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On the Highest Water Waves of Permanent Type

by

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Abstract

The highest gravity waves of permanent type on the water surface are treated hydrodynamically and calculated numerically for the 16 values of L/D, ranging from o to ∞, L being the length of the waves and D the mean depth of the uniform canal. Their characteristic values are tabulated and a representative wave form and the breaking index curve are shown graphically.

Introduction

About ten years ago one of the authors reported a series of numerical treatment of the nonlinear surface waves of permanent type, and above all calculated a few cases of extreme wave height [1-3][+]. By this method of calculation all the properties of permanent waves can be determined quantitatively, and especially the highest waves corresponding to every ratio of wave length to the mean depth of canal are to be studied, without encountering any difficulty, though calculation is not an easy task. The highest waves of the hydrodynamical theory, however, have been thought to be unrealisable [4], and we also have been thinking of them as unstable above a certain amplitude.

In the meantime the realisation of the theoretical extreme height is proved with an experimental arrangement of a slightly inclined bottom [5], which seemed to be in a good agreement with our previous results. In addition we happened recently to see the breaking index curve [6] of water waves composed by C. L. Bretschneider, which very clearly shows the occurrence of the theoretical maximum height.

This being the fact, irrespective of the waves of maximum height being unstable or not, to know the characteristics of the highest waves as accurately and precisely as possible is, we think, a necessary and significant task. This task being the solution of a nonlinear problem in an at first undefined region, our method in the following cannot be simple and can not pursue an analytical procedure to the end. At a certain point in the process we carried over the solution to a numerical treatment, and a large series of numerical computations, almost all of which were managed by an electronic computer, has been put into a consistent scheme of the characteristic numbers, which is the result aimed at concerning the waves of extreme height. A preliminary note being reported [7], the present paper is the full account of it.

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(+) As to the numbers in parentheses, see the list of references at the end of the paper.

1. Transfomation onto a unit circle

Our method of solution which has already been reported [3] will be given in this section again for better understandings. Its essence is a generalisation of the method used by T. Lévi-Civita [8] and K. J. Struik [9].

We observe the waves on the water from the coordinates system O-xy which follows after the permanent waves as fast as the waves, so that the wave form stands fixed relative to the axes, and water flows steadily from left to right (say). The origin O is at one wave crest, the x-axis is horizontal and directed to right, the y-axis vertical and upward (Fig. 1 a).

Let the wave-length be denoted by L and the mean depth by D. We define the wave velocity U by the formula :

$$UL= \int_{-L/2}^{L/2} u(x,y)dx=\varphi(L/2)-\varphi(-L/2), \qquad \cdots\cdots(1)$$

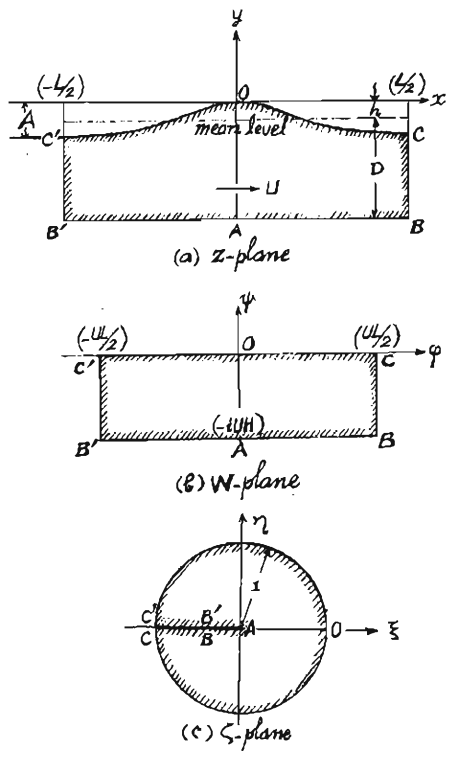

Fig. 1 Conformal transformation

whese φ is the potential function and $u(x,y)$ is the horizontal component of the flow velocity. The wave velocity thus defined is coincident with that of Stokes' waves when D tends to infinity, and with that of the solitary wave when L tends to infinity.

The complex potential function :

$$W(z)=\varphi+i\psi, \qquad \cdots\cdots(2)$$

in which ψ is the stream function and arbitrary constant is fixed so that $W(0)=0$, maps the physical z-plane onto the W-plane as shown in Fig. 1b, the same alphabetical letters indicating the same points of special significance. The distance OA of the W-plane is the flux through any sectional plane of the flow, and when we denote it by UH, by use of the above defined U, H is a potential depth (say) and is nearly equal to D, but differs a little in general.

Now we introduce the complete elliptic integral $K(k)$ and $K'=K(k')$, of the first kind with the modulus k and its complementary modulus $k'=\sqrt{1-k^2}$, and define the numerical value of k by the relation :

$$K'/K=2H/L. \qquad \cdots\cdots(3)$$

With k, k', K and K' thus determined we define a transformation :

$$sn\left(\frac{2K}{UL}W,k\right)=-i\ \frac{1-\zeta}{2\sqrt{\zeta}}, \qquad \cdots\cdots(4)$$

which maps the W-plane onto the ζ-plane. By this mapping the one wave length region of the W-plane, which is shown hatched in Fig. 1b, corresponds to the region interior to the unit circle about the origin of the ζ-plane, also shown hatched in Fig. 1c, the corresponding points being also indicated by the same letters. A special point ζ_B which corresponds to B and B' of W is defined, as easily follows from (4), as

$$\zeta_B = 1 - \frac{2}{k^2}\left(1 - \sqrt{1-k^2}\right). \qquad \cdots\cdots(5)$$

The transformation (4) is the first relation we aimed at, but as the required relation is that between the physical plane z and the unit circle region ζ, we have to find one more relation which connects z and W. As such one we have the complex velocity:

$$\frac{dW}{dz} = v\, e^{-i\theta}, \qquad \cdots\cdots(6)$$

where v and θ are the speed and the direction of flowing water respectively, θ being the angle measured upwards from the horizontal direction (to the right). We denote v/U by q and lnq by τ, and then (6) becomes

$$\frac{1}{U}\frac{dW}{dz} = qe^{-i\theta} = e^{-i\Omega(z)}, \qquad \cdots\cdots(7)$$

the field quantity $\Omega(z)$ being defined by the relation:

$$\Omega(z) = \theta + i\tau = iln\left(\frac{1}{U}\frac{dW}{dz}\right). \qquad \cdots\cdots(8)$$

Evidently $\Omega(z)$ is holomorphic at every interior point of the water region, and may have singular points at the boundary of the region.

As z has to be a certain function of ζ, holomorphic at every interior point of the cut unit circle, we have from (7)

$$\frac{1}{U}\frac{dW}{d\zeta}\frac{d\zeta}{dz} = e^{-i\Omega(\zeta)} \quad i.e.\ dz = \frac{1}{U}\frac{dW}{d\zeta}e^{i\Omega(\zeta)}d\zeta, \qquad \cdots\cdots(9)$$

where $\Omega(\zeta)$ is the transform of $\Omega(z)$ on the ζ-plane. On the other hand, differentiating (4) with regard to ζ, we have

$$\frac{1}{U}\frac{dW}{d\zeta} = \frac{L}{2K}\frac{1}{cn\left(\frac{2KW}{UL}\right)dn\left(\frac{2KW}{UL}\right)}\frac{i}{4}\frac{1+\zeta}{\sqrt{\zeta^3}}.$$

Also from (4) it results that

$$cn\left(\frac{2KW}{UL}\right) = \frac{1+\zeta}{2\sqrt{\zeta}},\quad dn\left(\frac{2KW}{UL}\right) = \sqrt{1+\frac{k^2(1-\zeta)^2}{4\zeta}},$$

and making use of these relations in the above, we have

$$\frac{1}{U}\frac{dW}{d\zeta} = i\frac{L}{4K}\frac{1}{\sqrt{\zeta^2+\frac{k^2}{4}\zeta(1-\zeta)^2}}. \qquad \cdots\cdots(10)$$

Finally combining (9) and (10), it results that

$$dz = i\frac{L}{4K}\frac{1}{\sqrt{\zeta^2+\frac{k^2}{4\zeta}(1-\zeta)^2}}e^{i\Omega(\zeta)}d\zeta, \qquad \cdots\cdots(11)$$

which is the required relation between z and ζ. In (11), however, $\Omega(\zeta)$ is presently unknown, and its determination is the essential point of our wave problem. When it is determined (11) can be integrated, giving $z=z(\zeta)$ which is our final object.

