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One-Dimensional Wave-Transfer Functions of the Linear Visco-Elastic Multi-Layered Half-Space

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Abstract

The most important problem in the earthquake response analysis of a structure is to suppose reasonable earthquake excitations depending on the seismicity and the dynamic characteristics of ground at the site of the structure.

In this paper, as one of the basic studies related to the supposition of random earthquake excitations for the dynamic aseismic design of structures, the authors will deal with the analytical expressions of the one-dimensional wave-transfer functions of a general class of linear visco-elastic, horizontally multi-layered half-space to vertically incident plane waves at the bottom boundary of the layered media through the half-space, and also discuss the properties of such wave-transfer functions in the complex plane.

Both the one-dimensional wave-transfer functions and the associate characteristic equation are expressed in the successive product forms involving some kind of symbolic operator, which are suitable for finding out the properties of those functions in the complex plane as well as for discussing the eigen-value problems of such layered media and also for carrying out the numerical calculation of the wave-transfer functions.

For the usually encountered linear visco-elastic layered half-space including the purely elastic case, it is found that the singular points of the wave-transfer functions consist of a finite number of branch points and a denumerably infinite number of poles having positive immaginary parts, which are zeros of the characteristic equation. And also, it is found that the wave-transfer functions are finite in the neighbourhood of the branch points and vanish in exponential order at infinity as far as the inner points of the layered media are concerned. These properties of the one-dimensional wave-transfer functions may guarantee the validity of the residue theorem in estimating the impulsive responses as well as the variances and co-variances of the random responses of the linear visco-elastic multi-layered half-space.

1. Introduction

One of the most important problems in estimating the response characteristics of a structure subjected to strong earthquakes is to suppose reasonable earthquake excitations according to the seismicity and the dynamic characteristics of ground at the site of the structure. As a rule the characteristics of an earthquake at a particular site are determined depending upon the properties of shocks at the origin and of the propagation pass of seismic waves. However, it has been pointed out by many investigators that the spectral characteristics of earthquakes at a site are remarkably influenced by the dynamic characteristics of the ground near the surface, which are determined by the ground structure and the physical properties of the media.[1)~8)]

Usually, in Japan, the ground structure at the site of structures is very complicated and it is likely to consist of a large number of alluvial and/or deluvial layers. And, the physical properties of the medium of each layer are also complex and different from each other. From the viewpoint of earthquake engineering, however, it is worthwhile to note that the ground structures are usually of a horizontally multi-layered type and that the velocities of bodily waves in the layers near the ground surface are sufficiently small compared with those in the depths of the crust and that they increase macroscopically with the depth, hence the direction of the propagation of seismic waves in the layers near the ground surface is considered to be approximately vertical. Taking account of the above-mentioned facts as well as the fact that the most destructive portion of seismic waves to structures is composed of the SH components of the seismic waves, the dynamic model of ground structure through which seismic waves propagate may be primarily considered as an elastic or a linear visco-elastic, horizontally multi-layered half-space subjected to vertically incident distortional waves at the bottom boundary adjacent to the half-space, as introduced by K. Sezawa and K. Kanai.[1),9)~12)]

When studying the dynamic characteristics of such a ground structure, it is important to consider a layered half-space in which the diffusion of wave energies in the strata to the subjacent half-space can occur. Hence, such a model of ground structure has damping characteristics even in the case of a perfectly elastic stratum on a half-space.[9),10)] As to the actual ground, there may exist various internal damping mechanisms in addition to the above-mentioned diffusive damping, although it is very difficult to describe them in explicit forms because of their variety and complexity. By assuming the Voigt type visco-elastic media K. Kanai obtained the amplitude magnification factor at the ground surface to the incident harmonic waves, namely the absolute value of the one-dimensional wave-transfer function at the ground surface to the vertically incident distortional waves in the case of one and two layered half-space.[1),11)]

In relation to the dynamic aseismic design of structures, it may be reasonable to suppose earthquake excitations as a stochastic ensemble in the future. As one of the most convenient and practical models of such a stochastic ensemble the quasi-stationary random process which is defined as the product of a deterministic function of time and an ergodic stationary random process was introduced by V. V. Bolotin.[13)] And, the general characteristics of the non-stationary responses of linear systems to the quasi-stationary random excitations and the simulation techniques to produce such excitations have been studied by several investigators including the authors.[14)~19)] Concerning the problem of how to suppose such a quasi-stationary random process depending upon the seismicity and the dynamic characteristics of the ground at the site of a structure, knowledge of the stationary and non-stationary random responses of the linear visco-elastic multi-layered media excited by a random incident waves propagated through the linear visco-elastic half-space seems to be useful for suggesting the spectral density or the auto-correlation function associated with the stationary random process and the deterministic function of time giving the envelope of the quasi-stationary random process.[14),15),18),20)] For this purpose, of course, it is desirable to express the dynamic characteristics of such ground in the form of the wave-transfer function of each part in the multi-layered media to the incident waves through the half-space.

In connexion with the above-mentioned problem, I. Herrera and E. Rosenblueth have already obtained the general expressions of the one-dimensional wave-transfer functions of the linear visco-elastic horizontally multi-layered half-space to the vertically incident waves at the bottom boundary adjacent to the half-space in the matrix forms for the case of an arbitrary number of layers and arbitrary linear visco-elasticity of each layer, and they have also presented an approximate formula in the integral form to evaluate the average values of the so-called pseudo-velocity response spectra of the movements at the ground surface of such a layered half-space excited by random incident waves.[21)] In this paper, the authors also deal with the similar problem as treated by I. Herrera and E. Rosenblueth, namely, the analytical expressions of the one-dimensional wave-transfer functions of a general class of the linear visco-elastic multi-layered half-space to incident waves to the layered media, and also study the properties of the one-dimensional wave-transfer functions in the complex plane. The main difference between the wave-transfer functions obtained by I. Herrera and E. Rosenblueth and those of the present paper is in their formal expressions, that is, in the former the wave-transfer functions were expressed in the matrix forms, while in the latter they are expressed in the successive product forms in terms of the scalar quantities including some kind of symbolic operator, which may be more suitable for finding out the properties of the wave-transfer functions in the complex plane as well as for discussing the associated eigen-value problem and also for carrying out the numerical evaluation of the wave-transfer functions than the matrix forms.

2. Fundamental equations and basic wave-transfer characteristics in the linear visco-elastic media

It is well-known that the fundamental equation of the wave propagation in a homogeneous isotropic linear visco-elastic medium is expressed as

$$\mu\left(\frac{\partial}{\partial\tau}\right)\nabla\cdot\nabla\boldsymbol{u}+\left(\lambda\left(\frac{\partial}{\partial\tau}\right)+\mu\left(\frac{\partial}{\partial\tau}\right)\right)\nabla\nabla\cdot\boldsymbol{u}-\rho\frac{\partial^2}{\partial\tau^2}\boldsymbol{u}+\boldsymbol{F}=\boldsymbol{O} \qquad \cdots(2.1)$$

in which $\boldsymbol{u}$ is a displacement vector, τ is time and $\mu(\partial/\partial\tau)$ and $\lambda(\partial/\partial\tau)$ are the generalized Lamé's constants represented by the rational function type differential operator, with respect to τ, having constant coefficients. And, ρ is the density, $\boldsymbol{F}$ is a body force vector, the symbol $\cdot$ designates the scalar product, and ∇ means the gradient operator defined as

$$\nabla=\frac{\partial}{\partial x_1}\boldsymbol{e}_1+\frac{\partial}{\partial x_2}\boldsymbol{e}_2+\frac{\partial}{\partial x_3}\boldsymbol{e}_3 \qquad \cdots(2.2)$$

where x_i and $\boldsymbol{e}_i$ denote the ith Cartesian co-ordinate and its associate unit vector, respectively. The strain and stress tensors may be expressed in the following forms:

$$\varepsilon=\frac{1}{2}(\partial\otimes u+u\otimes\partial) \qquad \cdots(2.3)$$

$$\sigma=2\mu\left(\frac{\partial}{\partial\tau}\right)\varepsilon+\lambda\left(\frac{\partial}{\partial\tau}\right)\nabla\cdot uE \qquad \cdots(2.4)$$

in which E is the 3×3 unit matrix and ∂ denotes a vector differential operator with respect to spatial co-ordinates and the symbols $\partial\otimes u$ and $u\otimes\partial$ in eq. (2.3) mean respectively the following 3×3 matrices:

$$\begin{aligned}\partial\otimes u&=[u_{j,i}]=\left[\frac{\partial u_j}{\partial x_i}\right]\\ u\otimes\partial&=[u_{i,j}]=\left[\frac{\partial u_i}{\partial x_j}\right]=(\partial\otimes u)^T,\ i,\ j=1,2,3\end{aligned} \qquad \cdots(2.5)$$

where u_i is the ith component of the displacement vector and T means the transposed matrix. The boundary conditions prescribed on the surface S enclosing the medium are given by the following forms according to the force and displacement types of boundary conditions, respectively:

$$\begin{aligned}&\text{force type;} && \sigma n=p && \text{on } S_1, && S_1\cap S_2=0\\ &\text{displacement type;} && u=q && \text{on } S_2, && S_1\cup S_2=S\end{aligned} \qquad \cdots(2.6)$$

where n is the unit vector of the outward normal on the sub-surface S_1 of S, and p and q are the prescribed distributed force vector and displacement vector on the sub-surfaces S_1 and S_2, respectively. To obtain the unique solution of the general dynamic problems, the initial conditions are to be given at all points of the medium including the boundary surface S as follows:

$$u=d \ \text{ and } \ \frac{\partial}{\partial\tau}u=v \quad \text{at } \tau=\tau_0 \quad \text{in } V\cup S \qquad \cdots(2.7)$$

in which d and v are the initial displacement and velocity at $\tau=\tau_0$, respectively and V represents the medium inside S. However, in determining the transfer functions by using the Fourier or Laplace transformations, it is not necessary to prescribe explicitly the initial conditions, otherwise they can be set to zero for all points of the medium.

By applying the Fourier transforms with respect to τ to eqs. (2.1), (2.3), (2.4) and (2.6), we obtain

$$\mu(j\omega)\nabla\cdot\nabla\tilde{u}+(\lambda(j\omega)+\mu(j\omega))\nabla\nabla\cdot\tilde{u}+\rho\omega^2\tilde{u}+\tilde{F}=O \qquad \cdots(2.8)$$

$$\varepsilon=\frac{1}{2}(\partial\otimes\tilde{u}+\tilde{u}\otimes\partial) \qquad \cdots(2.9)$$

$$\tilde{\sigma}=2\mu(j\omega)\tilde{\varepsilon}+\lambda(j\omega)\nabla\cdot\tilde{u}E \qquad \cdots(2.10)$$

and

$$\tilde{\sigma}n=\tilde{p} \ \text{ on } S_1, \qquad \tilde{u}=\tilde{q} \ \text{ on } S_2 \qquad \cdots(2.11)$$

in which

$$\tilde{u}\subset u,\ \tilde{\varepsilon}\subset\varepsilon,\ \tilde{\sigma}\subset\sigma \qquad \cdots(2.12)$$

$$\tilde{p}\subset p,\ \tilde{q}\subset q \qquad \cdots(2.13)$$

$$\mu(j\omega)\subset\mu\left(\frac{\partial}{\partial\tau}\right),\ \lambda(j\omega)\subset\lambda\left(\frac{\partial}{\partial\tau}\right),\ j=\sqrt{-1} \quad \cdots(2.14)$$

and ω denotes the frequency parameter. In the above equations, both $\mu(j\omega)$ and $\lambda(j\omega)$ are the rational functions with respect to $j\omega$ and represent the properties of the linear visco-elastic medium. It is well-known that $(j\omega)^{-1}\mu(j\omega)$ or $(j\omega)^{-1}\lambda(j\omega)$ and $(j\omega\mu(j\omega))^{-1}$ or $(j\omega\lambda(j\omega))^{-1}$ are the operators related to the stress relaxation and to the creep, respectively.

Introducing a scalar potential φ and a vector potential $\boldsymbol{\phi}$ and denoting their Fourier transforms as $\tilde{\varphi}$ and $\tilde{\boldsymbol{\phi}}$, respectively, the Fourier transform of the displacement vector $\boldsymbol{u}$ is expressed by

$$\tilde{\boldsymbol{u}}=\nabla\tilde{\varphi}+\nabla\times\tilde{\boldsymbol{\phi}} \quad \cdots(2.15)$$

where the symbol $\times$ denotes the vector product. Similarly representing the Fourier transform of the body force vector $\boldsymbol{F}$ in the form

$$\tilde{\boldsymbol{F}}=\nabla\tilde{F}_s+\nabla\times\tilde{\boldsymbol{F}}_v \quad \cdots(2.16)$$

and, substituting eqs. (2. 15) and (2. 16) in eqs. (2. 8)-(2. 10), we obtain the following set of equations:

$$\nabla[(\lambda(j\omega)+2\mu(j\omega))\nabla\cdot\nabla\tilde{\varphi}+\rho\omega^2\tilde{\varphi}+\tilde{F}_s] \\ +\nabla\times[\mu(j\omega)\nabla\cdot\nabla\tilde{\boldsymbol{\phi}}+\rho\omega^2\tilde{\boldsymbol{\phi}}+\tilde{\boldsymbol{F}}_v]=\boldsymbol{O} \quad \cdots(2.17)$$

and

$$\tilde{\boldsymbol{\varepsilon}}=\frac{1}{2}(\boldsymbol{\partial}\otimes(\nabla\tilde{\varphi}+\nabla\times\tilde{\boldsymbol{\phi}})+(\nabla\tilde{\varphi}+\nabla\times\tilde{\boldsymbol{\phi}})\otimes\boldsymbol{\partial}) \quad \cdots(2.18)$$

$$\tilde{\boldsymbol{\sigma}}=2\mu(j\omega)\tilde{\boldsymbol{\varepsilon}}+\lambda(j\omega)\nabla\cdot\nabla\tilde{\varphi}\boldsymbol{E} \quad \cdots(2.19)$$

By applying the operators $\nabla\cdot$ and $\nabla\times$ to eq. (2. 17) and taking account of the relations

$$\nabla\cdot\nabla\tilde{\varphi}=\nabla\cdot\tilde{\boldsymbol{u}},\quad \nabla\times\nabla\times\tilde{\boldsymbol{\phi}}=\nabla\times\tilde{\boldsymbol{u}} \quad \cdots(2.20)$$

$$\nabla\cdot\nabla\tilde{F}_s=\nabla\cdot\tilde{\boldsymbol{F}},\quad \nabla\times\nabla\times\tilde{\boldsymbol{F}}_v=\nabla\times\tilde{\boldsymbol{F}} \quad \cdots(2.21)$$

we obtain the fundamental equations concerning the dilatational and rotational waves as follows:

$$(\lambda(j\omega)+2\mu(j\omega))\nabla\cdot\nabla(\nabla\cdot\tilde{\boldsymbol{u}})+\rho\omega^2(\nabla\cdot\tilde{\boldsymbol{u}})+\nabla\cdot\tilde{\boldsymbol{F}}=0 \quad \cdots(2.22)$$

$$\mu(j\omega)\nabla\cdot\nabla(\nabla\times\tilde{\boldsymbol{u}})+\rho\omega^2(\nabla\times\tilde{\boldsymbol{u}})+\nabla\times\tilde{\boldsymbol{F}}=\boldsymbol{O} \quad \cdots(2.23)$$

And also, corresponding to the above two equations, we obtain

$$(\lambda(j\omega)+2\mu(j\omega))\nabla\cdot\nabla\tilde{\varphi}+\rho\omega^2\tilde{\varphi}+\tilde{F}_s=0 \quad \cdots(2.24)$$

$$\mu(j\omega)\nabla\cdot\nabla\tilde{\boldsymbol{\phi}}+\rho\omega^2\tilde{\boldsymbol{\phi}}+\tilde{\boldsymbol{F}}_v=\boldsymbol{O} \quad \cdots(2.25)$$

As found from eqs. (2. 15) and (2. 20) particularly confining the displacement vector $\boldsymbol{u}$ to either of the elements $\nabla\varphi$ and $\nabla\times\boldsymbol{\phi}$ in eqs. (2. 17)-(2. 19), we obtain directly the fundamental equations of the irrotational waves and of the equivolumental waves, respectively.

