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The Dynamic Theory of the Deformation of a Granular Solid Fully Saturated with a Liquid

By Yoshiaki FUKUO

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Abstract

The author, at first, derives the theory for the deformation of a granular solid saturated with a liquid, assuming that the liquid filling up the pore space is a Newtonian viscous fluid and the skeleton constituted by solid particles is a linear visco-elastic solid. The theory consists of three fundamental equations, that is, the equations of motion for pore liquid and solid skeleton, and the equation of continuity between the particles and the liquid. In the second place, a formal method is considered for solving the fundamental equations in a case where the liquid and the particles are incompressible in the gravitational field, and we have the respective equations of the dilatation, rotation in the solid skeleton and of the piezometric head of the liquid. Finally, it will be found that these respective equations obtained by a quasi-static treatment are accepted as the theory of three dimensional consolidation including TERZAGHI'S well-known equation as a special case, and are also recognized as the basic equations of motion of confined ground water in a visco-elastic aquifer. An example is shown for the motion of ground water in a simple model as seen in the figure.

Introduction

It is well known that soil mechanics has made great advances since the conception of pore pressure (Hydrostatische Überdruck) proposed by K. v. TERZAGHI. He considered that the grains and particles in soil, being more or less bound to each other by sticking force, constitute a skeleton of soil with elastic properties and that the skeleton supports the external burden together with the cooperation of the water filling up the pore space between the particles and he successfully solved the settlement of the soil layer with the good idea that a contraction of soil depends on the rate of squeezing out of pore water which neccessarily brings about the decrease of pore pressure. But he treated only a one-dimensional problem under constant load with a quasi-static method, that is, in such a way that a soil body is gradually deformed, the external force being equal to the result of the stress exerted in the skeleton and the pressure of pore water at any instance.

In Oct. 1940, M. A. BIOT[1] published the theory of three dimensional consolidation and developed the treatment of soil deformation for any arbitrary load variable with time. His paper contains much of interest to us but his treatment is also quasi-static method, although it is often applicable in engineering practice. Our expectations of his theory will be prescribed later in the discussion. Moreover, in Sep. 1963, M. MIKASA[2] published the useful theory of soft layer consolidation showing many suitable examples, especially taking account of finite strain. But he treated only a one dimensional and quasi-static problem in the same way as TERZAGHI.

Recently in soil mechanics, much attention has been paid to the correspondence of soil deformation caused by a vibrating agency in connection with the effective performance of engineering construction or safe protection from heavy damage such as an earthquake. It may be clear that the theory of consolidation must be improved and made into a dynamic one for the above requirements. We shall now derive in this paper the dynamic theory of consolidation, considering the rheological properties of soil.

1. Derivation of fundamental equations

Soil particles constitute the skeleton of the soil matrix, pushing and rubbing each other's contact portions against an external burden. Owing to the complexity of its structure, however, one could not expect the direct treatment of forces acting on each particle. In the same situation, it would also be quite impossible to deal quantitatively with the motion of pore water attending to tortuous and irregular pore space. Therefore we are obliged to consider the representation of motion averaged over a volume element of soil, which is taken to be large enough compared to the size of the pores so that it may be treated as homogeneous and at same time small enough compared to the scale of macroscopic phenomena in which we are interested so that it may be considered as infinitesimal in the mathematical treatment. It will be sufficient in soil mechanics to consider the average conditions over the volume of soil in the above sense.

(a) *Equation of motion of pore water*

The motion of water in pores is governed by the hydrodynamic equation of viscous fluid. We regard the pore water as a Newtonian fluid and denote by V (V_1, V_2, V_3) the particle velocity of pore water. The equation of motion of pore water is expressed by

$$\frac{DV}{Dt}=X-\frac{1}{\rho}\operatorname{grad} p-\left(\frac{1}{3}\eta-\kappa\right)\operatorname{grad}\theta+\eta\nabla^2 V \tag{1.1}$$

where t is time, X is external body force, ρ and p are the density and pressure of water, respectively, θ is the divergence of water flow, and η and κ are the kinematic viscosity of shear and bulk respectively; the dependences of which on density ρ are assumed to be slight.