As the wave form is symmetrical about the vertical line through the crest *i.e.* y-axis, it can easily be verified that the function $-i\Omega(z)=\tau-i\theta$ has conjugate complex values at every pair of symmetrical points $z=x+iy$ and $z=-x+iy$. To this pair of points, on the other hand, corresponds a pair of two points of the W-plane, situated symmetrically about the ψ-axis, by the due choice of the origin of W stated above. But now these two points becoming, by the transformation (4), a pair of conjugate points in the ζ-plane, we know that $-i\Omega(\zeta)$ has a pair of complex conjugate values at every pair of conjugate points in the unit circle $|\zeta|\leq 1$. Above all it takes real value on the real axis because of the horizontal velocity at points under crest or trough, or at the bottom, and all along the cut $-1\leq\zeta<0$ the values of the two sides coincide by reason of the symmetry of a wave. Thus the function $-i\Omega(\zeta)$, and consequently $\Omega(\zeta)$ itself, is not only holomorphic in the cut unit circle, but also in the unit circle without cut.

The precise functional form of $\Omega(\zeta)$ has to be determined by the condition prescribed on the boundary $|\zeta|=1$, which corresponds to the free surface condition of the physical plane. Along the streamline which constitutes the free surface of water a constant atmospheric pressure prevails, and by Bernoulli's equation we have

$$q^2+\frac{2g}{U^2}y=\text{const}, \qquad \cdots\cdots(12)$$

or differentiating this along the arc s of the streamline,

$$q\frac{dq}{ds}+\frac{g}{U^2}\sin\theta=0, \qquad \cdots\cdots(13)$$

where θ is the inclination of free surface (velocity).

When we take dz along the free surface this is equal to $dse^{i\theta}$ and has to correspond to $d\zeta=ie^{i\sigma}d\sigma$ on the unit circle $\zeta=e^{i\sigma}=\cos\sigma+i\sin\sigma$, σ being the arc length of the unit circle. The correspondence is given by (11), and by use of the expression (7) we easily arrive at the relation:

$$ds=-\frac{L}{4K}\frac{d\sigma}{q\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}. \qquad \cdots\cdots(14)$$

When we take this relation into (13), which is the surface condition in the physical plane, it can now be transformed into a condition on the unit circle $|\zeta|=1$ in the ζ-plane:

$$q^2\frac{dq}{d\sigma}=p\frac{\sin\theta(\sigma)}{\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}, \qquad \cdots\cdots(15)$$

where p is defined by

$$p=\frac{gL}{4KU^2}=\frac{gH}{2K'U^2}. \qquad \cdots\cdots(16)$$

(15) is the boundary condition, which is necessary, as cited above, for the determination of $\Omega(\zeta)$, and p is the eigenvalue which has to be determined simultaneously with $\Omega(\zeta)$. From p the wave velocity U follows at once.

Thus far our mathematical formulation is completed, and Stokes' waves and the solitary wave are easily seen to be the two extreme cases ($k=0$ and $k=1$) of our present formulation.

Now we have sufficient conditions for $\Omega(\zeta)$, and the problem of finding $\Omega(\zeta)$ is reduced to the determination of

$$\Omega(e^{i\sigma}) \equiv \theta(\sigma)+i\tau(\sigma) \qquad \cdots\cdots(17)$$

on the unit circle, by the well-known formula of Shwarz-Poisson:

$$\Omega(\zeta)=ia+\frac{1}{2\pi}\int_{-\pi}^{\pi}\theta(\sigma)\cdot\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\,d\sigma=b-\frac{1}{2\pi i}\int_{-\pi}^{\pi}\tau(\sigma)\cdot\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\,d\sigma, \quad \cdots(18)$$

where a and b are constants to be fixed. The determination of $\theta(\sigma)$, $\tau(\sigma)$ is then reduced to a nonlinear problem as follows:

For the sake of simplifying the actual computations we use $Q(\sigma)$ defined by

$$Q(\sigma)=\frac{q(\sigma)}{(3p)^{1/3}}=\frac{e^{\tau(\sigma)}}{(3p)^{1/3}} \quad i.e.\ \ lnQ(\sigma)=\tau(\sigma)-\frac{1}{3}ln(3p) \quad \cdots\cdots(19)$$

instead of $q(\sigma)$, or $\tau(\sigma)$, then (15) is changed into

$$\frac{dQ^3}{d\sigma}=\sin\theta(\sigma)\Big/\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)},$$

and when integrated it becomes

$$Q^3(\sigma)-Q^3(O)=\int_0^{\sigma}\frac{\sin\theta(\sigma')}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}\,d\sigma', \qquad \cdots\cdots(20)$$

which we use in place of (15). If we assume $\theta(\sigma)$ the corresponding value of $Q(\sigma)$ follows from (20), and $\tau(\sigma)$ is then fixed by (19) except an additive constant. This knowledge about $\tau(\sigma)$ is sufficient to determine the conjugate harmonic $\theta(\sigma)$, which has to be identical with the previously assumed $\theta(\sigma)$. We know then that this nonlinear problem is to be solved by an iteration procedure, details being deferred to the next section.

After $\theta(\sigma)$, $\tau(\sigma)$ have been determined, the eigenvalue p is fixed by means of any one equation which contains $q(\sigma)$, for by use of the relation $q(\sigma)=(3p)^{1/3}$ $Q(\sigma)$ we can bring p into an equation as a sole unknown quantity. Here we take as such an equation the expression of wave length. As the wave form is given by

$$dz=dse^{i\theta}=-\frac{L}{4K}\,\frac{e^{i\theta(\sigma)}}{q(\sigma)\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}\,d\sigma, \qquad \cdots\cdots(21)$$

which is nothing but (14), we take its real part and integrate it to

$$-\frac{L}{2}=-\frac{L}{4K}\int_0^{\sigma}\frac{\cos\theta(\sigma)}{q(\sigma)\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}\,d\sigma,$$

from which the equation aimed at follows at once:

$$(3p)^{1/3}=\frac{1}{2K}\int_0^{\pi}\frac{\cos\theta(\sigma)}{Q(\sigma)\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}\,d\sigma; \qquad \cdots\cdots(22)$$

i.e. p can be obtained by mere quadrature.

2. Calculation scheme for the highest waves

Our problem here is to calculate the highest waves, and for this purpose we employ an auxiliary function:

$$\Omega_0(\zeta)=i\frac{1}{3}\,ln\left(\frac{1-\zeta}{2}\right), \qquad \cdots\cdots(23)$$

i.e.

$$\Omega_0(e^{i\sigma})\equiv\theta_0(\sigma)+i\tau_0(\sigma)=\frac{\pi-\sigma}{6}+i\frac{1}{3}\,ln\sin\left(\frac{\sigma}{2}\right), \quad (\pi\geq\sigma\geq 0) \cdots\cdots(24)$$

which has the characteristic behavior of $\Omega(\zeta)$ in the neighborhood of the singular point $\zeta=1$, which corresponds to the angular crest. In reality $\theta_0(\sigma)$ jumps at $\sigma=0$ ($\zeta=1$) from $-30°$ to $+30°$, and $q(\sigma)=e^{\tau_0(\sigma)}$ vanishes continuously at $\sigma=0$ ($\zeta=1$).

With this auxiliary function we assume for the exact form of $\Omega(\zeta)$ the following:

$$\Omega(\zeta)\equiv\Omega_0(\zeta)+\Omega_r(\zeta), \quad \Omega_r(e^{i\sigma})\equiv\theta_r(\sigma)+i\tau_r(\sigma), \qquad \cdots\cdots(25)$$

i.e.

$$\theta(\sigma)\equiv\theta_0(\sigma)+\theta_r(\sigma), \quad \tau(\sigma)=\tau_0(\sigma)+\tau_r(\sigma). \qquad \cdots\cdots(26)$$

As $q(\sigma=0)=0$ now our principal equation (20) becomes

$$Q^3(\sigma)=\int_0^\sigma \frac{\sin(\theta_0(\sigma')+\theta_r(\sigma'))}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}d\sigma', \qquad \cdots\cdots(27)$$

and to treat this equation efficiently we introduce further functions $t(\sigma)$, $t_1(\sigma)$ by the definition

$$lnQ(\sigma)=t(\sigma), \quad t_1(\sigma)=t(\sigma)-\tau_0(\sigma). \qquad \cdots\cdots(28)$$

By the definitions (19) and (26) we have now

$$t(\sigma)=\tau(\sigma)-\frac{1}{3}ln(3p), \quad t_1(\sigma)=\tau_r(\sigma)-\frac{1}{3}ln(3p), \qquad \cdots\cdots(29)$$

and see $t(\sigma)$, $t_1(\sigma)$ being free of the eigenvalue, which is undetermined presently. Now (27) reduces to

$$t(\sigma)=\frac{1}{3}ln\int_0^\sigma \frac{\sin(\theta_0(\sigma')+\theta_r(\sigma'))}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}d\sigma',$$

and then finally

$$t_1(\sigma)=\frac{1}{3}ln\left\{\int_0^\sigma \frac{\sin(\theta_0(\sigma')+\theta_r(\sigma'))d\sigma'}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}\Big/\sin\left(\frac{\sigma}{2}\right)\right\}. \qquad \cdots\cdots(30)$$

