac{1}{2\pi}\int_{R^1{}_\infty}D(\nu\,;R^1{}_{\omega_1})D(\nu\,;R^1{}_{\omega_2})[S(\nu)]d\nu\cdots\cdots(37)$$

or

$$E(d\{J(\omega_1\,;R^1{}_{\tau_1'}\to R^1{}_\infty)\}d\{J(\omega_2\,;R^1{}_{\tau_2'}\to R^1{}_\infty)\}^*)=\frac{1}{2\pi}[S(\nu)]d\nu\cdots\cdots\cdots\cdots(38)$$

where

$$d\nu=d\omega_1\cap d\omega_2,\quad \nu\in\omega_1+d\omega_1\cap\omega_2+d\omega_2$$

Eq. (37) or (38) shows the well-known fact that the random vector, $\{J(\omega\,;R^1{}_\tau\to R^1{}_\infty)\}$ has non-correlative increments in the stationary process. Particularly in the case of the square frequency domain, ${}_sR^2{}_{\omega_1\omega_2}=R^2{}_{\omega_L\omega_U}$ eq. (37) reduces to

$$E(\{J(\omega_U\,;R^1{}_\tau\to R^1{}_\infty)-J(\omega_L\,;R^1{}_\tau\to R^1{}_\infty)\}\{J(\omega_U\,;R^1{}_\tau\to R^1{}_\infty)$$
$$-J(\omega_L\,;R^1{}_\tau\to R^1{}_\infty)\}^*)=\frac{1}{2\pi}\int_{R^1{}_{\omega_L\omega_U}}[S(\nu)]d\nu \qquad\cdots\cdots\cdots\cdots(39)$$

Substituting $\omega_{iU}=\infty$, $\omega_{iL}=-\infty$ in eq. (35) and taking into consideration

$$\{J(\infty\,;R^1{}_{\tau_i'})-J(-\infty\,;R^1{}_{\tau_i'})\}=\frac{1}{2\pi}\int_{R^1{}_\infty}\{F_{\xi_d}(\nu_i\,;R^1{}_{\tau_i})\}e^{j\nu_i\tau_{ic}}d\nu_i=\{\xi_d(\tau_{ic})\}\cdots(40)$$

we have the following equation identical to eq. (6):

$$E(\{\xi_d(\tau_{1c})\}\{\xi_d(\tau_{2c})\}^*)=\frac{1}{(2\pi)^2}\int_{R^2{}_\infty}[S(\omega_1,\omega_2\,;{}_rR^2{}_{\tau_1\tau_2})]e^{j(\omega_1\tau_{1c}-\omega_2\tau_{2c})}d\omega_1 d\omega_2$$
$$\cdots\cdots\cdots\cdots(41)$$

Next, we define the two kinds of one-dimensional local spectral density matrices from the two-dimensional spectral density matrix defined by eq. (4) as follows:

$$[S_1(\omega_1,\tau_2\,;R^2{}_{\tau_1\tau_2})]=E(\{F_{\xi_d}(\omega_1\,;R^1{}_{\tau_1}(\tau_2))\}D(\tau_2\,;R^1{}_{\tau_2})\{\xi_d(\tau_2)\}^*)$$

$$=\frac{1}{2\pi}\int_{R^1_\infty}[S(\omega_1,\omega_2\,;R^2_{\tau_1\tau_2})]e^{-j\omega_2\tau_2}d\omega_2$$

$$=\int_{R^1_\infty}[K(\tau_1,\tau_2\,;R^2_{\tau_1\tau_2})]e^{-j\omega_1\tau_1}d\tau_1 \qquad \cdots\cdots\cdots\cdots(42)$$

$$[S_2(\tau_1,\omega_2\,;R^2_{\tau_1\tau_2})]=E(D(\tau_1\,;R^1_{\tau_1})\{\xi_d(\tau_1)\}\{F_{\xi_d}(\omega_2\,;R^1_{\tau_2}(\tau_1))\}^*)$$

$$=\frac{1}{2\pi}\int_{R^1_\infty}[S(\omega_1,\omega_2\,;R^2_{\tau_1\tau_2})]e^{j\omega_1\tau_1}dw_1$$

$$=\int_{R^1_\infty}[K(\tau_1,\tau_2\,;R^2_{\tau_1\tau_2})]e^{j\omega_2\tau_2}d\tau_2 \qquad \cdots\cdots\cdots\cdots(43)$$

in which $R^1_{\tau_i}(\tau_j)$ is the one-dimensional domain with respect to τ_i as a function of τ_j. Inversely transforming eqs. (42) and (43), the local co-variance matrix can be expressed by the above defined one-dimensional local spectral matrices.

$$[K(\tau_1,\tau_2\,;R^2_{\tau_1\tau_2})]=\frac{1}{2\pi}\int_{R^1_\infty}[S_1(\omega_1,\tau_2\,;R^2_{\tau_1\tau_2})]e^{j\omega_1\tau_1}d\omega_1$$

$$=\frac{1}{2\pi}\int_{R^1_\infty}[S_2(\tau_1,\omega_2\,;R^2_{\tau_1\tau_2})]e^{-j\omega_2\tau_2}d\omega_2 \qquad \cdots\cdots\cdots\cdots(44)$$

In general, there exists the following relationship between the two-kinds of one-dimensional local spectral density matrices.

$$[S_1(\omega_1,\tau_2\,;R^2_{\tau_1\tau_2})]^*=[S_2(\tau_2,\omega_1\,;R^2_{\tau_2\tau_1})]$$

$$[S_2(\tau_1,\omega_2\,;R^2_{\tau_1\tau_2})]^*=[S_1(\omega_2,\tau_1\,;R^2_{\tau_2\tau_1})] \qquad \cdots\cdots\cdots\cdots(45)$$

In particular, where the real-valued process is concerned, the following equations are valid :

$$[S_1(\omega_1,\tau_2\,;R^2_{\tau_1\tau_2})]^*=[S_1(-\omega_1,\tau_2\,;R^2_{\tau_1\tau_2})]^T$$

$$[S_2(\tau_1,\omega_2\,;R^2_{\tau_1\tau_2})]^*=[S_2(\tau_1,-\omega_2\,;R^2_{\tau_1\tau_2})]^T \qquad \cdots\cdots\cdots\cdots(46)$$

If there exists a one-dimensional local spectral density matrix when $R^2_{\tau_1\tau_2}$ tends to R^2_∞, we call it the one-dimensional total spectral density matrix.

$$[S_{1\tau_1\tau_2}(\omega_1,\tau_2)]=[S_1(\omega_1,\tau_2\,;R^2_{\tau_1\tau_2}\to R^2_\infty)]=[S_1(\omega_1,\tau_2\,;R^2_\infty)]$$

or

$$[S_{2\tau_1\tau_2}(\tau_1,\omega_2)]=[S_2(\tau_1,\omega_2\,;R^2_{\tau_1\tau_2}\to R^2_\infty)]=[S_2(\tau_1,\omega_2\,;R^2_\infty)] \cdots\cdots\cdots\cdots(47)$$

Then, if the two-dimensional total spectral density matrix also exists, the following expressions are obtained:

$$[S_{1\tau_1\tau_2}(\omega_1,\tau_2)]=\frac{1}{2\pi}\int_{R^1_\infty}[S_{\tau_1\tau_2}(\omega_1,\omega_2)]e^{-j\omega_2\tau_2}d\omega_2$$

$$=\int_{R^1_\infty}[K(\tau_1,\tau_2)]e^{-j\omega_1\tau_1}d\tau_1 \qquad \cdots\cdots\cdots\cdots(48)$$

$$[S_{2\tau_1\tau_2}(\tau_1,\omega_2)]=\frac{1}{2\pi}\int_{R^1_\infty}[S_{\tau_1\tau_2}(\omega_1,\omega_2)]e^{j\omega_1\tau_1}d\omega_1$$

$$=\int_{R^1_\infty}[K(\tau_1,\tau_2)]e^{j\omega_2\tau_2}d\tau_2 \qquad \cdots\cdots\cdots\cdots(49)$$

and

$$[K(\tau_1, \tau_2)] = \frac{1}{2\pi}\int_{R^1_\infty}[S_{1\tau_1\tau_2}(\omega_1, \tau_2)]e^{j\omega_1\tau_1}d\omega_1$$
$$= \frac{1}{2\pi}\int_{R^1_\infty}[S_{2\tau_1\tau_2}(\tau_1, \omega_2)]e^{-j\omega_2\tau_2}d\omega_2 \qquad \cdots\cdots\cdots\cdots(50)$$

When the process is stationary, the one-dimensional total spectral density matrices are expressed by using the power spectral density $[S(\omega)]$.

$$[S_{1\tau_1\tau_2}(\omega_1, \tau_2)] = [S(\omega_1)]e^{-j\omega_1\tau_2}$$
$$[S_{2\tau_1\tau_2}(\tau_1, \omega_2)] = [S(\omega_2)]e^{j\omega_2\tau_1} \qquad \cdots\cdots\cdots\cdots(51)$$

Substituting eq. (51) for eq. (50), we obtain the Wiener-Kintchin relation in the stationary process.

$$[K(\tau_1, \tau_2)] = [R(\tau_1 - \tau_2)] = \frac{1}{2\pi}\int_{R^1_\infty}[S(\omega)]e^{j(\tau_1-\tau_2)\omega}d\omega \qquad \cdots\cdots\cdots\cdots(52)$$

The two-dimensional local spectral density matrix can be expressed in terms of the one-dimensional local spectral density matrices by inversely transforming eqs. (42) and (43).

$$[S(\omega_1, \omega_2\,;\, R^2_{\tau_1\tau_2})] = \int_{R^1_\infty}[S_1(\omega_1, \tau_2\,;\, R^2_{\tau_1\tau_2})]e^{j\omega_2\tau_2}d\tau_2$$
$$= \int_{R^1_\infty}[S_2(\tau_1, \omega_2\,;\, R^2_{\tau_1\tau_2})]e^{-j\omega_1\tau_1}d\tau_1 \qquad \cdots\cdots\cdots\cdots(53)$$

In particular, considering the square domain ${}_sR^2_{\tau_1\tau_2}$ and putting $\omega_1 = \omega_2 = \omega$ in eq. (53), the local energy spectral density matrix is expressed by

$$[S(\omega, \omega\,;\, {}_sR^2_{\tau_1\tau_2})] = \int_{R^1_\infty}[S_1(\omega, \tau_2\,;\, {}_sR^2_{\tau_1\tau_2})]e^{j\omega\tau_2}d\tau_2$$
$$= \int_{R^1_\infty}[S_2(\tau_1, \omega\,;\, {}_sR^2_{\tau_1\tau_2})]e^{-j\omega\tau_1}d\tau_1 \qquad \cdots\cdots\cdots\cdots(54)$$

Now, transforming the variables τ_1 and τ_2 into λ and τ by the equations

$$\lambda = \tau_1 - \tau_2, \quad \tau = \tau_2 \qquad \cdots\cdots\cdots\cdots(55)$$

we define the following one-dimensional local spectral density matrix :

$$[S(\omega, \tau\,;\, R^2_{\lambda\tau})] = [S_1(\omega, \tau\,;\, R^2_{\lambda+\tau\tau})]e^{j\omega\tau}$$
$$= [S_2(\lambda+\tau, \omega\,;\, R^2_{\lambda+\tau\tau})]e^{-j\omega(\lambda+\tau)}$$
$$= \int_{R^1_\infty}[K(\lambda+\tau, \tau\,;\, R^2_{\lambda+\tau\tau})]e^{-j\omega\lambda}d\lambda$$
$$= \int_{R^1_\lambda(\tau)}[K(\lambda+\tau, \tau\,;\, R^2_{\lambda+\tau\tau})]e^{-j\omega\lambda}d\lambda \qquad \cdots\cdots\cdots\cdots(56)$$

Inversely transforming the above equation, the local co-variance matrix is given by

$$[K(\lambda+\tau, \tau\,;\, R^2_{\lambda+\tau\tau})] = \frac{1}{2\pi}\int_{R^1_\infty}[S(\omega, \tau\,;\, R^2_{\lambda\tau})]e^{j\omega\lambda}d\omega \qquad \cdots\cdots\cdots\cdots(57)$$

Substituting $\lambda = 0$ in eq. (57), the spectral respresentation of the co-variance matrix at any given time τ in the non-stationary process is obtained in a similar form to that in the stationary process :

$$[K(\tau,\tau\,;R^2_{\lambda+\tau\tau})]=[K(\tau,\tau)]=\frac{1}{2\pi}\int_{R^1_\infty}[S(\omega,\tau\,;R^2_{\lambda\tau})]d\omega \qquad \cdots\cdots\cdots\cdots(58)$$
$$(\tau,\tau)\in R^2_{\lambda+\tau\tau}$$

If there exists a one-dimensional local spectral density matrix defined by eq. (56) in the case where $R^2_{\lambda\tau}\to R^2_\infty$, we call it also the one-dimensional total spectral density matrix as in eq. (47).

$$[S_{\lambda\tau}(\omega,\tau)]=[S(\omega,\tau\,;R^2_{\lambda\tau}\to R^2_\infty)]=[S(\omega,\tau\,;R^2_\infty)] \qquad \cdots\cdots\cdots\cdots(59)$$

From eqs. (47)$\sim$(50), (55) and (56), the following equations are obtainable:

$$\begin{aligned}[S_{\lambda\tau}(\omega,\tau)]&=[S_{1\lambda+\tau\tau}(\omega,\tau)]e^{j\omega\tau}=[S_{2\lambda+\tau\tau}(\tau,\omega)]e^{-j\omega(\lambda+\tau)}\\&=\frac{1}{2\pi}\int_{R^1_\infty}[S_{\lambda+\tau\tau}(\omega,\omega_2)]e^{j(\omega-\omega_2)\tau}d\omega_2\\&=\frac{1}{2\pi}\int_{R^1_\infty}[S_{\lambda+\tau\tau}(\omega_1,\omega)]e^{j(\omega_1-\omega)(\lambda+\tau)}d\omega_1\\&=\int_{R^1_\infty}[K(\lambda+\tau,\tau)]e^{-j\omega\lambda}d\lambda \qquad \cdots\cdots\cdots\cdots(60)\end{aligned}$$

and

$$[K(\lambda+\tau,\tau)]=\frac{1}{2\pi}\int_{R^1_\infty}[S_{\lambda\tau}(\omega,\tau)]e^{j\omega\lambda}d\omega \qquad \cdots\cdots\cdots\cdots(61)$$

It is easily shown from eqs. (51), (55) and (60) that when the process is stationary the one-dimensisoal total spectral density matrix defined by eq. (59) agrees with the power spectral density matrix in the stationary process, that is,

$$[S_{\lambda\tau}(\omega,\tau)]=[S(\omega)] \qquad \cdots\cdots\cdots\cdots(62)$$

and that eqs. (60) and (61) reduce to the Wiener-Kintchin theorem in the stationary process. From eqs. (42), (43) and (56), the two-dimensional local spectral density matrix can be expressed by

$$\begin{aligned}[S(\omega_1,\omega_2\,;R^2_{\lambda+\tau\tau})]&=\int_{R^1_\infty}[S(\omega_1,\tau\,;R^2_{\lambda\tau})]e^{-j(\omega_1-\omega_2)\tau}d\tau\\&=\int_{R^1_\tau}[S(\omega_1,\tau\,;R^2_{\lambda\tau})]e^{-j(\omega_1-\omega_2)\tau}d\tau \qquad \cdots\cdots\cdots\cdots(63)\end{aligned}$$

Choosing the square domain ${}_sR^2_{\lambda+\tau\tau}$ and substituting $\omega_1=\omega_2=\omega$ in the above equation, the local energy spectral density matrix is given by

$$[S(\omega,\omega\,;{}_sR^2_{\lambda+\tau\tau})]=\int_{R^1_{\tau_L\tau_U}}[S(\omega,\tau\,;{}_pR^2_{\lambda\tau})]d\tau \qquad \cdots\cdots\cdots\cdots(64)$$

where ${}_pR^2_{\lambda\tau}$ represents the transformed paralleogram domain on the $\lambda-\tau$ plane. From eqs. (58), (62) and (64) the one-dimensional local spectral density matrix defined by eq. (56) can be interpreted as the local power spectral density matrix in the non-stationary process.

It can easily be shown that the one-dimensional local spectral density matrices defined by eqs. (42), (43) and (56) are all the additive set functions of the relevant two-dimensional time domains. Therefore, as in the two-dimensional case, the following expressions of one-dimensional local spectral density matrices, that is the weighted averages of the corresponding total spectral density matrices, are obtainable:

$$[S_1(\omega_1,\tau_2\,;\,{}_rR^2{}_{\tau_1\tau_2})]=\frac{1}{2\pi}D(\tau_2\,;\,R^1{}_{\tau_2})Q(\omega_1\,;\,R^1{}_{\tau_1})\underset{\omega_1}{*}[S_{1\tau_1\tau_2}(\omega_1,\tau_2)] \qquad (65)$$

$$[S_2(\tau_1,\omega_2\,;\,{}_rR^2{}_{\tau_1\tau_2})]=\frac{1}{2\pi}D(\tau_1;\,R^1{}_{\tau_1})[S_{2\tau_1\tau_2}(\tau_1,\omega_2)]\underset{\omega_2}{*}Q^*(\omega_2\,;\,R^1{}_{\tau_2}) \qquad (66)$$

$$[S(\omega,\tau\,;\,{}_pR^2{}_{\lambda\tau})]=\frac{1}{2\pi}D(\tau\,;\,R^1{}_\tau)(Q(\omega\,;\,R^1{}_{\lambda+\tau})e^{j\omega\tau})\underset{\omega}{*}[S_{\lambda\tau}(\omega,\tau)] \qquad (67)$$

And also, transforming the time variables by

$$\tau_i'=\tau_i-\tau_{ic},\quad i=1,2\,;\quad \lambda'=\lambda-\lambda_c,\quad \tau'=\tau-\tau_c\,;\quad \lambda_c=\tau_{1c}-\tau_{2c},\quad \tau_c=\tau_{2c} \qquad (68)$$

we can obtain similar expressions related to the new variables, τ_i', λ' and τ'.

Finally, we introduce the one-dimensional local spectral density matrix which is Hermitian and tends to the power spectral density $[S(\omega)]$ in the stationary process.

$$\begin{aligned}[S_H(\omega,\tau\,;\,R^2{}_{\lambda\tau},\alpha)]&=\alpha[S(\omega,\tau\,;\,R^2{}_{\lambda\tau})]+\alpha^*[S(\omega,\tau\,;\,R^2{}_{\lambda\tau})]^*\\&=\int_{R^1{}_\lambda(\tau)}(\alpha[K(\lambda+\tau,\tau\,;\,R^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}+\alpha^*[K(\lambda+\tau,\tau\,;\,R^2{}_{\lambda+\tau\tau})]^*e^{j\omega\lambda})d\lambda \qquad (69)\end{aligned}$$

where α is a complex-valued constant. Performing an inverse transform in the above equation, we have

$$\begin{aligned}&\alpha[K(\lambda+\tau,\tau\,;\,R^2{}_{\lambda+\tau\tau})]+\alpha^*[K(-\lambda+\tau,\tau\,;\,R^2{}_{\lambda+\tau\tau})]^*\\&=\alpha[K(\lambda+\tau,\tau\,;\,R^2{}_{\lambda+\tau\tau})]+\alpha^*[K(\tau,-\lambda+\tau\,;\,R^2{}_{\tau\lambda+\tau})]\\&=\frac{1}{2\pi}\int_{R^1{}_\infty}[S_H(\omega,\tau\,;\,R^2{}_{\lambda\tau},\alpha)]e^{j\omega\lambda}d\omega \qquad (70)\end{aligned}$$

Now, representing a rectangular domain ${}_rR^2{}_{\tau_1\tau_2}$ as the sum of a left upper right triangular domain ${}_{t_1}R^2{}_{\tau_1\tau_2}$ and a right lower right triangular domain ${}_{t_2}R^2{}_{\tau_1\tau_2}$, that is,

$${}_rR^2{}_{\tau_1\tau_2}={}_{t_1}R^2{}_{\tau_1\tau_2}+{}_{t_2}R^2{}_{\tau_1\tau_2} \qquad (71)$$

we perform the following transformation of the time variables:

$$\lambda=\tau_1-\tau_2,\quad \tau=\tau_2\quad \text{for}\quad {}_{t_1}R^2{}_{\tau_1\tau_2}\qquad \tilde\lambda=\tau_2-\tau_1,\quad \tilde\tau=\tau_1\quad \text{for}\quad {}_{t_2}R^2{}_{\tau_1\tau_2} \qquad (72)$$