Consider a unit volume of the soil matrix in the sense stated above. Integrating eq. (1. 1) over the pore space σ of the unit volume and using the following notations

$$U\equiv\iiint_\sigma V\,dv \tag{1.2}$$

$$P\equiv\frac{1}{\sigma}\iiint_\sigma p\,dv \tag{1.3}$$

we have

$$\frac{DU}{Dt}=\sigma X-\frac{\sigma}{\rho}\operatorname{grad} P-\left(\frac{1}{3}\eta-\kappa\right)\iiint_\sigma \operatorname{grad}\theta\,dv+\eta\iiint_\sigma \nabla^2 V\,dv \tag{1.4}$$

where $\boldsymbol{U}$ is called "DARCY's velocity" or "specific flow rate" and P is the "pore pressure" proposed by TERZAGHI.

The pore pressure P is generally taken to be thermodynamic pressure and is determined by the density and temperature of water. In constant temperature, we can write

$$\log \frac{\rho}{\rho_0} = \beta(P - P_0) \tag{1.5}$$

where β is called isothermal compressibility. In the foregoing, ρ_0 and P_0 are the density and pressure in some reference state, say a state at rest. In the case where external body force $\boldsymbol{X}$ is gravitational force, it is convenient to introduce the quantity φ which is termed the "piezometric head"

$$\varphi \equiv \frac{P}{\rho g} + x_3 \tag{1.6}$$

where x_3-axis is taken as positive upward and g is gravitationl acceleration.

Appropriate expression to the last term in eq. (1. 4) is done by refering to DARCY's law governing the flow of water in a porous medium. He postulated the viscous force acting on water to be proportional to the flow velocity and introduced, as the proportional constant, physical quantity k which is called the coefficient of permeability of the soil and according to his expression, the viscous force is written by

$$\boldsymbol{F} \equiv \eta \iiint_\sigma \nabla^2 \boldsymbol{V} dv \equiv -\frac{\sigma g}{k} \iiint_\sigma \boldsymbol{V} dv$$

In next paragraph we shall treat the motion of soil particles together with the flow of pore water, and so it may be reasonable to assume that the viscous force $\boldsymbol{F}$ was proportional to the relative motion of water to soil particles, viz. $\left(\boldsymbol{V} - \frac{\partial \boldsymbol{u}}{\partial t}\right)$ where $\boldsymbol{u}$ is the mean displacement of soil particles as seen latter. From our assumption, it is possible to express its force as follows

$$\boldsymbol{F} \equiv -\frac{\sigma g}{k} \iiint_\sigma \left(\boldsymbol{V} - \frac{\partial \boldsymbol{u}}{\partial t}\right) dv = -\frac{\sigma g}{k} \left(\boldsymbol{U} - \sigma \frac{\partial \boldsymbol{u}}{\partial t}\right) \tag{1.7}$$

Pore water is regarded to be almost incompressible in engineering practice. In this case, inserting the expressions (1. 5), (1. 6) and (1. 7) into eq. (1.4) and regarding the inertia term of acceleration to be negligibly small, we have

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sigma g \left\{ \text{grad}\, \varphi + \frac{1}{k} \left(\boldsymbol{U} - \sigma \frac{\partial \boldsymbol{u}}{\partial t}\right)\right\} = 0 \tag{1.8}$$

Eq. (1. 8) is, of course, reduced to DARCY's law if the soil particle has no motion and the flow of water is not accelerated.

(*b*) *Equation of motion of soil skeleton*

We shall now pay attention to the motion of the soil skeleton. TERZAGHI, BIOT and MIKASA assumed the elastic isotropy of stress-strain relation for the soil skeleton. While we also accept the isotropy in order to avoid the trouble of mathematical presentation, we had better give up the elastic property of soil with reference to the results of many investigations postulating the nonelastic deformation of soil[3]. On the other hand, it may be clear that the nonlinear relation between stress and strain make it difficult to analyse the deformation quantitatively. We now regard it tentatively as a linear visco-elastic relation[4].