Taking x_1- and x_2- axis in a horizontal plane and x_3- axis downward in the vertical direction, and confining ourselves to the two-dimensional problem in which no quantities depend on x_2, the components, $\boldsymbol{\phi}_H=\psi_1 e_1+\psi_3 e_3$ and $\boldsymbol{\phi}_V=\psi_2 e_2$, of the vector potential $\boldsymbol{\phi}$ are concerned with the so-called SH and SV waves, respectively. The scalar potential φ, of course, is concerned with P waves. Consequently, the following sets of equations are obtained for the three kinds of waves, namely

$$\begin{aligned}&[(\lambda(j\omega)+2\mu(j\omega))\nabla\cdot\nabla+\rho\omega^2]\tilde{\varphi}+\tilde{F}_s=0\\ &\tilde{\boldsymbol{\varepsilon}}=\frac{1}{2}(\boldsymbol{\partial}\otimes\nabla\tilde{\varphi}+\nabla\tilde{\varphi}\otimes\boldsymbol{\partial})\\ &\tilde{\boldsymbol{\sigma}}=2\mu(j\omega)\tilde{\boldsymbol{\varepsilon}}+\lambda(j\omega)\nabla\cdot\nabla\tilde{\varphi}\boldsymbol{E} \qquad \text{for P waves}\end{aligned} \qquad \cdots(2.26)$$

$$\begin{aligned}&[\mu(j\omega)\nabla\cdot\nabla+\rho\omega^2]\begin{Bmatrix}\tilde{\psi}_1\\ \tilde{\psi}_3\end{Bmatrix}+\begin{Bmatrix}\tilde{F}_{v1}\\ \tilde{F}_{v3}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}\\ &\tilde{\boldsymbol{\varepsilon}}=\frac{1}{2}(\boldsymbol{\partial}\otimes\nabla\times(\tilde{\psi}_1 e_1+\tilde{\psi}_3 e_3)+\nabla\times(\tilde{\psi}_1 e_1+\tilde{\psi}_3 e_3)\otimes\boldsymbol{\partial})\\ &\tilde{\boldsymbol{\sigma}}=2\mu(j\omega)\tilde{\boldsymbol{\varepsilon}} \qquad \text{for SH waves}\end{aligned} \qquad \cdots(2.27)$$

and

$$\begin{aligned}&[\mu(j\omega)\nabla\cdot\nabla+\rho\omega^2]\tilde{\psi}_2+\tilde{F}_{v2}=0\\ &\tilde{\boldsymbol{\varepsilon}}=\frac{1}{2}(\boldsymbol{\partial}\otimes\nabla\times\tilde{\psi}_2 e_2+\nabla\times\tilde{\psi}_2 e_2\otimes\boldsymbol{\partial})\\ &\tilde{\boldsymbol{\sigma}}=2\mu(j\omega)\tilde{\boldsymbol{\varepsilon}} \qquad \text{for SV waves}\end{aligned} \qquad \cdots(2.28)$$

In what follows, we will consider the one-dimensional case where the dependence on the co-ordinate arises only with respect to the x_3-axis. And, for the sake of simplicity, neglecting the body foce vector $\boldsymbol{F}$ and rewritting x_3 by z, the sets of eqs. (2. 26)-(2. 28) reduce to the following forms:

$$\begin{aligned}&(\lambda(j\omega)+2\mu(j\omega))\tilde{u}_z^{(2)}+\rho\omega^2\tilde{u}=0\\ &\tilde{\varepsilon}=\tilde{u}_z^{(1)},\quad \tilde{\sigma}=(\lambda(j\omega)+2\mu(j\omega))\tilde{u}_z^{(1)}\end{aligned} \qquad \cdots(2.29)$$

where $\tilde{u}=\tilde{\varphi}_z^{(1)}$ for P waves

and

$$\begin{aligned}&\mu(j\omega)\tilde{u}_z^{(2)}+\rho\omega^2\tilde{u}=0\\ &\tilde{\varepsilon}=\frac{1}{2}\tilde{u}_z^{(1)},\quad \tilde{\sigma}=\mu(j\omega)\tilde{u}_z^{(1)}\end{aligned} \qquad \cdots(2.30)$$

where $\tilde{u}=\tilde{\psi}_{1z}^{(1)}$ for SH waves

$\tilde{u}=-\tilde{\psi}_{2z}^{(1)}$ for SV waves

In the above equations, $_z^{(i)}$ designates the ith derivative with respect to z. Hence, we have the following fundamental equations in the frequency domain for the one-dimensional wave propagation in the homogeneous, isotropic, linear visco-elastic medium:

$$p(j\omega)\tilde{u}_z^{(2)}(j\omega,z)+\rho\omega^2\tilde{u}(j\omega,z)=0 \qquad \cdots(2.31)$$

$$\tilde{\sigma}(j\omega,z)=p(j\omega)\tilde{u}_z^{(1)}(j\omega,z) \qquad \cdots(2\cdot 32)$$

in which $p(j\omega)$ stands for either $\lambda(j\omega)+2\mu(j\omega)$ or $\mu(j\omega)$.

Now, supposing the linear visco-elastic N-layered half-space as shown in

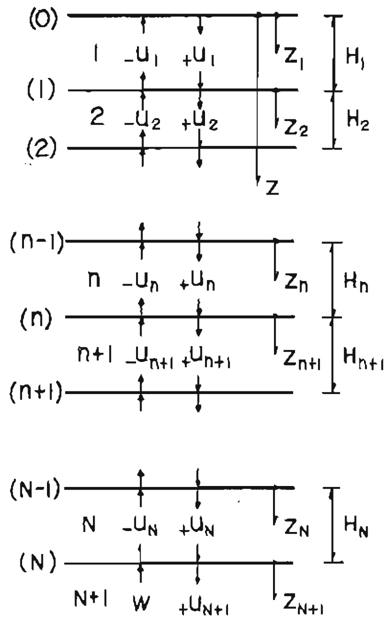

Fig. 1 Model of an N-layered half-space.

Fig. 1, the Fourier transform of the one-dimensional wave equation in the n-th homogeneous isotropic medium is expressed as follows:

$$p_n(j\omega)\tilde{u}_{n\,z_n}{}^{(2)}(j\omega,\,z_n) + \rho_n\omega^2\tilde{u}_n(j\omega,\,z_n)=0 \qquad \cdots(2.33)$$

$$\tilde{u}_n(j\omega,\,z_n)\sqsubset u_n(\tau,\,z_n),\quad n=1,\,2,\cdots\cdots,\,N,\,N+1 \qquad \cdots(2.34)$$

$$H_n\geqq z_n\geqq 0,\quad n=1,\,2,\cdots\cdots,\,N;\quad \infty>z_{N+1}\geqq 0 \qquad \cdots(2.35)$$

In the above equations, z_n is the co-ordinate associated with the nth medium, which has its origin at the $(n-1)$th boundary and is measured downward as shown in Fig. 1. And, $u_n(\tau,\,z_n)$ denotes the displacement of the point z_n, and H_n is the thickness of the nth layer.

The boundary conditions in the frequency domain which represent the condition of the free surface and the continuity of displacement and stress are expressed as follows:

$$\tilde{u}_{1\,z_1}{}^{(1)}(j\omega,\,0)=0 \qquad \cdots(2.36)$$

$$\tilde{u}_n(j\omega,\,H_n)=\tilde{u}_{n+1}(j\omega,\,0) \qquad \cdots(2.37)$$

$$p_n(j\omega)\tilde{u}_{n\,z_n}{}^{(1)}(j\omega,\,H_n)=p_{n+1}(j\omega)\tilde{u}_{n+1\,z_{n+1}}{}^{(1)}(j\omega,\,0)\quad n=1,\,2,\cdots\cdots,\,N \qquad \cdots(2.38)$$

The fundamental system of solutions of eq. (2.33) is given by

$$\tilde{u}_n{}^1(j\omega,\,z_n)=\exp(-j\kappa_n z_n),\quad \tilde{u}_n{}^2(j\omega,\,z_n)=\exp(j\kappa_n z_n) \qquad \cdots(2.39)$$

where

$$\kappa_n=\omega\sqrt{\frac{\rho_n}{p_n(j\omega)}}=\boldsymbol{R}(\kappa_n)+j\,\boldsymbol{I}(\kappa_n) \qquad \cdots(2.40)$$

and

$$\omega\boldsymbol{R}(\kappa_n)>0,\ \boldsymbol{I}(\kappa_n)\leqq 0 \quad \text{for } \boldsymbol{I}(\omega)=0 \qquad \cdots(2.41)$$

in which $\boldsymbol{R}$ and $\boldsymbol{I}$ designate the real and immaginary parts, respectively. It is noticed that for a rational function type $p(j\omega)$, the real part $\boldsymbol{R}(\kappa_n)$ is an even function of ω while the immaginary part $\boldsymbol{I}(\kappa_n)$ is an odd function of ω and also that for the actual media, $\boldsymbol{I}(\kappa_n)$ may always be chosen non-positive for all real ω by taking either of the two different branches of κ_n according to the sign of ω.

Since the general solution of eq. (2.33) is expressed as

$$\tilde{u}_n(j\omega, z_n) = A_n(j\omega)\tilde{u}_n^1(j\omega, z_n) + B_n(j\omega)\tilde{u}_n^2(j\omega, z_n) \quad \cdots(2.42)$$

the original general solution in the time domain is given by

$$u_n(\tau, z_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{A_n(j\omega)_+U_n(j\omega, \tau, z_n) + B_n(j\omega)_-U_n(j\omega, \tau, z_n)\}d\omega \quad \cdots(2.43)$$

in which

$$\begin{aligned} {}_+U_n(j\omega, \tau, z_n) &= \tilde{u}_n^1(j\omega, z_n)\exp(j\omega\tau) = \exp(j(\omega\tau - \kappa_n z_n)) \\ &= \exp\left\{jR(\kappa_n)\left(\frac{\omega}{R(\kappa_n)}\tau - z_n\right)\right\}\exp\{I(\kappa_n)z_n\} \end{aligned} \quad \cdots(2.44)$$

$$\begin{aligned} {}_-U_n(j\omega, \tau, z_n) &= \tilde{u}_n^2(j\omega, z_n)\exp(j\omega\tau) = \exp(j(\omega\tau + \kappa_n z_n)) \\ &= \exp\left\{jR(\kappa_n)\left(\frac{\omega}{R(\kappa_n)}\tau + z_n\right)\right\}\exp\{-I(\kappa_n)z_n\} \end{aligned} \quad \cdots(2.45)$$

By putting

$$v_n = \frac{\omega}{R(\kappa_n)} \quad \text{and} \quad l_n = -I(\kappa_n) \quad \cdots(2.46)$$

the above two quantities v_n and l_n for real ω represent the wave velocity and the attenuation constant in the nth medium, respectively. As found from eqs. (2. 44) and (2. 45), ${}_+U_n(j\omega, \tau, z_n)$ and ${}_-U_n(j\omega, \tau, z_n)$ are the forward and backward complex harmonic waves, respectively, both of which have the unit amplitude at the origin of the nth co-ordinate. The two constants $A_n(j\omega)$ and $B_n(j\omega)$ in eqs. (2. 42) and (2. 43) are to be determined according to the boundary conditions.

In the following, we will consider the basic wave-transfer characteristics of the one-dimensional waves in the linear visco-elastic media. At first, by taking into consideration eq. (2. 39), the general solution in the frequency domain given by eq. (2. 42) is expressed as

$$\tilde{u}_n(j\omega, z_n) = {}_+\tilde{u}_n(j\omega, z_n) + {}_-\tilde{u}_n(j\omega, z_n) \quad \cdots(2.47)$$

in which

$$\begin{aligned} {}_+\tilde{u}_n(j\omega, z_n) &= A_n(j\omega)\exp(-j\kappa_n z_n) \subset {}_+u_n(\tau, z_n) \\ {}_-\tilde{u}_n(j\omega, z_n) &= B_n(j\omega)\exp(j\kappa_n z_n) \subset {}_-u_n(\tau, z_n) \end{aligned} \quad \cdots(2.48)$$

and ${}_+u_n(\tau, z_n)$ and ${}_-u_n(\tau, z_n)$ represent the forward and backward waves, respectively. Hence the wave-transfer function associated with the wave propagation in the nth medium is obtained as follows:

$$\frac{{}_+\tilde{u}_n(j\omega, z_n')}{{}_+\tilde{u}_n(j\omega, z_n)} = \frac{{}_-\tilde{u}_n(j\omega, z_n)}{{}_-\tilde{u}_n(j\omega, z_n')} = \exp(-j\kappa_n(z_n' - z_n)) \quad \cdots(2.49)$$

where

$$\begin{aligned} &H_n \geqq z_n' \geqq z_n \geqq 0 \quad \text{for} \quad N \geqq n \geqq 1 \\ &\infty \geqq z_n' \geqq z_n \geqq 0 \quad \text{for} \quad n = N+1 \end{aligned} \quad \cdots(2.50)$$

The quantity given by eq. (2. 49) is a complex-valued function of the real ω and its absolute value and argument represent the amplitude and phase characteristics, respectively.

Next, adopting the $(n+1)$th co-ordinate for both the nth and $(n+1)$th media, the Fourier transform of the incident waves ${}_{+}u_n(\tau, z_{n+1})$ to the nth boundary through the nth medium, that of the reflected waves ${}_{-}u_n(\tau, z_{n+1})$ to the nth medium and of the refracted waves ${}_{+}u_{n+1}(\tau, z_{n+1})$ in the $(n+1)$th medium are expressed as follows:

$$\begin{aligned} {}_{+}\tilde{u}_n(j\omega, z_{n+1}) &= A_n(j\omega)\exp(-j\kappa_n z_{n+1}) \\ {}_{-}\tilde{u}_n(j\omega, z_{n+1}) &= B_n(j\omega)\exp(j\kappa_n z_{n+1}) \\ {}_{+}\tilde{u}_{n+1}(j\omega, z_{n+1}) &= A_{n+1}(j\omega)\exp(-j\kappa_{n+1} z_{n+1}) \end{aligned} \quad \cdots(2.51)$$

Taking account of eqs. (2.47) and (2.51) together with $B_{n+1}(j\omega)=0$, the boundary conditions given by eqs. (2.37) and (2.38) take the forms

$$\begin{aligned} A_n(j\omega)+B_n(j\omega) &= A_{n+1}(j\omega) \\ \alpha_n(A_n(j\omega)-B_n(j\omega)) &= A_{n+1}(j\omega) \end{aligned} \quad \cdots(2.52)$$

where

$$\alpha_n = \frac{p_n(j\omega)\kappa_n}{p_{n+1}(j\omega)\kappa_{n+1}} = \frac{\rho_n\kappa_{n+1}}{\rho_{n+1}\kappa_n} = \sqrt{\frac{\rho_n p_n(j\omega)}{\rho_{n+1}p_{n+1}(j\omega)}} = \boldsymbol{R}(\alpha_n)+j\boldsymbol{I}(\alpha_n) \quad \cdots(2.53)$$

$$\boldsymbol{R}(\alpha_n)>0 \quad \text{for} \quad \boldsymbol{I}(\omega)=0 \quad \cdots(2.54)$$

The quantity defined by eq. (2.53) is called the impedance ratio, which is generally a complex-valued function of the real ω, and its real part is expressed as

$$\boldsymbol{R}(\alpha_n) = \frac{\rho_n}{\rho_{n+1}}\left[\boldsymbol{R}(\kappa_{n+1})\boldsymbol{R}\left(\frac{1}{\kappa_n}\right) - \boldsymbol{I}(\kappa_{n+1})\boldsymbol{I}\left(\frac{1}{\kappa_n}\right)\right] \quad \cdots(2.55)$$

Since from eq. (2.41), it is found that $\boldsymbol{R}(\kappa_{n+1})\boldsymbol{R}(1/\kappa_n)>0$ and $\boldsymbol{I}(\kappa_{n+1})\boldsymbol{I}(1/\kappa_n) \leqq 0$ for the real ω, the real part $\boldsymbol{R}(\alpha_n)$ of the impedance ratio should be positive for the real ω.

By solving eq. (2.52), the wave-transfer functions associated with the reflected and refracted waves to the normally incident plane waves at the nth boundary through the nth medium are obtained as follows:

$$\frac{{}_{-}\tilde{u}_n(j\omega, 0)}{{}_{+}\tilde{u}_n(j\omega, 0)} = \frac{B_n(j\omega)}{A_n(j\omega)} = b_n(j\omega) \equiv b_n = \frac{\alpha_n - 1}{\alpha_n + 1} \quad \cdots(2.56)$$

and

$$\frac{{}_{+}\tilde{u}_{n+1}(j\omega, 0)}{{}_{+}\tilde{u}_n(j\omega, 0)} = \frac{A_{n+1}(j\omega)}{A_n(j\omega)} = 1 + b_n \equiv a_n{}' = \frac{2\alpha_n}{\alpha_n + 1} \quad \cdots(2.57)$$

Similarly, considering the incident waves at the nth boundary through the $(n+1)$th medium, the wave-transfer functions of the reflected and refracted waves to the normally incident plane waves are determined as

$$\frac{{}_{+}\tilde{u}_{n+1}(j\omega, 0)}{{}_{-}\tilde{u}_{n+1}(j\omega, 0)} = -b_n \equiv b_n{}' = \frac{1-\alpha_n}{\alpha_n + 1} \quad \cdots(2.58)$$

and

$$\frac{{}_{-}\tilde{u}_n(j\omega, 0)}{{}_{-}\tilde{u}_{n+1}(j\omega, 0)} = 1 - b_n \equiv a_n = \frac{2}{\alpha_n + 1} \quad \cdots(2.59)$$

In particular, for the free surface the wave-transfer function associated with the reflected waves to the normally incident plane waves is given as follows:

$$\frac{{}_{+}\tilde{u}_1(j\omega,\ 0)}{{}_{-}\tilde{u}_1(j\omega,\ 0)}=b_0'=1 \qquad \cdots(2.60)$$

3. One-dimensional wave-transfer functions of the linear visco-elastic multi-layered half-space

Supposing the linear visco-elastic horizontally N-layered half-space as shown in Fig. 1 and taking the vertically incident plane displacement waves

$$w(\tau)={}_{-}u_{N+1}(\tau,\ 0)\supset\tilde{w}(j\omega) \qquad \cdots(3.1)$$

at the Nth boundary through the half-space as an input and the displacement vector of the boundaries

$$\{u^n(\tau)\}=\{u_{n+1}(\tau,\ 0)\}\supset\{\tilde{u}^n(j\omega)\},\quad n=0,\ 1,\cdots\cdots,\ N \qquad \cdots(3.2)$$

as an output vector, we will consider the one-dimensional wave-transfer functions of the linear visco-elastic multi-layered half-space to the incident waves at the bottom boundary adjacent to the half-space.