Then the two right triangular domains, ${}_{t_1}R^2{}_{\tau_1\tau_2}$ and ${}_{t_2}R^2{}_{\tau_1\tau_2}$ are maped to the corresponding triangular domains, ${}_{t_1}R^2{}_{\lambda\tau}$ and ${}_{t_2}R^2{}_{\tilde\lambda\tilde\tau}$, respectively. In particular, if we select square domain ${}_sR^2{}_{\tau_1\tau_2}$, the two triangular domains ${}_{t_1}R^2{}_{\lambda\tau}$ and ${}_{t_2}R^2{}_{\tilde\lambda\tilde\tau}$ are reduced to the same right isosceles triangular domain ${}_tR^2$ which is on the left half-plane and has one side on the τ- or $\tilde\tau$-axis and other side parallel to λ- or $\tilde\lambda$-axis. And then, the integral over the square domain ${}_sR^2{}_{\tau_1\tau_2}$ is transformed to the following from:

$$\begin{aligned}\int_{{}_sR^2{}_{\tau_1\tau_2}}\cdot d\tau_1 d\tau_2&=\int_{{}_tR^2{}_{\lambda\tau}}\cdot d\lambda d\tau+\int_{{}_tR^2{}_{\tilde\lambda\tilde\tau}}\cdot d\tilde\lambda d\tilde\tau\\&=\int_{\tau_L}^{\tau_U}\cdot d\tau\int_{\tau_L-\tau}^{0}\cdot d\lambda+\int_{\tau_L}^{\tau_U}\cdot d\tilde\tau\int_{\tau_L-\tilde\tau}^{0}\cdot d\tilde\lambda \qquad (73)\end{aligned}$$

By making use of the above transformation, the two-dimensional local spectral density matrix can be expressed as follows:

$$[S(\omega_1,\omega_2\,;\,{}_sR^2{}_{\tau_1\tau_2})]=\int_{R^2{}_\infty}[K(\tau_1,\tau_2\,;\,{}_sR^2{}_{\tau_1\tau_2})]e^{-j(\omega_1\tau_1-\omega_2\tau_2)}d\tau_1 d\tau_2$$

$$=\int_{{}_tR^2{}_{\lambda\tau}}([K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]e^{-j(\omega_1(\lambda+\tau)-\omega_2\tau)}$$
$$+[K(\tau,\lambda+\tau\,;\,{}_sR^2{}_{\tau\lambda+\tau})]e^{-j(\omega_1\tau-\omega_2(\lambda+\tau))})d\lambda d\tau$$
$$=\int_{{}_tR^2{}_{\lambda\tau}}([K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]e^{-j(\omega_1(\lambda+\tau)-\omega_2\tau)}$$
$$+[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]^*e^{-j(\omega_1\tau-\omega_2(\lambda+\tau))})d\lambda d\tau \quad \cdots\cdots\cdots\cdots(74)$$

Substituting $\omega_1=\omega_2=\omega$ in eq. (74) and taking into consideration eq. (56), the local energy spectral density matrix can be expressed by using the Hermitian local power spectral density matrix which is obtainable by selecting $\alpha=1$ and $R^2{}_{\lambda\tau}={}_tR^2{}_{\lambda\tau}$ in eq. (69).

$$[S(\omega,\omega\,;\,{}_sR^2{}_{\tau_1\tau_2})]=\int_{{}_tR^2{}_{\lambda\tau}}([K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}$$
$$+[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]^*e^{j\omega\lambda})d\lambda d\tau$$
$$=\int_{\tau_L}^{\tau_U}d\tau\int_{\tau_L-\tau}^{0}d\lambda([K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}+[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]^*e^{j\omega\lambda})$$
$$=\int_{\tau_L}^{\tau_U}([S(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau})]+[S(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau})]^*)d\tau$$
$$=\int_{R^1{}_{\tau_L\tau_U}}[S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1)]d\tau \quad \cdots\cdots\cdots\cdots(75)$$

On the other hand, since the local energy spectral density matrix is substantially Hermitian, the following equation is obtained from eqs. (64) and (69) :

$$[S(\omega,\omega\,;\,{}_sR^2{}_{\tau_1\tau_2})]=\int_{R^1{}_{\tau_L\tau_U}}\left[S_H\left(\omega,\tau\,;\,{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]d\tau \quad \cdots\cdots\cdots\cdots(76)$$

Eqs. (64), (75) and (76) are the integral representations of the same local energy spectral density matrix with respect to ${}_sR^2{}_{\tau_1\tau_2}$, obtained by using the different local power spectral density matrices. It can be shown that the inverse Fourier transforms of these equations give the same result,

$$\frac{1}{2\pi}\int_{R^1{}_\infty}[S(\omega,\omega\,;\,{}_sR^2{}_{\lambda+\tau\tau})]e^{j\omega\lambda}d\omega=\int_{R^1{}_{\tau_L\tau_U}}[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]d\tau \quad \cdots\cdots(77)$$

by using of eqs. (57), (69) and the identities,

$$\int_{R^1{}_{\tau_L\tau_U}}([K(\lambda+\tau,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]+[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})])d\tau$$
$$=\frac{1}{2}\int_{R^1{}_{\tau_L\tau_U}}([K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]+[K(\tau,\tau-\lambda\,;\,{}_sR^2{}_{\tau\tau-\lambda})])d\tau$$
$$=\int_{R^1{}_{\tau_L\tau_U}}[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]d\tau \quad \cdots\cdots\cdots\cdots(78)$$

Substituting $\lambda=0$ in eq. (77) we obtain again eq. (13). The same equation can be obtained from eqs. (58) and (64).

The relation between the local co-variance matrix and the one-dimensional local power spectral density matrix introduced to eq. (75) can be obtained from eqs. (69) and (70) as follows:

$$[S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1)]=[S(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau})]+[S(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau})]^*$$
$$=\int_{\tau_L-\tau}^{0}([K(\lambda+\tau,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}+[K(\lambda+\tau,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]^*e^{j\omega\lambda})d\lambda$$

$$=\int_0^{\tau-\tau_L}([K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]e^{-j\omega\lambda}+[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]^*e^{j\omega\lambda})d\lambda$$
$$=\int_{\tau_L-\tau}^{0}[K(\tau+\lambda,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}d\lambda+\int_0^{\tau-\tau_L}[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]e^{-j\omega\lambda}d\lambda \quad\cdots\cdots\cdots\cdots(79)$$

and

$$[K(\tau+\lambda,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]+[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]$$
$$=\frac{1}{2\pi}\int_{R^1_\infty}[S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1]e^{j\omega\lambda}d\omega \quad\cdots\cdots\cdots\cdots(80)$$

From the definition of the local co-variance matrix given by eqs. (2) and (3) the first and the second terms of the left hand side of eq. (80) are zero for $\lambda>0$ and $\lambda<0$, respectively, and they take on the same value, that is, one-half of $[K(\tau,\tau)]$ for $\lambda=0$. Therefore eq. (80) can be rewritten as

$$[K(\tau+\lambda,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]=\frac{1}{2\pi}\int_{R^1_\infty}[S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1)]e^{j\omega\lambda}d\omega \qquad \text{for} \quad \lambda<0$$
$$[K(\tau,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]+[K(\tau,\tau\,;\,{}_{t_2}R^2{}_{\lambda+\tau\tau})]=[K(\tau,\tau\,;\,{}_tR^2{}_{\lambda+\tau\tau})]$$
$$=\frac{1}{2\pi}\int_{R^1_\infty}[S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1)]d\omega \qquad \text{for} \quad \lambda=0$$
$$[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]=\frac{1}{2\pi}\int_{R^1_\infty}[S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1)]e^{j\omega\lambda}d\omega \qquad \text{for} \quad \lambda>0 \quad\cdots\cdots(81)$$

In particular, for the real-valued non-stationary process, the following relations are valid:

$$[K(\lambda+\tau,\tau\,;\,R^2{}_{\lambda+\tau\tau})]^*=[K(\lambda+\tau,\tau\,;\,R^2{}_{\lambda+\tau\tau})]^T$$
$$[S(\omega,\tau\,;\,R^2{}_{\lambda\tau})]^*=[S(-\omega,\tau\,;\,R^2{}_{\lambda\tau})]^T \quad\cdots\cdots\cdots\cdots(82)$$

Therefore, if the coefficient α is real, the relation,

$$[S_H(\omega,\tau\,;\,R^2{}_{\lambda\tau},\alpha)]^T=[S_H(-\omega,\tau\,;\,R^2{}_{\lambda\tau},\alpha)] \quad\cdots\cdots\cdots\cdots(83)$$

is valid. By using the above relations, eqs. (79) and (80) can be written as the following equations for the real-valued non-stationary process:

$$[S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1)]=\int_0^{\tau-\tau_L}([K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]e^{-j\omega\lambda}$$
$$+[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]^Te^{j\omega\lambda})d\lambda$$
$$=\int_{\tau_L-\tau}^{0}[K(\tau+\lambda,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}d\lambda+\int_0^{\tau-\tau_L}[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]e^{-j\omega\lambda}d\lambda \quad\cdots\cdots\cdots\cdots(84)$$

and

$$[K(\tau+\lambda,\tau\,;\,{}_{t_1}R^2{}_{\lambda+\tau\tau})]+[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_{\tau\tau-\lambda})]$$
$$=\frac{1}{2\pi}\int_0^{\infty}([S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1)]e^{j\omega\lambda}+[S_H(\omega,\tau\,;\,{}_tR^2{}_{\lambda\tau},1)]^Te^{-j\omega\lambda})d\omega \quad\cdots\cdots\cdots(85)$$

In the stationary process, the triangular domains, ${}_{t_1}R^2{}_{\lambda+\tau\tau}$, ${}_{t_2}R^2{}_{\lambda+\tau\tau}$ and ${}_tR^2{}_{\lambda\tau}$ are replaced by a left upper half-plane, ${}_{t_1}R^2{}_\infty$, right lower half-plane, ${}_{t_2}R^2{}_\infty$ and a left half-plane, ${}_tR^2{}_\infty$, respectively. And eqs. (79) and (80) tend to the following equations:

$$[S_H(\omega,\tau\,;\,{}_tR^2{}_\infty,1)]=\int_0^\infty([R(\lambda)]e^{-j\omega\lambda}+[R(\lambda)]^*e^{j\omega\lambda})d\lambda$$
$$=\int_{R^1\infty}[R(\lambda)]e^{-j\omega\lambda}d\lambda=[S(\omega)] \qquad \cdots\cdots\cdots\cdots(86)$$

and

$$[K(\tau+\lambda,\tau\,;\,{}_{t_1}R^2{}_\infty)]+[K(\tau,\tau-\lambda\,;\,{}_{t_2}R^2{}_\infty)]$$
$$=(D(\lambda\,;\,{}_{t_1}R^2{}_\infty)+D(\lambda\,;\,{}_{t_2}R^2{}_\infty))[R(\lambda)]=[R(\lambda)]=\frac{1}{2\pi}\int_{R^1\infty}[S(\omega)]e^{j\omega\lambda}d\omega \qquad \cdots\cdots\cdots\cdots(87)$$

In the above discussions, it appears that the selection of a triangular domain, ${}_tR^2{}_{\lambda\tau}$ would make the convergence of inverse Fourier transforms given by eqs. (80), (81) and (85) poor in the neighbourhood of the boundary points $\lambda=0$ and $\lambda=\tau-\tau_L$. To avoid poor convergency at the point $\lambda=0$, we can define the other one-dimensional local power spectral density matrix by choosing $\alpha=1/2$ and the parallelogram domain ${}_pR^2{}_{\lambda\tau}$ which corresponds to the square domain ${}_sR^2{}_{\lambda+\tau\tau}$ in eq. (69). In this case, we can obtain the following pair of equations instead of eqs. (79) and (80).

$$\left[S_H\left(\omega,\tau\,;\,{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]=\frac{1}{2}\int_{\tau_L-\tau}^{\tau_U-\tau}([K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}$$
$$+[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]^*e^{j\omega\lambda})d\lambda$$
$$=\frac{1}{2}\int_{\tau_L-\tau}^{\tau_U-\tau}([K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}+[K(\tau,\lambda+\tau\,;\,{}_sR^2{}_{\tau\lambda+\tau})]e^{j\omega\lambda})d\lambda$$
$$=\frac{1}{2}\int_{\tau_L-\tau}^{\tau_U-\tau}[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}d\lambda+\frac{1}{2}\int_{\tau-\tau_U}^{\tau-\tau_L}[K(\tau,\tau-\lambda\,;\,{}_sR^2{}_{\tau\tau-\lambda})]e^{-j\omega\lambda}d\lambda \qquad \cdots\cdots\cdots\cdots(88)$$

and

$$\frac{1}{2}[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]+\frac{1}{2}[K(\tau-\lambda,\tau\,;\,{}_sR^2{}_{\tau-\lambda\tau})]^*$$
$$=\frac{1}{2}[K(\lambda+\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]+\frac{1}{2}[K(\tau,\tau-\lambda\,;\,{}_sR^2{}_{\tau\tau-\lambda})]$$
$$=\frac{1}{2\pi}\int_{R^1\infty}\left[S_H\left(\omega,\tau\,;\,{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]e^{j\omega\lambda}d\omega \qquad \cdots\cdots\cdots\cdots(89)$$

Substituting $\lambda=0$ in eq. (89), we have

$$[K(\tau,\tau\,;\,{}_sR^2{}_{\lambda+\tau\tau})]=\frac{1}{2\pi}\int_{R^1\infty}\left[S_H\left(\omega,\tau\,;\,{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]d\omega \qquad \cdots\cdots\cdots\cdots(90)$$

For the stationary process, replacing both ${}_sR^2{}_{\lambda+\tau\tau}$ and ${}_pR^2{}_{\lambda\tau}$ by $R^2{}_\infty$, eqs. (88) and (89) are (89) are reduced to

$$\left[S_H\left(\omega,\tau\,;\,R^2{}_\infty,\frac{1}{2}\right)\right]=\frac{1}{2}\int_{R^1\infty}([R(\lambda)]e^{-j\omega\lambda}+[R(\lambda)]^*e^{j\omega\lambda})d\lambda$$
$$=\frac{1}{2}([S(\omega)]+[S(\omega)]^*)=[S(\omega)] \qquad \cdots\cdots\cdots\cdots(91)$$

and

$$\frac{1}{2}([R(\lambda)]+[R(-\lambda)]^*)=[R(\lambda)]=\frac{1}{2\pi}\int_{R^1_\infty}[S(\omega)]e^{j\omega\lambda}d\omega \quad \cdots\cdots\cdots\cdots(92)$$

By the above discussions it is shown that the one-dimensional local power spectral density matrices defined by eqs. (79) and (88) are Hermitian matrices and reduce to the so-called power spectral density matrix in the stationary process. The former constitute a Fourier transform pair with the local co-variance matrix in a modified sense as shown in eqs. (79)~(81), but the inverse Fourier transform of the latter does not give directly the local co-variance matrix. To obtain the local co-variance matrix in the latter case, we must introduce the auxiliary spectral density matrix defined as

$$\begin{aligned}\left[S_H'\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]&=\frac{1}{2}\int_{\tau_L-\tau}^{\tau_U-\tau}([K(\lambda+\tau,\tau\,;{}_sR^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}\\&\qquad-[K(\lambda+\tau,\tau\,;{}_sR^2{}_{\lambda+\tau\tau})]^*e^{j\omega\lambda})d\lambda\\&=\frac{1}{2}\int_{\tau_L-\tau}^{\tau_U-\tau}[K(\lambda+\tau,\tau\,;{}_sR^2{}_{\lambda+\tau\tau})]e^{-j\omega\lambda}d\lambda\\&\qquad-\frac{1}{2}\int_{\tau-\tau_U}^{\tau-\tau_L}[K(\tau,\tau-\lambda\,;{}_sR^2{}_{\tau\tau-\lambda})]e^{-j\omega\lambda}d\lambda \quad\cdots\cdots\cdots\cdots(93)\end{aligned}$$

then,

$$\begin{aligned}&\frac{1}{2}[K(\lambda+\tau,\tau\,;{}_sR^2{}_{\lambda+\tau\tau})]-\frac{1}{2}[K(\tau,\tau-\lambda\,;{}_sR^2{}_{\tau\tau-\lambda})]\\&=\frac{1}{2\pi}\int_{R^1_\infty}\left[S_H'\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]e^{j\omega\lambda}d\omega \quad\cdots\cdots\cdots\cdots(94)\end{aligned}$$

By adding eq. (89) and eq. (94) or subtracting eq. (94) from eq. (89) we obtain

$$\begin{aligned}[K(\lambda+\tau,\tau\,;{}_sR^2{}_{\lambda+\tau\tau})]&=\frac{1}{2\pi}\int_{R^1_\infty}\left(\left[S_H\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]\right.\\&\qquad\left.+\left[S_H'\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]\right)e^{j\omega\lambda}d\omega\\ [K(\tau,\tau-\lambda\,;{}_sR^2{}_{\tau\tau-\lambda})]&=\frac{1}{2\pi}\int_{R^1_\infty}\left(\left[S_H\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]\right.\\&\qquad\left.-\left[S_H'\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]\right)e^{j\omega\lambda}d\omega \quad\cdots\cdots\cdots\cdots(95)\end{aligned}$$

It is noted that each equation in the above is valid for all λ in R^1_∞ and that they constitute a pair of transposed conjugate matrices if the sign of λ is changed in either of these equations, because the matrix defined by eq. (93) is a skew Hermitian matrix, that is,

$$\left[S_H'\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]^*=-\left[S_H'\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]$$

Applying the Fourier transform operator with respect to λ to each equation in (95) and considering eq. (56) the following equations are obtained:

$$\begin{aligned}[S(\omega,\tau\,;{}_pR^2{}_{\lambda\tau})]&=\left[S_H\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]+\left[S_H'\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]\\ [S(\omega,\tau\,;{}_pR^2{}_{\lambda\tau})]^*&=\left[S_H\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]-\left[S_H'\left(\omega,\tau\,;{}_pR^2{}_{\lambda\tau},\frac{1}{2}\right)\right]\end{aligned}$$
$$\cdots\cdots\cdots\cdots(96)$$

And, from the above equations we obtain

$$\left[S_H\left(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau}, \frac{1}{2}\right)\right] = \frac{1}{2}[S(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau})] + \frac{1}{2}[S(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau})]^*$$
$$\left[S_H'\left(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau}, \frac{1}{2}\right)\right] = \frac{1}{2}[S(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau})] - \frac{1}{2}[S(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau})]^*$$

The first equation in the above is the definition of the one-dimensional local Hermitian spectral density matrix, $[S_H(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau}, 1/2)]$ as given by eq. (69), and the second equation is considered as another definition of the auxiliary skew Hermitian matrix $[S_H'(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau}, 1/2)]$, given by using the one-dimensional local power spectral density matrix $[S(\omega, \tau\,;\, {}_{\nu}R^2{}_{\lambda\tau})]$ defined by eq. (56).

On the other hand, although the one-dimensional local power spectral density matrix defined by eq. (56) is not a Hermitian matrix, it constitutes a Fourier transform pair with the local co-variance matrix as shown in eqs. (56) and (57), and there is no difficulty respecting the convergency of the inverse Fourier transform at $\lambda=0$ in the evaluating of the cross-variance matrix at any time τ. Also, it is reduced to the power spectral density matrix in the stationary process if the process is stationary and the finite domain tends to the infinite full domain.