Consider again the volume element in the sense stated previously and take the average over the actual displacement $\boldsymbol{v}$ of soil particle contained in that volume. We define it as the displacement of skeleton $\boldsymbol{u}$, that is

$$\boldsymbol{u} \equiv \frac{1}{1-\sigma}\iiint_{(1-\sigma)} \boldsymbol{v}\, dv \tag{1.9}$$

Supposing that the difference $(\boldsymbol{v}-\boldsymbol{u})$ produces only a minor effect on the stress on the skeleton viz. effective stress and assuming the strain to be infinitesimally small, the strain on the skeleton is given by tensor $\boldsymbol{e}(e_{ij})$

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \quad (i,\ j=1,\ 2,\ 3) \tag{1.10}$$

Corresponding to this, the effective stress $\boldsymbol{\sigma}(\sigma_{ij})$ is exerted on the skeleton. According to the assumption of linear visco-elasticity, stress σ_{ij} is represented, as positive in compression, by

$$\left(1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}\right)\sigma_{ij}=-\left\{\lambda\left(1+\sum_{p=1}^{l}\alpha_p\frac{\partial^p}{\partial t^p}\right)\delta_{ij}\Theta+2\mu\left(1+\sum_{p=1}^{m}\beta_p\frac{\partial^p}{\partial t^p}\right)e_{ij}\right\} \tag{1.11}$$

where δ_{ij} is kronecker notation and Θ is the dilatation of the soil skeleton, that is

$$\Theta=\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right) \tag{1.12}$$

Introducing the operation with respect to time

$$\mathcal{L}\equiv\lambda\frac{\left(1+\sum_{p=1}^{l}\alpha_p\frac{\partial^p}{\partial t^p}\right)}{\left(1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}\right)},\quad \mathcal{M}\equiv\mu\frac{\left(1+\sum_{p=1}^{m}\beta_p\frac{\partial^p}{\partial t^p}\right)}{\left(1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}\right)} \tag{1.13, 1.14}$$

we can write the stress force per unit cubic element of soil as follows

$$\frac{\partial\sigma_{ij}}{\partial x_j}=-\left\{(\mathcal{L}+\mathcal{M})\frac{\partial\Theta}{\partial x_i}+\mathcal{M}\nabla^2 u_i\right\} \tag{1.15}$$

Furthermore, the skeleton is pushed by the pore pressure of surrounding water. This pressure action $\boldsymbol{f}_1$ may not produce any shearing strain by reason of the assumed isotropy and will be expressed by

$$\boldsymbol{f}_1=\iint_{(1-\sigma)surf} p\boldsymbol{n}da=-\iiint_{(1-\sigma)}\operatorname{grad} p dv=-(1-\sigma)\operatorname{grad} P \tag{1.16}$$

per unit volume of soil, where $\boldsymbol{n}$ is the outward unit vector normal to the surface element da of solid space.

In addition to the above forces, the skeleton tends to be dragged by the flow of pore water in its direction through the reaction of the viscous force acting on the pore water. This drag force f_2 will be expressed by

$$f_2 = -\rho \frac{\sigma g}{k}\left(\sigma \frac{\partial u}{\partial t} - U\right) \tag{1.17}$$

We can thus establish the equation of motion of the soil skeleton, that is

$$(1-\sigma)\rho_s \frac{\partial^2 u}{\partial t^2} = (1-\sigma)X + \nabla \cdot \sigma + f_1 + f_2$$

$$= (1-\sigma)X + (\mathcal{L}+\mathcal{M})\ \mathrm{grad}\ \Theta + \mathcal{M}\nabla^2 u - (1-\sigma)\ \mathrm{grad}\ P + \rho \frac{\sigma g}{k}\left(U - \sigma \frac{\partial u}{\partial t}\right) \tag{1.18}$$

where ρ_s is the density of soil particles and X is an external body force. In almost all cases with which we are concerned, force X is a gravitational force. Expressing the vertically upward unit vector by k, we have

$$(1-\sigma)\rho_s \frac{\partial^2 u}{\partial t^2} = -(1-\sigma)\rho_s g k + (\mathcal{L}+\mathcal{M})\ \mathrm{grad}\ \Theta + \mathcal{M}\nabla^2 u$$

$$-(1-\sigma)\ \mathrm{grad}\ P + \rho \frac{\sigma g}{k}\left(U - \sigma \frac{\partial u}{\partial t}\right) \tag{1.19}$$