On the other hand by Villat's formula for conjugate functions we have

$$\theta_r(\sigma)=\text{const.}+\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\tau_r(\sigma')-\tau_r(\sigma)\}\cot\frac{\sigma'-\sigma}{2}d\sigma',$$

and $\tau_r(\sigma)$ being an even function of σ this formula can easily be reduced to

$$\theta_r(\sigma)=\text{const.}-\frac{\sin\sigma}{\pi}\int_0^\pi \frac{\tau_r(\sigma')-\tau_r(\sigma)}{\cos\sigma'-\cos\sigma}d\sigma'.$$

And moreover, in this formula we can replace τ_r by t_1 by reason of the second relation of (29), and determine the undetermined constant by the condition $\theta_r(0)$

$=0$. Thus we arrive finally at

$$\theta_r(\sigma) = -\sin\sigma \frac{1}{\pi}\int_0^{\pi} \frac{t_1(\sigma') - t_1(\sigma)}{\cos\sigma' - \cos\sigma}\, d\sigma'. \qquad \cdots\cdots(31)$$

We have reached the point to determine $t_1(\sigma)$ and $\theta_r(\sigma)$, by the simultaneous nonlinear integral equations (30) (31). When their solutions are obtained, we have $\theta(\sigma)$, $Q(\sigma)$ by the relations

$$\theta(\sigma) = \theta_0(\sigma) + \theta_r(\sigma), \quad Q(\sigma) = e^{\tau_0(\sigma) + t_1(\sigma)}, \qquad \cdots\cdots(32)$$

the eigenvalue p by (22), and then the wave velocity and the flow velocity on the surface by

$$\frac{U}{\sqrt{gL}} = \frac{1}{\sqrt{4pK}} \qquad \cdots\cdots(33)$$

and

$$\tau(\sigma) = \tau_0(\sigma) + t_1(\sigma) + \frac{1}{3}\, ln(3p), \quad q(\sigma) = (3p)^{1/3} Q(\sigma). \qquad \cdots\cdots(34)$$

The correspondence between σ and a surface point is determined by the surface profile (x, y) as functlons of σ, which are obtained by integrating (21):

$$\frac{x(\sigma)}{L} + i\frac{y(\sigma)}{L} = -\frac{1}{4K(3p)^{1/3}}\int_0^{\sigma} \frac{\cos\theta(\sigma') + i\sin\theta(\sigma')}{Q(\sigma')\sqrt{1 - k^2\sin^2\left(\frac{\sigma'}{2}\right)}}\, d\sigma', \quad \cdots(35)$$

or by means of (3),

$$\frac{x(\sigma)}{H} + i\frac{y(\sigma)}{H} = -\frac{1}{2K'(3p)^{1/3}}\int_0^{\sigma} \frac{\cos\theta(\sigma') + i\sin\theta(\sigma')}{Q(\sigma')\sqrt{1 - k^2\sin^2\left(\frac{\sigma'}{2}\right)}}\, d\sigma'. \quad \cdots\cdots(36)$$

Also by Bernoulli's theorem (12), with the undetermined constant fixed to zero now, and by the definition (16) of p we obtain

$$q^2 + 4pK'\frac{y}{H} = 0. \qquad \cdots\cdots(37)$$

By means of this equation we are again able to obtain y/H from q^2, or rewritten as

$$Q^2(\sigma) + \frac{4}{3}K'(3p)^{1/3}\frac{y(\sigma)}{H} = 0, \qquad \cdots\cdots(37)'$$

$y(\sigma)/H$ from $Q^2(\sigma)$. This form of Bernoulli's theorem is nothing but the relation (27), for differentiating both sides of (27) by σ and integrating again from $\sigma=0$ to σ after multiplication by Q^{-1}, we obtain just (37'). Putting $\sigma=\pi$ we get an important formula from this:

$$\frac{A}{H} = \frac{3}{4K'(3p)^{1/3}}Q^2(\pi), \qquad \cdots\cdots(38)$$

which gives the height A of the crest above the trough, *c.f. Fig. 1a.* We have seen that there are two ways of obtaining y/H (or y/L). We can then use one of them for the proilfe calculation and the other as a check of the legitimacy of the numerical computation.

Let h be the height of the crest above the mean water surface, *c.f. Fig. 1a.* Then the definition of mean level surface is written as

$$h\frac{L}{2}=\int_0^\pi(-y(\sigma))(-dx(\sigma)),$$

and expressing $y(\sigma)$ in the integral by $Q^2(\sigma)$, by means of (37'), and $dx(\sigma)$ by means of the real part of (36), we easily find

$$\frac{h}{H}=\frac{3}{8KK'(3p)^{1/3}}\int_0^\pi\frac{\cos\theta(\sigma)Q(\sigma)}{\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}d\sigma, \qquad \cdots\cdots(39)$$

which enables us to calculate the position of mean surface.

Finally we require the mean depth D. Referring to *Figs.* 1*a* and 1*c*, it is evident that

$$h+D=A\dot{O}=\int_0^1 dy(\xi), \qquad \cdots\cdots(40)$$

where ξ is the point on the real axis of the ζ-plane, $1\geq\xi\geq0$. On this range of ξ

$$\Omega(\zeta)\,|_{\zeta=\xi}=\theta\,|_{\zeta=\xi}+i\tau\,|_{\zeta=\xi}$$

is imaginary, *i.e.* $\theta\,|_{\zeta=\xi}=0$. Let us denote $\tau\,|_{\zeta=\xi}$ by $\tau(\xi)$. Then by the formula (11) of dz we obtain

$$dz(\zeta)\,|_{\zeta=\xi}=idy(\xi)=\frac{iL}{4K}\frac{e^{-\tau(\xi)}}{\sqrt{\xi^2+\frac{k^2}{4}\xi(1-\xi)^2}}d\xi;$$

integrating both sides of this between $\xi=0$ to 1, and using it in (40) we find

$$\frac{h+D}{H}=\frac{1}{2K'}\int_0^1\frac{e^{-\tau(\xi)}}{\sqrt{\xi^2+\frac{k^2}{4}\xi(1-\xi)^2}}d\xi, \qquad \cdots\cdots(41)$$

where the relation $L/2KH=1/K'$ is used.

We take in the decomposition :

$$\tau(\zeta)=\tau_0(\zeta)+\tau_r(\zeta)=\tau_0(\zeta)+\frac{1}{3}\ln(3p)+t_1(\zeta), \qquad \cdots\cdots(42)$$

the notation being understood at once. By means of this decomposition we have

$$e^{-\tau(\xi)}=e^{-\tau_0(\xi)}(3p)^{-1/3}e^{-t_1(\xi)}=(3p)^{-1/3}\left(\frac{1-\xi}{2}\right)^{-1/3}e^{-t_1(\xi)},$$

and using this in (41) it results that:

$$\frac{h+D}{H}=\frac{1}{2K'(3p)^{1/3}}\int_0^1\frac{e^{-t_1(\xi)}}{\sqrt{\xi}\left(\frac{1-\xi}{2}\right)^{1/3}\sqrt{\xi+k^2\left(\frac{1-\xi}{2}\right)^2}}d\xi. \quad \cdots\cdots(43)$$

We thus have to calculate $t_1(\xi)$ for the purpose of obtaining D/H.