The several kinds of one-dimensional local spectral density matrices discussed above are all defined with reference to the finite two-dimensional time domains, and their relations to the local co-variance matrix defined in the relevant finite two-dimensional time domain are obtained. However, it is noted that the explicit time variable contained in these one-dimensional local power spectral density matrices can be considered as a parameter rather than a variable restricted in the finite domains. In other words, we need not take particular note of the boundary points with respect to the explicit time variable but only of the boundary points with respect to the implicit time variable. Then the two-dimensional time domains can be replaced by the one-dimensional time domains with respect to the implicit time variable. Since the boundary points with respect to the implicit time variable are generally expressed as the functions of the explicit time variable, the one-dimensional local spectral density matrices and the local co-variance matrix are symbolically rewritten as

$$\begin{aligned}
&[S_1(\omega_1, \tau_2\,;\, R^2{}_{\tau_1\tau_2})] \equiv D(\tau_2\,;\, R^1{}_{\tau_2})[S_1(\omega_1, \tau_2\,;\, R^1{}_{\tau_1}(\tau_2))]\\
&[S_2(\tau_1, \omega_2\,;\, R^2{}_{\tau_1\tau_2})] \equiv D(\tau_1\,;\, R^1{}_{\tau_1})[S_2(\tau_1, \omega_2\,;\, R^1{}_{\tau_2}(\tau_1))]\\
&[S(\omega, \tau\,;\, R^2{}_{\lambda\tau})] \equiv D(\tau\,;\, R^1{}_{\tau})[S(\omega, \tau\,;\, R^1{}_{\lambda}(\tau))]\\
&[S_H(\omega, \tau\,;\, R^2{}_{\lambda\tau}, \alpha)] \equiv D(\tau\,;\, R^1{}_{\tau})[S_H(\omega, \tau\,;\, R^1{}_{\lambda}(\tau), \alpha)] \qquad \cdots\cdots\cdots\cdots(97)
\end{aligned}$$

and

$$\begin{aligned}
[K(\lambda+\tau, \tau\,;\, R^2{}_{\lambda+\tau\tau})] &\equiv D(\tau\,;\, R^1{}_{\tau})[K(\lambda+\tau, \tau\,;\, R^1{}_{\lambda}(\tau))]\\
&= D(\tau\,;\, R^1{}_{\tau}) D(\lambda\,;\, R^1{}_{\lambda}(\tau))[K(\lambda+\tau, \tau)] \qquad \cdots\cdots\cdots\cdots(98)
\end{aligned}$$

Finally we will discuss the relationship between the one-dimensional local power spectral density matrices and the spectral density defined by D. G. Lampard for a real-valued non-stationary process[6]. By using eqs. (64), (75) and (76), the partial derivative with respect to the upper limit, τ_U of the square domain, of the local energy spectral density matrix can be expressed as follows :

$$\frac{\partial}{\partial \tau_U}[S(\omega, \omega\,;\, {}_sR^2{}_{\lambda+\tau\tau})] = [S(\omega, \tau_U\,;\, {}_pR^2{}_{\lambda\tau})] + \int_{R^1{}_{\tau_L\tau_U}} \frac{\partial}{\partial \tau_U}[S(\omega, \tau\,;\, {}_pR^2{}_{\lambda\tau})]d\tau$$
$$= [S(\omega, \tau_U\,;\, {}_pR^2{}_{\lambda\tau})] + [S(\omega, \tau_U\,;\, {}_pR^2{}_{\lambda\tau})]^*$$
$$= D(\tau_U\,;\, R^1{}_{\tau_U})[S_H(\omega, \tau_U\,;\, {}_pR^1{}_\lambda(\tau_U), 1)] \qquad \cdots\cdots(99)$$

$$\frac{\partial}{\partial \tau_U}[S(\omega, \omega\,;\, {}_sR^2{}_{\lambda+\tau\tau})] = [S_H(\omega, \tau_U\,;\, {}_tR^2{}_{\lambda\tau}, 1)] + \int_{R^1{}_{\tau_L\tau_U}} \frac{\partial}{\partial \tau_U}[S_H(\omega, \tau\,;\, {}_tR^2{}_{\lambda\tau}, 1)]d\tau$$
$$= D(\tau_U\,;\, R^1{}_{\tau_U})[S_H(\omega, \tau_U\,;\, {}_tR^1{}_\lambda(\tau_U), 1)] \qquad \cdots\cdots(100)$$

$$\frac{\partial}{\partial \tau_U}[S(\omega, \omega\,;\, {}_sR^2{}_{\lambda+\tau\tau})] = \left[S_H\left(\omega, \tau_U\,;\, {}_pR^2{}_{\lambda\tau}, \frac{1}{2}\right)\right] + \int_{R^1{}_{\tau_L\tau_U}} \frac{\partial}{\partial \tau_U}\left[S_H\left(\omega, \tau\,;\, {}_pR^2{}_{\lambda\tau}, \frac{1}{2}\right)\right]d\tau$$
$$= \left[S_H\left(\omega, \tau_U\,;\, {}_pR^2{}_{\lambda\tau}, \frac{1}{2}\right)\right] + \left[S_H\left(\omega, \tau_U\,;\, {}_pR^2{}_{\lambda\tau}, \frac{1}{2}\right)\right]^*$$
$$= 2\left[S_H\left(\omega, \tau_U\,;\, {}_pR^2{}_{\lambda\tau}, \frac{1}{2}\right)\right] = D(\tau_U\,;\, R^1{}_{\tau_U})[S_H(\omega, \tau_U\,;\, {}_pR^1{}_\lambda(\tau_U), 1)] \qquad \cdots\cdots(101)$$

Then it follows that

$$[S_H(\omega, \tau_U\,;\, {}_pR^1{}_\lambda(\tau_U), 1)] = [S_H(\omega, \tau_U\,;\, {}_tR^1{}_\lambda(\tau_U), 1)]$$
$$= [S_H(\omega, \tau_U\,;\, R^1{}_\lambda(\tau_U), 1)] \equiv [S_H(\omega, \tau_U\,;\, \tau_L)] \qquad \cdots\cdots(102)$$

Hence the following equations are obtained by making use of eqs. (79), (80), (88), (89) and (98).

$$[S_H(\omega, \tau_U\,;\, \tau_L)] = \int_0^{\tau_U-\tau_L} ([K(\tau_U, \tau_U-\lambda\,;\, R^1{}_{-\lambda}(\tau_U))]e^{j\omega\lambda} + [K(\tau_U, \tau_U-\lambda\,;\, R^1{}_{-\lambda}(\tau_U))]^* e^{j\omega\lambda})d\lambda$$
$$= \int_{\tau_L-\tau_U}^0 [K(\tau_U+\lambda, \tau_U\,;\, R^1{}_\lambda(\tau_U))]e^{-j\omega\lambda}d\lambda + \int_0^{\tau_U-\tau_L} [K(\tau_U, \tau_U-\lambda\,;\, R^1{}_{-\lambda}(\tau_U))]e^{-j\omega\lambda}d\lambda$$
$$= \int_{R^1{}_\infty} ([K(\tau_U+\lambda, \tau_U\,;\, R^1{}_\lambda(\tau_U))] + [K(\tau_U, \tau_U-\lambda\,;\, R^1{}_{-\lambda}(\tau_U))])e^{-j\omega\lambda}d\lambda \qquad \cdots\cdots(103)$$

and

$$[K(\tau_U+\lambda, \tau_U\,;\, R^1{}_\lambda(\tau_U))] + [K(\tau_U, \tau_U-\lambda\,;\, R^1{}_{-\lambda}(\tau_U))]$$
$$= \frac{1}{2\pi}\int_{R^1{}_\infty} [S_H(\omega, \tau_U\,;\, \tau_L)]e^{j\omega\lambda}d\omega \qquad \cdots\cdots(104)$$

where

$$R^1{}_\lambda(\tau_U) = [\tau_L-\tau_U, 0], \quad R^1{}_{-\lambda}(\tau_U) = -R^1{}_\lambda(\tau_U) = [0, \tau_U-\tau_L] \qquad \cdots\cdots(105)$$

As previously mentioned, the explicit time variable can be considered as a parameter, and we can replace τ_U by τ and ${}_sR^2{}_{\lambda+\tau\tau}$ by $R^2{}_{\tau_L\tau}$. Then, eqs. (99)~(105) can be expressed as

$$[S_H(\omega, \tau\,;\, \tau_L)] = \frac{\partial}{\partial \tau}[S(\omega, \omega\,;\, R^2{}_{\tau_L\tau})]$$
$$= \int_{R^1{}_\infty} ([K(\tau+\lambda, \tau\,;\, R^1{}_\lambda(\tau))] + [K(\tau, \tau-\lambda\,;\, R^1{}_{-\lambda}(\tau))])e^{-j\omega\lambda}d\lambda \qquad \cdots\cdots(106)$$

$$[K(\tau+\lambda, \tau\,;\, R^1{}_\lambda(\tau)] + [K(\tau, \tau-\lambda\,;\, R^1{}_{-\lambda}(\tau))]$$

$$=\frac{1}{2\pi}\int_{R^1_\infty}[S_H(\omega,\tau\,;\tau_L)]e^{j\omega\lambda}d\omega \qquad \cdots\cdots\cdots\cdots(107)$$

where

$$R^1_\lambda(\tau)=[\tau_L-\tau,0],\quad R^1_{-\lambda}(\tau)=-R^1_\lambda(\tau)=[0,\tau-\tau_L] \qquad \cdots\cdots\cdots\cdots(108)$$

The above equations determined at the boundary points τ_U are completely equivalent to eqs. (79) and (80) which are valid inside the two-dimensional triangular time domain. Of course, this is directly obtainable from eq. (21) and the property of the explicit time variable as a parameter. As shown in eq. (107), the inverse Fourier transform of the one-dimensional local power spectral density matrix given by eq. (106) converges to the value of the co-variance matrix on the orthogonal line-segments, $\tau_L<\tau_1\leqq\tau$, $\tau_2=\tau$ and $\tau_1=\tau$, $\tau_L<\tau_2\leqq\tau$ and one-half of the value of the co-variance matrix at the points $\tau_1=\tau_L$, $\tau_2=\tau$ and $\tau_1=\tau$, $\tau_2=\tau_L$. However, as previously mentioned, the convergency of the inverse Fourier transform in the neighbourhood of the points $\tau_1=\tau_2=\tau$ would be poor even if the convergence to the value of the co-variance matrix at $\tau_1=\tau_2=\tau$ might be guaranteed, since this points corresponds to one of the boundary points, $\lambda=0$, of the implicit time variable.

The spectral density matrix defined by D. G. Lampard corresponds to the limiting case of the one-dimensional local power spectral density matrix defined by eq. (106) taking $\tau_L\rightarrow-\infty$ for the case of the real-valued non-stationary process. Assuming the existence of the limit,

$$[\bar{S}_H(\omega,\tau)]=[S_H(\omega,\tau\,;-\infty)] \qquad \cdots\cdots\cdots\cdots(109)$$

we obtain the following expressions by using eqs. (84), (85), (106) and (107).

$$[\bar{S}_H(\omega,\tau)]=\int_0^\infty([K(\tau,\tau-\lambda\,;R^1_{0\infty})]e^{-j\omega\lambda}+[K(\tau,\tau-\lambda\,;R^1_{0\infty})]^Te^{j\omega\lambda})d\lambda$$
$$=\int_{R^1_\infty}([K(\tau+\lambda,\tau\,;R^1_{-\infty0})]+[K(\tau,\tau-\lambda\,;R^1_{0\infty})])e^{-j\omega\lambda}d\lambda \qquad \cdots\cdots\cdots(110)$$

$$[K(\tau,\tau-\lambda\,;R^1_{0\infty})]+[K(\tau,\tau+\lambda\,;R^1_{-\infty0})]^T$$
$$=[K(\tau+\lambda,\tau\,;R^1_{-\infty0})]+[K(\tau,\tau-\lambda\,;R^1_{0\infty})]$$
$$=\frac{1}{2\pi}\int_0^\infty([\bar{S}_H(\omega,\tau)]e^{j\omega\lambda}+[\bar{S}_H(\omega,\tau)]^Te^{-j\omega\lambda})d\omega$$
$$=\frac{1}{2\pi}\int_{R^1_\infty}[\bar{S}_H(\omega,\tau)]e^{j\omega\lambda}d\omega \qquad \cdots\cdots\cdots\cdots(111)$$

where

$$R^1_{-\infty0}=(-\infty,0],\quad R^1_{0\infty}=[0,\infty) \qquad \cdots\cdots\cdots\cdots(112)$$

3. Input and output relations of the local co-variance matrix and the local spectral density matrix

The deviation vector of the output response of a multi-input and -output linear discrete system with complex-valued time-variant coefficients subjected to a finite duration of a complex-valued non-stationary random input vector is given by the following equation:

$$\{\eta_d(\tau)\}=\{\eta(\tau)\}-E\{\eta(\tau)\}=\int_{-\infty}^\tau[g(\tau,\mu)]D(\mu\,;R^1_\mu)\{f_d(\mu)\}d\mu \qquad \cdots\cdots\cdots(113)$$

$$E\{\eta(\tau)\}=\int_{-\infty}^{\tau}[g(\tau,\mu)]D(\mu\,;R^1{}_{\mu})E\{f(\mu)\}d\mu+\{\eta(\tau)\}_i \quad \cdots\cdots\cdots\cdots(114)$$

$$\{f_d(\mu)\}=\{f(\mu)\}-E\{f(\mu)\} \quad \cdots\cdots\cdots\cdots(115)$$

where $\{\eta(\tau)\}$, $E\{\eta(\tau)\}$ and $\{\eta_d(\tau)\}$ are a complex-valued output random vector, its mean vector and its deviation vector, respectively, and $\{f(\mu)\}$, $E\{f(\mu)\}$, $\{f_d(\mu)\}$ are the corresponding input vectors. τ and μ are the real-valued time variables in $R^1{}_{\infty}$. $[g(\tau,\mu)]$ is the unit impulsive response matrix of the time-variant linear discrete system with complex-valued coefficients. $D(\mu\,;R^1{}_{\mu})$ is the cutoff operator defined by eq. (3). And $\{\eta(\tau)\}_i$ is the output response vector resulting from the initial conditions at μ_L which is the lower boundary of $R^1{}_{\mu}$. For the sake of simplicity, the initial conditions are assumed here to be deterministic. For the case where the initial conditions are random, the following equations should be used instead of eqs. (113) and (114).

$$\{\eta_d(\tau)\}=\int_{-\infty}^{\tau}[g(\tau,\mu)]D(\mu\,;R^1{}_{\mu})\{f_d(\mu)\}d\mu+[i(\tau,\mu_L)]\{i_d(\mu_L)\} \quad \cdots\cdots\cdots\cdots(116)$$

$$E\{\eta(\tau)\}=\int_{-\infty}^{\tau}[g(\tau,\mu)]D(\mu\,;R^1{}_{\mu})E\{f(\mu)\}d\mu+[i(\tau,\mu_L)]E\{i(\mu_L)\} \quad \cdots\cdots\cdots(117)$$

where $[i(\tau,\mu_L)]$ is a unit impulsive response matrix related to the initial conditions and $\{i(\mu_L)\}$, $E\{i(\mu_L)\}$ and $\{i_d(\mu_L)\}$ are an initial random vector at μ_L, its mean vector and its deviation vector, respectively. Under the assumption that $\{i(\mu_L)\}$ is a deterministic vector, we have

$$E\{i(\mu_L)\}=\{i(\mu_L)\},\quad \{i_d(\mu_L)\}=\{0\} \quad \cdots\cdots\cdots\cdots(118)$$

Then, eqs. (116) and (117) are reduced to eqs. (113) and (114) by making use of the notation,

$$\{\eta(\tau)\}_i=[i(\tau,\mu_L)]\{i(\mu_L)\} \quad \cdots\cdots\cdots\cdots(119)$$

The input and output relation of the local co-variance matrix defined in the two-dimensional time domain is easily determined from eqs. (2) and (113) as follows:

$$\begin{aligned}&[{}_0K(\tau_1,\tau_2\,;{}_{0r}R^2{}_{\tau_1\tau_2})]\\&=D(\tau_1,\tau_2\,;{}_{0r}R^2{}_{\tau_1\tau_2})\int_{-\infty}^{\tau_1}d\mu_1\int_{-\infty}^{\tau_2}d\mu_2[g(\tau_1,\mu_1)][{}_IK(\mu_1,\mu_2\,;{}_IR^2{}_{\mu_1\mu_2})][g(\tau_2,\mu_2)]^*\end{aligned} \quad \cdots\cdots\cdots\cdots(120)$$

where subscripts I and 0 denote the quantities with respect to the input and the output, respectively. Substituting ${}_{0r}R^2{}_{\tau_1\tau_2}=R^2{}_{\infty}$ in eq. (120), we obtain the co-variance matrix defined in $R^2{}_{\infty}$.

The input and output relation of the two-dimensional local spectral density matrix can be obtained from eq. (120), by making use of its correspondence to the local co-variance matrix as a double Fourier transform.

$$\begin{aligned}[{}_0S(\omega_1,\omega_2\,;{}_{0r}R^2{}_{\tau_1\tau_2})]=\frac{1}{(2\pi)^2}\int_{R^2{}_{\infty}}&[G(\omega_1,\omega_1'\,;{}_0R^1{}_{\tau_1})][{}_IS(\omega_1',\omega_2'\,;{}_IR^2{}_{\mu_1\mu_2})]\\&\cdot[G(\omega_2,\omega_2'\,;{}_0R^1{}_{\tau_2})]^*d\omega_1'd\omega_2'\end{aligned} \quad \cdots\cdots\cdots\cdots(121)$$

where

$$[G(\omega_i, \omega_i' ; {}_0R^1\tau_i)] = \int_{{}_0R^1\tau_i} e^{-j\omega_i\tau}d\tau\int_{-\infty}^{\tau}[g(\tau, \mu)]e^{j\omega_i'\mu}d\mu$$
$$= \int_{{}_0R^1\tau_i}[G(\omega_i' ; \tau)]e^{-j(\omega_i-\omega_i')\tau}d\tau \qquad \cdots\cdots\cdots\cdots(122)$$

and

$$[G(\omega_i' ; \tau)] = \int_{-\infty}^{\tau}[g(\tau, \mu)]e^{-j(\tau-\mu)\omega_i'}d\mu \qquad \cdots\cdots\cdots\cdots(123)$$

The quantity given by eq. (123) is the complex transfer function of the time-variant linear discrete system, which means the frequency characteristics at time τ.

As mentioned above, the local co-variance matrix of the output can be expressed as the inverse double Fourier transform of the two-dimensional local spectral density matrix of the output, that is,

$$[{}_0K(\tau_1, \tau_2 ; {}_{0r}R^2\tau_1\tau_2)] = \frac{1}{(2\pi)^2}\int_{R^2_\infty}[{}_0S(\omega_1, \omega_2 ; {}_{0r}R^2\tau_1\tau_2)]e^{j(\omega_1\tau_1-\omega_2\tau_2)}d\omega_1 d\omega_2 \quad \cdots\cdots(124)$$

Therefore, the co-variance matrix of the output, defined in R^2_∞ can be given by the following formula from eqs. (121) and (124) :

$$[{}_0K(\tau_1, \tau_2)] = [{}_0K(\tau_1, \tau_2 ; R^2_\infty)]$$
$$= \frac{1}{(2\pi)^2}\int_{R^2_\infty}[{}_0S(\omega_1, \omega_2 ; R^2_\infty)]e^{j(\omega_1\tau_1-\omega_2\tau_2)}d\omega_1 d\omega_2$$
$$= \frac{1}{(2\pi)^4}\int_{R^2_\infty}d\omega_1'd\omega_2'\int_{R^2_\infty}d\omega_1 d\omega_2[G(\omega_1, \omega_1' ; R^1_\infty)][{}_iS(\omega_1', \omega_2' ; {}_iR^2\mu_1\mu_2)]$$
$$\cdot[G(\omega_2, \omega_2' ; R^1_\infty)]^* e^{j(\omega_1\tau_1-\omega_2\tau_2)} \qquad \cdots\cdots\cdots\cdots(125)$$

On the other hand, substituting ${}_{0r}R^2\tau_1\tau_2 = R^2_\infty$ and the integral representation of the local co-variance matrix of the input similar to eq. (124), in eq. (120), we can obtain another expression of the co-variance matrix of the output as follows :

$$[{}_0K(\tau_1, \tau_2)] = \frac{1}{(2\pi)^2}\int_{R^2_\infty}[{}_0S(\omega_1', \omega_2' ; \tau_1, \tau_2)]e^{j(\omega_1'\tau_1-\omega_2'\tau_2)}d\omega_1'd\omega_2'$$
$$= \frac{1}{(2\pi)^2}\int_{R^2_\infty}[{}_0\tilde{S}(\omega_1', \omega_2' ; \tau_1, \tau_2)]d\omega_1'd\omega_2' \qquad \cdots\cdots\cdots\cdots(126)$$

where

$$[{}_0S(\omega_1', \omega_2' ; \tau_1, \tau_2)] = [G(\omega_1' ; \tau_1)][{}_iS(\omega_1', \omega_2' ; {}_iR^2\mu_1\mu_2)][G(\omega_2' ; \tau_2)]^* \quad \cdots\cdots\cdots(127)$$
$$[{}_0\tilde{S}(\omega_1', \omega_2' ; \tau_1, \tau_2)] = [X(\omega_1' ; \tau_1)][{}_iS(\omega_1', \omega_2' ; {}_iR^2\mu_1\mu_2)][X(\omega_2' ; \tau_2)]^* \cdots\cdots\cdots(128)$$

and

$$[X(\omega_i' ; \tau_i)] = [G(\omega_i' ; \tau_i)]e^{j\omega_i'\tau_i} = \int_{-\infty}^{\tau_i}[g(\tau_i, \mu)]e^{j\omega_i'\mu}d\mu \quad \cdots\cdots\cdots\cdots(129)$$

As regards the quantities given by eqs. (127) and (128), the following relations are valid :

$$[{}_0S(\omega_1', \omega_2' ; \tau_1, \tau_2)]^* = [{}_0S(\omega_2', \omega_1' ; \tau_2, \tau_1)]$$
$$[{}_0\tilde{S}(\omega_1', \omega_2' ; \tau_1, \tau_2)]^* = [{}_0\tilde{S}(\omega_2', \omega_1' ; \tau_2, \tau_1)] \qquad \cdots\cdots\cdots\cdots(130)$$

The quantity expressed by eq. (129) is the output response of the time-variant linear discrete system subjected to a complex harmonic excitation.