(c) *The equation of mass continuity*

Finally, we shall derive the equation of mass continuity per unit volume of soil. Suppose that a skeleton in any volume of soil had porosity σ_0 at an instance of no dilatation $\Theta=0$, and that the particles in it had density ρ_{so} at that time. Because the skeleton under consideration is to be framed by the same particles at any instance, the mass of the skeleton must be conserved, that is

$$\rho_s(1-\sigma)(1+\Theta) = \rho_{so}(1-\sigma_0) \tag{1.20}$$

since the dilatation Θ represents the volume increase of soil skeleton per unit initial volume and ρ_s is the density of particles at dilatation Θ. The volume of the skeleton is varied with time by external force and consequently, porosity σ is also varied. Eq. (1.20) gives us the relation between their time rates.

As our subject is soil fully saturated with pore water, the change of pore volume results in the flow of pore water into or out of the volume element. This situation is represented by

$$\frac{\partial}{\partial t}(\sigma\rho) = -\mathrm{div}(\sigma\rho V) = -\mathrm{div}(\rho U) \tag{1.21}$$

where ρ is the density of water, V is the particle velocity of water and U is the specific flow rate as defined previously.

Combining eq. (1.20) with eq. (1.21), we make

$$\frac{\partial}{\partial t}\left\{(1-\sigma)\rho_s + \sigma\rho\right\} = \frac{\partial}{\partial t}\left\{\frac{(1-\sigma_0)\rho_{so}}{1+\Theta}\right\} - \mathrm{div}(\rho U) \tag{1.22}$$

This is an equation which we expected to derive. Relation of ρ_s to ρ_{so} may be obtained from the consideration of the compression of soil particles due to the effective stress σ and pore pressure P, although we have little knowledge about it at present. However, in general, the compressibility of soil particles and water is small. Assuming both densities to be constant, we rewrite eq.

(1. 22) as

$$(1-\sigma_0)\frac{\partial\Theta}{\partial t}+\operatorname{div}\boldsymbol{U}=0 \qquad (1.23)$$

with good approximation neglecting the small quantity of order Θ^2.

Approximating the values σ in eqs. (1. 8) and (1. 19) to the value σ_0 after eq. (1. 23) and summarizing the fundamental equations in the case where the external body force is only gravitational force and the soil particles and pore water are incompressible, we have

$$(1-\sigma_0)\rho_{s0}\frac{\partial^2\boldsymbol{u}}{\partial t^2}=-(1-\sigma_0)(\rho_{s0}-\rho_0)g\boldsymbol{k}+(\mathcal{L}+\mathcal{M})\operatorname{grad}\Theta$$
$$+\mathcal{M}\nabla^2\boldsymbol{u}-(1-\sigma_0)\rho_0 g\operatorname{grad}\varphi+\rho_0\frac{\sigma_0 g}{k}\left(\boldsymbol{U}-\sigma_0\frac{\partial\boldsymbol{u}}{\partial t}\right) \qquad \text{(a)}$$

$$\frac{\partial\boldsymbol{U}}{\partial t}+\sigma_0 g\left\{\operatorname{grad}\varphi+\frac{1}{k}\left(\boldsymbol{U}-\sigma_0\frac{\partial\boldsymbol{u}}{\partial t}\right)\right\}=0 \qquad \text{(b)}$$

$$(1-\sigma_0)\frac{\partial\Theta}{\partial t}+\operatorname{div}\boldsymbol{U}=0 \qquad \text{(c)}$$

where

$$\Theta\equiv\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3} \qquad \text{(d)}$$

$$\varphi\equiv\frac{P}{\rho_0 g}+x_3 \qquad \text{(e)}$$

$$\mathcal{L}\equiv\lambda\frac{\left(1+\sum_{p=1}^{l}\alpha_p\frac{\partial^p}{\partial t^p}\right)}{\left(1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}\right)},\quad \mathcal{M}\equiv\mu\frac{\left(1+\sum_{p=1}^{m}\beta_p\frac{\partial^p}{\partial t^p}\right)}{\left(1+\sum_{p=1}^{n}\gamma_p\frac{\partial^p}{\partial t^p}\right)} \qquad \text{(f)}$$

In the remainder, we shall omit a subscript "$_0$" from the respective notations.