The imaginary part τ_r of a holomorphic function $\Omega_r(\zeta)$ is being given on a unit circle $\zeta=e^{i\sigma}$ as the function $\tau_r(\sigma)$, we can use the Schwarz-Poisson formula (18) to obtain $\Omega_r(\zeta)$ itself:

$$\Omega_r(\zeta)=b-\frac{1}{2\pi i}\int_{-\pi}^{\pi}\tau_r(\sigma)\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}d\sigma. \qquad \cdots\cdots(44)$$

At $\zeta=0$ this gives a complex value:

$\Omega_r(0)=b+imaginary\ value.$ ....(44)'

From physical considerations however it is evident $\theta(\zeta)|_{\zeta=0}=0$, and the assumed form of $\Omega_0(\zeta)$ assures the fact: $\theta_0(\zeta)\ |_{\zeta=0}=0$. Then $\theta_r(\zeta)\ |_{\zeta=0}$ must vanish, and then b in (44') also vanishes. Thus we have

$$\theta_r(\zeta)+i\tau_r(\zeta)=-\frac{1}{2\pi i}\int_{-\pi}^{\pi}\tau_r(\sigma)\ \frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\ d\sigma$$

$$=i\frac{1}{3}ln(3p)-\frac{1}{2\pi i}\int_{-\pi}^{\pi}t_1(\sigma)\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\ d\sigma,$$

i.e.

$$\theta_r(\zeta)+it_1(\zeta)=-\frac{1}{2\pi i}\int_{-\pi}^{\pi}t_1(\sigma)\ \frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}d\sigma, \qquad \text{..................(45)}$$

and putting $\zeta=\xi$ we finally have

$$t_1(\xi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}t_1(\sigma)\frac{e^{i\sigma}+\xi}{e^{i\sigma}-\xi}d\sigma=\frac{1}{\pi}\int_0^{\pi}t_1(\sigma)\frac{1-\xi^2}{1+\xi^2-2\xi\cos\sigma}d\sigma, \qquad \text{......(46)}$$

which gives $t_1(\xi)$ by means of $t_1(\sigma)$. Using (46) in (43) we know the value $(h+D)/H$, and subtracting the known value (39) of h/D, we arrive at the required value of D/H.

3. Numerical Computations (1)

Along with the preceding section we calculate $\theta_r(\sigma)$, $t_1(\sigma)$ of the highest waves for the following values of the parameter k:

k=[sin0°], sin20°, sin60°, [sin80°], sin87°, sin89.5°, sin90°;

the values in brackets are the cases which we have already calculated and reported [1-3], the main characteristics of these being indicated in Table 6 at the end of this paper, along with our new calculations. Table 1 shows their velocities and profiles of the free surface. The case where $k=sin\ 90°=1$ is also reported [2], but owing to its principal significance for the waves in shallow water, as will be seen in the following section, we have calculated it again here.

Formulae (30) and (31) are our present concern. $t_1(0)=0$ and $\theta_r(0)=\theta_r(\pi)=0$ being easily ascertained, we first calculate $\theta_r(\sigma)$ by integrating (31), starting from an arbitrary but appropriately chosen $t_1(\sigma)$. Integration is numerical and then stepwise, 23 points being taken at equal intervals between $\sigma=0°$ and 180°. Using $\theta_r(\sigma)$ thus calculated in (30) we obtain the renewed values of $t_1(\sigma)$ at these 25 points (including $\sigma=0°$ and 180°), one cycle of the iteration process being closed. With this renewed values of $t_1(\sigma)$ we begin the next cycle of iteration. Numerical integrations are all done with the modified Simpson rule:

$$\left.\begin{aligned}\int_0^h f(x)dx&=\frac{h}{12}\ \{5f(0)+8f(h)-f(2h)\},\\ \int_h^{2h} f(x)dx&=\frac{h}{12}\ \{-f(0)+8f(h)+5f(2h)\},\end{aligned}\right\} \qquad \text{..................(47)}$$

Table 1. Wave profile and surface velocity (1) ($k=0$ and sin 80°)

	$k=0.00000$				$k=\sin 80°$			
σ	θ	q	$-x/L$	$-y/L$	θ	q	$-x/L$	$-y/L$
0°	0.5236	0.0000	0.0000	0.0000	0.5236	0.0000	0.0000	0.0000
7.5°	0.4803	0.5409	0.0502	0.0277	0.4923	0.4435		
15	0.4500	0.6729	0.0808	0.0430	0.4698	0.5525	0.0491	0.0268
22.5	0.4240	0.7624	0.1070	0.0553	0.4544	0.6277		
30	0.3992	0.8317	0.1310	0.0658	0.4385	0.6893	0.0796	0.0417
37.5	0.3748	0.8885	0.1534	0.0749	0.4196	0.7405		
45	0.3514	0.9347	0.1747	0.0830	0.4020	0.7829	0.1067	0.0538
52.5	0.3292	0.9748	0.1953	0.0903	0.3865	0.8209		
60	0.3075	1.0099	0.2152	0.0969	0.3710	0.8567	0.1327	0.0644
67.5	0.2861	1.0405	0.2347	0.1028	0.3538	0.8896		
75	0.2653	1.0673	0.2536	0.1082	0.3366	0.9189	0.1589	0.0741
82.5	0.2451	1.0910	0.2724	0.1131	0.3205	0.9464		
90	0.2252	1.1122	0.2907	0.1175	0.3041	0.9736	0.1864	0.0832
97.5	0.2054	1.1309	0.3089	0.1214	0.2863	0.9994		
105	0.1860	1.1473	0.3268	0.1250	0.2680	1.0235	0.2161	0.0919
112.5	0.1669	1.1617	0.3446	0.1281	0.2495	1.0466		
120	0.1478	1.1744	0.3622	0.1310	0.2300	1.0697	0.2496	0.1004
127.5	0.1289	1.1853	0.3797	0.1334	0.2093	1.0922		
135	0.1103	1.1944	0.3970	0.1355	0.1870	1.1140	0.2894	0.1089
142.5	0.0919	1.2021	0.4144	0.1372	0.1632	1.1350		
150	0.0734	1.2085	0.4316	0.1387	0.1371	1.1551	0.3397	0.1171
157.5	0.0549	1.2134	0.4488	0.1398	0.1079	1.1739		
165	0.0366	1.2167	0.4659	0.1406	0.0756	1.1903	0.4080	0.1243
172.5	0.0183	1.2187	0.4830	0.1410	0.0393	1.2025		
180°	0.0000	1.2194	0.5001	0.1412	0.0000	1.2069	0.5000	0.1278

for we are required to conserve the number of points for which function values are assigned, after any cycles of iteration. The iteration process has been stopped when the renewed values of $t_1(\sigma)$ differ from the preceding ones by numbers less than 10^{-3}, taking the last $t_1(\sigma)$ as the required one. Ordinarily the number of iteration cycles were less than 10. Obtained results are expressed in $t(\sigma)$ and $\theta(\sigma)$ and tabulated in Table 2.

Caution is necessary regarding two points. Firstly, the indeterminate value at the point $\sigma'=\sigma$ in (31) is to be replaced by the limiting value:

$$\frac{t_1(\sigma')-t_1(\sigma)}{\cos\sigma'-\cos\sigma}\bigg|_{\sigma=\sigma'} = -\frac{t_1'(\sigma)}{\sin\sigma} = -\frac{t_1(\sigma+7.5°)-t_1(\sigma-7.5°)}{0.26180\sin\sigma}.$$

Secondly, in cases where $k\simeq 1$ the square root in (30) changes its value very rapidly when σ' approaches π, and 7.5° as an integration step becomes too rough. To estimate the integral in the interval 172.5°-180° we write this in the form:

Table 2 Surface velocity (k=sin 20°, sin 60°, sin 87°, sin 89.5°, sin 90°)