Both eqs. (125) and (126) are the spectral representations of the co-variance matrix of the output. Indeed, eq. (126) can be obtained from eq. (125) by using the following equation:

$$\int_{R^1_\infty}[G(\omega_i,\omega_i';R^1_\infty)]e^{j\omega_i\tau_i}d\omega_i=\int_{R^1_\infty}d\tau[G(\omega_i';\tau)]e^{j\omega_i'\tau}\int_{R^1_\infty}d\omega_i e^{-j(\tau-\tau_i)\omega_i}$$
$$=2\pi[G(\omega_i';\tau_i)]e^{j\omega_i'\tau_i} \qquad \cdots\cdots\cdots\cdots(131)$$

However, it should be noted that the output spectral density matrix used in eq. (125) does constitute a pair of Fourier transforms with the co-variance matrix in R^2_∞, but that used in eq. (126) does not. In the following, we will generally discuss the relations between the local co-variance matrix of the output and the two kinds of two-dimensional spectral density matrices given by eq. (121) and eq. (127), respectively. Denoting the integrand in eq. (121) as

$$[{}_0S(\omega_1,\omega_1';\omega_2,\omega_2';{}_IR^2_{\mu_1\mu_2},{}_{0\tau}R^2_{\tau_1\tau_2})]$$
$$=[G(\omega_1,\omega_1';{}_0R^1_{\tau_1})][{}_IS(\omega_1',\omega_2';{}_IR^2_{\mu_1\mu_2})][G(\omega_2,\omega_2';{}_0R^1_{\tau_2})]^* \cdots\cdots(132)$$

and putting

$$\omega_i''=\omega_i-\omega_i' \qquad \cdots\cdots\cdots\cdots(133)$$

in eq. (122), eq. (132) can be expressed as follows.

$$[{}_0S(\omega_1,\omega_1';\omega_2,\omega_2';{}_IR^2_{\mu_1\mu_2},{}_{0\tau}R^2_{\tau_1\tau_2})]$$
$$\equiv[{}_0\tilde{\tilde{S}}(\omega_1'',\omega_1';\omega_2'',\omega_2';{}_IR^2_{\mu_1\mu_2},{}_{0\tau}R^2_{\tau_1\tau_2})]$$
$$=[\tilde{\tilde{G}}(\omega_1'',\omega_1';{}_0R^1_{\tau_1})][{}_IS(\omega_1',\omega_2';{}_IR^2_{\mu_1\mu_2})][\tilde{\tilde{G}}(\omega_2'',\omega_2';{}_0R^1_{\tau_2})]^* \cdots\cdots\cdots(134)$$

where

$$[\tilde{\tilde{G}}(\omega_i'',\omega_i';{}_0R^1_{\tau_i})]=\int_{{}_0R^1_{\tau_i}}[G(\omega_i';\tau)]e^{-j\omega_i''\tau}d\tau \qquad \cdots\cdots\cdots\cdots(135)$$

Since the following equations are valid,

$$\frac{1}{2\pi}\int_{R^1_\infty}[G(\omega_i,\omega_i';{}_0R^1_{\tau_i})]e^{j\omega_i\tau_i}d\omega_i$$
$$=\frac{1}{2\pi}\int_{{}_0R^1_{\tau_i}}d\tau[G(\omega_i';\tau)]e^{j\omega_i'\tau}\int_{R^1_\infty}d\omega_i e^{-j(\tau-\tau_i)\omega_i}$$
$$=D(\tau_i;{}_0R^1_{\tau_i})[G(\omega_i';\tau_i)]e^{j\omega_i'\tau_i} \qquad \cdots\cdots\cdots\cdots(136)$$

and

$$\frac{1}{2\pi}\int_{R^1_\infty}[\tilde{\tilde{G}}(\omega_i'',\omega_i;{}_0R^1_{\tau_i})]e^{j\omega_i''\tau_i}d\omega_i''=\frac{1}{2\pi}\int_{{}_0R^1_{\tau_i}}[G(\omega_i';\tau)]\int_{R^1_\infty}d\omega_i''e^{-j(\tau-\tau_i)\omega_i''}$$
$$=D(\tau_i;{}_0R^1_{\tau_i})[G(\omega_i';\tau_i)] \qquad \cdots\cdots\cdots\cdots(137)$$

the inverse double Fourier transforms of eqs. (132) and (134) result in

$$D(\tau_1,\tau_2;{}_{0\tau}R^2_{\tau_1\tau_2})[{}_0S(\omega_1',\omega_2';\tau_1,\tau_2)]e^{j(\omega_1'\tau_1-\omega_2'\tau_2)}$$
$$\underset{\tau_1\tau_2}{\overset{\omega_1\omega_2}{\supset}}[{}_0S(\omega_1,\omega_1';\omega_2,\omega_2';{}_IR^2_{\mu_1\mu_2},{}_{0\tau}R^2_{\tau_1\tau_2})] \qquad \cdots\cdots\cdots\cdots(138)$$

and

$$D(\tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})[{}_0S(\omega_1', \omega_2'\,;\, \tau_1, \tau_2)] \overset{\omega_1''\omega_2''}{\underset{\tau_1\tau_2}{\supset}} [{}_0\overset{\approx}{S}(\omega_1'', \omega_1'\,;\, \omega_2'', \omega_2'\,;\, {}_IR^2_{\mu_1\mu_2}, {}_{0\tau}R^2_{\tau_1\tau_2})] \quad \cdots\cdots\cdots\cdots(139)$$

where the symbols $\supset$, $\overset{\omega_1\omega_2}{\underset{\tau_1\tau_2}{\supset}}$ and $\overset{\omega_1\omega_2}{\underset{\tau_1\tau_2}{\supset}}$ denote the double Fourier transform, the invers double Fourier transform and a pair of the double Fourier transforms, respectively. Therefore the local co-variance matrix can be expressed by making use of the quantities defined by eqs. (132) and (134) as follows:

$$[{}_0K(\tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})] \overset{\omega_1\omega_2}{\underset{\tau_1\tau_2}{\supset}} [{}_0S(\omega_1, \omega_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})]$$

$$= \frac{1}{(2\pi)^2} \overset{\omega_1'\omega_2'}{S^{-1}} [{}_0S(\omega_1, \omega_1'\,;\, \omega_2, \omega_2'\,;\, {}_IR^2_{\mu_1\mu_2}; {}_{0\tau}R^2_{\tau_1\tau_2})] \quad \cdots\cdots\cdots(140)$$

$$[{}_0K(\tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})] = \frac{1}{(2\pi)^2} \overset{\omega_1'\omega_2'}{S^{-1}} D(\tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})[{}_0S(\omega_1', \omega_2'\,;\, \tau_1, \tau_2)] e^{j(\omega_1'\tau_1 - \omega_2'\tau_2)}$$

$$= \frac{1}{(2\pi)^2} \overset{\omega_1'\omega_2'}{S^{-1}} \overset{\omega_1\omega_2}{\underset{\tau_1\tau_2}{\supset}} [{}_0S(\omega_1, \omega_1'\,;\, \omega_2, \omega_2'\,;\, {}_IR^2_{\mu_1\mu_2}, {}_{0\tau}R^2_{\tau_1\tau_2})] \quad \cdots\cdots\cdots\cdots(141)$$

and

$$[{}_0K(\tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})] \overset{\omega_1''\omega_2''}{\supset} [{}_0\overset{\approx}{S}(\omega_1'', \omega_2''\,;\, \tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})]$$

$$\overset{\omega_1'\omega_2'}{\underset{\tau_1\tau_2}{\supset}} [{}_0\overset{\approx}{S}(\omega_1'', \omega_1'\,;\, \omega_2'', \omega_2'\,;\, {}_IR^2_{\mu_1\mu_2}, {}_{0\tau}R^2_{\tau_1\tau_2})] \quad \cdots\cdots\cdots\cdots(142)$$

$$[{}_0K(\tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})] \overset{\omega_1'\omega_2'}{\supset} D(\tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})[{}_0S(\omega_1', \omega_2'\,;\, \tau_1, \tau_2)]$$

$$\overset{\omega_1''\omega_2''}{\underset{\tau_1\tau_2}{\supset}} [{}_0\overset{\approx}{S}(\omega_1'', \omega_1'\,;\, \omega_2'', \omega_2'\,;\, {}_IR^2_{\mu_1\mu_2}, {}_{0\tau}R^2_{\tau_1\tau_2})] \quad \cdots\cdots\cdots\cdots(143)$$

where the symbol $\overset{\omega_1\omega_2}{S^{-1}}$ denotes the integral operator with respect to ω_1 and ω_2 over the two-dimensional full domain R^2_∞, and the quantity, $[{}_0\overset{\approx}{S}(\omega_1'', \omega_2''\,;\, \tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})]$ appearing in eq. (142) is

$$[{}_0\overset{\approx}{S}(\omega_1'', \omega_2''\,;\, \tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})] = \frac{1}{(2\pi)^2} \int_{R^2_\infty} d\omega_1' d\omega_2' e^{j(\omega_1'\tau_1 - \omega_2'\tau_2)}$$

$$\cdot \int_{{}_{0\tau}R^2_{\tau_1\tau_2}} d\lambda_1 d\lambda_2 [{}_0S(\omega_1', \omega_2'\,;\, \lambda_1, \lambda_2)] e^{-j(\omega_1''\lambda_1 - \omega_2''\lambda_2)} \quad \cdots\cdots\cdots\cdots(144)$$

From eqs. (140) ~ (143), it is shown that the local co-variance matrix is determined by operating the double operator, $\frac{1}{(2\pi)^2} \overset{\omega_1'\omega_2'}{S^{-1}} \overset{\omega_1\omega_2}{\supset}$ to eq. (132) or $\overset{\omega_1'\omega_2'}{\supset} \overset{\omega_1''\omega_2''}{\supset}$ to eq. (134) and that the three kinds of two-dimensional spectral density matrices, $[{}_0S(\omega_1, \omega_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})]$, $[{}_0S(\omega_1', \omega_2'\,;\, \tau_1, \tau_2)]$ and $[{}_0\overset{\approx}{S}(\omega_1'', \omega_2''\,;\, \tau_1, \tau_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})]$ are obtainable in the intermediate step of operations by changing the order of the elemental operators of the double operators. And also, from these equations the following relations can be found:

$$[{}_0S(\omega_1, \omega_2\,;\, {}_{0\tau}R^2_{\tau_1\tau_2})] = \frac{1}{(2\pi)^2} \int_{{}_{0\tau}R^2_{\tau_1\tau_2}} e^{-j\omega_1\tau_1} \underset{\omega_1}{*} [{}_0S(\omega_1, \omega_2\,;\, \tau_1, \tau_2)] \underset{\omega_2}{*} e^{j\omega_2\tau_2} d\tau_1 d\tau_2 \quad \cdots\cdots\cdots\cdots(145)$$

$$[{}_0S(\omega_1, \omega_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})] = \frac{1}{(2\pi)^2}\int_{R^2_\infty} e^{-j\omega_1\tau_1} \underset{\omega_1}{*} [{}_0\tilde{\tilde{S}}(\omega_1, \omega_2 ; \tau_1, \tau_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})] \underset{\omega_2}{*} e^{j\omega_2\tau_2} d\tau_1 d\tau_2 \cdots\cdots (146)$$

Although the local co-variance matrix $[{}_0K(\tau_1, \tau_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})]$ associated with the output of the time-variant discrete linear system can be obtained by operating the inverse double Fourier transform to any one of the three kinds of two-dimensional spectral density matrices, defined in a finite domain ${}_{0r}R^2{}_{\tau_1\tau_2}$, $[{}_0S(\omega_1, \omega_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})]$, $D(\tau_1, \tau_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})[{}_0S(\omega_1', \omega_2' ; \tau_1, \tau_2)]$ and $[{}_0\tilde{\tilde{S}}(\omega_1'', \omega_2'' ; \tau_1, \tau_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})]$, these quantities are different from each other in their spectral expressions and have a different physical meaning. The two-dimensional local spectral density matrix, $[{}_0S(\omega_1, \omega_2\, {}_{0r}R^2{}_{\tau_1\tau_2})]$ which only constitutes a pair of double Fourier transforms with the local co-variance matrix is a set function of the two-dimensional time domain ${}_{0r}R^2{}_{\tau_1\tau_2}$ as well as a function of the output frequency variables (ω_1, ω_2). The two-dimensional spectral density matrix, $D(\tau_1, \tau_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})[{}_0S(\omega_1', \omega_2' ; \tau_1, \tau_2)]$ which contains time variables (τ_1, τ_2) as the parameters has a form expressed as the product of the cut-off operator associated with the domain ${}_{0r}R^2{}_{\tau_1\tau_2}$ and a function of the input frequency variables, (ω_1', ω_2') and the time variables (τ_1, τ_2). The two-dimensional spectral density matrix, $[{}_0\tilde{\tilde{S}}(\omega_1'', \omega_2'' ; \tau_1, \tau_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})]$, which depends on the time variables, (τ_1, τ_2) as well as the domain ${}_{0r}R^2{}_{\tau_1\tau_2}$, is a function of the difference frequency variables (ω_1'', ω_2'') between the output and input. It is clear that the local co-variance matrix expressed by eq. (120) is independent of the choice of the lower limits of the double integral in eq. (120) provided they are smaller than the relevant lower limits of the two-dimensional domain of input ${}_{I}R^2{}_{\mu_1\mu_2}$. The two-dimensional local spectral density matrix, $[{}_0S(\omega_1, \omega_2; {}_{0r}R^2{}_{\tau_1\tau_2})]$ expressed as eq. (121) is uniquely determined independently of the lower limits of the integral in spite of the dependence of the quantities similarly defined as eqs. (122) and (123) on the lower limit of integration. On the other hand, the other two-dimensional spectral density matrix expressed as eq. (127) or (144) may be influenced by the choice of the lower limit of integration. That is, similar expressions to these two-dimensional spectral density matrices can be obtained according to the choice of the lower limit of integration. For the general case of a time-variant linear discrete system, however, all of these types of spectral density matrices of the output can not be expressed in terms of the co-variance matrix or the local co-variance matrix of the output, whereas the two-dimensional local spectral density matrix $[{}_0S(\omega_1, \omega_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})]$ is given by the double Fourier transform of the local co-variance matrix $[{}_0K(\tau_1, \tau_2 ; {}_{0r}R^2{}_{\tau_1\tau_2})]$.

In particular, for the case of a time-invariant linear discrete system the complex transfer function given by eq. (123) is reduced to

$$[G(\omega_i' ; \tau)] = \int_{-\infty}^{\tau} [g(\tau, \mu)] e^{-j(\tau-\mu)\omega_i'} d\mu = \int_0^\infty [g(\lambda)] e^{-j\lambda\omega_i'} d\lambda = [G(j\omega_i')] \qquad \cdots\cdots\cdots\cdots (147)$$

Then, eqs. (122) and (135) result in

$$[G(\omega_i, \omega_i' ; {}_0R^1{}_{\tau_i})] = [G(j\omega_i')]\int_{{}_0R^1{}_{\tau_i}} e^{-j(\omega_i-\omega_i')\tau} d\tau$$

$$=[G(j\omega_i')]Q(\omega_i-\omega_i'\,;\,{}_0R^1{}_{\tau_i}) \qquad \cdots\cdots\cdots\cdots(148)$$

$$[\tilde{\tilde{G}}(\omega_i'',\omega_i'\,;\,{}_0R^1{}_{\tau_i}]=[G(j\omega_i')]Q(\omega_i''\,;\,{}_0R^1{}_{\tau_i}) \qquad \cdots\cdots\cdots\cdots(149)$$

where $Q(\omega\,;R^1{}_\tau)$ is defined by eq. (20).

By making use of eq. (147), the two-dimensional spectral density matrix defined by eq. (127) can be expressed as

$$[{}_0S(\omega_1',\omega_2'\,;\tau_1,\tau_2)]=[G(j\omega_1')][{}_IS(\omega_1',\omega_2'\,;\,{}_IR^2{}_{\mu_1\mu_2})][G(j\omega_2')]^* \equiv[{}_0S(\omega_1',\omega_2')] \qquad \cdots\cdots\cdots\cdots(150)$$

It is noted that the two-dimensional spectral density matrix $[{}_0S(\omega_1',\omega_2'\,;\tau_1,\tau_2)]$ does not depend upon the time variables (τ_1,τ_2) in the case of a time-invariant system.

On the other hand, the two-dimensional local spectral density matrix defined by eq. (121) is expressed in the following form by using eqs. (148) and (150) :

$$\begin{aligned}[{}_0S(\omega_1,\omega_2\,;\,{}_{0r}R^2{}_{\tau_1\tau_2})]&=\frac{1}{(2\pi)^2}\int_{R^2{}_\infty}Q(\omega_1-\omega_1'\,;\,{}_0R^1{}_{\tau_1})[G(j\omega_1')][{}_IS(\omega_1',\omega_2'\,;\,{}_IR^2{}_{\mu_1\mu_2})]\\&\quad\cdot[G(j\omega_2')]^*Q^*(\omega_2-\omega_2'\,;\,{}_0R^1{}_{\tau_2})d\omega_1'd\omega_2'\\&=\frac{1}{(2\pi)^2}Q(\omega_1\,;\,{}_0R^1{}_{\tau_1})\underset{\omega_1}{*}[{}_0S(\omega_1,\omega_2)]\underset{\omega_2}{*}Q^*(\omega_2\,;\,{}_0R^1{}_{\tau_2})\\&\overset{\omega_1\omega_2}{\underset{\tau_1\tau_2}{\subset}}[{}_0K(\tau_1,\tau_2\,;\,{}_{0r}R^2{}_{\tau_1\tau_2})]\end{aligned} \qquad \cdots\cdots\cdots\cdots(151)$$

When ${}_0R^1{}_{\tau_i}$ and ${}_{0r}R^2{}_{\tau_1\tau_2}$ tend to $R^1{}_\infty$ and $R^2{}_\infty$ respectively, the following equation is valid :

$$Q(\omega_i\,;\,R^1{}_\infty)=2\pi\delta(\omega_i)\subset 1 \qquad \cdots\cdots\cdots\cdots(152)$$

Therefore, eq. (151) is reduced to

$$\begin{aligned}[{}_0S(\omega_1,\omega_2\,;\,R^2{}_\infty)]&=[{}_0S_{\tau_1\tau_2}(\omega_1,\omega_2)]\\&=[G(j\omega_1)][{}_IS(\omega_1,\omega_2\,;\,{}_IR^2{}_{\mu_1\mu_2})][G(j\omega_2)]^*\\&=[{}_0S(\omega_1,\omega_2)]\overset{\omega_1\omega_2}{\underset{\tau_1\tau_2}{\subset}}[{}_0K(\tau_1,\tau_2)]\end{aligned} \qquad \cdots\cdots\cdots\cdots(153)$$

From eqs. (150) and (153) it is found that, for the case of a time-invariant linear discrete system, the two-dimensional spectral density matrix defined by eq. (127) is in accordance with the two-dimensional total spectral density matrix of the output which gives the co-variance matrix of the output by the inverse double Fourier transform.

Similarly, the two-dimensional spectral density matrix defined by eq. (144) can be reduced to the following form by making use of eqs. (150) and (153).

$$\begin{aligned}&[{}_0\tilde{\tilde{S}}(\omega_1'',\omega_2''\,;\tau_1,\tau_2\,;\,{}_{0r}R^2{}_{\tau_1\tau_2})]\\&=Q(\omega_1''\,;\,{}_0R^1{}_{\tau_1})Q^*(\omega_2''\,;\,{}_0R^1{}_{\tau_2})\frac{1}{(2\pi)^2}\int_{R^2{}_\infty}[{}_0S(\omega_1',\omega_2')]e^{j(\omega_1'\tau_1-\omega_2'\tau_2)}d\omega_1'd\omega_2'\\&=Q(\omega_1''\,;\,{}_0R^1{}_{\tau_1})Q^*(\omega_2''\,;\,{}_0R^1{}_{\tau_2})[{}_0K(\tau_1,\tau_2)]\overset{\omega_1''\omega_2''}{\subset}[{}_0K(\tau_1,\tau_2\,;\,{}_{0r}R^2{}_{\tau_1\tau_2})]\end{aligned} \qquad \cdots\cdots\cdots(154)$$

If ${}_0R^1{}_{\tau_i}$ and ${}_{0r}R^2{}_{\tau_1\tau_2}$ tend to $R^1{}_\infty$ and $R^2{}_\infty$, eq. (154) is reduced to

$$[{}_0\tilde{\tilde{S}}(\omega_1'', \omega_2''; \tau_1, \tau_2; R^2_\infty)] = (2\pi)^2\delta(\omega_1'')\delta(\omega_2'')[{}_0K(\tau_1, \tau_2)] \underset{\omega_1''\omega_2''}{\subset} [{}_0K(\tau_1, \tau_2)] \qquad \cdots\cdots(155)$$

Eqs. (154) and (155) show that, in the case of a time-invariant linear discrete system, the two-dimensional spectral density matrix defined by eq. (144) is expressed as the product of the co-variance matrix and the double Fourier transform of the two-dimensional time domain. Particularly, eq. (155) which means the concentration of the two-dimensional spectral density in the zero difference frequency, $(\omega_1'', \omega_2'') = (0, 0)$, for the case of the full domain R^2_∞, is compatible with the accordance of the two-dimensional total spectral density matrix $[{}_0S(\omega_1, \omega_2; R^2_\infty)]$ which concerns the output frequency and the two-dimensional spectral density matrix $[{}_0S(\omega_1', \omega_2')]$ related to the input frequency, as shown in eq. (153). But, if the input time domain ${}_IR^2_{\mu_1\mu_2}$ is finite and the lower limits of integration τ_{iL} are taken as $-\infty < \tau_{iL} < \mu_{iL}$, $(i=1, 2)$ in the first line of eq. (147), eqs. (153) and (155) are no longer valid even in the case of a time-invariant linear discrete system. In particular, when ${}_IR^2_{\mu_1\mu_2}$ tends to R^2_∞, eq. (153) represents the input and output relation of the two-dimensional total spectral density matrix under the assumption of its existence.

$$[{}_0S(\omega_1, \omega_2; R^2_\infty)] = [G(j\omega_1)][{}_IS(\omega_1, \omega_2; R^2_\infty)][G(j\omega_2)]^* \qquad \cdots\cdots(156)$$

The above equation is the result presented by J. S. Bendat[7].