2. Method for solving the fundamental equations

In this section, we shall seek for a formal method to solve the fundamental equations (a), (b) and (c)

Let us first examine, a static equilibrium state of the skeleton with a steady flow of pore water. Eqs. (a) (b) and (c) become

$$0=-(1-\sigma)(\rho_s-\rho)g\boldsymbol{k}+(\mathcal{L}+\mathcal{M})\operatorname{grad}\Theta$$
$$+\mathcal{M}\nabla^2\boldsymbol{u}-(1-\sigma)\rho g\operatorname{grad}\varphi+\rho\frac{\sigma g}{k}\boldsymbol{U} \qquad (2.1)$$

$$0=\boldsymbol{U}+k\operatorname{grad}\varphi \qquad (2.2)$$

$$0=\operatorname{div}\boldsymbol{U} \qquad (2.3)$$

From eqs. (2. 2) and (2. 3), we can solve the quantities $\boldsymbol{U}$ and φ satisfying the respective boundary conditions under consideration. Inserting these solutions into eq. (2. 1), we have

$$0=-(1-\sigma)(\rho_s-\rho)g\boldsymbol{k}+(\mathcal{L}+\mathcal{M})\operatorname{grad}\Theta+\mathcal{M}\nabla^2\boldsymbol{u}-\rho g\operatorname{grad}\varphi \qquad (2.4)$$

Now, we divide the displacement $\boldsymbol{u}$ into two parts, $\boldsymbol{u}_{01}$ and $\boldsymbol{u}_{02}$, which satisfy the equations

$$0=-(1-\sigma)(\rho_s-\rho)g\boldsymbol{k}+(\mathcal{L}+\mathcal{M})\operatorname{grad}\Theta_{01}+\mathcal{M}\nabla^2\boldsymbol{u}_{01} \tag{2.5}$$

$$0=(\mathcal{L}+\mathcal{M})\operatorname{grad}\Theta_{02}+\mathcal{M}\nabla^2\boldsymbol{u}_{02}-\rho g\operatorname{grad}\varphi \tag{2.6}$$

respectively, where

$$\Theta_{01}\equiv\operatorname{div}\boldsymbol{u}_{01} \tag{2.7}$$

$$\Theta_{02}\equiv\operatorname{div}\boldsymbol{u}_{02} \tag{2.8}$$

Eq. (2. 5) expresses that $\boldsymbol{u}_{01}$ is the displacement caused by the apparent weight of the soil skeleton in water without external load and eq. (2. 6) means that if $\boldsymbol{u}_{02}$ is uniform in the entire body of soil, grad φ, consequently $\boldsymbol{U}$, is zero, say conversely, if pore water flows, the skeleton of the soil must be strained to that extent.

When external load and/or piezometric head φ vary with time after the initial state, the strain on the skeleton and the velocity of flow begin to leave the static state. We proceed to investigate the unsteady motion.

Disolving the quantities $\boldsymbol{u}$, Θ, $\boldsymbol{U}$ and φ into the static part $\boldsymbol{u}_0=\boldsymbol{u}_{01}+\boldsymbol{u}_{02}$, $\Theta_0=\Theta_{01}+\Theta_{02}$, $\boldsymbol{U}_0$ and φ_0 and respective deviations $\boldsymbol{u}'$, Θ', $\boldsymbol{U}'$ and φ' from static one, we have for the deviating parts