By making use of the basic wave-transfer function associated with the one-dimensional wave propagation and those of the reflected and refracted waves to the normally incident plane waves, which are obtained in the preceding section, the following vector-matrix relation among the Fourier transforms of the displacement components due to the forward and backward waves in the $(n-1)$th, nth and $(n+1)$th media is obtained:

$$[{}_n\boldsymbol{L}]\{{}_n\boldsymbol{U}\}=\{{}_n\boldsymbol{H}\},\quad n=1,\ 2,\cdots\cdots,\ N \qquad \cdots(3.3)$$

in which

$$[{}_n\boldsymbol{L}]=\begin{bmatrix} a'_{n-1} & b'_{n-1} & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\exp(-j\kappa_nH_n) & 0 & 0 \\ 0 & 0 & \exp(-j\kappa_nH_n) & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -b_n & -a_n \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \cdots(3.4)$$

and

$$\{{}_n\boldsymbol{U}\}=\begin{Bmatrix} {}_{+}\tilde{u}_{n-1}(j\omega,\ H_{n-1}) \\ {}_{-}\tilde{u}_n(j\omega,\ 0) \\ {}_{+}\tilde{u}_n(j\omega,\ 0) \\ {}_{-}\tilde{u}_n(j\omega,\ H_n) \\ {}_{+}\tilde{u}_n(j\omega,\ H_n) \\ {}_{-}\tilde{u}_{n+1}(j\omega,\ 0) \end{Bmatrix} \quad \cdots(3.5),\qquad \{{}_n\boldsymbol{H}\}=\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{u}^{n-1}(j\omega) \\ \tilde{u}^n(j\omega) \end{Bmatrix} \qquad \cdots(3.6)$$

In the above equations, particularly for the cases of $n=1$ and $n=N$ the following equations are to be used:

$$a_0'=0,\quad {}_{-}\tilde{u}_{N+1}(j\omega,\ 0)=\tilde{w}(j\omega) \qquad \cdots(3.7)$$

For an arbitrary number of layers N, the one-dimensional wave-transfer function associated with the nth boundary displacement $u^n(\tau)$ to the incident displacement waves $w(\tau)$ at the bottom boundary adjacent to the half-space is defined as

$$G_N^n(j\omega)=\frac{\tilde{u}^n(j\omega)}{\tilde{w}(j\omega)},\quad n=0,1,\cdots\cdots,N \qquad \cdots(3.8)$$

By making use of eqs. (3.3)-(3.6), the following vector-matrix equation related to the above-defined wave-transfer functions G_N^n's is obtained for an arbitrary N:

$$[A]\{G\}=\{F\} \qquad \cdots(3.9)$$

in which $\{G\}$ denotes the $(N+1)\times 1$ column vector composed of $G_N^0, G_N^1, \cdots, G_N^N$, and the $(N+1)\times(N+1)$ matrix $[A]$ and the $(N+1)\times 1$ column vector $\{F\}$ are expressed in terms of the determinant ${}_n\Delta$ of the coefficient matrix $[{}_nL]$ in eq. (3.3) and its λ-th order minor determinate ${}^{(\lambda)}_{n}\Delta^{j_1 j_2\cdots j_\lambda}_{i_1 i_2\cdots i_\lambda}$ withr espect to the i_1-, i_2-,$\cdots$, i_λ-th rows and j_1-, j_2-,$\cdots j_\lambda$-th columns and the wave-transfer function a_N associated with the refracted waves to incident waves at the bottom bundary through the half-space, namely

$$[A]=\begin{pmatrix} a_0^0 a_1^0\, 0\cdots\cdots\cdots & & \cdots\cdots 0 \\ 0\; a_1^1 a_2^1\, 0\cdots\cdots & & \cdots\cdots 0 \\ \cdots\cdots\cdots\cdots & & \cdots\cdots \\ 0\;\cdots\cdots\cdots 0\; a_{n-1}^n a_n^n\, a_{n+1}^n\, 0 & & \cdots\cdots 0 \\ \cdots\cdots\cdots\cdots & & \\ 0\;\cdots\cdots\cdots & \cdots\cdots 0 & a_{N-1}^N a_N^N \end{pmatrix} \qquad \cdots(3.10)$$

and

$$\{G\}=\begin{Bmatrix} G_N^0 \\ G_N^1 \\ \vdots \\ G_N^{N-1} \\ G_N^N \end{Bmatrix} \quad \cdots(3.11), \qquad \{F\}=\begin{Bmatrix} f^0 \\ f^1 \\ \vdots \\ f^{N-1} \\ f^N \end{Bmatrix} \quad \cdots(3.12)$$

For the case where $N\geqq 3$, the elements of the matrix $[A]$ and the column vector $\{F\}$ are determined as follows:

$$a_0^0=1,\quad a_1^0=-\frac{{}^{(2)}_{1}\Delta^{12}_{56}+{}^{(2)}_{1}\Delta^{13}_{56}}{{}^{(1)}_{1}\Delta^{1}_{6}}$$

$$a_1^1=1-\frac{{}^{(2)}_{1}\Delta^{16}_{56}}{{}^{(1)}_{1}\Delta^{1}_{5}}-\frac{{}^{(1)}_{2}\Delta^{3}_{5}}{{}_2\Delta},\quad a_2^1=-\frac{{}^{(1)}_{2}\Delta^{3}_{6}}{{}_2\Delta}$$

$$a_{n-1}^n=-\frac{{}^{(1)}_{n}\Delta^{6}_{5}}{{}_n\Delta},\quad a_n^n=1-\frac{{}^{(1)}_{n}\Delta^{6}_{6}}{{}_n\Delta}-\frac{{}^{(1)}_{n+1}\Delta^{3}_{5}}{{}_{n+1}\Delta},\quad a_{n+1}^n=-\frac{{}^{(1)}_{n+1}\Delta^{3}_{6}}{{}_{n+1}\Delta}$$

$$a_{N-1}^N=-\frac{{}^{(2)}_{N}\Delta^{64}_{65}+{}^{(2)}_{N}\Delta^{65}_{66}}{{}^{(1)}_{N}\Delta^{6}_{6}},\quad a_N^N=1 \qquad \cdots(3.13)$$

and

$$f^0=f^1=\cdots\cdots=f^{N-1}=0,\quad f^N=\frac{a_N\left({}^{(2)}_{N}\Delta^{64}_{64}+{}^{(2)}_{N}\Delta^{65}_{64}\right)}{{}^{(1)}_{N}\Delta^{6}_{6}} \qquad \cdots(3.14)$$

For the case where $N=2$, the similar expressions as in eqs. (3.13) and (3.14) are obtained. However, in this case the elements a_1^1 and a_2^1 in eq. (3.13) should be replaced by the following equations:

$$a_1^1=1-\frac{{}^{(2)}_{1}\varDelta^{16}_{66}}{{}^{(1)}_{1}\varDelta^{1}_{5}}+\frac{{}^{(2)}_{2}\varDelta^{63}_{45}}{{}^{(1)}_{2}\varDelta^{6}_{4}},\quad a_2^1=-\frac{{}^{(2)}_{2}\varDelta^{63}_{46}}{{}^{(1)}_{2}\varDelta^{6}_{4}} \qquad \cdots(3.15)$$

Hence, for the case where $N\geqq 2$, denoting the determinant of the coefficient matrix $[A]$ in eq. (3.9) and its co-factor with respect to the $(N+1)$th row and the $(n+1)$th column as $\varDelta_N$ and ${}^{(1)}\varDelta_N{}^{n+1}_{N+1}$, respectively, the one-dimensional wave-transfer functions G_N^n's can be expressed as follows:

$$G_N^n=\frac{f^{N(1)}\varDelta_N{}^{n+1}_{N+1}}{\varDelta_N},\qquad n=0,1,\cdots\cdots,N \qquad \cdots(3.16)$$

In particular, for $n=0$, the wave-transfer function associated with the ground surface to incident waves is given by

$$G_N^0=\frac{f^{N(1)}\varDelta_N{}^{1}_{N+1}}{\varDelta_N}=\frac{(-1)^N f^N\prod\limits_{n=0}^{\tilde{1}} a_{n+1}^n}{\varDelta_N} \qquad \cdots(3.17)$$

For the case where $N=1$, the following equations are obtained instead of eqs. (3.13) and (3.14):

$$a_0^0=1,\quad a_1^0=0,\quad a_1^1=1 \qquad \cdots(3.18)$$

and

$$f^0=-\frac{a_1({}^{(3)}_{1}\varDelta^{162}_{564}+{}^{(3)}_{1}\varDelta^{163}_{564})}{{}^{(2)}_{1}\varDelta^{16}_{66}},\quad f^1=-\frac{a_1({}^{(3)}_{1}\varDelta^{164}_{564}+{}^{(3)}_{1}\varDelta^{165}_{564})}{{}^{(2)}_{1}\varDelta^{16}_{56}} \qquad \cdots(3.19)$$

Hence, in this case, the one-dimensional wave-transfer functions G_1^n's are expressed as

$$G_1^n=\frac{f^n}{a_n^n}=f^n,\quad n=0,1 \qquad \cdots(3.20)$$

Since the expressions of the one-dimensional wave-transfer functions of the boundary displacements to incident waves, which are given by eq. (3.16) or eq. (3.20), contain the determinant of the matrices $[{}_nL]$, $n=1,2,\cdots,N$ and their co-factors and higher order minor determinants, the numerical evaluation of them may be complicated and tedious for the case of a large N.

In what follows, it is verified that the one-dimensional wave-transfer functions G_N^n's associated with the boundary displacements to the incident displacement waves are uniformly expressed in the following successive product forms for an arbitrary number of layers $N\geqq 1$ and any number of boundaries $N\geqq n\geqq 0$:

$$G_N^n=\frac{\prod\limits_{i=n}^{N}(1-b_i)\prod\limits_{i=n+1}^{N}X_i\;{}_n\varDelta_c'}{{}_N\varDelta_c},\qquad n=0,1,\cdots\cdots,N \qquad \cdots(3.21)$$

where

$${}_n\varDelta_c'=1\prod_{i=1}^{n-1}\circ(1+\delta_{i-1}b_{i-1}\delta_i b_i X_i^2)\circ(1-\delta_{n-1}b_{n-1}X_n^2)\circ 1 \qquad \cdots(3.22)$$

$$_N\Delta_c = 1 \prod_{i=1}^{N} \circ (1 + \delta_{i-1} b_{i-1} \delta_i b_i X_i^2) \circ 1 \quad \cdots(3.23)$$

and

$$X_i = \exp(-j\kappa_i H_i) \quad \cdots(3.24)$$

In the above equations, the complex wave number κ_i, $i=1, 2, \cdots\cdots, N+1$ and the wave-transfer function b_i, $i=1, 2, \cdots\cdots, N$ associated with the reflected waves at the ith boundary to the incident waves through the ith medium are given in eqs. (2.40) and (2.56), respectively. As for b_0, it is assumed that

$$b_0 = -1 \quad \cdots(3.25)$$

As found from eq. (2.49), the quantity X_i, $i=1, 2, \cdots\cdots, N$ in eq. (3.24) means the transfer function associated with the one-dimensional wave propagation in the ith medium along the thickness H_i of the ith layer.

The symbols δ_i's, $i=-1, 0, 1, \cdots\cdots, N$, appearing in eqs. (3.22) and (3.23) denote the symbolic operators associated with the non-commutative multiplication denoted by the mark $\circ$, and the operational conventions with respect to the $\circ$ marked multiplication and the conventional multiplication between the symbols δ_i's and the scalar quantities, c and d, are defined as follows:

$$\begin{aligned} &c\delta_i = \delta_i c \\ &c \circ d = d \circ c = cd \\ &\delta_i \circ \delta_i = b_i^{-2} \\ &c \circ \delta_i \circ d = cd \\ &c \circ \delta_i \delta_j \circ d = cd, \quad j \geqq i+1, \quad i, j = 0, 1, \cdots\cdots, N \end{aligned} \quad \cdots(3.26)$$

In particular, δ_{-1} which appears by substituting $n=0$ in eq. (3.22) is defined as

$$\delta_{-1} = 0 \quad \cdots(3.27)$$

And also, the product symbol $\prod_{i=p}^{q}$ having the smaller upper limit q than the lower limit p is assumed to be unity for both the $\circ$ marked multiplication and the conventional multiplication, namely

$$\prod_{i=p}^{q} C_i = 1, \qquad 1 \prod_{i=p}^{q} \circ D_i \circ 1 = 1 \qquad \text{for} \quad q < p \quad \cdots(3.28)$$

The proof of the validity of eqs. (3.21)-(3.23) can be made by the so-called method of mathematical induction. At first, in the case where $N=1$ the following expressions which are consistent with eqs. (3.21)-(3.23) are obtained by making use of eqs. (3.4), (3.19) and (3.20).

$$G_1^0 = \frac{2(1-b_1)X_1}{_1\Delta_c}, \quad G_1^1 = \frac{(1-b_1)(1+X_1^2)}{_1\Delta_c} \quad \cdots(3.29)$$

$$_1\Delta_c = 1 - b_1 X_1^2 \quad \cdots(3.30)$$

Secondly, assuming the validity of the expressions given by eqs. (3.21)-(3.23) for G_N^0 and G_N^N, we will show the validity of the expressions for G_{N+1}^0. When the number of layers increases from N to $N+1$, we consider the feed-back system as shown in Fig. 2, in which the input and the output are taken as the incident waves to the $(N+1)$th boundary through the half-space and the incident waves arriving at the Nth boundary through the $(N+1)$th medium, respectively. Thus, considering the forward and backward gains which are obtained respectively by making use of the basic wave-transfer functions associated with the one-dimensional wave propagation, the reflected and refracted waves to incident waves in the forms

$$A=(1-b_{N+1})X_{N+1},$$
$$B=\frac{b_{N+1}X_{N+1}}{1-b_{N+1}}(G_N^N-1) \qquad \cdots(3.31)$$

the one-dimensional wave-transfer function G_{N+1}^0 is expressed in terms of G_N^0 and G_N^N as follows:

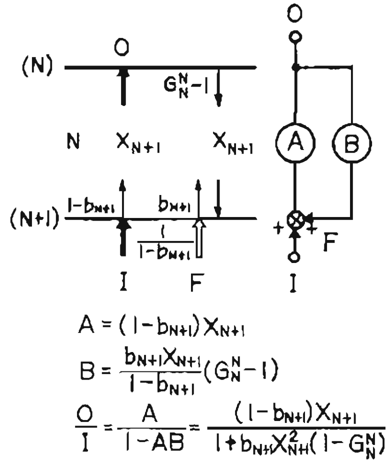

Fig. 2 Formation of a feed-back system.

$$G_{N+1}^0=\frac{G_N^0(1-b_{N+1})X_{N+1}}{[1+b_{N+1}X_{N+1}^2(1-G_N^N)]} \qquad \cdots(3.32)$$

Taking into consideration the following expressions

$$G_N^0=\frac{2\prod_{i=1}^{N}(1-b_i)\prod_{i=1}^{N}X_i}{{}_N\Delta_c}, \quad G_N^N=\frac{(1-b_N)\,{}_N\Delta'_c}{{}_N\Delta_c} \qquad \cdots(3.33)$$

and

$$\begin{aligned}{}_N\Delta_c-(1-b_N)\,{}_N\Delta_c' &= 1\prod_{i=1}^{N-1}\circ(1+\delta_{i-1}b_{i-1}\delta_ib_iX_i^2)\circ(b_N+\delta_{N-1}b_{N-1}X_N^2)\circ 1\\ &= {}_N\Delta_c\delta_Nb_N\circ 1 \end{aligned} \qquad \cdots(3.34)$$

$${}_{N+1}\Delta_c={}_N\Delta_c(1+\delta_Nb_N\delta_{N+1}b_{N+1}X_{N+1}^2)\circ 1={}_N\Delta_c(1+\delta_Nb_Nb_{N+1}X_{N+1}^2)\circ 1$$

which are obtained from eqs. (3.21)-(3.23) and (3.26)-(3.28) the one-dimensional wave-transfer function G_{N+1}^0 given by eq. (3.32) takes the form obtained by substituting $n=0$ and replacing N by $N+1$ in eq. (3.21).