The input and output relations of the one-dimensional local spectral density matrices defined by eqs. (42), (43) and (56) respectively, are found from eq. (121) and their respective definitions as follows:

$$\begin{aligned}[{}_0S_1(\omega_1, \tau_2; {}_{0r}R^2_{\tau_1\tau_2})] &= \frac{1}{2\pi}\int_{R^1_\infty}[{}_0S(\omega_1, \omega_2; {}_{0r}R^2_{\tau_1\tau_2})]e^{-j\omega_2\tau_2}d\omega_2 \\ &= \frac{1}{2\pi}D(\tau_2; {}_0R^1_{\tau_2})\int_{R^1_\infty}d\omega_1'[G(\omega_1, \omega_1'; {}_0R^1_{\tau_1})]\int_{-\infty}^{\tau_2}d\nu[{}_IS_1(\omega_1', \nu; {}_IR^2_{\mu_1\mu_2})] \\ &\quad \cdot[g(\tau_2, \nu)]^* \qquad \cdots\cdots(157)\end{aligned}$$

$$\begin{aligned}[{}_0S_2(\tau_1, \omega_2; {}_{0r}R^2_{\tau_1\tau_2})] &= \frac{1}{2\pi}\int_{R^1_\infty}[{}_0S(\omega_1, \omega_2; {}_{0r}R^2_{\tau_1\tau_2})]e^{j\omega_1\tau_1}d\omega_1 \\ &= \frac{1}{2\pi}D(\tau_1; {}_0R^1_{\tau_1})\int_{R^1_\infty}d\omega_2'\int_{-\infty}^{\tau_1}d\nu[g(\tau_1, \nu)][{}_IS_2(\nu, \omega_2'; {}_IR^2_{\mu_1\mu_2})] \\ &\quad \cdot[G(\omega_2, \omega_2'; {}_0R^1_{\tau_2})]^* \qquad \cdots\cdots(158)\end{aligned}$$

and

$$\begin{aligned}[{}_0S(\omega, \tau; {}_{0p}R^2_{\lambda\tau})] &= [{}_0S_1(\omega, \tau; {}_{0r}R^2_{\lambda+\tau\tau})]e^{j\omega\tau} \\ &= \frac{1}{2\pi}D(\tau; {}_0R^1_\tau)e^{j\omega\tau}\int_{R^1_\infty}d\omega'[G(\omega, \omega'; {}_0R^1_{\lambda+\tau})]\int_{-\infty}^{\tau}d\nu e^{-j\omega'\nu} \\ &\quad \cdot[{}_IS(\omega', \nu; {}_IR^2_{\kappa\nu})][g(\tau, \nu)]^* \qquad \cdots\cdots(159)\end{aligned}$$

To obtain the input and output relation of the one-dimensional local Hermitian spectral density matrix defined by eq. (69), we must consider the auxiliary skew Hermitian matrix which is similar to that defined by eq. (93),

$$\begin{aligned}[{}_0S_H'(\omega, \tau; {}_{0p}R^2_{\lambda\tau}, \alpha)] &= \alpha[{}_0S(\omega, \tau; {}_{0p}R^2_{\lambda\tau})] - \alpha^*[{}_0S(\omega, \tau; {}_{0p}R^2_{\lambda\tau})]^* \\ [{}_IS_H'(\omega', \nu; {}_IR^2_{\kappa\nu}, \alpha)] &= \alpha[{}_IS(\omega', \nu; {}_IR^2_{\kappa\nu})] - \alpha^*[{}_IS(\omega', \nu; {}_IR^2_{\kappa\nu})]^* \qquad \cdots\cdots(160)\end{aligned}$$

and a pair of conjugate linear operators.

$$H=\frac{1}{2\pi}D(\tau\,;{}_0R^1{}_\tau)e^{j\omega\tau}\int_{R^1{}_\infty}d\omega'[G(\omega,\omega'\,;{}_0R^1{}_{\lambda+\tau\tau})]\int_{-\infty}^{\tau}d\nu e^{-j\omega'\nu}[g(\tau,\nu)]^*$$

$$H^*=\frac{1}{2\pi}D(\tau\,;{}_0R^1{}_\tau)e^{-j\omega\tau}\int_{R^1{}_\infty}d\omega'[G(\omega,\omega'\,;{}_0R^1{}_{\lambda+\tau})]^*\int_{-\infty}^{\tau}d\nu e^{j\omega'\nu}[g(\tau,\nu)] \quad\cdots\cdots\cdots\cdots(161)$$

Then the input and output relations of the one-dimensional local spectral density matrix $[S_H]$ and the auxiliary spectral density matrix $[S_H']$ are expressed as follows:

$$[{}_0S_H(\omega,\tau\,;{}_{0p}R^2{}_{\lambda\tau},\alpha)]=(H+H^*)[{}_IS_H(\omega',\nu\,;{}_IR^2{}_{\kappa\nu},\alpha/2)] + (H-H^*)[{}_IS_H'(\omega',\nu\,;{}_IR^2{}_{\kappa\nu},\alpha/2)] \quad\cdots\cdots\cdots\cdots(162)$$

$$[{}_0S_H'(\omega,\tau\,;{}_{0p}R^2{}_{\lambda\tau},\alpha)]=(H-H^*)[{}_IS_H(\omega',\nu\,;{}_IR^2{}_{\kappa\nu},\alpha/2)] + (H+H^*)[{}_IS_H'(\omega',\nu\,;{}_IR^2{}_{\kappa\nu},\alpha/2)] \quad\cdots\cdots\cdots\cdots(163)$$

where

$$[{}_IS_H(\omega',\nu\,;{}_IR^2{}_{\kappa\nu},\alpha/2)]=(1/2)[{}_IS_H(\omega',\nu\,;{}_IR^2{}_{\kappa\nu},\alpha)]$$

$$[{}_IS_H'(\omega',\nu\,;{}_IR^2{}_{\kappa\nu},\alpha/2)]=(1/2)[{}_IS_H'(\omega',\nu\,;{}_IR^2{}_{\kappa\nu},\alpha)] \quad\cdots\cdots\cdots\cdots(164)$$

From the two-dimensional spectral density matrix of the output defined by eq. (127) we can define the following two kinds of one-dimensional spectral density matrices for the output in the same way as we define the one-dimensional local spectral density matrices from the two-dimensional local spectral density matrix. And they can be related to their relevant one-dimensional local spectral density matrices for the input as follows:

$$[{}_0S_1(\omega_1'\,;\tau_1,\tau_2)]=\frac{1}{2\pi}\int_{R^1{}_\infty}[{}_0S(\omega_1',\omega_2'\,;\tau_1,\tau_2)]e^{-j\omega_2'\tau_2}d\omega_2'$$

$$=[G(\omega_1'\,;\tau_1)]\int_{-\infty}^{\tau_2}d\nu[{}_IS_1(\omega_1',\nu\,;{}_IR^2{}_{\mu_1\mu_2})][g(\tau_2,\nu)]^* \quad\cdots\cdots\cdots(165)$$

$$[{}_0S_2(\omega_2'\,;\tau_1,\tau_2)]=\frac{1}{2\pi}\int_{R^1{}_\infty}[{}_0S(\omega_1',\omega_2'\,;\tau_1,\tau_2)]e^{j\omega_1'\tau_1}d\omega_1'$$

$$=\int_{-\infty}^{\tau_1}d\nu[g(\tau_1,\nu)][{}_IS_2(\nu,\omega_2'\,;{}_IR^2{}_{\mu_1\mu_2})][G(\omega_2'\,;\tau_2)]^* \quad\cdots\cdots\cdots(166)$$

As regards the two-dimensional spectral density matrix defined by eq. (144), similar relations are obtained, that is,

$$[{}_0\tilde{\tilde{S}}_1(\omega_1''\,;\tau_1,\tau_2\,;{}_{0\tau}R^2{}_{\tau_1\tau_2})]=\frac{1}{2\pi}\int_{R^1{}_\infty}[{}_0\tilde{\tilde{S}}(\omega_1'',\omega_2''\,;\tau_1,\tau_2\,;{}_{0\tau}R^2{}_{\tau_1\tau_2})]e^{-j\omega_2''\tau_2}d\omega_2''$$

$$=\frac{1}{2\pi}D(\tau_2\,;{}_0R^1{}_{\tau_2})\int_{R^1{}_\infty}d\omega_1'[\tilde{\tilde{G}}(\omega_1'',\omega_1'\,;{}_0R^1{}_{\tau_1})]\int_{-\infty}^{\tau_2}d\nu[{}_IS_1(\omega_1',\nu\,;{}_IR^2{}_{\mu_1\mu_2})]\cdot[g(\tau_2,\nu)]^*e^{j\omega_1'\tau_1} \quad\cdots\cdots\cdots\cdots(167)$$

$$[{}_0\tilde{\tilde{S}}_2(\omega_2''\,;\tau_1,\tau_2\,;{}_{0\tau}R^2{}_{\tau_1\tau_2})]=\frac{1}{2\pi}\int_{R^1{}_\infty}[{}_0\tilde{\tilde{S}}(\omega_1'',\omega_2''\,;\tau_1,\tau_2\,;{}_{0\tau}R^2{}_{\tau_1\tau_2})]e^{j\omega_1''\tau_1}d\omega_1''$$

$$=\frac{1}{2\pi}D(\tau_1\,;{}_0R^1{}_{\tau_1})\int_{R^1{}_\infty}d\omega_2'\int_{-\infty}^{\tau}d\nu[g(\tau_1,\nu)][{}_IS(\nu,\omega_2'\,;{}_IR^2{}_{\mu_1\mu_2})]\cdot[\tilde{\tilde{G}}(\omega_2'',\omega_2'\,;{}_0R^1{}_{\tau_2})]^*e^{-j\omega_2'\tau_2} \quad\cdots\cdots\cdots\cdots(168)$$

Particularly for the case of a time-invariant linear discrete system, the one-dimensional total spectral density matrices for the output are expressed by using eqs. (148), (152) and (157)～(159) as follows:

$$[{}_0S_1(\omega_1, \tau_2; R^2_\infty)] = [G(j\omega_1)]\int_{-\infty}^{\tau_2} d\nu [{}_IS_1(\omega_1, \nu; {}_IR^2_{\mu_1\mu_2})][g(\tau_2-\nu)]^*$$

$$= [G(j\omega_1)][{}_IS(\omega_1, \tau_2; {}_IR^2_{\mu_1\mu_2})] \underset{\tau_2}{*} [g(\tau_2)]^*$$

$$\equiv [{}_0S_1(\omega_1, \tau_2)] \overset{\omega_1}{\underset{\tau_1}{\subset}} [{}_0K(\tau_1, \tau_2)] \qquad \cdots\cdots\cdots\cdots(169)$$

$$[{}_0S_2(\tau_1, \omega_2; R^2_\infty)] = [g(\tau_1)] \underset{\tau_1}{*} [{}_IS(\tau_1, \omega_2; {}_IR^2_{\mu_1\mu_2})][G(j\omega_2)]^*$$

$$\equiv [{}_0S_2(\tau_1, \omega_2)] \overset{\omega_2}{\underset{\tau_2}{\subset}} [{}_0K(\tau_1, \tau_2)] \qquad \cdots\cdots\cdots\cdots(170)$$

$$[{}_0S(\omega, \tau; R^2_\infty)] = [G(j\omega)][{}_IS(\omega, \tau; {}_IR^2_{\kappa\nu})] \underset{\tau}{*} ([g(\tau)]^* e^{j\omega\tau})$$

$$\equiv [{}_0S(\omega, \tau)] \overset{\omega}{\underset{\lambda}{\subset}} [{}_0K(\lambda+\tau, \tau)] \qquad \cdots\cdots\cdots\cdots(171)$$

Similarly, from eqs. (165)～(168) the following expressions are obtained:

$$[{}_0S_1(\omega_1'; \tau_1, \tau_2)] = [G(j\omega_1')][{}_IS_1(\omega_1', \tau_2; {}_IR^2_{\mu_1\mu_2})] \underset{\tau_2}{*} [g(\tau_2)]^*$$

$$= [{}_0S_1(\omega_1', \tau_2)] \overset{\omega_1'}{\underset{\tau_1}{\subset}} [{}_0K(\tau_1, \tau_2)] \qquad \cdots\cdots\cdots\cdots(172)$$

$$[{}_0S_2(\omega_2'; \tau_1, \tau_2)] = [{}_0S_2(\tau_1, \omega_2')] \overset{\omega_2'}{\underset{\tau_2}{\subset}} [{}_0K(\tau_1, \tau_2)] \qquad \cdots\cdots\cdots\cdots(173)$$

$$[{}_0\tilde{\tilde{S}}_1(\omega_1''; \tau_1, \tau_2)] = \delta(\omega_1'')\int_{R^1_\infty} d\omega_1' [G(j\omega_1')]([{}_IS_1(\omega_1', \tau_2; {}_IR^2_{\mu_1\mu_2})] \underset{\tau_2}{*} [g(\tau_2)]^*) e^{j\omega_1'\tau_1}$$

$$= \delta(\omega_1'')\int_{R^1_\infty} [{}_0S_1(\omega_1'; \tau_2)] e^{j\omega_1'\tau_1} d\omega_1'$$

$$= 2\pi\delta(\omega_1'')[{}_0K(\tau_1, \tau_2)] \overset{\omega_1''}{\subset} [{}_0K(\tau_1, \tau_2)] \qquad \cdots\cdots\cdots\cdots(174)$$

$$[{}_0\tilde{\tilde{S}}_2(\omega_2''; \tau_1, \tau_2)] = 2\pi\delta(\omega_2'')[{}_0K(\tau_1, \tau_2)] \overset{\omega_2''}{\subset} [{}_0K(\tau_1, \tau_2)] \qquad \cdots\cdots\cdots\cdots(175)$$

In the special case of a time-invariant linear discrete system subjected to a stationary input, the well-known input and output relation of the power spectral density matrix in the stationary process is obtainable from eq. (153) or any one of the eqs. (169)～(173), if ${}_IR^2_{\mu_1\mu_2}$ is replaced by R^2_∞ and eqs. (28), (51) and (62) which are valid in the stationary process are taken into consideration.

$$[{}_0S(\omega)] = [G(j\omega)][{}_IS(\omega)][G(j\omega)]^* \qquad \cdots\cdots\cdots\cdots(176)$$

4. Linear discrete systems subjected to quasi-stationary inputs

A quasi-stationary input vector $\{f(\mu)\}$, is defined as the product of a deterministic matrix $[a(\mu)]$, each element of which is a deterministic time function defined in R^1_∞ and a stationary input vector $\{\psi(\mu)\}$[4].

$$\{f(\mu)\}=[a(\mu)]\{\psi(\mu)\} \qquad \cdots\cdots\cdots\cdots(177)$$

Hence, the mean and the deviation vector of the quasi-stationary input vector are given by

$$E\{f(\mu)\}=[a(\mu)]E\{\psi(\mu)\},\quad \{f_d(\mu)\}=[a(\mu)]\{\psi_d(\mu)\} \qquad \cdots\cdots\cdots\cdots(178)$$

For a time-variant linear discrete system having an impulsive response matrix $[g(\tau,\mu)]$, we define the following modified impulsive response matrix $[g_m(\tau,\mu)]$, by right-multiplying the deterministic matrix $[a(\mu)]$, to the original impulsive response matrix $[g(\tau,\mu)]$.

$$[g_m(\tau,\mu)]=[g(\tau,\mu)][a(\mu)] \qquad \cdots\cdots\cdots\cdots(179)$$

Hence, the new system characterized by the above defined modified impulsive response matrix $[g_m(\tau,\mu)]$, may be subjected to a stationary input vector having the following co-variance matrix $[_IR(\lambda)]$, and the corresponding power spectral density matrix $[_IS(\omega)]$.

$$[_IS(\omega)]\underset{\lambda}{\overset{\omega}{\subset}}[_IR(\lambda)]=E\{\psi_d(\lambda+\mu)\}\{\psi_d(\mu)\}^* \qquad \cdots\cdots\cdots\cdots(180)$$

Substituting eqs. (179) and (180) in eq. (120), the local co-variance matrix of the output is given by

$$[_0K(\tau_1,\tau_2\,;\,_{0r}R^2{}_{\tau_1\tau_2})]=D(\tau_1,\tau_2\,;\,_{0r}R^2{}_{\tau_1\tau_2})\int_{-\infty}^{\tau_1}d\mu_1\int_{-\infty}^{\tau_2}d\mu_2[g_m(\tau_1,\mu_1)]$$
$$\cdot[_IR(\mu_1-\mu_2)][g_m(\tau_2,\mu_2)]^* \qquad \cdots\cdots\cdots\cdots(181)$$

The two-dimensional local spectral density matrix of the output is obtained from eq. (121) as follows by making use of eq. (28):

$$[_0S(\omega_1,\omega_2\,;\,_{0r}R^2{}_{\tau_1\tau_2})]=\frac{1}{2\pi}\int_{R^1_\infty}[G_m(\omega_1,\omega'\,;\,_0R^1{}_{\tau_1})][_IS(\omega')]$$
$$\cdot[G_m(\omega_2,\omega'\,;\,_0R^1{}_{\tau_2})]^*d\omega' \qquad \cdots\cdots\cdots\cdots(182)$$

Similarly the two-dimensional spectral density matrices of the output defined by eqs. (127), (128) and (144) are expressed as

$$[_0S(\omega_1',\omega_2'\,;\,\tau_1,\tau_2)]=2\pi\delta(\omega_1'-\omega_2')[G_m(\omega_1'\,;\,\tau_1)][_IS(\omega_1')][G_m(\omega_2'\,;\,\tau_2)]^* \qquad \cdots\cdots\cdots\cdots(183)$$

$$[_0\tilde{S}(\omega_1',\omega_2'\,;\,\tau_1,\tau_2)]=2\pi\delta(\omega_1'-\omega_2')[X_m(\omega_1'\,;\,\tau_1)][_IS(\omega_1')][X_m(\omega_2'\,;\,\tau_2)]^* \qquad \cdots\cdots\cdots\cdots(184)$$

$$[_0\tilde{\tilde{S}}(\omega_1'',\omega_2''\,;\,\tau_1,\tau_2\,;\,_{0r}R^2{}_{\tau_1\tau_2})]=\frac{1}{2\pi}\int_{R^1_\infty}[\tilde{\tilde{G}}_m(\omega_1'',\omega'\,;\,_0R^1{}_{\tau_1})][_IS(\omega')]$$
$$\cdot[\tilde{\tilde{G}}(\omega_2'',\omega'\,;\,_0R^1{}_{\tau_2})]e^{j(\tau_1-\tau_2)\omega'}d\omega' \qquad \cdots\cdots\cdots\cdots(185)$$

Substituting eqs. (183) and eq. (184) into eq. (126) the co-variance matrix of the output is expressed as follows:

$$[_0K(\tau_1,\tau_2)]=\frac{1}{2\pi}\int_{R^1_\infty}[G_m(\omega'\,;\,\tau_1)][_IS(\omega')][G_m(\omega'\,;\,\tau_2)]^*e^{j(\tau_1-\tau_2)\omega'}d\omega'$$
$$=\frac{1}{2\pi}\int_{R^1_\infty}[X_m(\omega'\,;\,\tau_1)][_IS(\omega')][X_m(\omega'\,;\,\tau_2)]^*d\omega' \qquad \cdots\cdots\cdots\cdots(186)$$

By making use of eq. (131), the same expression as above can be obtained by taking the inverse double Fourier transform of eq. (182) and replacing ${}_{0r}R^2{}_{\tau_1\tau_2}$ by $R^2{}_{\infty}$.