$$(1-\sigma)\rho_s\frac{\partial^2\boldsymbol{u}'}{\partial t^2}=(\mathcal{L}+\mathcal{M})\operatorname{grad}\Theta' +\mathcal{M}\nabla^2\boldsymbol{u}'-(1-\sigma)\rho g\operatorname{grad}\varphi'+\rho\frac{\sigma g}{k}\left(\boldsymbol{U}'-\sigma\frac{\partial\boldsymbol{u}'}{\partial t}\right) \tag{2.9}$$

$$\frac{\partial\boldsymbol{U}'}{\partial t}+\sigma g\left\{\operatorname{grad}\varphi'+\frac{1}{k}\left(\boldsymbol{U}'-\sigma\frac{\partial\boldsymbol{u}'}{\partial t}\right)\right\}=0 \tag{2.10}$$

$$(1-\sigma)\frac{\partial\Theta'}{\partial t}+\operatorname{div}\boldsymbol{U}'=0 \tag{2.11}$$

We can now derive the equation for only the dilatation Θ in the following manner. In the remainder, let us omit the prime on each quantity. Taking the divergences of eqs. (2. 9) and (2. 10) and time derivative of eq. (2. 11)

$$(1-\sigma)\rho_s\frac{\partial^2\Theta}{\partial t^2}=(\mathcal{L}+2\mathcal{M})\nabla^2\Theta-(1-\sigma)\rho g\nabla^2\varphi+\rho\frac{\sigma g}{k}\left(\operatorname{div}\boldsymbol{U}-\sigma\frac{\partial\Theta}{\partial t}\right) \tag{2.12}$$

$$\frac{\partial}{\partial t}(\operatorname{div}\boldsymbol{U})+\sigma g\left\{\nabla^2\varphi+\frac{1}{k}\left(\operatorname{div}\boldsymbol{U}-\sigma\frac{\partial\Theta}{\partial t}\right)\right\}=0 \tag{2.13}$$

$$(1-\sigma)\frac{\partial^2\Theta}{\partial t^2}+\frac{\partial}{\partial t}(\operatorname{div}\boldsymbol{U})=0 \tag{2.14}$$

Combining eq. (2. 13) with eqs. (2. 11) and (2. 14),

$$\nabla^2\varphi=\frac{1-\sigma}{\sigma g}\frac{\partial^2\Theta}{\partial t^2}+\frac{1}{k}\frac{\partial\Theta}{\partial t} \tag{2.15}$$

Substituting from eqs. (2. 14) and (2. 15) into eq. (2. 12),

$$(1-\sigma)\left[\rho_s+\frac{1-\sigma}{\sigma}\rho\right]\frac{\partial^2\Theta}{\partial t^2}+\frac{\rho g}{k}\frac{\partial\Theta}{\partial t}=(\mathcal{L}+2\mathcal{M})\nabla^2\Theta \tag{2.16}$$

This is the equation we want to derive.

In the second place, we shall derive the equation for the rotation

$$\Omega \equiv \mathrm{rot}\,\boldsymbol{u}$$

Taking the rotation of eqs. (2.9) and (2.10),

$$(1-\sigma)\rho_s \frac{\partial^2 \boldsymbol{\Omega}}{\partial t^2} = \mathcal{M}\nabla^2\boldsymbol{\Omega} + \rho \frac{\sigma g}{k}\left(\mathrm{rot}\,\boldsymbol{U} - \sigma \frac{\partial \boldsymbol{\Omega}}{\partial t}\right) \tag{2.17}$$

$$\frac{\partial}{\partial t}\left(\mathrm{rot}\,\boldsymbol{U}\right) + \frac{\sigma g}{k}\,\mathrm{rot}\,\boldsymbol{U} - \frac{\sigma^2 g}{k}\frac{\partial \boldsymbol{\Omega}}{\partial t} = 0 \tag{2.18}$$

From eq (2.18) we have

$$\mathrm{rot}\,\boldsymbol{U} = \frac{\sigma^2 g}{k} e^{-\sigma g t/k}\int_0^t e^{-\sigma g t/k}\frac{\partial \boldsymbol{\Omega}}{\partial \tau}d\tau \tag{2.19}$$

where, we can put rot $\boldsymbol{U}=0$ at $t=0$ as $\boldsymbol{U}$ is the deviating velocity from the steady state.