			k=sin2o°		k=sin60°		k=sin87°		k=sin89.5°		k=1.00000	
σ	τ_0	θ_0	t	θ	t	θ	t	θ	t	θ	t	θ
0°	$-\infty$	0.5236	$-\infty$	0.5236	$-\infty$	0.5236	$-\infty$	0.5236	$-\infty$	0.5326	$-\infty$	0.5236
7.5°	-0.9091	0.5018	-0.9216	0.4823	-0.9192	0.4896	-0.9171	0.4960	-0.9170	0.4964	-0.9170	0.4964
15	-0.6787	0.4800	-0.7013	0.4526	-0.6964	0.4650	-0.6924	0.4760	-0.6922	0.4767	-0.6922	0.4766
22.5	-0.5448	0.4582	-0.5758	0.4259	-0.5684	0.4428	-0.5625	0.4580	-0.8621	0.4589	-0.5621	0.4588
30	-0.4505	0.4363	-0.4892	0.4010	-0.4790	0.4220	-0.4710	0.4410	-0.4706	0.4421	-0.4706	0.4420
37.5	-0.3783	0.4145	-0.4239	0.3773	-0.4105	0.4020	-0.4005	0.4246	-0.4000	0.4260	-0.4000	0.4259
45	-0.3202	0.3927	-0.3720	0.3545	-0.3553	0.3826	-0.3430	0.4087	-0.3424	0.4103	-0.3424	0.4101
52.5	-0.2719	0.3709	-0.3295	0.3325	-0.3092	0.3635	-0.2944	0.3930	-0.2937	0.3948	-0.2937	0.3947
60	-0.2311	0.3491	-0.2939	0.3111	-0.2698	0.3447	-0.2522	0.3775	-0.2514	0.3796	-0.2514	0.3794
67.5	-0.1959	0.3273	-0.2636	0.2901	-0.2353	0.3259	-0.2148	0.3620	-0.2139	0.3643	-0.2139	0.3641
75	-0.1654	0.3054	-0.2375	0.2695	-0.2049	0.3071	-0.1812	0.3464	-0.1801	0.3489	-0.1801	0.3487
82.5	-0.1388	0.2836	-0.2149	0.2493	-0.1777	0.2883	-0.1504	0.3307	-0.1491	0.3334	-0.1492	0.3331
90	-0.1155	0.2618	-0.1953	0.2293	-0.1532	0.2692	-0.1219	0.3146	-0.1205	0.3176	-0.1206	0.3173
97.5	-0.0951	0.2400	-0.1781	0.2096	-0.1311	0.2498	-0.0953	0.2982	-0.0937	0.3015	-0.0938	0.3011
105	-0.0772	0.2182	-0.1631	0.1901	-0.1111	0.2300	-0.0701	0.2812	-0.0683	0.2849	-0.0684	0.2845
112.5	-0.0615	0.1964	-0.1501	0.1707	-0.0929	0.2097	-0.0462	0.2636	-0.0441	0.2676	-0.0443	0.2672
120	-0.0480	0.1745	-0.1388	0.1514	-0.0764	0.1889	-0.0232	0.2452	-0.0208	0.2496	-0.0210	0.2491
127.5	-0.0363	0.1527	-0.1291	0.1323	-0.0617	0.1676	-0.0009	0.2258	0.0018	0.2306	0.0016	0.2301
135	-0.0264	0.1309	-0.1209	0.1133	-0.0486	0.1455	0.0209	0.2051	0.0241	0.2104	0.0239	0.2098
142.5	-0.0182	0.1091	-0.1141	0.0943	-0.0373	0.1227	0.0424	0.1827	0.0462	0.1887	0.0459	0.1879
150	-0.0116	0.0873	-0.1087	0.0754	-0.0278	0.0993	0.0639	0.1581	0.0685	0.1650	0.0682	0.1641
157.5	-0.0065	0.0655	-0.1045	0.0565	-0.0202	0.0751	0.0855	0.1304	0.0914	0.1387	0.0910	0.1374
165	-0.0029	0.0436	-0.1015	0.0376	-0.0146	0.0504	0.1076	0.0980	0.1157	0.1087	0.1150	0.10653
172.5	-0.0007	0.0218	-0.0997	0.0188	-0.0113	0.0253	0.1299	0.0553	0.1460	0.0676	0.1425	0.06700
180°	0.0000	0.0000	-0.0991	0.0000	-0.0101	0.0000	0.1422	0.0000	0.1726	0.0000	0.1828	0.00000

$$I(\varepsilon)=\int_{\pi-\varepsilon}^{\pi}\frac{F(\sigma)}{\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}d\sigma, \qquad \cdots\cdots(48)$$

and $F(\sigma)$, being here $\sin\theta(\sigma)$, is approximated by

$$F(\sigma)=A\sin\frac{\sigma}{2}+B\sin\sigma+C\sin\sigma\cos\frac{\sigma}{2}, \qquad \cdots\cdots(48)'$$

where constants A, B and C are determined by the values $F(165°)$, $F(172.5°)$ and $F(180°)$. Then (48) is easily evaluated and gives an integration formula:

$$I(\epsilon)=\frac{2A}{k}\ ln\ \frac{\sqrt{1-k^2\cos^2\left(\frac{\epsilon}{2}\right)}+k\sin\left(\frac{\epsilon}{2}\right)}{k'}+\frac{4B}{k^2}\left\{\sqrt{1-k^2\cos^2\left(\frac{\epsilon}{2}\right)}-k'\right\}$$
$$+\frac{2C}{k^3}\left\{\sin\left(\frac{\epsilon}{2}\right)\sqrt{1-k^2\cos^2\left(\frac{\epsilon}{2}\right)}-\frac{k'^2}{k}\sinh^{-1}\left(\frac{k}{k'}\sin\left(\frac{\epsilon}{2}\right)\right)\right\}. \quad \cdots\cdots\cdots\cdots(48)''$$

Now $\epsilon=7.5°$ and $A=0$, and from B, C determined as above we easily obtained the integral values for cases $k=\sin 87°$ and $\sin 89.5°$; for cases $k=\sin 20°$ and $\sin 60°$ such caution is unnecessary.

The case where $k=\sin 90°$ we have already treated separately and far more accurately in the same line of numerical treatment [10]. Our present calculation, which is rather rough in comparison, coincided well with this accurate one except for the one value $t_1(\pi)$, and then we corrected this one value. The numericals given in the last two columns of Table 2 are the results thus obtained.

We then consider the evaluation of p, $x(\sigma)/L$ and $y(\sigma)/L$, (22) and (35). As the solitary wave ($k=\sin 90°$) and its neighboring long waves are to be studied in the next section we here take up cases where

$k=\sin 20°, \sin 60°, \sin 87°$ and $\sin 89.5°$,

For the highest waves $Q(\sigma)$ vanishes at the initial point $\sigma=0$, and then integrals are improper at $\sigma=0$. In the neighborhood of this point

$$Q(\sigma)=e^{t(\sigma)}=e^{\tau_0(\sigma)+t_1(\sigma)}=\sin^{1/3}\left(\frac{\sigma}{2}\right)e^{t_1(\sigma)},$$

and making use of $t_1(0)=0$ we have $Q(\sigma)^{-1}\simeq 2^{1/3}\sigma^{-1/3}$. We then have the approximate expressions:

$$\left.\begin{aligned}\frac{\cos\theta(\sigma)}{Q(\sigma)\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}&=2^{1/3}\sigma^{-1/3}\cos 30°\,(1+c_1\sigma+c_2\sigma^2),\\ \frac{\sin\theta(\sigma)}{Q(\sigma)\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}&=2^{1/3}\sigma^{-1/3}\sin 30°\,(1+s_1\sigma+s_2\sigma^2),\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(49)$$

where constants c_1, c_2, s_1 and s_2 are to be determined so as to secure the equalities of the equations at $\sigma=7.5°$ and $15°$. With these expressions the necessary integrals between $\sigma=0°$ and $15°$ are obtained. Near the other end of the integration range integrands change abruptly because of the factor $\left(1-k^2\sin^2\frac{\sigma}{2}\right)^{-1/2}$, and we apply the integration formula (48-48") taking for $F(\sigma)$:

$$F(\sigma)=\begin{Bmatrix}\cos\theta/Q\\ \sin\theta/Q\end{Bmatrix}=A\sin\frac{\sigma}{2}+B\sin\sigma+C\sin\sigma\cos\frac{\sigma}{2}.$$

At the middle range 15°-165° the modified Simpson rule being sufficient, we thus complete the numerical evaluations. In reality we calculate p only, and not x/L and y/L, in these four cases of k, but the same evalution method will be applied in the next section for $k=\sin 89°54'$ and $\sin 90°$, without any notice. Obtained values of p are inscribed in Table 6 at the end of the paper, under the heading $p_1(=p_{min})$, and the values of the required complete integral are given in the second column of Table 3 in the rows of corresponding k-values.

Turning to the computation of h/H and D/H, the former is obtained by (39) which requires the evaluation of the integral:

$$\int_0^\pi \frac{\cos\theta(\sigma)Q(\sigma)}{\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}d\sigma.$$

This can be treated just as well as the case of p; $\sigma=0$ not being improper now, caution is necessary only for the range $\sigma=165°\sim180°$. Obtained values are given in Table 3.