The one-dimensional local spectral density matrices of the output are obtained from eqs. (157), (158) and (159) by taking into account of eqs. (51) and (62) as follows:

$$[{}_0S_1(\omega_1, \tau_2\,;\,{}_{0r}R^2{}_{\tau_1\tau_2})] = \frac{1}{2\pi}D(\tau_2\,;\,{}_0R^1{}_{\tau_2})\int_{R^1{}_\infty} d\omega'[G_m(\omega_1, \omega'\,;\,{}_0R^1{}_{\tau_1})][{}_IS(\omega')] \cdot\int_{-\infty}^{\tau_2} d\nu[g_m(\tau_2, \nu)]^* e^{-j\omega'\nu}$$
$$= \frac{1}{2\pi}D(\tau_2\,;\,{}_0R^1{}_{\tau_2})\int_{R^1{}_\infty}[G_m(\omega_1, \omega'\,;\,{}_0R^1{}_{\tau_1})][{}_IS(\omega')][X_m(\omega'\,;\,\tau_2)]^* d\omega' \quad\cdots\cdots(187)$$

$$[{}_0S_2(\tau_1, \omega_2\,;\,{}_{0r}R^2{}_{\tau_1\tau_2})] = \frac{1}{2\pi}D(\tau_1\,;\,{}_0R^1{}_{\tau_1})\int_{R^1{}_\infty} d\omega'\int_{-\infty}^{\tau_1} d\nu[g_m(\tau_1, \nu)]e^{j\omega'\nu} \cdot[{}_IS(\omega')][G_m(\omega_2, \omega'\,;\,{}_0R^1{}_{\tau_2})]^*$$
$$= \frac{1}{2\pi}D(\tau_1\,;\,{}_0R^1{}_{\tau_1})\int_{R^1{}_\infty}[X_m(\omega'\,;\,\tau_1)][{}_IS(\omega')][G_m(\omega_2, \omega'\,;\,{}_0R^1{}_{\tau_2})]^* d\omega' \quad\cdots\cdots(188)$$

and

$$[{}_0S(\omega, \tau\,;\,{}_{0p}R^2{}_{\lambda\tau})] = \frac{1}{2\pi}D(\tau\,;\,{}_0R^1{}_\tau)e^{j\omega\tau}\int_{R^1{}_\infty} d\omega'[G_m(\omega, \omega'\,;\,{}_0R^1{}_{\lambda+\tau})][{}_IS(\omega')] \cdot\int_{-\infty}^{\tau} d\nu[g_m(\tau, \nu)]^* e^{-j\omega'\nu}$$
$$= \frac{1}{2\pi}D(\tau\,;\,{}_0R^1{}_\tau)e^{j\omega\tau}\int_{R^1{}_\infty}[G_m(\omega, \omega'\,;\,{}_0R^1{}_{\lambda+\tau})][{}_IS(\omega')][X_m(\omega'\,;\,\tau)]^* d\omega' \quad\cdots\cdots\cdots\cdots(189)$$

From eqs. (86), (91) and (162), the one-dimensional local Hermitian spectral density matrices of the output defined by eq. (69) are given by the following equations, because the one-dimensional total skew Hermitian spectral density matrices are zero in the stationary input:

$$[{}_0S_H(\omega, \tau\,;\,{}_{0t}R^2{}_{\lambda\tau}, 1)] = \frac{1}{2}(H_t + H_t^*)[{}_IS(\omega')] \quad\cdots\cdots\cdots\cdots(190)$$

$$H_t = \frac{1}{2\pi}D(\tau\,;\,{}_{0t}R^1{}_\tau)e^{j\omega\tau}\int_{R^1{}_\infty} d\omega'[G_m(\omega, \omega'\,;\,{}_{0t}R^1{}_{\lambda+\tau})] \cdot\int_{-\infty}^{\tau} d\nu e^{-j\omega'\nu}[g_m(\tau, \nu)]^* \quad\cdots\cdots\cdots\cdots(191)$$

$$[{}_0S_H(\omega, \tau\,;\,{}_{0p}R^2{}_{\lambda\tau}, 1/2)] = \frac{1}{2}(H_p + H_p^*)[{}_IS(\omega')] \quad\cdots\cdots\cdots\cdots(192)$$

$$H_p = \frac{1}{2\pi}D(\tau\,;\,{}_{0p}R^1{}_\tau)e^{j\omega\tau}\int_{R^1{}_\infty} d\omega'[G_m(\omega, \omega'\,;\,{}_0R^1{}_{\lambda+\tau})] \cdot\int_{-\infty}^{\tau} d\nu e^{-j\omega'\nu}[g_m(\tau, \nu)]^* \quad\cdots\cdots\cdots\cdots(193)$$

And also, by making use of eq. (51), the one-dimensional spectral density matrices of the output defined by eqs. (165)～(168) are obtained as follows:

$$[{}_0S_1(\omega_1'\,;\,\tau_1, \tau_2)] = [G_m(\omega_1'\,;\,\tau_1)][{}_IS(\omega_1')][X_m(\omega_1'\,;\,\tau_2)]^* \quad\cdots\cdots\cdots\cdots(194)$$

$$[{}_0S_2(\omega_2'\,;\tau_1,\tau_2)]=[X_m(\omega_2'\,;\tau_1)][{}_IS(\omega_2')][G_m(\omega_2'\,;\tau_2)]^* \quad\cdots\cdots\cdots\cdots(195)$$

$$[{}_0\tilde{\tilde{S}}_1(\omega_1''\,;\tau_1,\tau_2\,;{}_{0\tau}R^2{}_{\tau_1\tau_2})]=\frac{1}{2\pi}D(\tau_2\,;{}_0R^1{}_{\tau_2})]\int_{R^1_\infty}d\omega'[\tilde{\tilde{G}}_m(\omega_1'',\omega'\,;{}_0R^1{}_{\tau_1})]\cdot[{}_IS(\omega')]\int_{-\infty}^{\tau_2}d\nu[g_m(\tau_2,\nu)]^*e^{j(\tau_1-\nu)\omega'}$$
$$=\frac{1}{2\pi}D(\tau_2\,;{}_0R^1{}_{\tau_2})\int_{R^1_\infty}[\tilde{\tilde{G}}_m(\omega_1'',\omega'\,;{}_0R^1{}_{\tau_1})][{}_IS(\omega')][X_m(\omega'\,;\tau_2)]^*e^{j\tau_1\omega'}d\omega' \quad\cdots\cdots\cdots\cdots(196)$$

$$[{}_0\tilde{\tilde{S}}_2(\omega_2''\,;\tau_1,\tau_2\,;{}_{0\tau}R^2{}_{\tau_1\tau_2})]=\frac{1}{2\pi}D(\tau_1\,;{}_0R^1{}_{\tau_1})\int_{R^1_\infty}d\omega'\int_{-\infty}^{\tau_1}d\nu[g_m(\tau_1,\nu)]e^{j\omega'\nu}\cdot[{}_IS(\omega')][\tilde{\tilde{G}}_m(\omega_2'',\omega'\,;{}_0R^1{}_{\tau_2})]^*e^{-j\tau_2\omega'}$$
$$=\frac{1}{2\pi}D(\tau_1\,;{}_0R^1{}_{\tau_1})\int_{R^1_\infty}[X_m(\omega'\,;\tau_1)][{}_IS(\omega')][\tilde{\tilde{G}}_m(\omega_2'',\omega'\,;{}_0R^1{}_{\tau_2})]^*e^{-j\tau_2\omega'}d\omega' \quad\cdots\cdots\cdots\cdots(197)$$

Moreover, if we define the following two kinds of one-dimensional spectral density matrices based upon the two-dimensional spectral density matrix given by eq. (128):

$$[{}_0\tilde{S}_1(\omega_1'\,;\tau_1,\tau_2)]=\frac{1}{2\pi}\int_{R^1_\infty}[{}_0\tilde{S}(\omega_1',\omega_2'\,;\tau_1,\tau_2)]d\omega_2' \quad\cdots\cdots\cdots\cdots(198)$$

$$[{}_0\tilde{S}_2(\omega_2'\,;\tau_1,\tau_2)]=\frac{1}{2\pi}\int_{R^1_\infty}[{}_0\tilde{S}(\omega_1',\omega_2'\,;\tau_1,\tau_2)]d\omega_1' \quad\cdots\cdots\cdots\cdots(199)$$

$$[{}_0K(\tau_1,\tau_2)]=\frac{1}{2\pi}\int_{R^1_\infty}[{}_0\tilde{S}_1(\omega_1'\,;\tau_1,\tau_2)]d\omega_1'=\frac{1}{2\pi}\int_{R^1_\infty}[{}_0\tilde{S}_2(\omega_2'\,;\tau_1,\tau_2)]d\omega_2' \quad\cdots\cdots\cdots\cdots(200)$$

these two quantities defined by eqs. (198) and (199) become identical in the case of a stationary input. And they are expressed in the following form, as can easily be seen from eqs. (28) and (128):

$$[{}_0\tilde{S}_1(\omega'\,;\tau_1,\tau_2)]=[{}_0\tilde{S}_2(\omega'\,;\tau_1,\tau_2)]=[X_m(\omega'\,;\tau_1)][{}_IS(\omega')][X_m(\omega'\,;\tau_2)]^* \quad\cdots\cdots(201)$$

The spectral representation of the co-variance matrix of the output, as given by eqs. (200) and (201), has already been shown by V. V. Bolotin[4)5)].

In particular, when the impulsive response matrix of the modified linear system $[g_m(\tau,\mu)]$, has the form of a time-invariant linear discrete system $[g_m(\tau-\mu)]$, and the finite time domains related to the output tend to infinite full domains, the spectral density matrices expressed as in eqs. (182)～(185) and eqs. (187)～(199) are reduced in the following forms:

$$[{}_0S(\omega_1,\omega_2\,;R^2{}_\infty)]=[{}_0S(\omega_1,\omega_2\,;\tau_1,\tau_2)]$$
$$=2\pi\delta(\omega_1-\omega_2)[G_m(j\omega_1)][{}_IS(\omega_1)][G_m(j\omega_2)]^* \quad\cdots\cdots\cdots\cdots(202)$$

$$[{}_0\tilde{S}(\omega_1,\omega_2\,;\tau_1,\tau_2)]=2\pi\delta(\omega_1-\omega_2)[X_m(j\omega_1\,;\tau_1)][{}_IS(\omega_1)][X_m(j\omega_2\,;\tau_2)]^* \quad\cdots\cdots\cdots\cdots(203)$$

where

$$[X_m(j\omega_i\,;\tau_i)]=[G_m(j\omega_i)]e^{j\omega_i\tau_i}$$
$$[{}_0\tilde{\tilde{S}}(\omega_1,\omega_2\,;\tau_1,\tau_2)]=(2\pi)^2\delta(\omega_1)\delta(\omega_2)[{}_0K(\tau_1-\tau_2)] \quad\cdots\cdots\cdots\cdots(204)$$

and

$$[{}_0S_1(\omega_1, \tau_2 ; R^2_\infty)] = [{}_0S_1(\omega_1 ; \tau_1, \tau_2)] = [G_m(j\omega_1)][{}_IS(\omega_1)][X_m(j\omega_1 ; \tau_2)]^*$$
$$[{}_0S_2(\tau_1, \omega_2 ; R^2_\infty)] = [{}_0S_2(\omega_2 ; \tau_1, \tau_2)] = [X_m(j\omega_2 ; \tau_1)][{}_IS(\omega_2)][G_m(j\omega_2)]^* \qquad \cdots\cdots(205)$$

$$[{}_0S(\omega, \tau ; R^2_\infty)] = [{}_0S_H(\omega, \tau ; {}_tR^2_\infty, 1)] = [{}_0S_H(\omega, \tau ; R^2_\infty, 1/2)]$$
$$= [G_m(j\omega)][{}_IS(\omega)][G_m(j\omega)]^* \qquad \cdots\cdots(206)$$

$$[{}_0\tilde{\tilde{S}}_1(\omega ; \tau_1, \tau_2)] = [{}_0\tilde{\tilde{S}}_2(\omega ; \tau_1, \tau_2)] = 2\pi\delta(\omega)[{}_0K(\tau_1 - \tau_2)] \qquad \cdots\cdots(207)$$

$$[{}_0\tilde{S}_1(\omega ; \tau_1, \tau_2)] = [{}_0\tilde{S}_2(\omega ; \tau_1, \tau_2)] = [X_m(j\omega ; \tau_1)][{}_IS(\omega)][X_m(j\omega ; \tau_2)]^* \qquad \cdots\cdots(208)$$

However, if the original system is a time-invariant linear discrete system, the deterministic matrix $[a(\mu)]$ in eq. (179) should be a constant matrix in order that the modified system is a time-invariant linear discrete system. Hence, the expressions given by eqs. (202)~(208) are nothing but the input and output relations of the various spectral density matrices considered in the case of a stationary process which are adaptable only to a time-invariant linear discrete system subjected to stationary input.

As a problem of the quasi-stationary input, the modified impulsive response matrix should be considered to be a time-variant type whatever the original linear system is. Thus the complex transfer matrix of the modified system is generally given by the following expression by making use of eqs. (123) and (179):

$$[G_m(\omega_t' ; \tau_t)] = \frac{1}{2\pi}[G(\omega_t' ; \tau_t)] \underset{\omega_t'}{*} e^{-j\tau_t\omega_t'}[A(-j\omega_t')] \qquad \cdots\cdots(209)$$

where

$$[A(j\omega_t')] \overset{\omega_t'}{\underset{\mu_t}{\subset}} [a(\mu_t)] \qquad \cdots\cdots(210)$$

Particularly when the original system is a time-invariant linear discrete system, the above equation is written as follows by using eq. (147).

$$[G_m(\omega_t' ; \tau_t)] = \frac{1}{2\pi}[G(j\omega_t')] \underset{\omega_t'}{*} e^{-j\tau_t\omega_t'}[A(-j\omega_t')]$$
$$= \frac{1}{2\pi} e^{-j\tau_t\omega_t'} \int_{R^1_\infty} [G(js)][A(j(s-\omega_t'))] e^{j\tau_t s} ds \qquad \cdots\cdots(211)$$

Therefore, in the case of a time-invariant linear discrete system subjected to a quasi-stationary input, the local co-variance matrix and the various spectral density matrices of the output which are given by eqs. (181)~(201) are written by making use of the following expressions:

$$[g_m(\tau_t, \mu_t)] = [g(\tau_t - \mu_t)][a(\mu_t)] \qquad \cdots\cdots(212)$$

$$[X_m(\omega_t' ; \tau_t)] = [G_m(\omega_t' ; \tau_t)] e^{j\omega_t'\tau_t}$$
$$= \frac{1}{2\pi} \int_{R^1_\infty} [G(js)][A(j(s-\omega_t'))] e^{j\tau_t s} ds \qquad \cdots\cdots(213)$$

$$[G_m(\omega_i, \omega_i' ; {}_0R^1{}_{\tau_i})] = \frac{1}{2\pi}\int_{R^1{}_\infty} ds[G(js)][A(j(s-\omega_i'))] \cdot \int_{{}_0R^1{}_{\tau_i}} d\tau_i e^{-j(\omega_i - s)\tau_i} \quad \cdots\cdots\cdots\cdots(214)$$

$$[\tilde{\tilde{G}}_m(\omega_i'', \omega_i' ; {}_0R^1{}_{\tau_i})] = \frac{1}{2\pi}\int_{R^1{}_\infty} ds[G(js)][A(j(s-\omega_i'))] \cdot \int_{{}_0R^1{}_{\tau_i}} d\tau_i e^{-j(\omega_i'' + \omega_i' - s)\tau_i} \quad \cdots\cdots\cdots\cdots(215)$$

In particular, when the finite time domain tends to the full time domain, eqs. (214) and (215) are expressed as follows:

$$[G_m(\omega_i, \omega_i' ; R^1{}_\infty)] = [G(j\omega_i)][A(j(\omega_i - \omega_i'))] \quad \cdots\cdots\cdots\cdots(216)$$

$$[\tilde{\tilde{G}}_m(\omega_i'', \omega_i' ; R^1{}_\infty)] = [G(j(\omega_i'' + \omega_i'))][A(j\omega_i'')] \quad \cdots\cdots\cdots\cdots(217)$$

Hence the various spectral density matrices of the output, each of which is definable in the relevant full time domain, are written as follows in the case of a time-invariant linear discrete system subjected to a quasi-stationary input:

$$[{}_0S(\omega_1, \omega_2 ; R^2{}_\infty)] = \frac{1}{2\pi}[G(j\omega_1)]\int_{R^1{}_\infty}[A(j(\omega_1 - \omega'))][{}_IS(\omega')] \cdot [A(j(\omega_2 - \omega'))]^* d\omega' [G(j\omega_2)]^* \quad \cdots\cdots\cdots\cdots(218)$$

$$[{}_0S(\omega_1', \omega_2'; \tau_1, \tau_2)] = 2\pi\delta(\omega_1' - \omega_2')[G_m(\omega_1'; \tau_1)][{}_IS(\omega_1')][G_m(\omega_2'; \tau_2)]^* \cdots\cdots(219)$$

$$[{}_0\tilde{S}(\omega_1', \omega_2' ; \tau_1, \tau_2)] = 2\pi\delta(\omega_1' - \omega_2')[X_m(\omega_1' ; \tau_1)][{}_IS(\omega_1')][X_m(\omega_2' ; \tau_2)]^* \quad \cdots\cdots\cdots\cdots(220)$$

$$[{}_0\tilde{\tilde{S}}(\omega_1'', \omega_2'' ; \tau_1, \tau_2 ; R^2{}_\infty)] = \frac{1}{2\pi}\int_{R^1{}_\infty}[G(j(\omega_1'' + \omega'))][A(j\omega_1'')][{}_IS(\omega')] \cdot [A(j\omega_2'')]^*[G(j(\omega_2'' + \omega'))]^* e^{j(\tau_1 - \tau_2)\omega'} d\omega' \quad \cdots\cdots\cdots\cdots(221)$$

$$[{}_0S_1(\omega_1, \tau_2 ; R^2{}_\infty)] = \frac{1}{2\pi}[G(j\omega_1)]\int_{R^1{}_\infty}[A(j(\omega_1 - \omega'))][{}_IS(\omega')][X_m(\omega' ; \tau_2)]^* d\omega' \quad \cdots\cdots\cdots\cdots(222)$$

$$[{}_0S_2(\tau_1, \omega_2 ; R^2{}_\infty)] = \frac{1}{2\pi}\int_{R^1{}_\infty}[X_m(\omega' ; \tau_1)][{}_IS(\omega')][A(j(\omega_2 - \omega'))]^* d\omega' [G(j\omega_2)]^* \quad \cdots\cdots\cdots\cdots(223)$$

$$[{}_0S(\omega, \tau; R^2{}_\infty)] = \frac{1}{2\pi}e^{j\omega\tau}[G(j\omega)]\int_{R^1{}_\infty}[A(j(\omega - \omega'))][{}_IS(\omega')][X_m(\omega'; \tau)]^* d\omega' \quad \cdots\cdots\cdots\cdots(224)$$

$$[{}_0S_H(\omega, \tau ; R^2{}_\infty, 1/2)] = \frac{1}{4\pi}\Big(e^{j\omega\tau}[G(j\omega)]\int_{R^1{}_\infty}[A(j(\omega - \omega'))][{}_IS(\omega')] \cdot [X_m(\omega' ; \tau)]^* d\omega' + e^{-j\omega\tau}\int_{R^1{}_\infty}[X_m(\omega' ; \tau)][{}_IS(\omega')][A(j(\omega - \omega'))]^* d\omega' [G(j\omega)]^*\Big) \quad \cdots\cdots\cdots\cdots(225)$$

$$[{}_0S_1(\omega_1' ; \tau_1, \tau_2)] = [G_m(\omega_1' ; \tau_1)][{}_IS(\omega_1')][X_m(\omega_1' ; \tau_2)]^* \quad \cdots\cdots\cdots\cdots(226)$$

$$[{}_0S_2(\omega_2' ; \tau_1, \tau_2)] = [X_m(\omega_2' ; \tau_1)][{}_IS(\omega_2')][G_m(\omega_2' ; \tau_2)]^* \quad \cdots\cdots\cdots\cdots(227)$$

$$[{}_0\tilde{\tilde{S}}_1(\omega_1'' ; \tau_1, \tau_2 ; R^2{}_\infty)] = \frac{1}{2\pi}\int_{R^1{}_\infty}[G(j(\omega_1'' + \omega'))][A(j\omega_1'')][{}_IS(\omega')]$$

$$\cdot[X_m(\omega'\,;\,\tau_2)]^* e^{j\tau_1\omega'} d\omega' \qquad \cdots\cdots\cdots\cdots(228)$$

$$[{}_0\tilde{\tilde{S}}_2(\omega_2''; \tau_1, \tau_2\,;\,R^2{}_\infty)] = \frac{1}{2\pi}\int_{R^1{}_\infty}[X_m(\omega'\,;\,\tau_1)][{}_IS(\omega')][A(j\omega_2'')]^* \cdot[G(j(\omega_2''+\omega'))]^* e^{-j\tau_2\omega'} d\omega' \qquad \cdots\cdots\cdots\cdots(229)$$

where

$$[X_m(\omega'\,;\,\tau)] = \frac{1}{2\pi}\int_{R^1{}_\infty}[G(js)][A(j(s-\omega'))]e^{j\tau s}ds$$

$$[G_m(\omega'\,;\,\tau)] = e^{-j\omega'\tau}[X_m(\omega', \tau)] = \frac{1}{2\pi}\int_{R^1{}_\infty}[G(js)][A(j(s-\omega'))]e^{j(s-\omega')\tau}ds \qquad \cdots\cdots\cdots\cdots(230)$$

In the special case where the deterministic matrix $[a(\mu)]$ is a constant matrix, eqs. (218)$\sim$(229) are reduced to the input and output relations of the spectral density matrices in a stationary process by making use of the following equations:

$$[A(j\omega')] = 2\pi\delta(\omega')[a],\ [G_m(\omega'\,;\,\tau)] = [G(j\omega')][a] \qquad \cdots\cdots\cdots\cdots(231)$$

$$[X_m(\omega', \tau)] = [G(j\omega')][a]e^{j\tau\omega'} = [X(j\omega')][a] \qquad \cdots\cdots\cdots\cdots(232)$$

In the substitution of eq. (231) in eqs. (221), (228) and (229) it is noted that the following expression can be used:

$$[\tilde{\tilde{G}}_m(\omega_i'', \omega_i'\,;\,R^1{}_\infty)] = 2\pi\delta(\omega_i'')[G(j(\omega_i''+\omega_i'))] = 2\pi\delta(\omega_i'')[G(j\omega_i')]$$

In the above, the problem of a time-variant linear discrete system subjected to a quasi-stationary input is discussed by replacing it by a modified time-variant linear discrete system subjected to a stationary input. Of course, this problem can be directly treated as a special case of a time-variant linear discrete system subjected to non-stationary input discussed in the preceding section. In general, in order to make use of the input and output relations of the local co-variance matrix and the local spectral density matrices, we introduce the modified deterministic matrix of the quasi-stationary input which is defined by

$$[a_m(\mu\,;\,{}_IR^1{}_\mu)] = D(\mu\,;\,{}_IR^1{}_\mu)[a(\mu)] \qquad \cdots\cdots\cdots\cdots(233)$$

Then the local co-variance matrix and the local spectral density matrices are determined as follows:

$$[{}_IK(\mu_1, \mu_2\,;\,{}_{II}R^2{}_{\mu_1,\mu_2})] = D(\mu_1, \mu_2\,;\,{}_{II}R^2{}_{\mu_1\mu_2})[a(\mu_1)][{}_IR(\mu_1-\mu_2)][a(\mu_2)]^* = [a_m(\mu_1\,;\,{}_IR^1{}_{\mu_1})][{}_IR(\mu_1-\mu_2)][a_m(\mu_2\,;\,{}_IR^1{}_{\mu_2})]^* \qquad \cdots\cdots\cdots(234)$$

$$[{}_IS(\omega_1', \omega_2'\,;\,{}_{II}R^2{}_{\mu_1\mu_2})] = \frac{1}{2\pi}\int_{R^1{}_\infty}[A_m(j(\omega_1'-s)\,;\,{}_IR^1{}_{\mu_1})][{}_IS(s)] \cdot[A_m(j(\omega_2'-s)\,;\,{}_IR^1{}_{\mu_2})]^* ds \qquad \cdots\cdots\cdots\cdots(235)$$

$$[{}_IS(\omega', \nu\,;\,{}_{IP}R^2{}_{x\nu})] = \frac{1}{2\pi}D(\nu\,;\,{}_IR^1{}_\nu)e^{j\omega'\nu}\int_{R^1{}_\infty}[A_m(j(\omega'-s)\,;\,{}_IR^1{}_{x+\nu})] \cdot[{}_IS(s)][a(\nu)]^* e^{-js\nu}ds \qquad \cdots\cdots\cdots\cdots(236)$$

and so on, where

$$[A_m(j\omega'\,;\,{}_IR^1{}_\mu)] \overset{\omega'}{\underset{\mu}{\subset}} [a_m(\mu\,;\,{}_IR^1{}_\mu)] \qquad \cdots\cdots\cdots\cdots(237)$$

By substituting these quantities in the general input and output relations given in the preceding section, we can generally obtain the local co-variance matrix and the one- or two-dimensional local spectral density matrices of the output of a time-variant linear discrete system subjected to a quasi-stationary input.