In many cases, we assume the dependence of variables on time as form $e^{\alpha t}$ to solve the transient problem. Put $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ in the forms

$$\boldsymbol{U} \equiv \boldsymbol{U}^* e^{\alpha t}, \quad \Omega \equiv \Omega^* e^{\alpha t} \tag{2.20, 2.21}$$

then

$$\mathrm{rot}\,\boldsymbol{U}^* = \frac{\alpha\sigma^2 g}{\sigma g+\alpha k}\left[1-\exp\left(-\frac{\sigma g+\alpha k}{k}t\right)\right]\boldsymbol{\Omega}^* \tag{2.22}$$

The term $\boldsymbol{\Omega}^* \exp\left(-\frac{\sigma g+\alpha k}{k}t\right)$ represents the effect of initial rotation on the sequential motion of the soil skeleton and will disappear exponentially with time as its expression.

If we obtain the dilatation Θ and rotation $\boldsymbol{\Omega}$ from eqs. (2.16) and (2.17). we can find the deviation of displacement $\boldsymbol{u}$ by solving the following equation

$$\nabla^2\boldsymbol{u} = \mathrm{grad}\,\Theta - \mathrm{rot}\,\boldsymbol{\Omega} \tag{2.23}$$

with the convenience of the Heaviside operation for partial differentiation.

As the particular case which interests us, we shall pick out the quasi-static motion. Neglecting the accelerated terms in eqs. (2.15) and (2.16), we find

$$\frac{\partial\Theta}{\partial t} = k\nabla^2\varphi \tag{2.24}$$

$$\frac{\partial\Theta}{\partial t} = \frac{k}{\rho g}(\mathcal{L}+2\mathcal{M})\nabla^2\Theta \tag{2.25}$$

Differentiating eq. (2.25) with respect to time and inserting eq. (2.24) into this, we make

$$\nabla^2\left[\frac{\partial\varphi}{\partial t} - \frac{k}{\rho g}(\mathcal{L}+2\mathcal{M})\nabla^2\varphi\right] = 0 \tag{2.26}$$

This may be regarded as the basic equation of three dimensional consolidation. It is worthy of note that if $\varphi_1(t, x_i)$ is a solution of eq. (2.26), $\varphi_1(t, x_i)+F_1(x_i)+F_2(t)$, where $\nabla^2 F_1=0$ is also the solution within the limits of quasi-static transition, in physical words, the progress of consolidation is not affected by the non-divergent flow of pore water and/or the gradual change of the water head in the entire region of the soil medium. When the non-divergent flow has already been included in the static part of piezometric head φ_0 considered previously, eq. (2. 26) is reduced to

$$\frac{\partial \varphi}{\partial t} = \frac{k}{\rho g}(\mathcal{L}+2\mathcal{M})\nabla^2\varphi \tag{2.27}$$

In the case of the elastic skeleton, operations $\mathcal{L}$ and $\mathcal{M}$ are reduced to LAME's constant λ and μ, respectively. Let us examine the special case of a column of soil supporting a load and confined laterally in a rigid cylinder so that no lateral expansion can occur. Eq. (2. 27) is then rewritten as

$$\frac{\partial \varphi}{\partial t} = \frac{k}{\rho g}(\lambda+2\mu)\frac{\partial^2 \varphi}{\partial x_3^2}$$

By comparing this with TERZAGHI's well-known equation, we can see the relation of operations $\mathcal{L}$ and $\mathcal{M}$ to the value a which is termed the coefficient of compressibility in soil mechanics, $(\mathcal{L}+2\mathcal{M}) = (1+e)/a = 1/(1-\sigma)a$.

Example

Eqs. (2. 24) (2. 25) and (2. 26) are also taken as the basic equations for subsidence related closely to the flow of confined ground water through the visco-elastic aquifer. We shall it here in a simple example.

Let us consider the motion of confined water caused by tidal oscillation of sea water over the area of seashore $0 \leq x_1 \leq a$ in a visco-elastic aquifer as shown in Fig. 1, where no flow of water exists through the surfaces $x_1=0$ (contact surface between sea and aquifer) $x_3=0$ and h (lower and upper boundary surfaces of aquifer).