Finally, in order to complete the proof of the validity of eqs. (3.21)-(3.23), it suffices to show the validity of these equations for the one-dimensional wave-transfer function G_N^{n+1} under the assumption that eqs. (3.21)-(3.23) are valid for all G_N^i's in which $i\leq n$. In general, in the case where $N\geq 2$ and $N-1\geq n\geq 0$ the following recurrence formula is obtained from eqs. (3.9)-(3.14):

$$G_N^{n+1}=-\left\{\frac{a_{n-1}^n}{a_{n+1}^n}G_N^{n-}+\frac{a_n^n}{a_{n+1}^n}G_N^n\right\} \qquad \cdots(3.35)$$

The numerators and denominators of the coefficients in the above equation which are given by eq. (3.13) for the case where $N\geq 3$ and by eqs. (3.13) and (3.15) for the case of $N=2$, respectively, are expressed in terms of b_n's and X_n's as follows:

$$a_{-1}^0=0, \quad a_0^0=1, \quad a_1^0=-\frac{2X_1}{1+X_1^2}, \quad a_0^1=0$$

$$a_1^1 = -\frac{X_1^2X_2^2 - b_1(X_1^2 + X_2^2) + 1}{(1-b_1)(1+X_1^2)(1-X_2^2)}, \quad a_2^1 = \frac{X_2}{1-X_2^2} \qquad \cdots(3.36)$$

for $n=0,\ 1$

and

$$a_{n-1}^n = \frac{(1+b_n)X_n}{(1-b_n)(1-X_n^2)}, \quad a_n^n = \frac{X_n^2X_{n+1}^2 - b_n(X_n^2 - X_{n+1}^2) - 1}{(1-b_n)(1-X_n^2)(1-X_{n+1}^2)}$$

for $N-1 \geqq n \geqq 2$

$$a_{n+1}^n = \frac{X_{n+1}}{1-X_{n+1}^2} \qquad \cdots(3.37)$$

Hence, for the case where $n=0$ and $n=1$, the one-dimensional wave-transfer functions G_N^1 and G_N^2 are obtained respectively in the following forms by using the first equation of (3. 33) and by substituting eq. (3. 36) in eq. (3. 35) :

$$G_N^1 = -\frac{a_0^0}{a_1^0}G_N^0 = \frac{1+X_1^2}{2X_1} \frac{2\prod_{i=1}^{N}(1-b_i)\prod_{i=1}^{N}X_i}{{}_N\Delta_c} = \frac{\prod_{i=1}^{N}(1-b_i)\prod_{i=2}^{N}X_i(1+X_1^2)}{{}_N\Delta_c} \qquad \cdots(3.38)$$

for $n=0$

$$G_N^2 = -\frac{a_1^1}{a_2^1}G_N^1 = \frac{X_1^2X_2^2 - b_1X_1^2 - b_1X_2^2 + 1}{(1-b_1)(1+X_1^2)X_2} \frac{\prod_{i=1}^{N}(1-b_i)\prod_{i=2}^{N}X_i(1+X_1^2)}{{}_N\Delta_c}$$

$$= \frac{\prod_{i=2}^{N}(1-b_i)\prod_{i=3}^{N}X_i(1-b_1X_1^2 - b_1X_2^2 + X_1^2X_2^2)}{{}_N\Delta_c} \qquad \cdots(3.39)$$

for $n=1$

And for the case where $N-1 \geqq n \geqq 2$, by making use of eqs. (3. 21), (3. 22), (3. 35) and (3. 37) the one-dimensional wave-transfer function G_N^{n+1} is expressed by

$$G_N^{n+1} = -\frac{(1+b_n)X_n(1-X_{n+1}^2)}{(1-b_n)X_{n+1}(1-X_n^2)} \frac{\prod_{i=n-1}^{N}(1-b_i)\prod_{i=n}^{N}X_i\ {}_{n-1}\Delta_c'}{{}_N\Delta_c}$$

$$-\frac{X_n^2X_{n+1}^2 - b_n(X_n^2 - X_{n+1}^2) - 1}{(1-b_n)(1-X_n^2)X_{n+1}} \frac{\prod_{i=n}^{N}(1-b_i)\prod_{i=n+1}^{N}X_i\ {}_n\Delta_c'}{{}_N\Delta_c}$$

$$= \frac{\prod_{i=n+1}^{N}(1-b_i)\prod_{i=n+2}^{N}X_i\prod_{i=1}^{n-2}\circ(1+\delta_{i-1}b_{i-1}\delta_ib_iX_i^2)}{{}_N\Delta_c} \qquad \cdots(3.40)$$

$$\circ[1 + b_{n-1}b_nX_n^2 - b_nX_{n+1}^2 - b_{n-1}X_n^2X_{n+1}^2$$
$$+ \delta_{n-2}b_{n-2}X_{n-1}^2(b_{n-1} + b_nX_n^2 - b_{n-1}b_nX_{n+1}^2 - X_n^2X_{n+1}^2)]$$

By taking into consideration eqs. (3. 22) and (3. 26), it is found that all the expressions in eqs. (3. 38), (3. 39) and (3. 40) are reducible to the form given by eq. (3. 21).

Hence from the above discussions, the validity of the successive product forms of the one-dimensional wave-transfer functions G_N^n's which are given by eqs. (3. 21)-(3. 28) is verified for the general cases where $N \geqq 1$ and $N \geqq n \geqq 0$. By using these formal expressions, for instance, the one-dimensional wave-transfer functions for the cases of $N=1, 2, 3$ are determined as follows:

$$G_1^0=\frac{2(1-b_1)X_{1\,0}\Delta_c'}{{}_1\Delta_c},\quad G_1^1=\frac{(1-b_1)_1\Delta_c'}{{}_1\Delta_c} \qquad \cdots(3.41)$$

$$G_2^0=\frac{2(1-b_1)(1-b_2)X_1X_{2\,0}\Delta_c'}{{}_2\Delta_c},\quad G_2^1=\frac{(1-b_1)(1-b_2)X_{2\,1}\Delta_c'}{{}_2\Delta_c}$$

$$G_2^2=\frac{(1-b_2)_2\Delta_c'}{{}_2\Delta_c} \qquad \cdots(3.42)$$

$$G_3^0=\frac{2(1-b_1)(1-b_2)(1-b_3)X_1X_2X_{3\,0}\Delta_c'}{{}_3\Delta_c},\quad G_3^1=\frac{(1-b_1)(1-b_2)(1-b_3)X_2X_{3\,1}\Delta_c'}{{}_3\Delta_c}$$

$$G_3^2=\frac{(1-b_2)(1-b_3)X_{3\,2}\Delta_c'}{{}_3\Delta_c},\quad G_3^3=\frac{(1-b_3)_3\Delta_c'}{{}_3\Delta_c} \qquad \cdots(3.43)$$

in which

$$ {}_1\Delta_c=1-b_1X_1^2,\quad {}_2\Delta_c=1-b_1X_1^2+b_1b_2X_2^2-b_2X_1^2X_2^2$$
$$ {}_3\Delta_c=1-b_1X_1^2+b_1b_2X_2^2+b_2b_3X_3^2-b_2X_1^2X_2^2+b_1b_3X_2^2X_3^2$$
$$ -b_1b_2b_3X_1^2X_3^2-b_3X_1^2X_2^2X_3^2 \qquad \cdots(3.44)$$

and

$$ {}_0\Delta_c'=1,\quad {}_1\Delta_c'=1+X_1^2,\quad {}_2\Delta_c'=1-b_1(X_1^2+X_2^2)+X_1^2X_2^2$$
$$ {}_3\Delta_c'=1-b_1X_1^2+b_1b_2X_2^2-b_2X_3^2-b_2X_1^2X_2^2-b_1X_2^2X_3^2$$
$$ +b_1b_2X_1^2X_3^2+X_1^2X_2^2X_3^2 \qquad \cdots(3.45)$$

By making use of eqs. (3.21) and (3.28) the transfer function associated with the displacement of the nth boundary to that of the Nth boundary which is the bottom boundary of the N-layered media is expressed as

$$G'^n_N=\frac{G_N^n}{G_N^N}=\frac{\prod_{i=n}^{N-1}(1-b_i)\prod_{i=n+1}^{N}X_{i\,n}\Delta_c'}{{}_N\Delta_c'},\quad n=0, 1,\cdots\cdots, N \qquad \cdots(3.46)$$

It is noted that the transfer functions defined by the above equation do not depend on the property of the linear visco-elastic half-space in spite of the use of the wave-transfer functions associated with the linear visco-elastic multi-layered half-space. In general, the ratio of the wave-transfer function of the nth boundary to that of the mth boundary is given by

$$G'^n_m=\frac{G_N^n}{G_N^m}=\frac{\prod_{i=n}^{m-1}(1-b_i)\prod_{i=n+1}^{m}X_{i\,n}\Delta_c'}{{}_m\Delta_c'}\quad \text{for } m\geqq n\geqq 0 \qquad \cdots(3.47)$$

$$=\frac{{}_n\Delta_c'}{\prod_{i=m}^{n-1}(1-b_i)\prod_{i=m+1}^{n}X_{i\,m}\Delta_c'}\quad \text{for } n\geqq m\geqq 0 \qquad \cdots(3.48)$$

which is also obtained as the ratio between G'^n_N and G'^m_N defined by eq. (3.46). As found from eq. (3.46) the quantities given by eqs. (3.47) and (3.48) are the transfer function associated with the m-layered media excited at the bottom boundary and the inverse of such a transfer function associated with the n-layered media, respectively, neither of which depends on the properties of the media having numbers greater than the maximum of n and m.

This is, of course, a property peculiar to the one-dimensional wave propagation in the linear visco-elastic media having the free surface.

Although it is difficult to find precisely the physical meaning of the symbolic operators appearing in eqs. (3. 22) and (3. 23), a meaning of such operators may be interpreted by eliminating ${}_N\Delta_c'$ from the second equation of (3. 33) and the first equation of (3. 34) as follows:

$$ {}_N\Delta_c\delta_N\circ 1 = {}_N\Delta_c\left(\frac{G_N^N-1}{-b_N}\right) \qquad \cdots(3.49)$$

The above quantity in parentheses means the ratio of the Fourier transform of the outgoing waves from the Nth boundary in the $(N+1)$th medium to that of the reflected waves at the Nth boundary to the $(N+1)$th medium. Similarly, the symbolic quantities $-\delta_N b_N\circ 1$ and $b_N(1-\delta_N)\circ 1$, which are multiplied by ${}_N\Delta_c$, mean the wave-transfer function of the outgoing waves from the Nth boundary in the $(N+1)$th medium and that of the refracted waves at the Nth boundary to the $(N+1)$th medium subjected to the incident waves at the Nth boundary through the $(N+1)$th medium, respectively. In the above explanation, the Nth boundary does not necessarily mean the bottom boundary of the multi-layered half-space, but is considered as an arbitrary boundary. Hence, in eq. (3. 49) N may be replaced by an arbitrary integer n.

Comparing the relation obtained by substituting eq. (3. 49) in the first equation of (3. 34)

$$ {}_N\Delta_c-(1-b_N)\,{}_N\Delta_c' = {}_{N-1}\Delta_c[b_N+X_N^2(1-G_{N-1}^{N-1})] \qquad \cdots(3.50)$$

with the equivalent expression determined by using eqs. (3. 23) and the first equation of (3. 34) as

$$ {}_N\Delta_c\delta_N b_N\circ 1 = {}_{N-1}\Delta_c[\circ\delta_N b_N\circ 1+\delta_{N-1}b_{N-1}\delta_N b_N\circ\delta_N b_N X_N^2\circ 1] \qquad \cdots(3.51)$$

it is found that the following relations associated with the $\circ$ marked multiplication are to be valid:

$$\delta_N\circ\delta_N = b_N^{-2}, \quad 1\circ\delta_N\circ 1 = 1 \qquad \cdots(3.52)$$

and

$$1\circ\delta_i\delta_{N-1}\circ\delta_N\circ 1 = 1 \qquad \text{for } i\leqq N-2$$

or

$$\delta_{N-1}\circ\delta_N = 1 \qquad \cdots(3.53)$$

The operational rules shown in eq. (3. 52) are the same as the conventions associated with the $\circ$ marked multiplication of the symbolic operator δ_i, which are given by the third and the fourth equation in (3. 26), whereas the operational rule shown in eq. (3. 53) corresponds to the set of operational conventions related to the $\circ$ marked and ordinary multiplications between the symbolic operators δ_i's and the scalar quantities which are given by the first, second and fifth equations in (3. 26).

In the following we will compare the one-dimensional wave-transfer functions of the multi-layered half-space subjected to the incident waves propagated through the half-space and the transfer functions of the multi-layered media excited at the bottom boundary, which are given by eq. (3. 21) and eq. (3. 46), respectively, with the corresponding transfer functions which are expressed in the following successive product matrices forms based on the techniques used by I. Herrera and E. Rosenblueth :[21]

$$G_N^0=\frac{2}{\{1\ \ -j\}T_N T_{N-1}\cdots\cdots T_2 T_1\{{}^1_0\}} \qquad \cdots(3.54)$$

$$G_N^n=\frac{2\{1\ 0\}T_n T_{n-1}\cdots\cdots T_2 T_1\{{}^1_0\}}{\{1\ \ -j\}T_N T_{N-1}\cdots\cdots T_2 T_1\{{}^1_0\}}, \quad n=1,2,\cdots\cdots,N \qquad \cdots(3.55)$$

and

$$G'^0_N=\frac{1}{\{1\ 0\}T_N T_{N-1}\cdots\cdots T_2 T_1\{{}^1_0\}} \qquad \cdots(3.56)$$

$$G'^n_N=\frac{\{1\ 0\}T_n T_{n-1}\cdots\cdots T_2 T_1\{{}^1_0\}}{\{1\ 0\}T_N T_{N-1}\cdots\cdots T_2 T_1\{{}^1_0\}}, \quad n=1,2,\cdots\cdots,N \qquad \cdots(3.57)$$

in which

$$T_n=\begin{pmatrix} \cos\kappa_n H_n & \sin\kappa_n H_n \\ -\alpha_n\sin\kappa_n H_n & \alpha_n\cos\kappa_n H_n \end{pmatrix} \qquad \cdots(3.58)$$

In the above matrix concerned with the nth layer, κ_n and α_n are defined by eqs. (2. 40) and (2. 53), respectively, and H_n denotes the thickness of the nth layer.

Taking into consideration $b_0=-1$ and ${}_0\Delta_c'=1$, and substituting $n=0$ in eqs. (3. 21) and (3. 46) we obtain

$$G_N^0=\frac{2\prod_{i=1}^{N}(1-b_i)X_i}{{}_N\Delta_c} \qquad \cdots(3.59)$$

$$G'^0_N=\frac{2\prod_{i=1}^{N}(1-b_i)X_i}{(1-b_N)\,{}_N\Delta'_c} \qquad \cdots(3.60)$$

Comparing each of the above equations with eq. (3. 54) and eq. (3. 56), respectively, the following relations are obtained :

$${}_N\Delta_c=\prod_{i=1}^{N}(1-b_i)X_i\{1\ \ -j\}T_N T_{N-1}\cdots\cdots T_2 T_1\{{}^1_0\} \qquad \cdots(3.61)$$

$${}_N\Delta_c'=\frac{2}{1-b_N}\prod_{i=1}^{N}(1-b_i)X_i\{1\ 0\}T_N T_{N-1}\cdots\cdots T_2 T_1\{{}^1_0\} \qquad \cdots(3.62)$$

and by using the above equations, the following equations are obtained corresponding to the first equation of (3. 34) and eq. (3. 49).

$$\delta_N b_N\circ 1=\frac{{}_N\Delta_c-(1-b_N)\,{}_N\Delta'_c}{{}_N\Delta_c}=-\frac{\{1\ j\}T_N T_{N-1}\cdots\cdots T_2 T_1\{{}^1_0\}}{\{1\ \ -j\}T_N T_{N-1}\cdots\cdots T_2 T_1\{{}^1_0\}}=1-G_N^N \qquad \cdots(3.63)$$

By comparing the transfer functions expressed in the successive product forms including the symbolic operators δ_i's which are given by eqs. (3. 21)-(3. 28) and (3. 46) and those expressed in the successive product forms in terms of matrices as in eqs. (3. 54)-(3. 58), it is found that the former expressions are simpler in carrying out the numerical calculations than the latter.

Finally, it is shown that the one-dimensional wave-transfer function of the

λ th and μ th derivative, with respect to the time and spatial co-ordinate, respectively, of the displacement of an arbitraray point in the linear visco-elastic N-layered half-space subjected to the incident displacement waves at the bottom boundary can be expressed in terms of the one-dimensional wave-transfer functions of the boundaries G_N^n's.

By making use of eqs. (2. 47) and (2. 48) concerned with the nth medium or the wave-transfer function associated with the wave propagation in the nth medium given by eq. (2. 49) we obtain

$$\tilde{u}_n(j\omega, z_n) = {}_+\tilde{u}_n(j\omega, 0)\exp(-j\kappa_n z_n) + {}_-\tilde{u}_n(j\omega, H_n)\exp(-j(z_n - H_n)) \quad \cdots(3.64)$$

Defining the one-dimensional wave-transfer function as

$$G_N{}^{n\lambda\mu}(j\omega, z_n) = \frac{\tilde{u}_{n\tau}{}^{(\lambda)}{}_{z_n}{}^{(\mu)}(j\omega, z_n)}{\tilde{w}(j\omega)}, \qquad \lambda+\mu=0,\ 1,\ 2 \quad \cdots(3.65)$$

the following expressions are obtained by using eqs. (3. 3)-(3. 8) and (3. 24).

$$G_N{}^{1\lambda\mu}(j\omega, z_1) = \frac{(j)^{\lambda+\mu}\omega^\lambda \kappa_1{}^\mu X_1}{1+X_1{}^2}[\exp(j\kappa_1 z_1) + (-1)^\mu \exp(-j\kappa_1 z_1)]G_N^1$$
$$= \frac{(j)^{\lambda+\mu}\omega^\lambda \kappa_1{}^\mu}{2}[\exp(j\kappa_1 z_1) + (-1)^\mu \exp(-j\kappa_1 z_1)]G_N^0 \quad \cdots(3.66)$$

$$G_N{}^{n\lambda\mu}(j\omega, z_n) = -\frac{(j)^{\lambda+\mu}\omega^\lambda \kappa_n{}^\mu}{1-X_n{}^2}[\{X_n{}^2\exp(j\kappa_n z_n) - (-1)^\mu \exp(-j\kappa_n z_n)\}G_N^{n-1}$$
$$- X_n\{\exp(j\kappa_n z_n) - (-1)^\mu \exp(-j\kappa_n z_n)\}G_N{}^n], \qquad n=2,\ 3, \cdots\cdots, N \quad \cdots(3.67)$$

$$G_N{}^{N+1\,\lambda\mu}(j\omega, z_{N+1}) = (j)^{\lambda+\mu}\omega^\lambda \kappa_{N+1}{}^\mu[(-1)^\mu \exp(-j\kappa_{N+1} z_{N+1})G_N^N$$
$$+ \exp(j\kappa_{N+1} z_{N+1}) - (-1)^\mu \exp(-j\kappa_{N+1} z_{N+1})] \quad \cdots(3.68)$$

In the successive product form in terms of matrices, the above one-dimensional wave-transfer functions are written as follows:

$$G_N{}^{1\lambda\mu}(j\omega, z_1) = (j)^\lambda \omega^\lambda \kappa_1{}^\mu \frac{2\{\cos\kappa_1 z_1\ \sin\kappa_1 z_1\}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^\mu \begin{Bmatrix}1\\0\end{Bmatrix}}{\{1\ \ -j\}T_N T_{N-1}\cdots\cdots T_2 T_1 \begin{Bmatrix}1\\0\end{Bmatrix}} \quad \cdots(36.9)$$

$$G_N{}^{n\lambda\mu}(j\omega, z_n) = (j)^\lambda \omega^\lambda \kappa_n{}^\mu \frac{2\{\cos\kappa_n z_n\ \sin\kappa_n z_n\}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^\mu T_{n-1}\cdots T_1 \begin{Bmatrix}1\\0\end{Bmatrix}}{\{1\ \ -j\}T_N T_{N-1}\cdots\cdots T_2 T_1 \begin{Bmatrix}1\\0\end{Bmatrix}}$$
$$n=2,\ 3, \cdots\cdots, N,\ N+1 \quad \cdots(3.70)$$

For the case where $\lambda=\mu=0$, for example, the one-dimensional wave-transfer functions given by eq. (3. 67) or eq. (3. 70) for $n=2, 3, \cdots\cdots, N$ are also expressed in the following form:

$$G_N{}^n(j\omega, z_n) = G_N^{n-1}\cos\kappa_n z_n + (\mathrm{cosec}\,\kappa_n H_n G_N^n - \cot\kappa_n H_n G_N^{n-1})\sin\kappa_n z_n, \qquad n=2,\ 3, \cdots, N \quad \cdots(3.71)$$

4. Singular points of the one-dimensional wave-transfer functions and their properties in the complex plane

In order to evaluate the impulsive responses of the linear visco-elastic multi-layered half-space and the co-variances or the spectral densities of such a dynamic system subjected to stationary or non-stationary random excitations including the so-called quasi-stationary random excitations, not to mention the general characteristics of the one-dimensional wave-transfer functions, it is useful to study the properties of the singular points of the transfer functions and their behaviour at these points and infinity in the complex plane.