To obtain the latter *i.e.* D/H we require the values of $t_1(\xi)$ whose functional form is defined by (45). This definition can be rewritten as

$$t_1(\xi) = C(\xi)\int_0^\pi \frac{t_1(\sigma)}{\mu(\xi)-\cos\sigma}d\sigma \quad (0<\xi<1), \qquad \cdots\cdots(50)$$

where

$$C(\xi)=\frac{1-\xi^2}{2\pi\xi}, \quad \mu(\xi)=\frac{1+\xi^2}{2\xi}; \qquad \cdots\cdots(50)'$$

when $\xi=0$

$$t_1(0)=\frac{1}{\pi}\int_0^\pi t_1(\sigma)d\sigma, \qquad \cdots\cdots(51)$$

and when $\xi=1$

$$t_1(\xi)\,|_{\xi=1}=t_1(\sigma)\,|_{\sigma=0}=0. \qquad \cdots\cdots(51)'$$

Table 3 Auxiliary integrals (1)

$k(k')$	$\int_0^\pi \frac{\cos\theta d\sigma}{Q\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}$	$\int_0^\pi \frac{\cos\theta\cdot Q d\sigma}{\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}$
sin20°	4.365	2.401
sin60°	5.302	3.508
sin87°	8.942	8.514
sin89.5°	11.904	12.807
(0.17453×10^{-2})	14.585	16.689
(0.14544×10^{-3})	18.728	22.652
(0.48481×10^{-5})	24.40	30.82
(〃 $\times10^{-7}$)	32.07	41.88
(〃 $\times10^{-9}$)	39.74	52.93
(〃 $\times10^{-13}$)	55.08	75.05
(〃 $\times10^{-21}$)	85.77	119.27
(〃 $\times10^{-31}$)	124.13	174.56
(〃 $\times10^{-51}$)	200.85	285.13
(0.00000)	∞	∞

When ξ is not 1 but is near it the denominator of the integrand changes abruptly in the vicinity of $\sigma=0$, and we have to use an analogous integration method as (48-48"). $t_1(\sigma)$ being expressed as

$$t_1(\sigma) = A\sin\sigma + B\sin^2\sigma,$$

where constants A, B are to be determined by the values $t_1(7.5°)$ and $t_1(15°)$, the integral can be evaluated as

$$C(\xi)\int_0^{15°} \frac{A\sin\sigma+B\sin^2\sigma}{\mu(\xi)-\cos\sigma}d\sigma=\bar{A}(\xi)t_1(7.5°)+B(\xi)t_1(15°), \qquad \cdots\cdots(52)$$

at the starting integration range $\sigma=0°\sim15°$. The remaining range is well managed by the Simpson rule. In such a way $t_1(\xi)$ is given in the interval $\xi=0\sim1$ at the points of equal distance 0.05, for the four values of k.

Finally the integral in (43) has to be evaluated. Both ends of integration of this integral are improper. Then near $\xi=0$ we put

$$\left\{e^{t_1(\xi)}\left(\frac{1-\xi}{2}\right)^{1/3}\sqrt{\xi+\frac{k^2}{4}(1-\xi)^2}\right]^{-1}\doteqdot F_1(\xi)=A_1+B_1\xi+C_1\xi^2, \quad\cdots\cdots\cdots\cdots\cdots(53)$$

determining the constants A_1, B_1, C_1 by the values of $F_1(\xi)$ at $\xi=0$, 0,05, and 0.1. Near $\xi=1$ we put

$$\left\{e^{t_1(\xi)}\sqrt{\xi}\sqrt{\xi+\frac{k^2}{4}(1-\xi)^2}\right\}^{-1}\doteqdot F_2(\xi)=A_2+B_2(1-\xi)+C_2(1-\xi)^2, \quad\cdots\cdots(54)$$

determining A_2, B_2, C_2 by the values of F_2 at $\xi=1.00$, 0.95 and 0.9. By means of these expressions integration is easily done, the former between $\xi=0\sim0.1$, the latter between $\xi=0.9\sim1$, and the middle part by the Simpson rule. Making use of these integral values D/H is easily evaluated, which is indicated, along with h/D, in Table 6, in the reciprocal value H/D.

4. Numerical computations (2)

Here we consider the case where $k=1$ or is very approximate to it. Making use of the new notations;

$$\pi/2-\vartheta=\vartheta',\quad \sin\vartheta'=k',\quad \text{and}\quad K(k')=K', \quad\cdots\cdots\cdots\cdots\cdots\cdots(55)$$

we take up the following 10 as representative cases:

$\vartheta'=360$, 30, 1, 10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}, 10^{-26}, 10^{-46}, and 0 (*sec.*).

For these values of ϑ' we have the values of k, k', K, K' and L/H, listed in Table 6 columns 2~6.

Through these ten cases differences in $t_1(\sigma)$ and $\theta_r(\sigma)$ are effected by the parameter k^2 in the integral

$$I(\sigma)=\int_0^\sigma \frac{\sin\theta(\sigma')}{\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}d\sigma'.$$

Replacing k^2 in this integral by means of $k'^2=\sin^2\vartheta'$ this integral becomes

$$I(\sigma)=\int_0^\sigma \frac{\sin\theta(\sigma')}{\cos\left(\frac{\sigma'}{2}\right)}F(\sigma',\vartheta')d\sigma', \quad\cdots\cdots\cdots\cdots\cdots\cdots(56)$$

where

$$F(\sigma',\vartheta')=1\Big/\sqrt{1+\left(\sin\vartheta'\tan\frac{\sigma'}{2}\right)^2}=1\Big/\sqrt{1+\left(\sin\vartheta'\cot\frac{\epsilon'}{2}\right)^2}, \quad\cdots\cdots(56)'$$

in which we replaced σ' by $\pi-\epsilon'$.

Since ϑ' is so small an angle that $F(\sigma',\vartheta')$ is very nearly equal to 1 through the whole integration range except in the neighborhood of $\sigma=\pi$, where it abruptly drops to zero. If we then assume $F=1$ all along the integration range, there will be an error in the value of the integral, which is largest when $\vartheta'=360''=6'$. To estimate this error when $\vartheta'=6'$ we have the following data:

$$\lim_{\sigma'\to\pi}\frac{\sin\theta(\sigma')}{\cos\left(\frac{\sigma'}{2}\right)}\simeq1.3,\qquad I(\pi)\simeq1.7,$$

$$F(179°,6')=0.9804,\quad\text{and}\quad\int_{179°}^{180°}(1-F)d\sigma'\simeq0.0007;$$

we know then an over-estimation of 0.04% at most, which changes the value of $t_1(\sigma)$ in the fourth decimal place, which is within the limit of accuracy of our numerical treatment. Thus we obtain $t_1(\sigma)$, $\theta_r(\sigma)$ all common to the cases where $\vartheta' \leq 6'$, and equal to the case of the solitary wave $\vartheta'=0$, which is already done in the preceding section. This fact much simplifies our calculation.

Turning to the calculation of p we see

$q(\pi)=(3p)^{1/3}Q(\pi)$,

and

$Q(\pi)=e^{\tau_0(\pi)+t_1(\pi)}=e^{0.1828}=1.2005$

by Table 2 last column. For the solitary wave $q(\pi)$ is 1, as is evident from physical consideration, and we have

$(3p)^{1/3}=0.8330 \quad i.e. \quad p=0.1927.$

When k is not exactly 1 we have recourse to (22), which is to be approximated now by

$$(3p)^{1/3}=\frac{1}{2K}\left\{\int_0^{172.5^\circ}\frac{\cos\theta}{Q\cos\left(\frac{\sigma}{2}\right)}d\sigma+\int_{172.5^\circ}^{180^\circ}\frac{\cos\theta}{Q\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}d\sigma\right\}\cdots\cdots\cdots\cdots(57)$$

The same sort of integral occurs in the calculation of wave profile:

$$\left.\begin{aligned}-\frac{x(\sigma)}{H}&=\frac{1}{\pi(3p)^{1/3}}\int_0^\sigma\frac{\cos\theta(\sigma')}{Q(\sigma')\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}}d\sigma',\\-\frac{y(\sigma)}{H}&=\frac{1}{2\pi p}q^2(\sigma)=\frac{3}{2\pi(3p)^{1/3}}Q^2(\sigma),\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(58)$$

where we used the approximate value $K'=\pi/2$. In the integral of this expression $\sqrt{1-k^2\sin^2\left(\frac{\sigma'}{2}\right)}$ ist to be replaced by $\cos\left(\frac{\sigma'}{2}\right)$ up to $\sigma'=172.5^\circ$, and above this angle the integral has to be evaluated by the formula (48-48''). $\sigma=0$ is an improper end of the integral and procedure (49) has to be used. In this manner two cases ($\vartheta'=6'$ and $0'$) are taken up and Table 4 gives the results.