In particular, the two-dimensional total spectral density matrix of the output $[{}_0S(\omega_1, \omega_2 ; R^2_\infty)]$ of a time-invariant linear discrete system subjected to the original quasi-stationary input is obtained in the same expression to that of eq. (218) by substituting eq. (235) in eq. (153). As shown in eq. (153), in the case of a time-invariant system, the two-dimensional spectral density matrix $[{}_0S(\omega_1', \omega_2' ; \tau_1, \tau_2)]$, defined by eq. (127), has the same form as the two-dimensional total spectral density matrix $[{}_0S(\omega_1, \omega_2 ; R^2_\infty)]$ and it does not depend upon the time variables τ_1 and τ_2. However, the expression given by eq. (219) which is derived by using the modified time-variant linear discrete system contains the time variables even in the case of a time-invariant linear discrete system, hence it is different from the expression of $[{}_0S(\omega_1, \omega_2 ; R^2_\infty)]$ although both of them are maped to the co-variance matrix of the output by the inverse double Fourier transform. In similar fashion, the one-dimensional total spectral density matrices, $[{}_0S_1(\omega_1, \tau_2 ; R^2_\infty)]$, $[{}_0S(\tau_1, \omega_2 ; R^2_\infty)]$ and $[{}_0S(\omega, \tau ; R^2_\infty)]$ of a time-invariant linear discrete system are obtained from eqs. (169), (170) and (171) in the same expressions as eqs. (222), (223) and (224) by making use of the corresponding one-dimensional total spectral density matrices of the original quasi-stationary input which are given by the following forms, respectively :

$$[{}_IS_1(\omega_1', \nu ; R^2_\infty)] = \frac{1}{(2\pi)^2}\int_{R^1_\infty} d\omega'[A(j\omega_1' - \omega'))][{}_IS(\omega')] \cdot \int_{R^1_\infty} ds[A(j(s-\omega'))]^* e^{-j\nu s} \qquad \cdots\cdots\cdots\cdots(238)$$

$$[{}_IS_2(\kappa, \omega_2' ; R^2_\infty)] = \frac{1}{(2\pi)^2}\int_{R^1_\infty} d\omega' \int_{R^1_\infty} ds[(A(j(s-\omega'))] e^{j\kappa s}[{}_IS(\omega')] \cdot [A(j(\omega_2' - \omega'))]^* \qquad \cdots\cdots\cdots\cdots(239)$$

$$[{}_IS(\omega', \nu ; R^2_\infty)] = ej\omega'\nu[{}_IS_1(\omega', \nu ; R^2_\infty)] \qquad \cdots\cdots\cdots\cdots(240)$$

As in the two-dimensional case, it can be shown that for a time-invariant linear discrete system, the one-dimensional spectral density matrix, $[{}_0S_1(\omega_1' ; \tau_1, \tau_2)]$ and $[{}_0S_2(\omega_2' ; \tau_1, \tau_2)]$, are reduced to the same expressions as those of $[{}_0S_1(\omega_1, \tau_2 ; R^2_\infty)]$ and $[{}_0S_2(\tau_1, \omega_2 ; R^2_\infty)]$ which are given by eqs. (222) and (223) respectively. However, the expressions of eqs. (226) and (227) which are derived by using the modified time-variant linear discrete system are different from the expression of $[{}_0S_1(\omega_1, \tau_2 ; R^2_\infty)]$ and $[{}_0S_2(\tau_1, \omega_2 ; R^2_\infty)]$ except for the special case of a time-invariant linear discrete system subjected to a stationary input. This discrepancy between the total spectral density matrices of the output and the corresponding spectral density matrices of the output represented by using the input frequency parameter substantially arises from the possible variety of definition of the latter spectral density matrices. In general, the co-variance matrix of the output of a time-variant linear discrete system subjected to the finite duration of a non-stationary input can be expressed in terms of the modified impulsive response matrix and the co-variance matrix of the non-stationary input as follows, as in the case of the quasi-stationary input :

$$[{}_0K(\tau_1, \tau_2)] = \int_{\tau_{1L}}^{\tau_1} d\mu_1 \int_{\tau_{2L}}^{\tau_2} d\mu_2 [g_m(\tau_1, \mu_1 ; {}_IR^1{}_{\mu_1})][{}_IK(\mu_1, \mu_2)][g_m(\tau_2, \mu_2 ; R^1{}_{\mu_2})]^*$$
$$-\infty < \tau_{iL} \leqq \mu_{iL} \qquad \cdots\cdots\cdots\cdots(241)$$

where

$$[g_m(\tau_i, \mu_i ; {}_IR^1{}_{\mu_i})] = D(\mu_i ; {}_IR^1{}_{\mu_i})[g(\tau_i, \mu_i)] \qquad \cdots\cdots\cdots\cdots(242)$$

From eqs. (6) and (241), the two-dimensional spectral density matrix with respect to the input parameters is given by

$$[{}_0S(\omega_1', \omega_2' ; \tau_1, \tau_2)] = [G_m(\omega_1' ; \tau_1 ; {}_IR^1{}_{\mu_1}, \tau_{1L})][{}_IS(\omega_1', \omega_2' ; R^2{}_\infty)]$$
$$\cdot [G_m(\omega_2' ; \tau_2 ; {}_IR^1{}_{\mu_2}, \tau_{2L})]^* \qquad \cdots\cdots\cdots\cdots(243)$$

where

$$[G_m(\omega_i' ; \tau_i ; {}_IR^1{}_{\mu_i}, \tau_{iL})] = \int_{\tau_{iL}}^{\tau_i} [g_m(\tau_i, \mu_i ; {}_IR^1{}_{\mu_i})] e^{-j(\tau_i-\mu_i)\omega_i'} d\mu_i$$
$$-\infty < \tau_{iL} \leqq \mu_{iL} \qquad \cdots\cdots\cdots\cdots(244)$$

This expression clearly depends on the lower limits of the integral, τ_{iL}'s, and hence it does not always agree with the expression given by eq. (127). if either of the input time domains, ${}_IR^1{}_{\mu i}$'s is finite, even in the case where the original system is a time-invariant linear discrete system. Only in the case where a time-invariant linear discrete system and the input domain ${}_IR^2{}_{\mu_1\mu_2} = R^2{}_\infty$ are considered do the two kinds of the two-dimensional spectral density matrices given by eqs. (127) and (243) become identical. After all, the spectral density matrices of the output expressed by using the input frequency parameter are rather limited and vague spectral notions in the sense that they are only applicable to the output of a linear system and that they may have different forms depending upon the possible expression of the co-variance matrix of the output. And also it is a disadvantageous property of these kinds of spectral density matrices that although they can be maped to the co-variance matrix of the output by applying the relevant inverse Fourier transform operators, the operators deriving them from the co-variance matrix can not always be found. On the other hand, the local or the total spectral density matrices are the general spectal notions which are adaptable to any non-stationary random vector and each of these kinds of spectral density matrices constitutes a relevant pair of Fourier transforms with the local co-variance matrix or the co-variance matrix of the random vector.

In the above we have considered a linear discrete system subjected to a quasi-stationary random input vactor. In the remainder of this section we shall consider a similar case as the above in which a linear discrete system is applied to a locally stationary random input vector introduced by R. A. Silverman[9]. The co-variance matrix of the locally stationary input vector may be expressed by the following form:

$$[{}_IK(\mu_1, \mu_2)] = \left[a\left(\frac{\mu_1+\mu_2}{2}\right)\right][{}_IR(\mu_1-\mu_2)]\left[a\left(\frac{\mu_1+\mu_2}{2}\right)\right]^* \qquad \cdots\cdots\cdots\cdots(245)$$

Hence the two-dimensional total spectral density matrix of this random input is given by

$$[{}_IS(\omega_1', \omega_2' ; R^2{}_\infty)]$$
$$= \int_{R^2{}_\infty} \left[a\left(\frac{\mu_1+\mu_2}{2}\right)\right][{}_IR(\mu_1-\mu_2)]\left[a\left(\frac{\mu_1+\mu_2}{2}\right)\right]^* e^{-j(\omega_1'\mu_1-\omega_2'\mu_2)} d\mu_1 d\mu_2$$

$$=\int_{R^2_\infty}[a(\mu_1)][{}_lR(\mu_2)][a(\mu_1)]^* e^{-j(\omega_1'-\omega_2')\mu_1} e^{-j\frac{(\omega_1'+\omega_2')}{2}\mu_2} d\mu_1 d\mu_2$$

$$=\frac{1}{2\pi}\int_{R^1_\infty}[A(j(\omega_1'-\omega_2'-s))]\Big[{}_lS\Big(\frac{\omega_1'+\omega_2'}{2}\Big)\Big][A(-js)]^* ds \qquad \cdots\cdots\cdots\cdots(246)$$

Similarly the one-dimensional total spectral density matrix is expressed by

$$[{}_lS(\omega',\nu;R^2_\infty)]=\int_{R^1_\infty}\Big[a\Big(\frac{\mu}{2}+\nu\Big)\Big][{}_lR(\mu)]\Big[a\Big(\frac{\mu}{2}+\nu\Big)\Big]^* e^{-j\omega'\mu}d\mu$$

$$=\frac{1}{\pi^2}(e^{2j\nu\omega'}[A(2j\omega')])\underset{\omega'}{*}[{}_lS(\omega')]\underset{\omega'}{*}(e^{2j\nu\omega'}[A(-2j\omega')]^*)$$

$$=\frac{1}{(2\pi)^2}e^{j\omega'\nu}\int_{R^1_\infty}d\omega_2' e^{-j\nu\omega_2'}\int_{R^1_\infty}ds[A(j(\omega'-\omega_2'-s))]\Big[{}_lS\Big(\frac{\omega'+\omega_2'}{2}\Big)\Big]\cdot[A(-js)]^* \qquad \cdots\cdots\cdots\cdots(247)$$

Therefore the local co-variance matrix and the one-and two-dimensional local spectral density matrices of the output of a time-variant linear discrete system subjected to the above random process are easily obtainable according to the general input and output relations described in the preceding section by making use of the modified impulsive response matrix $[g_m(\tau,\mu;{}_lR^1_\mu)]$ defined by eq. (242).

5. Conclusive Remarks

In relation to the statistical design method of anti-earthquake structures for moderately intense excitations, some basic studies on the statistical quantities in the non-stationary stochastic processes are described in this paper, and the input and output relations of such statistical quantities in the case of a multi-input and output, time-variant, linear discrete system are presented.

At first, the local co-variance matrix of the complex-valued non-stationary stochastic process, which is defined as the product of the two-dimensional cut-off operator and the co-variance matrix, is introduced as the basic statistical quantity in the time domain. And then, the two-dimensional local spectral density matrix and the several kinds of one-dimensional local spectral density matrices, which are defined as the conjugate double and the single Fourier transforms of the local co-variance matrix respectively, are presented as the basic quantities in the frequency domain. It is shown that the appropriately defined one-dimensional local spectral density matrices are equivalent to the power spectral density matrices in the non-stationary stochastic process in the sense that the integrals of these one-dimensional spectral density matrices over the finite time domain result in the local energy spectral density matrix which is obtained from the two-dimensional local spectral density matrix defined in the finite square domain by equating the two frequency variables. And also it is found that as the limiting case of the one-dimensional Hermitian local power spectral density matrix introduced in this paper, the spectral density matrix of a non-stationary stochastic process which is presented by D. G. Lampard is obtainable. Moreover, it is shown that the one-or two-dimensional local spectral density matrices can be expressed as the weighted averages of the corresponding one-or two-dimensional total spectral density matrices defined in the respective full time domains.

Secondly, for the multi-input and-output linear discrete system having the complex-valued time-variant coefficients, the input and output relations of the local co-variance matrix and the one-or two-dimensional local spectral density matrices are presented. In particular, for the case of a linear discrete system having real-valued time-invariant coefficients, the input and output relation of the two-dimensional total spectral density matrix is found to be reduced to the result presented by J. S. Bendat. In these relations, the one-or two-dimensional local spectral density matrices of the input and output are used to constitute the respective pairs of the single or the double Fourier transforms with the local co-variance matrices of the input and output. However, particularly in regard to the output of a linear system, the spectral density matrices which give formally the co-variance matrix of the output by the inverse Fourier transform operators, but have different expressions from the relevant total spectral density matrices of the output, are obtainable. It is shown that these kinds of spectral density matrices expressed in terms of the input frequency parameters may have different forms depending upon the possible expressions of the co-variance matrix of the output, except for the special case of a time-invariant linear discrete system subjected to the infinite duration of a random input. In these aspects, it is suggested that the local or the total spectral density matrices defined as the Fourier transforms of the local co-variance matrix or the co-variance matrix are the most reasonable and generally applicable spectral notions in the non-stationary stochastic process. Here it is noted that the local co-variance matrix is exactly in accordance with the co-variance matrix of the non-stationary process in the defining time domain and the value of the co-variance matrix inside the domain can be uniquely determined from the spectral density matrices defined in any domains containing the defining domain by the relevant inverse Fourier transforms. And also it is noted that local spectral density matrices of a non-stationary process are expressed by the weighted average of the corresponding total spectral density matrices and in general the local spectral desity matrices are given by the convolution of the Fourier transform of the defining domain and the local spectral density matrices defined in any domain containing the defining domain. As a rule, since the local co-variance matrix and the corresponding local spectral density matrices associated with a prescribed domain can be found from those defined in any domain containing the defining domain, the larger the defining domain the more information is obtainable in both the time and the frequency domain. However, it will not always be possible to acquire a perfect knowledge of a non-stationary process in the full time domain. And also, if such knowledge were available, the convergence of integral transforms defined in the infinite time domain would not always be guaranteed for the general non-stationary process even in the generalized sense. Moreover, the time-dependence of the spectral characteristics of a non-stationary process may not be precisely detected from the local spectral density matrices defined in the large time domain. On the other hand, it is noted that as the defining time domain becomes smaller the resolvability of the frequency characteristics may decrease. Hence, in order to estimate the time-dependence of the spectral characteristics of a non-stationary process, it is important to select a pertinent defining domain of the local spectral density matrices. Here, it is mentioned that for the case where the co-variance matrix in the full time domain can be predicted, the one-dimensional total spectral density matrices containing a time parameter present at least one aspect of the time-dependent spectral characteristics of the non-stationary process under the condition of their existence. And also it should be noted that the local co-variance matrix and the one-and two-dimensional local spectral density matrices constitute the respective pair of Fourier transforms and that they are all the additive set functions of the defining time domain. This shows that the local spectral density matrices defined in a domain give the complete knowledge of the co-variance matrix inside the defining domain and that the sum of the local spectral density matrices defined in the disjunctive domains gives the local spectral density matrix associated with the sum of the domains. For instance, if the value of the co-variance matrix $[K(\tau_1, \tau_2)]$ is neglegible in the domain $|\tau_1-\tau_2|>a$ or if we restrict a priori our interest in the domain $|\tau|=|\tau_2|<\infty$, $|\lambda|=|\tau_1-\tau_2|\leqq a$, the required information in the domain between the two parallel straight lines, $\tau_2=\tau_1\pm a$, or in the infinite strip region being parallel to the τ-axis and having the width $2a$ in the direction of λ, can be determined from the local co-variance matrix and the local spectral density matrices defined in a series of square domains each side of which is parallel to the τ_1-or τ_2-axis and has the side-length $4a$, and which have their centers at a series of points, $\tau=\tau_1=\tau_2=2ma+b$, where m takes the values, 0, ± 1, ± 2,... and b is an arbitrary constant. That is, from the local co-variance matrix and the local spectral density matrices defined in the series of square domains, we can determine those associated with a series of the disjunctive parallelogram domains enclosed by the lines, $\tau_2=\tau_1\pm a$ and $\tau_2=(2m\pm 1)a+b$. And, by adding these quantities, the local co-variance matrix and the local spectral density matrices in the sum of the parallelogram domains and taking the limiting case, those in the infinite strip region, are obtainable. And also once the information in a broad domain is found the local quantities associated with any domain contained within the domain are uniquely determined. In the above discussion, in the case where only the local co-variance matrix and the one-dimensional spectral density matrices are concerned, the defining domain of such quantities is reduced to the one-dimensional time domain $|\tau_1-\tau_2|\leqq a$ or $|\lambda|\leqq a$, because the variable τ_2 or τ can be considered as a parameter.

Finally, as the non-stationary input process most applicable to earthquake engineering the quasi-stationary random process introduced by V. V. Bolotin as well as the locally stationary random process presented by R. A. Silverman are considered and the basic statistical quantities presented in this paper, such as the local co-variance matrix and the local or total spectral density matrices, with respect to the output response vector of a multi-input and-output, time-variant, linear discrete system subjected to these input random vectors, are estimated. And it is shown that as a special case of the time-invariant linear discrete system subjected to a stationary input random vector, the input and output relations of the co-variance matrix and the one-or two-dimensional total spectral density matrices are reduced to the well-known results in the stationary random process.

In this paper, since the analysis of a non-stationary stochastic process has been carried out mainly from the mathematical point of view, the averaging operator always means the ensemble average and the time domain considered is not always restricted to the past time domain. However, in relation to an experimental technique using a realizable filter with a finite time constant, the so-called short-time auto-correlation functions and power spectra have been introduced by R. M. Fano[10] and were extended by M. A. Schroeder and B. S. Atal[11]. Since these investigations were made for the purpose of the analysis of a sample random function of a non-stationary stochastic process, the averaging operator has meant always the time average and the time domain considered has been always restricted to the past time domain. In the appendix which follows these notions will be expressed in matrix forms as the short-time correlation matrix and the short-time power spectral density matrix. And it will be shown that the ensemble averages of this shot-time correlation matrix and power spectral density matrix are expressed as the weighted averages of the local co-variance matrix and the local spectral density matrices defined in the semi-infinite square domain $R^2_{-\infty\tau}$.