As found from eqs. (3.21)-(3.28) the singular points of the one-dimensional wave-transfer functions $G_N^{n'}$'s consist of the poles which are zeros of the denominator ${}_N\Delta_c$ of eq. (3.21) and the branch points appearing in the transfer functions b_n's associated with the reflected waves to incident waves, which are given by eq. (2.56) together with eqs. (2.53) and (2.54), and the complex wave numbers κ_n's given by eqs. (2.40) and (2.41). Of course, in the case of the perfectly elastic media, both b_n and κ_n contain no branch points and all b_n's reduce to real-valued constants and all κ_n's are real numbers for a real frequency parameter ω. Hence, in this case, the singular points of the one-dimensional wave-transfer functions are only poles. If the differential operators $p_n(j\omega)$, $n=1, 2, \cdots, N, N+1$, which represent the visco-elastic properties of the media, are of the rational function type, the branch points arising from κ_n and b_n consist of the poles and zeros of the operator $p_n(j\omega)$. Hence, in general, the branch points of the wave-transfer functions are the complex numbers. However, for the usually encountered stable visco-elastic operators, the branch points may have positive immaginary parts.

As regards the poles of the wave-transfer functions, namely, the zeros of the denominator of eq. (3.21), ${}_N\Delta_c$, which are also generally the complex numbers having positive immaginary parts for the stable visco-elastic operators, the so-called eigen-value problems associated with the relevant dynamic systems may be closely related to these singular points.

The eigen-value problem associated with the linear visco-elastic multi-layered half-space is prescribed by the homogeneous equation of (3.9) together with eqs. (3.10) and (3.11). The characteristic equation is given by

$$\Delta_N=\det[A]=0 \qquad \text{for } N\geqq 2 \qquad \cdots(4.1)$$

On the other hand, the following relation exists between Δ_N and ${}_N\Delta_c$ given by eq. (3.23):

$$\Delta_N=\frac{(-1)^{N+1}{}_N\Delta_c}{\prod_{i=1}^{N-1}(1-b_i)(1+X_1^2)\prod_{i=2}^{N-1}(1-X_i^2)(1+b_NX_N^2)} \qquad \text{for } N\geqq 2 \qquad \cdots(4.2)$$

Since there is no common factor between the numerator and denominator of the above equation and since the denominator is finite in the complex plane, at least except infinity, because of eqs. (3.24), (2.40), (2.53) and (2.56), it is found that eq. (4.1) is equivalent to

$${}_N\Delta_c=0 \qquad \cdots(4.3)$$

where the explicit form of ${}_N\Delta_c$ is given by eq. (3.23) or eq. (3.61).

In the following, at first, it is shown that the zeros of ${}_N\Delta_c$ do not exist on the real axis. Taking into consideration eq. (2.54) the following inequality is obtained from eq. (2.56):

$$|b_n|=\sqrt{\frac{|\alpha_n|^2-2R(\alpha_n)+1}{|\alpha_n|^2+2R(\alpha_n)+1}}<1 \qquad \text{for } I(\omega)=0 \qquad \cdots(4.4)$$

On the other hand, from the second inequality in eq. (2. 41) and eq. (3. 24), the following inequality is obtained:

$$|X_n^c| = |X_n|^c \leqq 1 \qquad \text{for } I(\omega) = 0 \qquad \cdots(4.5)$$

where c is an arbitrary positive real number. Then, by making use of the method of mathematical induction it can be verified that eq. (4. 3) has no real zeros. At first, for the case where $N=1$, the characteristic equation given by eq. (3. 30) has no zeros on the real axis, because if such zeros exist, the equation

$$|b_1||X_1^2| = 1 \qquad \cdots(4.6)$$

should be valid, which contradicts eqs. (4. 4) and (4. 5).

Next, to prove the above-mentioned statement, it suffices to show that ${}_{N+1}\Delta_c$ has no zeros on the real axis under the assumption of its validity for ${}_N\Delta_c$. Based on the identity

$${}_{N+1}\Delta_c = {}_N\Delta_c[1 + b_{N+1} X_{N+1}^2 (1 - G_N^N)] \qquad \cdots(4.7)$$

which is obtained by using eqs. (3. 23), (3. 26) and (3. 49), it can be shown that if ${}_N\Delta_c$ has no real zeros and if ${}_{N+1}\Delta_c$ has such zeros, the following equation should be valid:

$$|b_{N+1}||X_{N+1}^2||1 - G_N^N| = 1 \qquad \text{for } I(\omega) = 0 \qquad \cdots(4.8)$$

Considering eqs. (4. 4) and (4. 5) the above equation requires

$$|1 - G_N^N| > 1 \qquad \text{for } I(\omega) = 0 \qquad \cdots(4.9)$$

In terms of the Fourier transform of the incident waves, ${}_{-}\tilde{u}_{N+1}(j\omega, 0)$, at the Nth boundary through the half-space and that of the divergent waves, ${}_{+}\tilde{u}_{N+1}(j\omega, 0)$, from the Nth boundary to the half-space, eq. (4. 9) is rewritten as

$$\left| \frac{{}_{+}\tilde{u}_{N+1}(j\omega, 0)}{{}_{-}\tilde{u}_{N+1}(j\omega, 0)} \right| > 1 \qquad \text{for } I(\omega) = 0 \qquad \cdots(4.10)$$

Considering the energy fluces of the incident and divergent waves at the Nth boundary, the above inequality means

$$\frac{1}{2} \frac{\rho_{N+1}\omega^3}{R(\kappa_{N+1})} |{}_{-}\tilde{u}_{N+1}(j\omega, 0)|^2 < \frac{1}{2} \frac{\rho_{N+1}\omega^3}{R(\kappa_{N+1})} |{}_{+}\tilde{u}_{N+1}(j\omega, 0)|^2 \qquad \text{for } I(\omega) = 0 \quad \cdots(4.11)$$

This is a contradiction compared with the physical fact that whether the dynamic system is perfectly elastic or visco-elastic the energy flux of incident waves can not be smaller than that of divergent waves for a stationary state. Hence, instead of eq. (4. 9) the following inequality is valid on the real axis:

$$|1 - G_N^N| \leqq 1 \qquad \text{for } I(\omega) = 0 \qquad \cdots(4.12)$$

Therefore, all the zeros of ${}_N\Delta_c$, namely, all the eigen-values associated with the multi-layered half-space should have non-zero immaginary parts for an arbitrary number of layers N.

It is noted that for the continuous dynamic system having discontinuous boundary surfaces as considered here, there exists a denumerably infinite number of complex eigen-values $\{_\nu\omega\}$ and the corresponding complex eigen-functions $\{_\nu\varphi_N(z)\}$ in which z is the common spatial co-ordinate to the multi-layered half-space measured downward from the free surface, namely

$$z=\sum_{i=1}^{n-1}H_i+z_n, \quad n=1, 2, \cdots\cdots, N, \ N+1 \qquad \cdots(4.13)$$

where the summation symbol $\sum_{i=p}^{q}$ having the smaller upper limit q than the lower limit p is assumed to be zero.

$$\sum_{i=p}^{q}C_i\equiv 0 \qquad \text{for } q<p \qquad \cdots(4.14)$$

Taking into account eqs. (3.66), (3.67) and (3.68) in which the terms concerning the incident waves are neglected, the eigen-function $_\nu\varphi_N(z)$ corresponding to the eigen-value $_\nu\omega$ is expressed in terms of the complex eigen-vector $\{_\nu\varphi_N{}^n\}$ which is determined as a non-trivial vector solution of the homogeneous equation related to eq. (3.9) which has the singular coefficient matrix $[A]$ associated with the eigen-value $_\nu\omega$[14),22)-25)]. For instance, in the case where the multiplicity of the eigen-value $_\nu\omega$ is one, the complex eigen-vector $\{_\nu\varphi_N{}^n\}$ is expressed in the following form:

$$\begin{aligned} &\{_\nu\varphi_N{}^n\}=\{{}^{(1)}\Delta_N{}_{N+1}^{n+1}(j_\nu\omega)\}, \qquad n=0, 1, \cdots, N \\ &{}^{(1)}\Delta_N{}_{N+1}^{N+1}(j_\nu\omega)\neq 0 \end{aligned} \qquad \cdots(4.15)$$

in which ${}^{(1)}\Delta_N{}_{N+1}^{n+1}(j_\nu\omega)$ denotes the co-factor with respect to the $(N+1)$th row and the $(n+1)$th column of the matrix $[A]$ substituted by the eigen-value $_\nu\omega$. Then, the eigen-function $_\nu\varphi_N(z)$ associated with the displacement of the linear visco-elastic N-layered half-space is obtained by substituting $\lambda=\mu=0$, $\omega={}_\nu\omega$ and eq. (4.13) in eqs. (3.66), (3.67) and (3.68), neglecting the terms due to the incident waves and replacing $\{G_N^n\}$ by $\{_\nu\varphi_N{}^n\}$ as follows:

$$\begin{aligned} _\nu\varphi_N(z) &= \frac{_\nu X_1}{_\nu X_1{}^2+1}[\exp(j_\nu\kappa_1 z)+\exp(-j_\nu\kappa_1 z)]_\nu\varphi_N{}^1 \\ &= \frac{1}{2}[\exp(j_\nu\kappa_1 z)+\exp(-j_\nu\kappa_1 z)]_\nu\varphi_N{}^0 \end{aligned}$$

$$\text{for } H_1\geqq z\geqq 0 \qquad \cdots(4.16)$$

$$\begin{aligned} _\nu\varphi_N(z)=\frac{1}{_\nu X_n{}^2-1}[&\{_\nu X_n{}^2\exp(-j_\nu\kappa_n\sum_{i=1}^{n-1}H_i)\exp(j_\nu\kappa_n z) \\ &-\exp(j_\nu\kappa_n\sum_{i=1}^{n-1}H_i)\exp(-j_\nu\kappa_n z)\}_\nu\varphi_N{}^{n-1} \\ &-{}_\nu X_n\{\exp(-j_\nu\kappa_n\sum_{i=1}^{n-1}H_i)\exp(j_\nu\kappa_n z) \\ &-\exp(j_\nu\kappa_n\sum_{i=1}^{n-1}H_i)\exp(-j_\nu\kappa_n z)\}_\nu\varphi_N{}^n] \end{aligned}$$

$$\text{for } n=2, 3, \cdots\cdots, N, \text{ namely, } \sum_{i=1}^{N}H_i\geqq z\geqq H_1 \qquad \cdots(4.17)$$

$$_\nu\varphi_N(z)=\exp(j_\nu\kappa_{N+1}\sum_{i=1}^{N}H_i)\exp(-j_\nu\kappa_{N+1}z)_\nu\varphi_N{}^N \qquad \text{for } z\geqq\sum_{i=1}^{N}H_i \qquad \cdots(4.18)$$

in which ${}_\nu\kappa_n$ and ${}_\nu X_n$ denote the values of eqs. (2.40) and (3.24) when substituting by $\omega={}_\nu\omega$, respectively, namely,

$$ {}_\nu\kappa_n={}_\nu\omega\sqrt{\frac{\rho_n}{p_n(j_\nu\omega)}},\quad {}_\nu X_n=\exp(-j_\nu\kappa_n H_n) \qquad\cdots(4.19)$$

It is noted that for the stable multi-layered half-space the eigen-values are complex numbers with positive immaginary parts and that the system of eigen-vectors, or that of eigen-functions, may be of a type of complex generalized orthogonal system even in the case of perfectly elastic media, not to mention visco-elastic media, because of the presence of the diffusive energy to the half-space in addition to the energy dissipation due to viscosity.[14),22)-25)] For instance, in the case where the multiplicity is one for all the eigen-values, the following generalized ortho-normal condition may be valid for the system of the complex eigen-vectors :[14),24),25)]

$$[\{{}_\lambda\psi_{Nn}\}_N,\ \{{}_\mu\varphi_N{}^n\}_N]=\{{}_\lambda\psi_{Nn}\}_N\frac{[A(j_\lambda\omega)]-[A(j_\mu\omega)]}{(j_\lambda\omega)^2-(j_\mu\omega)^2}\{{}_\mu\varphi_N{}^n\}_N=\delta_\mu{}^\lambda \qquad\cdots(4.20)$$

In particular, for $\lambda=\mu=\nu$, the above equation means

$$[\{{}_\nu\psi_{Nn}\}_N,\ \{{}_\nu\varphi_N{}^n\}_N]=\{{}_\nu\psi_{Nn}\}_N\frac{[A_{j\omega}{}^{(1)}(j_\nu\omega)]}{2j_\nu\omega}\{{}_\nu\varphi_N{}^n\}_N=1 \qquad\cdots(4.21)$$

In eqs. (4.20) and (4.21), the symbol $\delta_\mu{}^\lambda$ denotes Kronecker's delta, the subscript N outside parentheses indicates a normalized eigen-vector, $[A_{j\omega}{}^{(p)}(j_\nu\omega)]$ represents the value of the pth derivative of the matrix $[A]$, which is defined by eq. (3.10), with respect to $j\omega$ when substituting by $\omega={}_\nu\omega$, and the row vector $\{{}_\nu\psi_{Nn}\}$ denotes the adjoint eigen-vector of the column eigen-vector $\{{}_\nu\varphi_N{}^n\}$, which are defined as a set of the non-trivial vector solutions of the following homogeneous equations :

$$[A(j_\nu\omega)]\{{}_\nu\varphi_N{}^n\}=\{0\},\ \{{}_\nu\psi_{Nn}\}[A(j_\nu\omega)]=\{0\},\qquad n=0,1,\cdots\cdots,N \qquad\cdots(4.22)$$

Under the assumption ${}^{(1)}\Delta_N{}^{N+1}_{N+1}(j_\nu\omega)\neq0$, a solution of the first equation of (4.22) is given by eq. (4.15) and a solution of the second equation is expressed as

$$\{{}_\nu\psi_{Nn}\}=\{{}^{(1)}\Delta_N{}^{N+1}_{n+1}(j_\nu\omega)\},\ {}^{(1)}\Delta_N{}^{N+1}_{N+1}(j_\nu\omega)\neq0,\quad n=0,1,\cdots\cdots,N \qquad\cdots(4.23)$$

By using the set of complex eigen-vectors which are given by eq. (4.15) and (4.23), respectively, the operator defined by the first and second terms in eq. (4.21) takes the form

$$\begin{aligned}[\{{}_\nu\psi_{Nn}\},\ \{{}_\nu\varphi_N{}^n\}]&=[\{{}^{(1)}\Delta_N{}^{N+1}_{n+1}(j_\nu\omega)\},\ \{{}^{(1)}\Delta_N{}^{n+1}_{N+1}(j_\nu\omega)\}]\\&=\frac{{}^{(1)}\Delta_N{}^{N+1}_{N+1}(j_\nu\omega)\Delta_{N\,j\omega}{}^{(1)}(j_\nu\omega)}{2j_\nu\omega}\neq0\end{aligned} \qquad\cdots(4.24)$$

Then, the set of the two complex normalized eigen-vectors, $\{{}_\nu\varphi_N{}^n\}_N$ and $\{{}_\nu\psi_{Nn}\}_N$, associated with the simple eigen-value ${}_\nu\omega$ can be expressed as follows :

$$\{{}_\nu\varphi_N{}^n\}_N=\frac{\{{}_\nu\varphi_N{}^n\}}{[\{{}_\nu\psi_{Nn}\},\ \{{}_\nu\varphi_N{}^n\}]^{1/2}}=\frac{\{{}^{(1)}\Delta_N{}^{n+1}_{N+1}(j_\nu\omega)\}}{[\{{}^{(1)}\Delta_N{}^{N+1}_{n+1}(j_\nu\omega)\},\ \{{}^{(1)}\Delta_N{}^{n+1}_{N+1}(j_\nu\omega)\}]^{1/2}} \qquad\cdots(4.25)$$

$$\{{}_\nu\psi_{Nn}\}_N=\frac{\{{}_\nu\psi_{Nn}\}}{[\{{}_\nu\psi_{Nn}\},\ \{{}_\nu\varphi_N{}^n\}]^{1/2}}=\frac{\{{}^{(1)}\Delta_N{}^{N+1}_{n+1}(j_\nu\omega)\}}{[\{{}^{(1)}\Delta_N{}^{N+1}_{n+1}(j_\nu\omega)\},\ \{{}^{(1)}\Delta_N{}^{n+1}_{N+1}(j_\nu\omega)\}]^{1/2}} \qquad\cdots(4.26)$$

in which the square root means either of the two branches.