For $\vartheta=6'$ the complete integral value is 14.585, as seen in Table 4, and using this value in (57) we obtain

$p=0.2792.$

p being known $-x/H$ is calculated by means of Table 4. Obtained values are tabulated in Table 5, for the cases where $\vartheta'=6'$ and $0'$, being changed however in $-x/D$, explanation being deferred a little. Values of $q(\sigma)=(3p)^{1/3}Q(\sigma)$ are also given in Table 5, detailed by interpolation. The second formula of (58) then gives $-y/H$,

Table 4 Auxiliary integral (2)

σ	$\int_0^\sigma\frac{\cos\theta d\sigma}{Q\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}$	
0°	0.0000	
7.5°	0.4264	
15	0.6835	
30	1.1070	
45	1.4837	
60	1.8459	
75	2.2109	
90	2.5936	
105	3.0111	
120	3.488	
135	4.007	
150	4.840	
157.5	5.372	
165	6.108	
172.5	7.325	
	($\vartheta'=6'$)	($\vartheta'=0$)
175	8.023	〃
176.25	8.514	〃
177.5	9.201	〃
178.75	10.368	〃
179.5	11.902	〃
180	14.585	∞

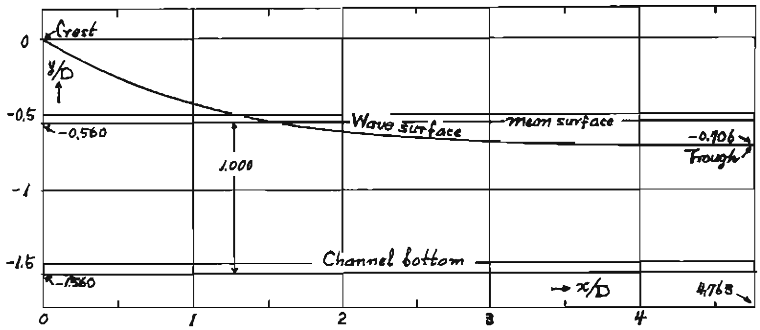

Fig. 2 Wave profile ($k=\sin 89°54'$)

Table 5 Wave profile and surface velocity (2) ($k=\sin 89°54'$, $\sin 90°$)

		$k=\sin 89°54'$			$k=1.00000$		
σ	θ	q	$-x/D$	$-y/D$	q	$-x/D$	$-y/D$
0°	0.5236	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7.5°	0.4964	0.3768	0.1392	0.0783	0.3330	0.1629	0.0916
15	0.4766	0.4718	0.2232	0.1227	0.4169	0.2612	0.1436
22.5	0.4588	0.5373			0.4748		
30	0.4420	0.5888	0.3615	0.1911	0.5203	0.4230	0.2236
37.5	0.4259	0.6319			0.5584		
45	0.4101	0.6693	0.4845	0.2469	0.5915	0.5670	0.2890
52.5	0.3947	0.7027			0.6210		
60	0.3794	0.7331	0.6027	0.2962	0.6478	0.7054	0.3467
67.5	0.3641	0.7611			0.6725		
75	0.3487	0.7872	0.7219	0.3416	0.6957	0.8449	0.3998
82.5	0.3331	0.8120			0.7175		
90	0.3173	0.8356	0.8469	0.3849	0.7384	0.9911	0.4504
97.5	0.3011	0.8583			0.7584		
105	0.2845	0.8803	0.9832	0.4271	0.7779	1.1506	0.4999
112.5	0.2672	0.9018			0.7969		
120	0.2491	0.9231	1.1389	0.4697	0.8157	1.333	0.5497
127.5	0.2301	0.9442			0.8343		
135	0.2098	0.9654	1.328	0.5137	0.8531	1.554	0.6012
142.5	0.1879	0.9869			0.8721		
150	0.1641	1.0091	1.581	0.5613	0.8917	1.850	0.6568
157.5	0.1374	1.0324	1.754	0.5875	0.9123	2.053	0.6875
165	0.10653	1.0575	1.995	0.6164	0.9345	2.334	0.7214
172.5	0.06700	1.0870	2.392	0.6513	0.9606	2.799	0.7623
175		1.0994	2.620	0.6662	0.9716	3.066	0.7798
176.25		1.1065	2.780	0.6749	0.9778	3.253	0.7898
177.5		1.1141	3.005	0.6842	0.9845	3.516	0.8007
178.75		1.1224	3.386	0.6944	0.9919	3.962	0.8128
179.5°		1.1279	3.886	0.7012	0.9967	4.548	0.8206
180°	0.00000	1.1316	4.763	0.7058	1.0000	∞	0.8261

which is also, indicated in Table 5, also changed in $-y/D$. Wave profile in the case $\vartheta'=6'$ is shown in Fig. 2, as a representative one.

Through all the cases of k, *i.e.* ϑ or ϑ', p has been obtained by means of (22), or (57), required integral values being calculated as above, and tabulated in Table 3 column 2; the values of p are in Table 6 column p_1. The wave profile is not calculated, except in the two cases given above (and the previous cases where $\vartheta=0°$ and 80° given in Table 1). Instead we have only calculated A/H, *i.e.* the maximum, value of $-y/H$, which is given by (38), and for the present cases where $\vartheta'\leq 6'$, which reduces to

$$\frac{A}{H}=\frac{0.6381}{(3p)^{1/3}}\,; \qquad \cdots\cdots(59)$$

the results are in Table 6 column A/H.

The last constants of the highest waves are h and D. These are obtainable by means of (39), (43). In the present cases K' is approximated by $\pi/2$ and the formulae are rewritten as

$$\frac{h}{H}=\frac{0.2387}{(3p)^{1/3}K}\left\{\int_0^{172.5°}\frac{\cos\theta\cdot Q}{\cos\left(\frac{\sigma}{2}\right)}\,d\sigma+\int_{172.5°}^{180°}\frac{\cos\theta\cdot Q}{\sqrt{1-k^2\sin^2\left(\frac{\sigma}{2}\right)}}\,d\sigma,\right\} \cdots\cdots(60)$$

$$\frac{h+D}{H}=\frac{1}{(3p)^{1/3}}\cdot\frac{1}{\pi}\int_0^1\frac{e^{-t_1(\xi)}}{\sqrt{\xi}\left(\frac{1-\xi}{2}\right)^{1/3}\sqrt{\xi+k^2\left(\frac{1-\xi}{2}\right)^2}}\,d\xi. \qquad \cdots\cdots(61)$$

The first integral in (60) is evaluated by the Simpson rule which gives 6.426, and the second by method (48-48″) through the interpolation formula:

$$\cos\theta(\sigma)Q(\sigma)=1.2005\sin\frac{\sigma}{2}-0.4368\sin\sigma+1.1356\sin\sigma\cos\frac{\sigma}{2},$$

as $t_1(\sigma)$, and then $Q(\sigma)$, is common to all present cases. The integral values

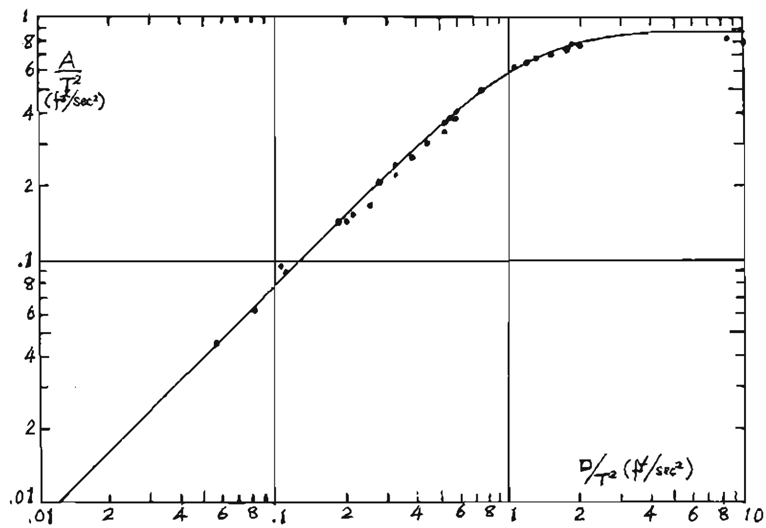

Fig. 3 Breaking index curve

between $\sigma=0$ and π thus obtained are listed in Table 3 column 3, and values of h/H, calculated therefrom are in Table 6.

The integral value of (61) is independent of the parameter k, or ϑ', because $t_1(\xi)$ is as well common to all the present cases as $t_1(\sigma)$, *c.f.* (46), and the factor $\sqrt{\xi+k^2\left(\frac{1-\xi}{2}\right)^2}$ is in reality independent of k, and eqnal to $\frac{1}{2}(1+\xi)$ within our approximation. Therefore we have

$$\frac{h+D}{H}=\frac{1}{(3p)^{1/3}}\left[(3p)^{1/3}\cdot\frac{h+D}{H}\right]_{k=1}$$
$$=\frac{1}{(3p)^{1/3}}\left[(3p)^{1/3}\left(\frac{A}{H}+1\right)\right]_{k=1}=\frac{1.5211}{(3p)^{1/3}}, \quad \cdots\cdots\cdots\cdots(62)$$

from which we have D/H, which is tabulated in Table 6 in the reciprocal form H/D. Making use of this value, ordinary normalisation by the mean depth D is secured for all lengths concerning waves, which we also see in Table 6.