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Appendix ; Short-time correlation matrix and power spectral density matrix of a non-stationary random process

In this appendix, we extend the short-time autocorrelation functions and power spectra of a random sample functions, introduced by R. M. Fano, M. R. Schroeder and B. S. Atal, to the matrix formulae which concern a random sample vector and show their relations to the local co-variance matrix and the local spectral density matrices of a non-stationary stochastic process. First, we introduce a weighted random process $\{\chi(\mu, \tau)\}$ which is defined as the product of a deterministic weighting function $w(\tau-\mu)$ and a non-stationary stochastic process $\{\xi(\mu)\}$. Here, the fuction $w(\mu)$ is characterized as zero in the domain $(-\infty, 0)$ and when μ tends to ∞. Then, for a random sample function $\{\chi(\mu, \tau)\}_s$, the energy spectral density matrix of the function $[G(\omega, \tau)]$ and its Fourier transform $[\varphi(\lambda, \tau)]$ are expressed as follows :

$$[G(\omega, \tau)]=\Big(\int_{R^1_\infty}\{\chi(\mu, \tau)\}_s e^{-j\omega\mu}d\mu\Big)\Big(\int_{R^1_\infty}\{\chi(\mu, \tau)\}_s e^{-j\omega\mu}d\mu\Big)^*$$
$$=\int_{R^2_\infty} w(\tau-\mu_1)\{\xi(\mu_1)\}_s\{\xi(\mu_2)\}_s^* w^*(\tau-\mu_2)e^{-j(\mu_1-\mu_2)\omega}d\mu_1 d\mu_2$$
$$=\int_{R^2_\infty} D(\mu_1, \mu_2\,;\,R^2_{-\infty\tau})w(\tau-\mu_1)\{\xi(\mu_1)\}_s\{\xi(\mu_2)\}_s^* w^*(\tau-\mu_2)e^{-j(\mu_1-\mu_2)\omega}d\mu_1 d\mu_2$$
$$=\frac{1}{(2\pi)^2}\Big((e^{-j\omega\tau}W(-\omega))\underset{\omega}{*}\{F_\xi(\omega\,;\,R^1_{-\infty\tau})\}_s\Big)\Big(\{F_\xi(\omega\,;\,R^1_{-\infty\tau})\}_s^*\underset{\omega}{*}(e^{-j\omega\tau}W(-\omega))^*\Big) \qquad \cdots\cdots\cdots\cdots(a\cdot1)$$

$$[\varphi(\lambda, \tau)]=\int_{R^1_\infty} D(\mu\,;\,R^1_{-\infty\tau})D(\mu-\lambda\,;\,R^1_{-\infty\tau})w(\tau-\mu)\{\xi(\mu)\}_s\{\xi(\mu-\lambda)\}_s^* w^*(\tau-\mu+\lambda)d\mu$$
$$=\int_{R^1_\infty} D(\mu\,;\,R^1_{-\infty\, min(\tau,\tau+\lambda)})w(\tau-\mu)\{\xi(\mu)\}_s\{\xi(\mu-\lambda)\}_s^* w^*(\tau-\mu+\lambda)d\mu \qquad \cdots\cdots\cdots\cdots(a\cdot2)$$

where

$$\{\chi(\mu, \tau)\}=w(\tau-\mu)\{\xi(\mu)\}=D(\mu\,;\,R^1_{-\infty\tau})w(\tau-\mu)\{\xi(\mu)\} \qquad \cdots\cdots\cdots\cdots(a\cdot3)$$
$$w(\mu)\underset{\mu}{\overset{\omega}{\rightleftharpoons}}W(\omega)$$

From the above definitions of $[G(\omega, \tau)]$ and $[\varphi(\lambda, \tau)]$ the following relations are valid :

$$[G(\omega, \tau)]\underset{\lambda}{\overset{\omega}{\rightleftharpoons}}[\varphi(\lambda,\tau)] \qquad \cdots\cdots\cdots\cdots(a\cdot4)$$
$$[G(\omega,\tau)]^*=[G(\omega,\tau)],\ [\varphi(\lambda,\tau)]^*=[\varphi(-\lambda,\tau)] \qquad \cdots\cdots\cdots\cdots(a\cdot5)$$

By making use of eqs. (a·1) and (a·2) the short-time correlation matrix $[\varphi_\tau(\lambda)]$ and the short-time power spectral density matrix $[G_\tau(\omega)]$ of the original sample function $\{\xi(\mu)\}_s$ can be defined as the following weighted time averages so as to be consistent with the definitions of the short-time auto-correlation functions and power spectra given by R. M. Fano.

$$[\varphi_\tau(\lambda)]=v^{-1}(\lambda)[\varphi(\lambda, \tau)] \qquad \cdots\cdots\cdots\cdots(a\cdot6)$$
$$[G_\tau(\omega)]=v^{-1}(0)[G(\omega, \tau)] \qquad \cdots\cdots\cdots\cdots(a\cdot7)$$

where

$$v(\lambda)=\int_{R^1_\infty} D(\mu\,;\,R^1_{-\infty\, min(\tau,\tau+\lambda)})w(\tau-\mu)w^*(\tau-\mu+\lambda)d\mu$$
$$=\int_{max(0,-\lambda)}^{\infty} w(\mu)w^*(\mu+\lambda)d\mu \qquad \cdots\cdots\cdots\cdots(a\cdot8)$$
$$v(0)=\int_{R^1_\infty} D(\mu\,;\,R^1_{-\infty\tau})\,|w(\tau-\mu)|^2 d\mu$$
$$=\int_0^{\infty}|w(\mu)|^2 d\mu \qquad \cdots\cdots\cdots\cdots(a\cdot9)$$

It is noted that eqs. (a·8) and (a·9) represent the integrals of the weighting functions in eq. (a·2) and in the integral of eq. (a·1) with respect to $\omega/2\pi$, respectively. From eqs. (a·1), (a·2) and (a·4), the latter integral gives the energy matrix of the weighted sample function.

$$\frac{1}{2\pi}\int_{R^1_\infty}[G(\omega, \tau)]d\omega=[\varphi(0, \tau)]=\int_{R^1_\infty}\{\chi(\mu, \tau)\}_s\{\chi(\mu, \tau)\}_s^* d\mu$$
$$=\int_{R^1_\infty} D(\mu\,;\,R^1_{-\infty\tau})w(\tau-\mu)\{\xi(\mu)\}_s\{\xi(\mu)\}_s^* w^*(\tau-\mu)d\mu \qquad \cdots\cdots\cdots\cdots(a\cdot10)$$

By making use of eq. (a·5) and the relation,

$$v^*(\lambda)=v(-\lambda),\quad v^*(0)=v(0) \qquad \cdots\cdots\cdots\cdots(a\cdot11)$$

which is derived from eqs. (a·8) and (a·9), the following relations are obtained for the short-time correlation matrix and power spectral density matrix:

$$[\varphi_\tau(\lambda)]^* = [\varphi_\tau(-\lambda)], \quad [G_\tau(\omega)]^* = [G_\tau(\omega)] \qquad \cdots\cdots\cdots\cdots(a\cdot 12)$$

And, by substituting eqs. (a·6) and (a·7) in eq. (a·4), the correspondence between the short-time correlation matrix and power spectral density matrix is found as follows:

$$[G_\tau(\omega)] \overset{\omega}{\underset{\lambda}{\subset}} \frac{v(\lambda)}{v(0)}[\varphi_\tau(\lambda)] \qquad \cdots\cdots\cdots\cdots(a\cdot 13)$$

that is,

$$[G_\tau(\omega)] = \int_{R^1_\infty} \frac{v(\lambda)}{v(0)}[\varphi_\tau(\lambda)]e^{-j\omega\lambda}d\lambda \qquad \cdots\cdots\cdots\cdots(a\cdot 14)$$

$$[\varphi_\tau(\lambda)] = \frac{v(0)}{2\pi v(\lambda)}\int_{R^1_\infty}[G_\tau(\omega)]e^{j\omega\lambda}d\omega \qquad \cdots\cdots\cdots\cdots(a\cdot 15)$$

Assuming that the mean vector of a non-stationary stochastic process $\{\xi(\mu)\}$ is zero, the ensemble average of the short-time correlation matrix is given by the following weighted average of the local co-variance matrix defined in the semi-infinite square domain, $R^2_{-\infty\tau}$:

$$E[\varphi_\tau(\lambda)] = v^{-1}(\lambda)E[\varphi(\lambda, \tau)] \qquad \cdots\cdots\cdots\cdots(a\cdot 16)$$

$$\begin{aligned} E[\varphi(\lambda, \tau)] &= \int_{R^1_\infty}[K_\chi(\mu, \mu-\lambda\,;\, R^2_{-\infty\tau})]d\mu \\ &= \int_{R^1_\infty} D(\mu\,;\, R^1_{-\infty\, min(\tau, \tau+\lambda)})[K_\chi(\mu, \mu-\lambda)]d\mu \\ &= \int_{R^1_\infty} D(\mu\,;\, R^1_{-\infty\, min(\tau, \tau+\lambda)})w(\tau-\mu)[K_\xi(\mu, \mu-\lambda)]w^*(\tau-\mu+\lambda)d\mu \\ &= \int_{R^1_\infty} w(\tau-\mu)[K_\xi(\mu, \mu-\lambda\,;\, R^2_{-\infty\tau})]w^*(\tau-\mu+\lambda)d\mu \end{aligned} \qquad \cdots\cdots\cdots\cdots(a\cdot 17)$$

where the subscripts χ and ξ denote the quantities concerning with $\{\chi(\mu, \tau)\}$ and $\{\xi(\mu)\}$, respectively. Similarly, the ensemble average of the shot-time power spectral density matrix can be expressed by the weighted average of the two-dimensional local spectral density matrix defined in the square domain $R^2_{-\infty\tau}$, as follows:

$$E[G_\tau(\omega)] = v^{-1}(0)E[G(\omega, \tau)] \qquad \cdots\cdots\cdots\cdots(a\cdot 18)$$

$$\begin{aligned} E[G(\omega, \tau)] &= [S_\chi(w, w\,;\, R^2_{-\infty\tau})] \\ &= \frac{1}{(2\pi)^2}(e^{-j\omega_1\tau}W(-\omega_1)) \underset{\omega_1}{*} [S_\xi(\omega_1, \omega_2\,;\, R^2_{-\infty\tau})] \\ &\qquad \underset{\omega_2}{*} (e^{-j\omega_2\tau}W(-\omega_2))^* \,|\, \omega_1=\omega_2=\omega \\ &= \frac{1}{(2\pi)^2}\int_{R^1_\infty} d\omega_1' \int_{R^1_\infty} d\omega_2' e^{j\omega_1'\tau}W(\omega_1'-\omega)[S_\xi(\omega_1', \omega_2'\,;\, R^2_{-\infty\tau})]e^{-j\omega_2'\tau}W^*(\omega_2'-\omega) \end{aligned}$$

$$\cdots\cdots\cdots\cdots(a\cdot 19)$$

The ensemble average of the short-time spectral density matrix is also expressed as the weighted average of the one-dimensional local spectral density matrix defined in the semi-infinite parallelogram domain ${}_pR^2{}_{\lambda\mu}$ in the $\lambda-\mu$ plane which is transformed from the square domain $R^2_{-\infty\tau}$ in the $\mu_1-\mu_2$ plane.

$$E[G_\tau(\omega)] = v^{-1}(0)E[G(\omega, \tau)]$$

$$\begin{aligned} E[G(\omega, \tau)] &= \int_{R^1_\infty} E[\varphi(\lambda, \tau)]e^{-j\omega\lambda}d\lambda \\ &= \int_{R^1_\infty} d\lambda e^{-j\omega\lambda}\int_{R^1_\infty} d\mu[K_\chi(\mu+\lambda, \mu\,;\, R^2_{-\infty\tau})] \\ &= \int_{R^1_\infty} [S_\chi(\omega, \mu\,;\, {}_pR^2{}_{\lambda\mu})]d\mu \end{aligned}$$

$$=\frac{1}{2\pi}\int_{R^1_\infty}d\mu(e^{-j(\tau-\mu)\omega}W(-\omega))\underset{\omega}{*}[S_\xi(\omega,\mu\,;\,{}_pR^2{}_{\lambda\mu})]w^*(\tau-\mu)$$

$$=\frac{1}{2\pi}\int_{R^1_\infty}d\mu\int_{R^1_\infty}d\omega' e^{-j(\tau-\mu)(\omega-\omega')}W(\omega'-\omega)[S_\xi(w',\mu\,;\,{}_pR^2{}_{\lambda\mu})]w^*(\tau-\mu)$$

$$\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 20)$$

By substituting $\lambda=0$ in the correspondence,

$$E[G(\omega,\tau)]\underset{\lambda}{\overset{\omega}{\subset}}E[\varphi(\lambda,\tau)],\quad E[G_\tau(\omega)]\underset{\lambda}{\overset{\omega}{\subset}}\frac{v(\lambda)}{v(0)}E[\varphi_\tau(\lambda)]\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 21)$$

and making use of eqs. (a·16) and (a·17), we obtain the integral of the short-time power spectral density matrix over the full frequency domain R^1_∞ as follows:

$$\frac{1}{2\pi}\int_{R^1_\infty}E[G_\tau(\omega)]d\omega=E[\varphi_\tau(0)]=v^{-1}(0)E[\varphi(0,\tau)]$$

$$=\frac{1}{v(0)}\int_{R^1_\infty}D(\mu\,;\,R^1_{-\infty\tau})|w(\tau-\mu)|^2[K_\xi(\mu,\mu)]d\mu\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 22)$$

Since the right-hand side of the above equation represents the weighted time average of the co-variance matrix of a stochastic process $\{\xi(\mu)\}$, over the semi-infinite time domain $R^1_{-\infty\tau}$, the ensemble average of the short-time power spectral density matrix has meaning as the averaged power spectral density matrix over the same domain.

For the case considered by R. M. Fano where the physically realizable weighting function is given by

$$w(\mu)=\alpha e^{-\alpha\mu}s(\mu)\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 23)$$

in which $s(\mu)$ denotes a step function, the matrices $[\varphi(\lambda,\tau)]$ and $[G(\omega,\tau)]$ are obtained as the following expressions by using eqs. (a·1), (a·2) and (a·3):

$$[\varphi(\lambda,\tau)]=\alpha^2e^{-\alpha\lambda}\int_{-\infty}^{min(\tau,\tau+\lambda)}e^{-2\alpha(\tau-\mu)}\{\xi(\mu)\}\{(\mu-\lambda)\}^*d\mu\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 24)$$

$$[G(\omega,\tau)]=\alpha^2\left(\int_{-\infty}^{\tau}e^{-\alpha(\tau-\mu)}\{\xi(\mu)\}e^{-j\omega\mu}d\mu\right)\left(\int_{-\infty}^{\tau}e^{-\alpha(\tau-\mu)}\{\xi(\mu)\}e^{-j\omega\mu}d\mu\right)^*\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 25)$$

On the other hand, from eqs. (a·8) and (a·9), the integral of the weighting functions $v(\lambda)$ and $v(0)$ are given by

$$v(\lambda)=\alpha^2e^{-\alpha\lambda}\int_{max(0,-\lambda)}^{\infty}e^{-2\alpha\mu}d\mu=\frac{\alpha e^{-\alpha|\lambda|}}{2},\quad v(0)=\frac{\alpha}{2}\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 26)$$

respectively. Hence the short-time correlation matrix $[\varphi_\tau(\lambda)]$ and power spectral density matrix $[G_\tau(\omega)]$ are expressed as follows by using eqs. (a·6), (a·7) and (a·24)∼(a·26):

$$[\varphi_\tau(\lambda)]=2\alpha e^{-\alpha(\lambda-|\lambda|)}\int_{-\infty}^{min(\tau,\tau+\lambda)}e^{-2\alpha(\tau-\mu)}\{\xi(\mu)\}\{\xi(\mu-\lambda)\}^*d\mu\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 27)$$

$$[G_\tau(\omega)]=2\alpha\left(\int_{-\infty}^{\tau}e^{-\alpha(\tau-\mu)}\{\xi(\mu)\}e^{-j\omega\mu}d\mu\right)\left(\int_{-\infty}^{\tau}e^{-\alpha(\tau-\mu)}\{\xi(\mu)\}e^{-j\omega\mu}d\mu\right)^*\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 28)$$

And, from eqs. (a·14) and (a·15) the relationship between the short-time correlation matrix and power spectral density matrix is given by the following set of equations:

$$[G_\tau(\omega)]=\int_{R^1_\infty}e^{-\alpha|\lambda|}[\varphi_\tau(\lambda)]e^{-j\omega\lambda}d\lambda$$

$$[\varphi_\tau(\lambda)]=\frac{e^{\alpha|\lambda|}}{2\pi}\int_{R^1_\infty}[G_\tau(\omega)]e^{j\omega\lambda}d\omega\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 29)$$

On the other hand, the ensemble averages of the short-time correlation matrix and power spectral density matrix are obtained from eqs. (a·16)∼(a·19) as follows:

$$E[\varphi_\tau(\lambda)]=2\alpha e^{-\alpha(\lambda-|\lambda|)}\int_{R^1_\infty}e^{-2\alpha(\tau-\mu)}[K_\xi(\mu,\mu-\lambda\,;\,R^2_{-\infty\tau})]d\mu\qquad\cdots\cdots\cdots\cdots(\mathrm{a}\cdot 30)$$

$$E[G_\tau(\omega)]=\frac{2\alpha}{(2\pi)^2}\frac{e^{-j\omega_1\tau}}{\alpha-j\omega_1}\underset{\omega_1}{*}[S_\xi(\omega_1,\omega_2\,;\,R^2_{-\infty\tau})]\underset{\omega_2}{*}\frac{e^{j\omega_2\tau}}{\alpha+j\omega_2}\Bigg|_{\omega_1=\omega_2=\omega}$$

$$=\frac{2\alpha}{(2\pi)^2}\int_{R^1_\infty}d\omega_1'\int_{R^1_\infty}d\omega_2'\frac{e^{j\omega_1'\tau}}{\alpha-j(\omega-\omega_1')}[S_\xi(\omega_1',\ \omega_2'\ ;\ R^2_{-\infty\tau})]\ \frac{e^{-j\omega_2'\tau}}{\alpha+j(\omega-\omega_2')} \quad\cdots\cdots\cdots\cdots(a\cdot 31)$$

As another example, for the case where the weighting function is the so-called rectangular window with the width T,

$$w(\mu)=s(\mu)-s(\mu-T),\quad T>0 \quad\cdots\cdots\cdots\cdots(a\cdot 32)$$

the matrices $[\varphi(\lambda,\ \tau)]$ and $[G(\omega,\ \tau)]$ and the function $v(\lambda)$ are respectively determined as follows:

$$[\varphi(\lambda,\ \tau)]=\int_{max(\tau-T,\ \tau-T+\lambda)}^{min(\tau,\tau+\lambda)}\{\xi(\mu)\}\{\xi(\mu-\lambda)\}^*d\mu \quad\cdots\cdots\cdots\cdots(a\cdot 33)$$

$$[G(\omega,\ \tau)]=\left(\int_{\tau-T}^{\tau}\{\xi(\mu)\}e^{-j\omega\mu}d\mu\right)\left(\int_{\tau-T}^{\tau}\{\xi(\mu)\}e^{-j\omega\mu}d\mu\right)^* \quad\cdots\cdots\cdots\cdots(a\cdot 34)$$

$$v(\lambda)=\int_{max(\tau-T,\ \tau-T+\lambda)}^{min(\tau,\tau+\lambda)}d\mu=T-|\lambda| \quad\cdots\cdots\cdots\cdots(a\cdot 35)$$

Therefore the short-time correlation matrix and power spectral density matrix are respectively expressed as the following formulae:

$$[\varphi_\tau(\lambda)]=\frac{1}{T-|\lambda|}\int_{max(\tau-T,\ \tau-T+\lambda)}^{min(\tau,\tau+\lambda)}\{\xi(\mu)\}\{\xi(\mu-\lambda)\}^*d\mu \quad\cdots\cdots\cdots\cdots(a\cdot 36)$$

$$[G_\tau(\omega)]=\frac{1}{T}\left(\int_{\tau-T}^{\tau}\{\xi(\mu)\}e^{-j\omega\mu}d\mu\right)\left(\int_{\tau-T}^{\tau}\{\xi(\mu)\}e^{-j\omega\mu}d\mu\right)^* \quad\cdots\cdots\cdots\cdots(a\cdot 37)$$

And the relationship of the short-time correlation matrix and the power spectral density matrix is given by

$$[G_\tau(\omega)]=\int_{R^1_\infty}\frac{T-|\lambda|}{T}[\varphi_\tau(\lambda)]e^{-j\omega\lambda}d\lambda$$

$$[\varphi_\tau(\lambda)]=\frac{1}{2\pi}\ \frac{T}{T-|\lambda|}\int_{R^1_\infty}[G_\tau(\omega)]e^{j\omega\lambda}d\omega \quad\cdots\cdots\cdots\cdots(a\cdot 38)$$

The ensemble averages of the short-time correlation matrix and power spectral density matrix are respectively determined as the following expressions:

$$E[\varphi_\tau(\lambda)]=\frac{1}{T-|\lambda|}\int_{R^1_\infty}[K_\xi(\mu,\ \mu-\lambda\ ;\ R^2_{\tau-T\tau})]d\mu \quad\cdots\cdots\cdots\cdots(a\cdot 39)$$

$$E[G_\tau(\omega)]=\frac{1}{T}[S_\xi(\omega,\ \omega\ ;\ R^2_{\tau-T\tau})] \quad\cdots\cdots\cdots\cdots(a\cdot 40)$$

In the above discussions, when both the stochastic process $\{\xi(\mu)\}$ and the weighting function $w(\mu)$ are real-valued time functions, the following relations are valid, as found from eqs. (a·4), (a·5) and (a·11):

$$[\varphi(\lambda,\ \tau)]^T=[\varphi(-\lambda,\ \tau)],\quad [G(\omega,\ \tau)]^T=[G(-\omega,\ \tau)] \quad\cdots\cdots\cdots\cdots(a\cdot 41)$$

$$v(\lambda)=v(-\lambda) \quad\cdots\cdots\cdots\cdots(a\cdot 42)$$

Hence the short-time correlation matrix and power spectral density matrix together with their ensemble averages have the same properties as those in eq. (a·41), that is,

$$[\varphi_\tau(\lambda)]^T=[\varphi_\tau(-\lambda)],\quad E[\varphi_\tau(\lambda)]^T=E[\varphi_\tau(-\lambda)]$$

$$[G_\tau(\omega)]^T=[G_\tau(-\omega)],\quad E[G_\tau(\omega)]^T=E[G_\tau(-\omega)] \quad\cdots\cdots\cdots\cdots(a\cdot 43)$$

This equation shows that the short-time auto-correlation functions and power spectra defined by R. M. Fano and their ensemble averages, which are given by the diagonal elements of the relevant matrices in eq. (a·43), are all even functions of their respective arguments.