Next, we will show that for a set of visco-elastic operators $p_n(\partial/\partial\tau)$, $n=1, 2, \cdots\cdots, N, N+1$, all of which have real coefficients, there exists a pair of zeros, ${}_\nu\omega$ and ${}_{-\nu}\omega$, which are symmetric with respect to the immaginary axis. Since all the coefficients of ${}_N\Delta_c$ expressed as a function of $j\omega$ are real numbers, the following equations are valid for the eigen-value ${}_\nu\omega$.

$$ {}_N\Delta_c^*(j_\nu\omega) = {}_N\Delta_c(-j_\nu\omega^*) = 0 \qquad \cdots(4.27)$$

where the superscript $*$ denotes the complex-conjugate. The above equations show that a set of eigen-values $({}_\nu\omega, {}_{-\nu}\omega)$ exists both of which have the same absolute value $|{}_\nu\omega|$ and are expressed in the following forms:

$$\begin{aligned} {}_\nu\omega &= \boldsymbol{R}({}_\nu\omega) + j\boldsymbol{I}({}_\nu\omega) \\ {}_{-\nu}\omega &= -{}_\nu\omega^* = -\boldsymbol{R}({}_\nu\omega) + j\boldsymbol{I}({}_\nu\omega), \qquad \boldsymbol{I}({}_\nu\omega) \neq 0 \end{aligned} \qquad \cdots(4.28)$$

From eqs. (4.25), (4.26) and (4.28) it is found that the two sets of normalized eigen-vectors which correspond to the eigen-values ${}_\nu\omega$ and ${}_{-\nu}\omega$, respectively, constitute the following two pairs of complex-conjugate vectors:

$$\begin{aligned} &\{{}_\nu\varphi_N{}^n\}_N \quad \text{for } {}_\nu\omega \quad \text{and } \{{}_\nu\varphi_N{}^{n*}\}_N \quad \text{for } {}_{-\nu}\omega \\ &\{{}_\nu\psi_{Nn}\}_N \quad \text{for } {}_\nu\omega \quad \text{and } \{{}_\nu\psi_{Nn}{}^*\}_N \quad \text{for } {}_{-\nu}\omega \end{aligned} \qquad (\cdots 4.29)$$

By making use of eq. (3.16) and the expression

$$ f^N = \frac{(1-b_N)(1-X_N{}^2)}{1+b_N X_N{}^2} \qquad \cdots(4.30)$$

the impulsive response $\{\boldsymbol{g}\}$ of the multi-layered half-space to the incident waves at the bottom boundary through the half-space is expressed as

$$ \{\boldsymbol{g}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\boldsymbol{G}\}\exp(j\omega\tau)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\{{}^{(1)}\Delta_N{}_{N+1}^{n+1}\}}{{}_f\Delta_N}\exp(j\omega\tau)d\omega \qquad \cdots(4.31)$$

where

$$ {}_f\Delta_N = \frac{\Delta_N}{f^N} = \frac{(-1)^{N+1}{}_N\Delta_c}{\prod_{i=1}^{N}(1-b_i)(1+X_1{}^2)\prod_{i=2}^{N}(1-X_i{}^2)} \qquad \cdots(4.32)$$

As shown in the later part in this section, the so-called residue theorem may be applicable to evaluation of the infinite integral in eq. (4.31). Therefore, assuming that all the eigen-values are simple and taking into consideration eqs. (4.15), (4.25), (4.28) and (4.29), the impulsive response $\{\boldsymbol{g}\}$ may be expressed in terms of a set of complex eigen-vectors and that of complex eigen-values as follows:

$$\begin{aligned} \{\boldsymbol{g}\} &= j\Big(\sum_{\nu=-\infty}^{-1} + \sum_{\nu=1}^{\infty}\Big)R\{\boldsymbol{G}(j_\nu\omega)\}\exp(j_\nu\omega\tau) = -2\sum_{\nu=1}^{\infty}\boldsymbol{I}(R\{\boldsymbol{G}(j_\nu\omega)\}\exp(j_\nu\omega\tau)) \\ &= j\Big(\sum_{\nu=-\infty}^{-1} + \sum_{\nu=1}^{\infty}\Big)R\Big(\frac{\{{}^{(1)}\Delta_N{}_{N+1}^{n+1}(j_\nu\omega)\}}{{}_f\Delta_N(j_\nu\omega)}\Big)\exp(j_\nu\omega\tau) \\ &= \Big(\sum_{\nu=-\infty}^{-1} + \sum_{\nu=1}^{\infty}\Big)\frac{\{{}^{(1)}\Delta_N{}_{N+1}^{n+1}(j_\nu\omega)\}}{{}_f\Delta_N{}_{j\omega}{}^{(1)}(j_\nu\omega)}\exp(j_\nu\omega\tau) = 2\sum_{\nu=1}^{\infty}\boldsymbol{R}\Big(\frac{\{{}_\nu\varphi_N{}^n\}}{{}_f\Delta_N{}_{j\omega}{}^{(1)}(j_\nu\omega)}\exp(j_\nu\omega\tau)\Big) \end{aligned}$$

$$=2\sum_{\nu=1}^{\infty}R\left(\frac{[\{{}_\nu\psi_{Nn}\},\ \{{}_\nu\varphi_N{}^n\}]^{1/2}}{{}_r\Delta_{N\,j\omega}{}^{(1)}(j_\nu\omega)}\{{}_\nu\varphi_N{}^n\}_N\exp(j_\nu\omega\tau)\right) \quad \cdots(4.33)$$

in which $R(G(j_\nu\omega))$ denotes the residue of a function $G(j\omega)$ at $\omega={}_\nu\omega$. From eqs. (4.28) and (4.33), it is found that for the stable multi-layered half-space, all the eigen-values shold be complex numbers having positive immaginary parts.

In the case of the linear visco-elastic multi-layered media excited at the bottom boundary, the fundamental equation governing the transfer functions $G'^{n'}_N$s, $n=0, 1, \cdots\cdots, N-1$, is given by

$$[A']\{G'\}=\{F'\} \quad \cdots(4.34)$$

in which for $N\geqq 2$

$$[A']=\begin{pmatrix} a_0^0\, a_1^0\, 0\cdots\cdots\cdots & \cdots\cdots\cdots 0 \\ 0\ a_1^1\, a_2^1\, 0\cdots\cdots & \cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ 0\ \cdots\cdots\cdots\ 0\ a_{n-1}^n\, a_n^n\, a_{n+1}^n\, 0 & \cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ 0\ \cdots\cdots\cdots & \cdots\cdots\cdots 0\ a_{N-2}^{N-1}a_{N-1}^{N-1} \end{pmatrix} \quad \cdots(4.35)$$

and

$$\{G'\}=\begin{Bmatrix} G'^0_N \\ G'^1_N \\ \vdots \\ G'^{N-2}_N \\ G'^{N-1}_N \end{Bmatrix},\ G'^N_N=1 \quad \cdots(4.36), \qquad \{F'\}=\begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f'^{N-1} \end{Bmatrix},\ f'^{N-1}=-a_N^{N-1} \quad \cdots(4.37)$$

The eigen-value problem associated with the homogeneous equation of (4.34) is, of course, concerned with the one-dimensional dynamic problem of the linear visco-elastic multi-layered plate. As in the case of a multi-layered half-space, the characteristic equation given by $\det[A']=0$ is identical to the equation

$$_N\Delta_c'=0 \quad \cdots(4.38)$$

in which $_N\Delta_c'$ is contained as the denominator in eq. (3.46) giving the explicit expressions $G'^{n'}_N$s for $n=0, 1, \cdots\cdots, N$ and $N\geqq 1$, and is defined by eq. (3.22) and eqs. (3.24)-(3.28).

By making use of eqs. (3.22), (3.23) and (3.49), $_N\Delta_c'$ is written in the following form:

$$_N\Delta_c'={}_{N-1}\Delta_c[1-X_N{}^2(1-G^{N-1}_{N-1})], \qquad N\geqq 2 \quad \cdots(4.39)$$

Substituting the second equation of (3.33) in eq. (4.39) and equating the parentheses to zero, we have

$$1-X_N{}^2(1-G^{N-1}_{N-1})=\frac{(1-X_N{}^2)_{N-1}\Delta_c+(1-b_{N-1})X_N{}^2{}_{N-1}\Delta'_c}{_{N-1}\Delta_c}=\frac{_N\Delta'_c}{_{N-1}\Delta_c}=0 \quad \cdots(4.40)$$

The above equation means that the displacement at the bottom of the multi-layered media is zero, because $X_N(G^{N-1}_{N-1}-1)$ and $X_N{}^{-1}$ represent the transfer function of the arriving waves and of the emitting waves at the bottom boundary when considering the incident waves at the $(N-1)$th boundary through the Nth medium as the input. From eq. (3. 50) substituted by the second equation of (3. 33) and eq. (4. 40) the following simultaneous recurrence formulae for ${}_N\Delta_c$ and ${}_N\Delta_c'$ are obtained:

$$\begin{aligned} {}_N\Delta_c &= (1+b_N X_N^2)\,{}_{N-1}\Delta_c - b_N(1-b_{N-1})X_N^2\,{}_{N-1}\Delta'_c \\ {}_N\Delta'_c &= (1-X_N^2)\,{}_{N-1}\Delta_c + (1-b_{N-1})X_N^2\,{}_{N-1}\Delta'_c \end{aligned} \qquad \text{for } N \geqq 1 \qquad \cdots(4.41)$$

in which

$$b_0 = -1, \quad {}_0\Delta_c = {}_0\Delta_c' = 1 \qquad \cdots(4.42)$$

Although it is shown that ${}_N\Delta_c$ has no zeros on the real axis, ${}_N\Delta'_c$ may have real zeros in the case of the perfectly elastic media. In fact, for the case where $N=1$, the characteristic equation of a perfectly elastic single-layered medium

$${}_1\Delta_c' = 1 + X_1^2 = 1 + \exp\left(-j\frac{2\omega H_1}{v_1}\right) = 0, \quad v_1 = \sqrt{\frac{p_1}{\rho_1}} \qquad \cdots(4.43)$$

has the following set of a denumerably infinite number of real zeros:

$$\{{}_\nu\omega'\} = \left\{\frac{(2\nu+1)\pi v_1}{H_1}\right\}, \qquad \nu = 0, \pm 1, \pm 2, \cdots\cdots \qquad \cdots(4.44)$$

or

$$\{{}_\nu\omega', {}_{-\nu}\omega'\} = \left\{\frac{(2\nu-1)\pi v_1}{H_1}, \ -\frac{(2\nu-1)\pi v_1}{H_1}\right\}, \qquad \nu = 1, 2, \cdots\cdots \qquad \cdots(4.45)$$

In the case where internal damping due to viscosity exists, however, all the zeros of ${}_N\Delta'_c$ are complex numbers, because in eq. (4. 40) both inequalities, $|1-G_{N-1}^{N-1}|<1$ and $|X_N^2|<1$ are valid for any real ω except zero, and because zero can neither be a zero of ${}_N\Delta_c$ nor ${}_N\Delta'_c$. The latter reason is easily found from eq. (4. 4) and the following equations which are valid for $\omega=0$.

$${}_N\Delta_c(0) = \prod_{i=1}^{N}(1-b_i), \quad {}_N\Delta'_c(0) = \prod_{i=0}^{N-1}(1-b_i) = 2\prod_{i=1}^{N-1}(1-b_i) \qquad \cdots(4.46)$$

and

$$G_N^n(0) = 2, \quad G'^n_N(0) = 1 \qquad \cdots(4.47)$$

As found from eqs. (3. 16) and (3. 46), the transfer functions G'^{n}_N's of the multi-layered media subjected to the excitations at the bottom boundary are expressed in terms of the co-factors of the matrix $[A]$ given by eq. (3. 10) as follows:

$$G'^n_N = \frac{{}^{(1)}\Delta_N{}^{n+1}_{N+1}}{{}^{(1)}\Delta_N{}^{N+1}_{N+1}}, \qquad n = 0, 1, \cdots\cdots, N \qquad \cdots(4.48)$$

in which ${}^{(1)}\Delta_N{}^{N+1}_{N+1}$ is equal to the determinant of the matrix $[A']$ given by eq. (4. 35).

As in the eigen-value problem associated with eq. (3. 9), the eigen-vectors $\{{}_\nu\varphi'_N{}^n\}$ corresponding to the eigen-value ${}_\nu\omega'$, which is a zero of ${}_N\Delta'_c$, is written as

$$\{{}_\nu\varphi'{}_N{}^n\}=\{{}^{(2)}\Delta_N{}^{n+1}_{N}{}^{N+1}_{N+1}(j_\nu\omega')\},\quad n=0,\,1,\cdots\cdots,\,N-1$$
$$ {}^{(2)}\Delta_N{}^{N}_{N}{}^{N+1}_{N+1}(j_\nu\omega')\neq 0, \qquad \cdots(4.49)$$

and the eigen-function ${}_\nu\varphi'{}_N(z)$ is expressed in terms of $\{{}_\nu\varphi'{}_N{}^n\}$ as follows:

$$\begin{aligned}{}_\nu\varphi'{}_N(z)&=\frac{{}_\nu X_1'}{{}_\nu X_1'^2+1}[\exp(j_\nu\kappa_1'z)+\exp(-j_\nu\kappa_1'z)]{}_\nu\varphi'{}_N{}^1\\&=\frac{1}{2}[\exp(j_\nu\kappa_1'z)+\exp(-j_\nu\kappa_1'z)]{}_\nu\varphi'{}_N{}^0\end{aligned}$$
$$\text{for } H_1\geq z\geq 0 \qquad \cdots(4.50)$$

$$\begin{aligned}{}_\nu\varphi'{}_N(z)=\frac{1}{{}_\nu X_n'^2-1}[&\{{}_\nu X_n'^2\exp(-j_\nu\kappa_n'\sum_{i=1}^{n-1}H_i)\exp(j_\nu\kappa_n'z)\\&-\exp(j_\nu\kappa_n'\sum_{i=1}^{n-1}H_i)\exp(-j_\nu\kappa_n'z)\}{}_\nu\varphi'{}_N{}^{n-1}\\&-{}_\nu X_n'\{\exp(-j_\nu\kappa_n'\sum_{i=1}^{n-1}H_i)\exp(j_\nu\kappa_n'z)\\&-\exp(j_\nu\kappa_n'\sum_{i=1}^{n-1}H_i)\exp(-j_\nu\kappa_n'z)\}{}_\nu\varphi'{}_N{}^n]\end{aligned}$$
$$\text{for } n=2,\,3,\cdots\cdots,\,N,\text{ namely, }\sum_{i=1}^{N}H_i\geq z\geq H_1 \qquad \cdots(4.51)$$

in which

$${}_\nu\varphi'{}_N{}^N=0 \qquad \cdots(4.52)$$

and

$${}_\nu\kappa_n'={}_\nu\omega'\sqrt{\frac{\rho_n}{p_n(j_\nu\omega')}},\quad {}_\nu X_n'=\exp(-j_\nu\kappa_n'H_n) \qquad \cdots(4.53)$$

As easily found by comparing eq. (4. 15) with eq. (4. 48), the complex eigen-vector $\{{}_\nu\varphi_N{}^n\}$ normalized by the condition, ${}_\nu\varphi_N{}^N=1$ can be obtained by substituting the complex eigen-value ${}_\nu\omega$ in the expression of $G'^{n'}_N$'s given by eq. (3. 46) or eq. (4. 48). Similarly, the eigen-vector $\{{}_\nu\varphi'{}_N{}^n\}$ normalized by the condition, ${}_\nu\varphi'{}_N{}^{N-1}=1$ is obtained by substituting the eigen-value ${}_\nu\omega'$ in the following reduced transfer functions;

$$\begin{aligned}G''^{n}_{N-1}&=\frac{G'^{n}_N}{G'^{N-1}_N}=\frac{{}^{(1)}\Delta_N{}^{n+1}_{N+1}}{{}^{(1)}\Delta_N{}^{N}_{N+1}}=G'^{N}_{N-1}=\frac{{}^{(2)}\Delta_N{}^{n+1}_{N}{}^{N+1}_{N+1}}{{}^{(2)}\Delta_N{}^{N}_{N}{}^{N+1}_{N+1}}\\&=\frac{\prod_{i=n}^{N-2}(1-b_i)\prod_{i=n+1}^{N-1}X_{i\,n}\Delta'_c}{{}_{N-1}\Delta_c'},\quad n=0,\,1,\cdots\cdots,\,N-1\end{aligned} \qquad \cdots(4.54)$$

which mean the transfer functions of the $(N-1)$-layered media subjected to the excitations at the bottom boundary as shown in eq. (3. 46) or eq. (3. 47). Generally, from eqs. (3. 46) and (3. 47) the following relations are obtained:

$$G''^{n'}_{m'}=\frac{G'^{n'}_m}{G'^{m'}_m}=\frac{G^{n'}_N}{G^{m'}_N}=G'^{n'}_{m'}=\frac{{}^{(1)}\Delta_N{}^{n'+1}_{N+1}}{{}^{(1)}\Delta_N{}^{m'+1}_{N+1}}=\frac{{}^{(N-m'+1)}\Delta_N{}^{n'+1}_{m'+1}{}^{m'+2}_{m'+2}{}^{\cdots}_{\cdots}{}^{N+1}_{N+1}}{{}^{(N-m'+1)}\Delta_N{}^{m'+1}_{m'+1}{}^{m'+2}_{m'+2}{}^{\cdots}_{\cdots}{}^{N+1}_{N+1}}$$
$$N\geq m\geq m'+1\text{ and }N\geq m'\geq n' \qquad \cdots(4.55)$$

From the above equation it is found that the eigen-vector and eigen-function of the N-layered half-space or those of the m-layered media, which are normalized by the value at the m'th boundary, can be obtained by substituting the relevant eigen-value to the reduced transfer functions of the m'-layered media subjected to the excitations at the bottom boundary as far as the inequalities in eq. (4. 55) are taken into account.