Conclusions

To represent our results graphically we selected three items: wave form $(x/D,\ y/D)$, breaking index curve following C. L. Bretschneider's, and wave velocity $\left(U\Big/\sqrt{\frac{gL}{4K}},\ L/H\right)$. As is already explained Fig. 2 is the highest wave profile of the case $\vartheta'=6'$, or in other words $L/D=9.526$, which seems common in a canal experiment. Such a wave is very similar to the solitary wave except for a finite wave length, as is known from preceding calculations, a fact

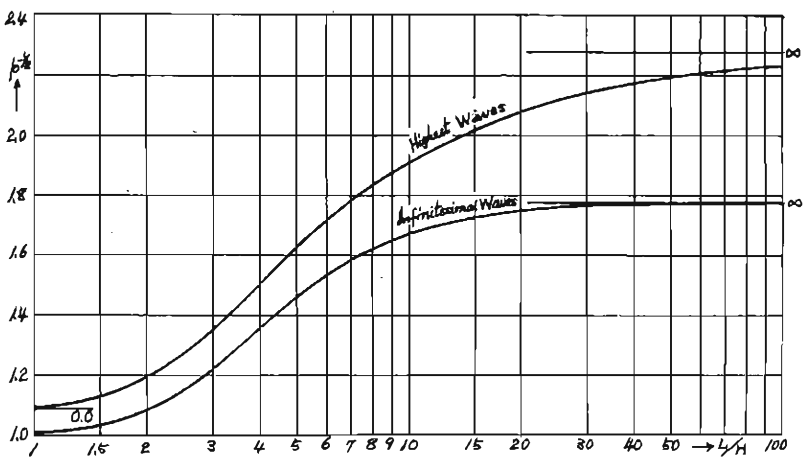

Fig. 4 Existence region of permanent waves

which has been utilised so far.

As is easily seen

$$\frac{1}{T^2}=\left(\frac{U}{L}\right)^2=\frac{1}{L}\cdot\frac{U^2}{L}=\frac{1}{L}\cdot\frac{g}{4pK},$$

and then we have

$$\frac{A}{T^2}=\frac{A}{L}\cdot\frac{g}{4pK},\quad \frac{D}{T^2}=\frac{D}{L}\cdot\frac{g}{4pK}, \quad \cdots\cdots\cdots\cdots(63)$$

Table 6 Characteristic numbers of permanent waves.

ϑ (ϑ')	k	k'	K	K'	L/H	$p_{max}=p_0$	$p_0^{-1/2}$	$p_{min}=p_1$	$p_1^{-1/2}$
0°	0.00000	1.00000	1.57080(=$\pi/2$)	∞	0.0000	1.0000	1.0000	0.8381	1.0923
20°	0.34202	0.93969	1.62003	2.5046	1.2937	0.9697	1.0155	0.8150	1.1077
60	0.86603	0.50000	2.1565	1.6858	2.5585	0.7392	1.1631	0.6193	1.2708
80	0.98481	0.17365	3.1534	1.5828	3.9846	0.5426	1.3576	0.4519	1.4840
87	0.99863	0.52336×10^{-1}	4.3387	1.5719	5.5204	0.4449	1.4993	0.3648	1.6556
89°30′	0.99996	0.87265×10^{-2}	6.1278	1.57083	7.8020	0.3843	1.6132	0.3054	1.8093
(6′)	1.00000	0.17453×10^{-2}	7.7371	1.57080	9.8511	0.3604	1.666	0.2792	1.893
(30″)	1.00000	0.14544×10^{-3}	10.2220	1.57080	13.015	0.3427	1.708	0.2563	1.979
(1″)	1.00000	0.48481×10^{-5}	13.619	1.57080	17.346	0.3323	1.735	0.2395	2.044
(0.01″)	1.00000	0.48481×10^{-7}	18.228	1.57080	23.209	0.3261	1.752	0.2268	2.100
($1''\times10^{-4}$)	1.00000	0.48481×10^{-9}	22.834	1.57080	29.073	0.3233	1.759	0.2196	2.134
($1''\times10^{-8}$)	1.00000	0.48481×10^{-13}	32.044	1.57080	40.800	0.3208	1.766	0.2116	2.174
($1''\times10^{-16}$)	1.00000	0.48481×10^{-21}	50.465	1.57080	64.254	0.3193	1.770	0.2046	2.211
($1''\times10^{-26}$)	1.00000	0.48481×10^{-31}	73.490	1.57080	93.571	0.3188	1.771	0.2008	2.232
($1''\times10^{-46}$)	1.00000	0.48481×10^{-51}	119.542	1.57080	152.206	0.3185	1.772	0.1976	2.249
(0)	1.00000	0.00000	∞	1.57080	∞	0.3183	1.773	0.1927	2.278

(continued on the following page)

Table 6 Characteristic numbers of permanent waves. (*continued*)

ϑ (ϑ')	h/H	A/H	A/L	H/D	L/D	h/D	A/D	$A/T^2\left(\frac{ft}{sec^2}\right)$	$D/T^2\left(\frac{ft}{sec^2}\right)$
0°	0.0000	0.0000	0.1412	1.0000	0.0000	0.0000	0.0000	0.8622	∞
20°	0.1223	0.1823	0.1409	0.9824	1.271	0.1201	0.1791	0.8578	4.790
60	0.2394	0.3547	0.1386	0.9744	2.493	0.2333	0.3456	0.8344	2.414
80	0.3516	0.5090	0.1277	0.9664	3.851	0.3398	0.4919	0.7206	1.4650
87	0.4408	0.6153	0.1115	0.9609	5.304	0.4236	0.5912	0.5657	0.9567
89°30′	0.5289	0.6942	0.08897	0.9627	7.511	0.5092	0.6683	0.3821	0.5718
(6′)	0.5795	0.7300	0.07410	0.9670	9.526	0.5604	0.7059	0.2757	0.3906
(30″)	0.6304	0.7512	0.05771	0.9708	12.635	0.6120	0.7293	0.1771	0.2429
(1″)	0.6735	0.7683	0.04430	0.9757	16.924	0.6571	0.7496	0.1092	0.14565
(0.01″)	0.7089	0.7823	0.03371	0.9799	22.74	0.6946	0.7666	0.06553	0.08548
(1″×10^{-4})	0.7309	0.7908	0.02720	0.9831	28.58	0.7185	0.7774	0.04360	0.05608
(1″×10^{-8})	0.7569	0.8006	0.01962	0.9872	40.28	0.7472	0.7904	0.02326	0.02943
(1″×10^{-16})	0.7813	0.8097	0.01260	0.9914	63.70	0.7746	0.8028	0.00982	0.01222
(1″×10^{-26})	0.7950	0.8148	0.008708	0.9940	93.01	0.7902	0.8099	0.00474	0.00586
(1″×10^{-46})	0.8068	0.8191	0.005382	0.9962	151.62	0.8037	0.8160	0.00183	0.00224
(0)	0.8261	0.8261	0.000000	1.0000	∞	0.8261	0.8261	0.00000	0.00000

where p takes the least value p_1 for the highest waves. A graph of A/T^2 ft/sec^2 as a function of D/T^2 ft/sec^2 is the breaking index curve of Bretschneider. Our theoretical values for this curve are inserted in the last two columns of Table 6, and indicated in Fig. 3. Small black circles in this figure are experimental points, inscribed in the Bretschneider diagram, their origins and weights of reliability are not discriminated here. Our diagram is almost coincident with Bretschneider's experimental one but for a small deflection in the neighborhood of $D/T^2=1$ ft/sec^2.

Fig. 4 gives the velocity of highest waves $U\Big/\sqrt{\frac{gL}{4K}}$ *i.e.* $1/\sqrt{p_1}$, as a function of L/H, this curve bounding the existence region of permanent waves from above. The lower bound is $1/\sqrt{p_0}$, where p_0 is the parameter value of waves of infinitesimal amplitude. p_0 is the maximum value of p, and defined by

$$p_0=\frac{gL}{4KU^2},\qquad U^2=\frac{gL}{2\pi}\tanh\left(2\pi\frac{H}{L}\right); \qquad \cdots\cdots(64)$$

from which we obtain

$$p_0=\frac{\pi}{2K}\coth\left(\pi\frac{K'}{K}\right), \qquad \cdots\cdots(64)'$$

and

$$p_0^{-1/2}=\left[U\Big/\sqrt{\frac{gL}{4K}}\right]_{min}=\left\{\frac{2K}{\pi}\tanh\left(\frac{\pi K'}{K}\right)\right\}^{1/2}. \qquad \cdots\cdots(65)$$

In this formula H can of course be replaced by D. Calculated values are inserted in Table 6 column $p_0^{-1/2}$.

We see from this figure that wave velocity is almost constant irrespective of its amplitude when water depth is large, *i.e.* wave velocity is determined almost solely by wave length, this characteistic having been adopted in and proved relevant to statistics of the sea. On the contrary, in the case of very shallow channel water, such statistics of waves are expected to show some discrepancies, for wave velocity in this case is really a function of A/H, as well as L/H.

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