In what follows, we will discuss the definite properties of the singular points of the one-dimensional wave-transfer functions of the linear visco-elastic multi-layered half-space. Since it is difficult to describe precisely the properties of the singular points for the general class of linear visco-elastic media, we will confine ourselves to the special classes which seem to be frequently encountered in practice. First, we will consider the case where the type of visco-elasticity is common to all the layers and the half-space, namely

$$p_n(j\omega) = p_{n0}q(j\omega), \qquad n=1, 2, \cdots\cdots, N, N+1 \qquad \cdots(4.56)$$

where p_{n0}'s are the real-valued distribution coefficients of visco-elasticity. In this case, the complex wave number κ_n given by eqs. (2. 40) and (2. 41), and the wave-transfer function associated with the reflected waves to the incident waves at the nth boundary through the nth medium b_n defined by eqs. (2. 53), (2. 54) and (2. 56) are respectively expressed as follows:

$$\kappa_n = \frac{\omega}{v_{n0}}\sqrt{\frac{1}{q(j\omega)}}, \quad \omega R(\kappa_n)>0 \qquad \text{for } I(\omega)=0 \qquad \cdots(4.57)$$

$$b_n = \frac{\sqrt{\rho_n p_{n0}} - \sqrt{\rho_{n+1} p_{n+10}}}{\sqrt{\rho_n p_{n0}} + \sqrt{\rho_{n+1} p_{n+10}}} = \frac{\rho_n v_{n0} - \rho_{n+1} v_{n+10}}{\rho_n v_{n0} + \rho_{n+1} v_{n+10}} \qquad \cdots(4.58)$$

in which v_{n0}'s are the distribution coefficients of wave velocity defined as

$$v_{n0} = \sqrt{p_{n0}/\rho_n} \qquad \cdots(4.59)$$

From the above equations, it is found that for this special class, κ_n's are expressed as frequency-dependent complex numbers, while b_n's become frequency-independent real numbers.

Transforming the frequency parameter ω to a new parameter λ by the equation

$$\lambda = \omega\sqrt{1/q(j\omega)} \qquad \cdots(4.60)$$

it can be shown that the problem concerned with the above-mentioned special class of linear visco-elastic media are reduced to the problem for the perfectly elastic case. Hence, if the singular points of the one-dimensional wave-transfer functions for the perfectly elastic media are known, those for the linear visco-elastic media are obtained by using the inverse transformation of eq. (4. 60) which is denoted as

$$\omega = l(\lambda) \qquad \cdots(4.61)$$

For instance, in the case of the three element model of the linear visco-elastic media as shown in Fig. 3, the operator $q(j\omega)$ in eq. (4. 56) is expressed in the

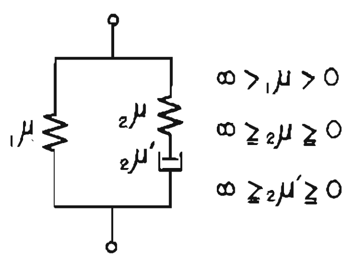

Fig. 3 Three element model of a linear visco-elastic medium.

form

$$q(j\omega)=\frac{1+jc\omega}{1+jd\omega} \qquad \cdots(4.62)$$

where

$$c=\frac{{}_2\mu'}{{}_1\mu}+\frac{{}_2\mu'}{{}_2\mu}=d\frac{{}_1\mu+{}_2\mu}{{}_1\mu}, \qquad d=\frac{{}_2\mu'}{{}_2\mu}$$

$$\infty>{}_1\mu>0,\ \infty\geqq{}_2\mu,\ {}_2\mu'\geqq 0,\ \infty\geqq c\geqq d\geqq 0 \qquad \cdots(4.63)$$

and the inverse transformation is determined as follows:

$$l(\lambda)=\frac{j}{3d}+\exp\left(j\frac{\pi}{6}\right)\sqrt[3]{-A+B}+\exp\left(-j\frac{\pi}{6}\right)\sqrt[3]{A+B} \qquad \cdots(4.64)$$

$$A=\frac{1}{2d}\left\{\frac{2}{27d^2}-\left(\frac{c}{3d}-1\right)\lambda^2\right\}$$

$$B=\lambda\sqrt{\frac{1}{27d^4}+\left(\frac{1}{4}-\frac{c}{6d}-\frac{c^2}{108d^2}\right)\frac{\lambda^2}{d^2}+\frac{c^3\lambda^4}{27d^3}} \qquad \cdots(4.65)$$

In particular, in the case where ${}_2\mu$ tends to infinity the three element model is reduced to the so-called Voigt model, and the coefficients in the operator $q(j\omega)$ given by eq. (4. 62) become

$$c=\frac{{}_2\mu'}{{}_1\mu}, \qquad d=0 \qquad \cdots(4.66)$$

In this case, by considering the higher terms with respect to d^{-1} in eq. (4. 65) it is found that eq. (4. 64) is reduced to

$$l(\lambda)\cong\frac{j}{3d}+\exp\left(j\frac{7\pi}{6}\right)\frac{1}{3d}\sqrt[3]{1-9d\left\{\frac{c\lambda^2}{2}+\lambda\sqrt{\frac{1}{3}-\frac{c^2\lambda^2}{12}}\right\}}$$

$$+\exp\left(-j\frac{\pi}{6}\right)\frac{1}{3d}\sqrt[3]{1-9d\left\{\frac{c\lambda^2}{2}-\lambda\sqrt{\frac{1}{3}-\frac{c^2\lambda^2}{12}}\right\}}$$

$$\cong\frac{jc\lambda^2}{2}+\lambda\sqrt{1-\frac{c^2\lambda^2}{4}} \qquad \cdots(4.67)$$

which is, of course, the inverse transformation of eq. (4. 60) for the Voigt model's operator $q(j\omega)=1+jc\omega$.

In the case where ${}_2\mu'=\infty$ the model consists of two parallel springs and the two coefficients in eq. (4. 62) and their ratio become

$$c=d=\infty, \quad \frac{c}{d}=\frac{{}_1\mu+{}_2\mu}{{}_1\mu} \qquad \cdots(4.68)$$

Substituting the above equations in eqs. (4. 64) and (4. 65) the following inverse transformation is obtained for the operator $q(j\omega)=c/d$.

$$l(\lambda)=\exp\left(j\frac{\pi}{6}\right)\sqrt{\frac{c}{3d}}\lambda+\exp\left(-j\frac{\pi}{6}\right)\sqrt{\frac{c}{3d}}\lambda=\sqrt{\frac{c}{d}}\lambda \qquad \cdots(4.69)$$

In this case, multiplying a constant by the distribution coefficients p_{n0}'s, the operator $q(j\omega)$ can be reduced to unity.

In the case where either of ${}_2\mu$ or ${}_2\mu'$ is zero, the model has only a spring and the operator $q(j\omega)$ becomes unity.

Henceforth, we can assume without loss of generality that the inequality

$$\infty > c > d \geqq 0 \qquad \cdots(4.70)$$

is valid for the three element model shown in Fig. 3. By substituting eq. (4.62) in eq. (4.57) and by taking into consideration eq. (4.70), the complex wave number is expressed in the following form:

$$\kappa_n = \frac{\omega}{v_{n0}}\sqrt{\frac{1+jd\omega}{1+jc\omega}}$$
$$\omega R(\kappa_n) > 0, \quad I(\kappa_n) < 0 \qquad \text{for } I(\omega)=0 \qquad \cdots(4.71)$$

Hence, in this case, the branch points of the one-dimensional wave-transfer functions G_N^n's, which are defined by eqs. (3.21)-(3.23), (3.25)-(3.28), (4.58) and (4.71), consist of a zero and a pole of the operator $q(j\omega)$ given by eq. (4.62), namely

$${}_1\omega_b = \frac{j}{c} \quad \text{and} \quad {}_2\omega_b = \frac{j}{d} \qquad \cdots(4.72)$$

each of which has a positive immaginary part.

As previously mentioned, if the singular points of the one-dimensional wave-transfer functions of the perfectly elastic multi-layered half-space, which may be composed of a denumerably infinite number of poles $\{{}_\nu\lambda, {}_{-\nu}\lambda\}$, having the positive immaginary parts, are known, the poles $\{{}_\nu\omega, {}_{-\nu}\omega\}$, which may also have the positive immaginary parts, of the linear visco-elastic multi-layered half-space can easily be determined by using eqs. (4.64) and (4.65).

Next, we will consider the properties of the one-dimensional wave-transfer functions in the neighbourhood of the branch points and at infinity for the case of multi-layered half-space which is composed of the visco-elastic media represented by the three element models as shown in Fig. 3. Here, the visco-elastic operators $p_n(j\omega)$'s are assumed to be different in each medium and given by

$$p_n(j\omega) = p_{n0}\frac{1+jc_n\omega}{1+jd_n\omega}, \qquad n=1, 2, \cdots\cdots, N, N+1 \qquad \cdots(4.73)$$
$$\infty > c_n > d_n \geqq 0$$

Hence, in this case, there exist $2(N+1)$ branch points $\{{}_1\omega_b, {}_2\omega_b\} = \{j/c_n, j/d_n\}$ and a denumerably infinite number of poles $\{{}_\nu\omega, {}_{-\nu}\omega\}$.

Representing the complx parameter ω in the form

$$\omega = \delta + j\gamma \qquad \cdots(4.74)$$

the complex wave number associated with the nth medium is expressed as follows:

$$\kappa_n = \frac{\omega}{v_{n0}}\sqrt{\frac{1+jd_n\omega}{1+jc_n\omega}} = R(\kappa_n) + jI(\kappa_n) \qquad \cdots(4.75)$$

$$R(\kappa_n) = \frac{1}{v_{n0}\sqrt{E_n}} \frac{\delta(C_n + \sqrt{C_n^2+D_n^2}) + \gamma D_n}{\sqrt{2}\sqrt{C_n + \sqrt{C_n^2+D_n^2}}}$$

$$I(\kappa_n)=\frac{1}{v_{n0}\sqrt{E_n}}\frac{\gamma(C_n+\sqrt{C_n^2+D_n^2})-\delta D_n}{\sqrt{2}\sqrt{C_n+\sqrt{C_n^2+D_n^2}}} \qquad \cdots(4.76)$$

and

$$|\kappa_n|^2=\frac{(\delta^2+\gamma^2)\sqrt{C_n^2+D_n^2}}{v_{n0}^2E_n} \qquad \cdots(4.77)$$

where

$$\begin{aligned} C_n&=c_nd_n\left[\delta^2+\left(\gamma-\frac{1}{c_n}\right)\left(\gamma-\frac{1}{d_n}\right)\right] \\ D_n&=(c_n-d_n)\delta \\ E_n&=c_n^2\left[\delta^2+\left(\gamma-\frac{1}{c_n}\right)^2\right]\geqq 0 \end{aligned} \qquad \cdots(4.78)$$

and

$$\begin{aligned} C_n^2+D_n^2&=c_n^2d_n^2\left[\delta^2+\left(\gamma-\frac{1}{c_n}\right)\left(\gamma-\frac{1}{d_n}\right)\right]^2+(c_n-d_n)^2\delta^2 \\ &=c_n^2d_n^2(\delta^2+\gamma^2)\left[\delta^2+\left(\gamma-\frac{1}{c_n}-\frac{1}{d_n}\right)^2-\frac{2}{c_nd_n}\right] \\ &\quad+c_nd_n\left(2\gamma-\frac{1}{c_n}\right)\left(2\gamma-\frac{1}{d_n}\right) \end{aligned} \qquad \cdots(4.79)$$

Here, it is noted that the following inequalities are valid:

$$C_n+\sqrt{C_n^2+D_n^2}\geqq 0, \quad \delta D_n\geqq 0 \qquad \cdots(4.80)$$

In particular, in the case where $d_n=0$, eqs. (4.78) and (4.79) are reduced respectively to the following equations:

$$\begin{aligned} C_n&=1-c_n\gamma \\ D_n&=c_n\delta \\ E_n&=c_n^2\left[\delta^2+\left(\gamma-\frac{1}{c_n}\right)^2\right] \end{aligned} \qquad \cdots(4.81)$$

and

$$C_n^2+D_n^2=E_n^2 \qquad \cdots(4.82)$$

Since κ_n given by eq. (4.75) is analytic except the two branch points j/c_n and j/d_n, the function X_n defined by eq. (3.24) is also analytic except for these points. On the other hand, the wave-transfer function b_n, associated with the reflected waves at the nth boundary to incident waves, is expressed as

$$b_n=\frac{\rho_nv_{n0}\sqrt{(1+jc_n\omega)(1+jd_{n+1}\omega)}-\rho_{n+1}v_{n+10}\sqrt{(1+jc_{n+1}\omega)(1+jd_n\omega)}}{\rho_nv_{n0}\sqrt{(1+jc_n\omega)(1+jd_{n+1}\omega)}+\rho_{n+1}v_{n+10}\sqrt{(1+jc_{n+1}\omega)(1+jd_n\omega)}} \cdots(4.83)$$

and its squared absolute value is written in the form

$$|b_n|^2=\frac{|\alpha_n|^2-2R(\alpha_n)+1}{|\alpha_n|^2+2R(\alpha_n)+1}=\frac{(R(\alpha_n)-1)^2+I^2(\alpha_n)}{(R(\alpha_n)+1)^2+I^2(\alpha_n)} \qquad \cdots(4.84)$$

in which

$$\alpha_n=\frac{\rho_nv_{n0}}{\rho_{n+1}v_{n+10}}\sqrt{\frac{(1+jc_n\omega)(1+jd_{n+1}\omega)}{(1+jc_{n+1}\omega)(1+jd_n\omega)}} \qquad \cdots(4.85)$$

From eq. (4. 83) it is found that the branch points of the function b_n consist of j/c_n, j/d_n, j/c_{n+1} and j/d_{n+1}. And, from eqs. (2. 54), (4. 84) and (4. 85), it is shown that if the pertinent cuts excluding the four branch points are considered, the equation $\alpha_n = -1$ can not be valid in the complex ω-plane. Then, the function b_n is one-valued, analytic and bounded in the complex plane with the above-mentioned cuts. Hence, the one-dimensional wave-transfer functions G_N^{n}'s defined by eqs. (3. 21)-(3. 28) are one-valued analytic functions in the complex ω-plane with the cuts excluding the finite number of branch points $\{j/c_n, j/d_n\}$, $n=1, 2, \cdots\cdots, N, N+1$ except for a denumerably infinite number of poles which are zeros of ${}_N\Delta_c$ given by eq. (3. 23).

In the following it is shown that the one-dimensional wave-transfer functions G_N^{n}'s are finite in the neighbourhood at any branch point, j/c or j/d, of the set $\{j/c_n, j/d_n\}$ in which

$$\begin{aligned} c &= c_{n_1} = c_{n_2} = \cdots\cdots = c_{n_p} \\ d &= d_{n_1} = d_{n_2} = \cdots\cdots = d_{n_q}, \end{aligned} \qquad N+1 \geqq p,\ q \geqq 1 \qquad \cdots(4.86)$$

As regards the branch point j/d, it is easily found that the values of b_n's and X_n's, $n=1, 2, \cdots\cdots N, N+1$, at this point are finite, hence, the one-dimensional wave-transfer functions G_N^{n}'s are also finite at the branch point j/d as well as in its neighbourhood.

To find the properties of the one-dimensional wave-transfer functions G_N^{n}'s at the branch point j/c, we consider the neighbourhood enclosed by an arbitrarily small circle around the branch point. Supposing an arbitrarily small positive number ε we set

$$\omega = \varepsilon \exp(j\theta) + \frac{j}{c} \qquad \cdots(4.87)$$

namely,

$$\delta = \varepsilon \cos\theta, \quad \gamma = \varepsilon \sin\theta + \frac{1}{c} \qquad \cdots(4.88)$$

Substituting the above equations in eq. (4. 78) we obtain

$$\begin{aligned} C &= cd\varepsilon^2 - (c-d)\sin\theta \cdot \varepsilon \\ D &= (c-d)\cos\theta \cdot \varepsilon \\ E &= c^2\varepsilon^2 \end{aligned} \qquad \cdots(4.89)$$

Hence, when ε tends to zero eqs. (4. 76) and (4. 77) are asymptotically expressed as follows:

$$R(\kappa_n) \cong \frac{\cos\theta \cdot \sqrt{c-d}}{\sqrt{2}\sqrt{1-\sin\theta}\, c^2 v_{n0}} \frac{1}{\sqrt{\varepsilon}}, \qquad I(\kappa_n) \cong \frac{\sqrt{1-\sin\theta}\sqrt{c-d}}{\sqrt{2}\, c^2 v_{n0}} \frac{1}{\sqrt{\varepsilon}} \qquad \cdots(4.90)$$

and

$$|\kappa_n|^2 \cong \frac{c-d}{c^4 v_{n0}^2} \frac{1}{\varepsilon} \qquad \cdots(4.91)$$

By using eq. (4. 90) the function X_n given by eq. (3. 24) is expressed as

$$X_n \cong \exp\left(\frac{\sqrt{1-\sin\theta}\sqrt{c-d}}{\sqrt{2}\,c^2 v_{n0}}\frac{H_n}{\sqrt{\varepsilon}}\right)\exp\left(-j\frac{\cos\theta\cdot\sqrt{c-d}}{\sqrt{2}\sqrt{1-\sin\theta}\,c^2 v_{n0}}\frac{H_n}{\sqrt{\varepsilon}}\right)$$

$$n=n_1,\ n_2,\cdots\cdots,\ n_p \qquad \cdots(4.92)$$

From the above equation, it is found that when ε tends to zero X_n's for $n=n_1, n_2,\cdots\cdots, n_p$ diverge with exponential order except for $\theta=\pi/2$.

By taking account of the fact that b_n's for $n=1, 2,\cdots\cdots, N, N+1$ and X_n's for $n\neq n_1, n_2,\cdots\cdots, n_p$ are finite and non-zero at the branch point, j/c, and by substituting eq. (4. 92) in eqs. (3. 21)-(3. 23), the order of the one-dimensional wave-transfer functions G_N^n's in the neighbourhood of the branch point is determined as follows:

$$O\left(\left|G_N^n\left(j\varepsilon\cos\theta-\left(\varepsilon\sin\theta+\frac{1}{c}\right)\right)\right|\right)$$

$$=O\left(\exp\left(-\frac{\sqrt{1-\sin\theta}\sqrt{c-d}}{\sqrt{2}\,c^2}\frac{1}{\sqrt{\varepsilon}}\sum_{\substack{i=n_l\\ N\geq n_l\geq n+1}}\frac{H_i}{v_{i0}}\right)\right)$$

$$n=0,\ 1,\cdots\cdots,\ N \qquad \cdots(4.93)$$

where $\{n_l\}=n_1, n_2,\cdots\cdots, n_p$ and the convention given by eq. (4. 14) is to be adopted in the summation for $n=N$. From the above equation, it is found that in general the one-dimensional wave-transfer functions G_N^n's are finite in the neighbourhood of the branch point j/c and that except the case where $n=N$ and the case where $\theta=\pi/2$ the absolute values of G_N^n's vanish in exponential order as ε tends to zero.

Finally, to examine the properties of the one-dimensional wave-transfer functions G_N^n's at infinity, we consider a circle having its center at the origin of the complex ω-plane and an infinitely large radius.

$$\omega=R\exp(j\theta) \qquad \cdots(4.94)$$

namely,

$$\delta=R\cos\theta,\quad \gamma=R\sin\theta \qquad \cdots(4.95)$$

In the case where d_n not equal to zero, by substituting eq. (4. 95) in eq. (4. 78) we obtain

$$\begin{aligned} C_n&=c_n d_n R^2-\sin\theta\cdot(c_n+d_n)R+1\\ D_n&=\cos\theta\cdot(c_n-d_n)R\\ E_n&=c_n^2R^2-2\sin\theta\cdot c_nR+1\end{aligned} \qquad \cdots(4.96)$$

Hence, when R tends to infinity eqs. (4. 76) and (4. 77) are asymptotically expressed as

$$\boldsymbol{R}(\kappa_n)\cong\cos\theta\cdot\sqrt{\frac{d_n}{c_n}}\frac{R}{v_{n0}},\quad \boldsymbol{I}(\kappa_n)\cong\sin\theta\cdot\sqrt{\frac{d_n}{c_n}}\frac{R}{v_{n0}} \qquad \cdots(4.97)$$

and

$$|\kappa_n|^2\cong\frac{d_n}{c_n}\frac{R^2}{v_{n0}^2} \qquad \cdots(4.98)$$

Substituting eq. (4. 97) in eq. (3. 24) the asymptotic formula of the function X_n

is given by

$$X_n \cong \exp\left(\sin\theta\cdot\sqrt{\frac{d_n}{c_n}}\frac{H_n}{v_{n0}}R\right)\exp\left(-j\cos\theta\cdot\sqrt{\frac{d_n}{c_n}}\frac{H_n}{v_{n0}}R\right)$$
$$\infty > c_n > d_n > 0,\quad R\to\infty \qquad \cdots(4.99)$$

From the above equation it is found that in the case where d_n not equal to zero, the absolute value of X_n diverges in exponential order in the upper half-plane, vanishes in the same order in the lower half-plane and remains to be unity on the real axis when R tends to infinity.

Similarly, in the case where d_n is zero, eqs. (4.76) and (4.77) are expressed in the following forms as R tends to infinity:

$$R(\kappa_n) \cong \frac{\cos\theta}{\sqrt{1-\sin\theta}\sqrt{2c_n}}\frac{\sqrt{R}}{v_{n0}},\qquad I(\kappa_n) \cong -\frac{\sqrt{1-\sin\theta}}{\sqrt{2c_n}}\frac{\sqrt{R}}{v_{n0}} \qquad \cdots(4.100)$$

and

$$|\kappa_n|^2 \cong \frac{1}{c_n}\frac{R}{v_{n0}^2} \qquad \cdots(4.101)$$

Hence, the asymptotic formula of the function X_n is determined as

$$X_n \cong \exp\left(-\frac{\sqrt{1-\sin\theta}}{\sqrt{2c_n}}\frac{H_n}{v_{n0}}\sqrt{R}\right)\exp\left(-j\frac{\cos\theta}{\sqrt{1-\sin\theta}\sqrt{2c_n}}\frac{H_n}{v_{n0}}\sqrt{R}\right)$$
$$\infty > c_n > 0,\quad d_n = 0,\quad R\to\infty \qquad \cdots(4.102)$$

From the above equation it is found that in the case where d_n is zero, the absolute value of X_n vanishes in exponential order in the complex ω-plane except the positive immaginary axis on which it takes the value of unity when R tends to infinity.

By taking into consideration the above-mentioned facts as well as the fact that all b_n's are finite at infinity the order of the one-dimensional wave-transfer functions G_N^n's at infinity is determined as follows:

$$O(|G_N^n(jR\cos - R\sin\theta)|)$$
$$= O\left(\exp\left(-|\sin\theta|R\sum_{\substack{i=n+1\\ d_i\neq 0}}^{N}\sqrt{\frac{d_i}{c_i}}\frac{H_i}{v_{i0}} - \frac{\sqrt{1-\sin\theta}}{\sqrt{2}}\sqrt{R}\sum_{\substack{i=n+1\\ d_i=0}}^{N}\frac{1}{\sqrt{c_i}}\frac{H_i}{v_{i0}}\right)\right)$$
$$\cdots(4.103)$$

where in the summation for $n=N$ the convention given by eq. (4.14) is adopted. Hence, it is found that when R tends to infinity, the absolute values of the one-dimensional wave-transfer function G_N^n's for $n=0, 1, \cdots\cdots, N-1$ vanish in exponential order while the absolute value of G_N^N remains finite in the complex ω-plane, at least except the real and posotive immaginary axes on which the absolute values $|G_N^n|$'s for $n=0, 1, \cdots\cdots, N$ are able to be finite and oscillatory if a certain condition is satisfied.

For the special case where both c_n and d_n are zero, that is, the perfectly elastic medium, no branch points exist and the quantities given by eq. (4.78) or eq. (4.81) become

$$C_n = E_n = 1,\quad D_n = 0 \qquad \cdots(4.104)$$

Then, eqs. (4.76) and (4.77) take the forms

$$R(\kappa_n)=\frac{\cos\theta}{v_{n0}}R,\quad I(\kappa_n)=\frac{\sin\theta}{v_{n0}}R \qquad \cdots(4.105)$$

and

$$|\kappa_n|^2=\frac{1}{v_{n0}^2}R^2 \qquad \cdots(4.106)$$

Hence, in this case, the function X_n is expressed as follows:

$$X_n=\exp\left(\sin\theta\frac{H_n}{v_{n0}}R\right)\exp\left(-j\cos\theta\frac{H_n}{v_{n0}}R\right)$$
$$c_n=d_n=0 \qquad \cdots(4.107)$$

By comparing the above equation with eq. (4.99), it is found that the behaviour of the function X_n at infinity for the perfectly elastic medium is similar to that of X_n for the linear visco-elastic medium in which $\infty>c_n>d_n>0$.

Therefore, all the preceding discussions about the behaviour, in the neighbourhood of the branch points and at infinity, of the one-dimensional wave-transfer functions G_N^n's are valid for the general multi-layered half-space which is composed of the linear visco-elastic media represented by the three element model shown in Fig. 3, including the Voigt type media and the perfectly elastic media. Also, based on eqs. (3.66)-(3.68) it is found that the singular points of the one-dimensional wave-transfer function at an arbitrary point in the linear visco-elastic multi-layered half-space are the same as those of G_N^n's and its properties in the neighbourhood of the branch points and at infinity are similar to those of G_N^n's.

As for the one-dimensional transfer functions G'^n_N's of the multi-layered media, which consist of the linear visco-elastic media characterized by the three element model, subjected to the excitations at the bottom boundary, similar discussions as previously mentioned on the singular points and the properties of such functions at the singular points and infinity can easily be taken up based on eqs. (3.22) and (3.46) together with the known properties of the one-dimensional wave-transfer functions G_N^n's.

5. Concluding remarks

As one of the basic studies related to the supposition of a reasonable model of random earthquake excitations according to the dynamic characteristics of the ground at the site of a structure, the analytical expressions of the one-dimensional wave-transfer functions of a general class of linear visco-elastic, horizontally multi-layered half-space to vertically incident plane waves at the bottom boundary through the half-space are obtained and the singular points of such wave-transfer functions and their properties in the complex plane are discussed.

The one-dimensional wave-transfer function associated with an arbitrary point in the multi-layered half-space is defined as the ratio of the Fourier transform of the response at the point to that of the incident waves at the bottom boundary, and it can be expressed in terms of the wave-transfer functions associated with wave propagation and such one-dimensional wave-transfer functions associated with the adjacent boundaries.

By making use of the method of mathematical induction and the concept of feed-back loop, it is verified that the one-dimensional wave transfer functions associated with the boundaries in the linear visco-elastic multi-layered half-space as well as the characteristic equation of such a dynamic system are expressed in the successive product forms in terms of the wave-transfer functions associated with wave propagation and with the reflected waves at the boundary to incident waves and some kind of symbolic operator related to a certain multiplication.

Based on the above-mentioned product forms involving the symbolic operators, the eigen-value problem of the linear visco-elastic multi-layered half-space and the properties of the one-dimensional wave-transfer functions in the complex plane, especially, the behaviour of such functions in the neighbourhood of the singular points and at infinity, are discussed.

As a result, for the usually encountered multi-layered half-space which is composed of the linear visco-elastic media represented by the three element model including the so-called Voigt type media and the perfectly elastic media, it is shown that the singular points of the one-dimensional wave-transfer functions consist of a finite number of branch points on the positive immaginary axis and the two sets of a denumerably infinite number of poles with positive immaginary parts, both of which are zeros of the characteristic equation and symmetrical with each other with respect to the immaginary axis. And also, it is found that the one-dimensional wave-transfer functions of the multi-layered half-space are one-valued analytic and bounded in the complex plane with the cuts excluding the branch points with positive immaginary parts except for the poles which also exist in the upper half-plane and that the wave-transfer functions vanish in exponential order at infinity for almost all points in the full complex plane as far as the inner points of the multi-layered media above the half-space are concerned. However, it is noticed that the one-dimensional wave-transfer function associated with the bottom boundary adjacent to the half-space takes the non-zero finite values at infinity. These properties of the one-dimensional wave-transfer functions may guarantee the physically realizable condition of the transfer function and also assure the applicability of the residue theorem in expressing the impulsive responses as well as the variances and co-variances of the random responses in the linear vsco-elastic multi-layered half-space in the forms of infinite series expansions.

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References

1) Kanai, K., "Relation between the Nature of Surface Layer and the Amplitudes of Earthquake Motions," Bulletin of the Earthquake Research Institute, Univ. of Tokyo, Vol. 30, Part 1, March, 1952, pp. 31-37.

2) Kanai, K., "Semi-Emperical Formula for the Seismic Characteristics of the Ground,"

Bulletin of the Earthquake Research Institute, Univ. of Tokyo, June, 1957, pp. 309-325.

3) Kanai, K., Tanaka, T., and Yoshizawa, S., "Comparative Studies of Earthquake Motions on the Ground and Underground (Multiple Reflection Problem)," Bulletin of the Earthquake Research Institute, Univ. of Tokyo, Vol. 37, 1959, pp. 53-87.
4) Bustamante, J. I., "Response Spectra of Earthquakes on Very Soft Clay," Bulletin of the Seismological Society of America, Vol. 54, No. 3, June, 1964, pp. 855-866.
5) Herrera, I., Rosenblueth, E., and Rascon, O. A., "Earthquake Spectrum Prediction for the Valley of Mexico," Proceedings of the 3rd World Conference on Earthquake Engineering, Vol. 1, 1965, pp. I-61-74.
6) Yoshikawa, S., Shima, M., and Irikura, K., "Vibrational Characteristics of the Ground Investigated by Several Methods," Bulletin of the Disaster Prevention Research Institute, Kyoto Univ., Vol. 16, Part 2, Jun., 1967, pp. 1-16.
7) Yoshikawa, S., Shima, M., Goto, N., and Irikura, K., "Vibrational Characteristics of the Ground in the Region of Matsushiro," Annuals of the Disaster Prevention Research Institute, Kyoto Univ., No. 10, A, March, 1967, pp. 189-199. (in Japanese)
8) Yoshikawa, S., Shima, M., and Irikura, K., "Vibrational Characteristics of the Ground in the Region of Matsushiro II," Annuals of the Disaster Prevention Research Institute, Kyoto Univ., No. 11, A, March, 1968, pp. 179-192. (in Japanese)
9) Sezawa, K., "Possibility of the Free Oscillations of the Surface Layer Excited by the Seismic Waves," Bulletin of the Earthquake Research Institute, Univ. of Tokyo, Vol. 8, 1930, pp. 1-11.
10) Sezawa, K., and Kanai, K., "Decay Constants of Seismic Vibrations of a Surface Layer," Bulletin of the Earthquake Research Institute, Univ. of Tokyo, Vol. 13, Part 2, 1935, pp. 251-265.
11) Kanai, K., "The Effect of Solid Viscosity of Surface Layer on the Earthquake Movements," Bulletin of the Earthquake Research Institute, Univ. of Tokyo, Vol. 28, Parts 1-2, 1950, pp. 31-35.
12) Kanai, K., "The Requisite Conditions for the Predominant Vibration of Ground," Bulletin of the Earthquake Research Institute, Univ. of Tokyo, Vol. 38, Part 3, 1957, pp. 457-471.
13) Bolotin, V. V., "Statistical Theory of the Aseismic Design of Structures," Proceedings of the 2nd World Conference on Earthquake Engineering, 1960, pp. 1365-1374.
14) Tanabashi, R., Kobori, T., and Minai, R., "Aseismic Design and Earthquake Response of Structure," Annuals of Disaster Prevention Research Institute of Kyoto Univ., No. 5, B, March, 1962, pp. 1-32. (in Japanese)
15) Kobori, T., and Minai, R., "Non-Stationary Response of the Linear System to Random Excitation," Bulletin of the Disaster Prevention Research Institute, Kyoto Univ., Vol. 16, Part 2, No. 111, Jan., 1967, pp. 37-80.
16) Shinozuka, M., "Simulation of Nonstationary Random Process," Proceedings of the American Society of Civil Engineers, No. EM1, Feb., 1967, pp. 11-40.
17) Housner, G. W., and Jennings, P. C., "Generation of Artificial Earthquakes," Proceedings of the American Society of Civil Engineers, No. EM1, Feb., 1964, pp. 113-150.
18) Kobori, T. and Minai, R., "Response Spectra of Quasi-Stationary Random Excitations," Bulletin of the Disaster Prevention Research Institute, Kyoto Univ., Vol. 17, Part 3, March, 1968, pp. 45-94.
19) Kobori, T., Minai, R., Inoue, Y., and Takeuchi, Y., "On Spectral Characteristics of Artificial Earthquakes Generated for Structural Response Analysis," Annuals of the Disaster Prevention Research Institute, Kyoto Univ., No. 11, A, March, 1968, pp. 369-403. (in Japanese)
20) Kobori, T., and Minai, R., "Non-Stationary Random Excitations of a Linear Discrete System," Proceedings of the 16th Japan National Congress for Appl. Mech., 1966,

pp. 324-332.
21) Herrera, I., and Resenblueth, E., "Response Spectra in Stratified Soil," Proceedings of the 3 rd World Conference on Earthquake Engineering, Vol. 1, 1965, pp. I-44-60.
22) Kobori, T., and Minai, R., "Study on Unstationary Vibration of Building Structure with Plastic Deformation of Substructure," Proceedings of the 2 nd World Conference on Earthquake Engineering, Vol. II, 1960, pp. 1085-1104.
23) Kobori, T., and Minai, R., "On Some Boundary Value Problems of Vibrating Elastic Body in One-dimension," Proceedings of the 10 th Japan National Congress for Appl. Mech., 1960, pp. 319-324.
24) Kobori, T., and Minai, R., "Eigenvalue Problem in Earthquake Response Analysis," Proceedings of the 2 nd Japan Earthquake Engineering Symposium, 1962, pp. 147-152. (in Japanese)
25) Kobori, T., Minai, R., and Inoue, Y., "Earthquake Response of the Structure Considering the Effect of Ground Compliance, 2 nd Report," Annuals of the Disaster Prevention Research Institute, Kyoto Univ., No. 8, March, 1965, pp. 193-218. (in Japanese)