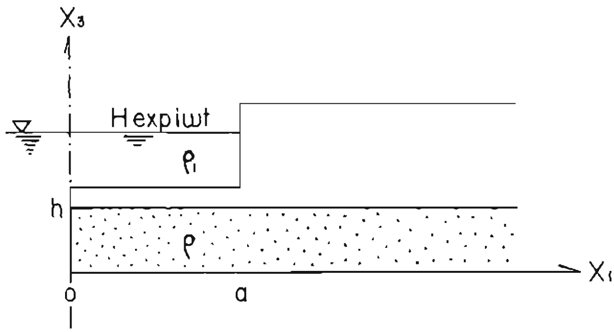

Fig. 1. Idealized cross section of visco-elastic aquifer where the confined water is moved by tidal oscillation of sea water on the area of seashore $0 \leq x_1 \leq a$.

We have the following equations for the deviation from the static state

$$\frac{\partial \Theta}{\partial t} = \frac{k}{\rho g}(\mathcal{L}+2\mathcal{M})\nabla^2\Theta \tag{E·1}$$

$$\frac{\partial \Theta}{\partial t} = k\nabla^2\varphi \tag{E·2}$$

with the boundary conditions

$$u_1=0, \qquad \frac{\partial u_3}{\partial x_1}=0, \qquad \frac{\partial \varphi}{\partial x_1}=0 \qquad \text{at } x_1=0 \tag{E.3}$$

$$u_1=0, \qquad u_3=0, \qquad \varphi=0 \qquad \text{at } x_1=\infty \tag{E.4}$$

$$u_1=0, \qquad u_3=0, \qquad \frac{\partial \varphi}{\partial x_3}=0 \qquad \text{at } x_3=0 \tag{E.5}$$

$$\sigma_{33}+\rho g\varphi=\rho_1 g f(x_1) \exp i\omega t, \qquad \frac{\partial \varphi}{\partial x_3}=0 \qquad \text{at } x_3=h \tag{E.6}$$

where ρ_1 is the density of sea water and $f(x_1)$ is

$$\begin{aligned} f(x_1)&=H \qquad 0\leq x\leq a \\ &=0 \qquad a<x \end{aligned} \tag{E.7}$$

Now, suppose that the dependent variables φ, Θ and $\boldsymbol{u}$ are forced to be a steady periodic motion owing to the tidal oscillation of sea water level $H \exp i\omega t$, we can put

$$\Theta=\Theta^*(x_1, x_3) \exp i\omega t \tag{E.8}$$

$$\boldsymbol{u}=\boldsymbol{u}^*(x_1, x_3) \exp i\omega t \tag{E.9}$$

$$\varphi=\varphi^*(x_1, x_3) \exp i\omega t \tag{E.10}$$

Taking $u_1^*=0$ and u_3^*, Θ^*, φ^* to be symmetric with respect to x_1-axis on the basis of boundary conditions, the differential eq. (E. 1) becomes

$$i\omega\Theta^*=\frac{k}{\rho g}\left\{\mathcal{L}(i\omega)+2\mathcal{M}(i\omega)\right\}\nabla^2\Theta^*$$

and we obtain

$$\Theta^*=\frac{2}{\pi}\int_0^\infty d\alpha \int_0^\infty d\lambda\left[(Ae^{\beta x_3}+Be^{-\beta x_3})\cos\alpha x_1 \cos\alpha\lambda\right]$$

$$u_3^*=\int\Theta^* dz=\frac{2}{\pi}\int_0^\infty d\alpha\int d\lambda\left[\frac{1}{\beta}(Ae^{\beta x_3}-Be^{-\beta x_3})\cos\alpha x_1\cos\alpha\lambda\right]+F(x_1)$$

where $F(x_1)$ is an arbitrary function of only x_1 and

$$\beta^2-\alpha^2=\frac{i\omega\rho g}{k\{\mathcal{L}(i\omega)+2\mathcal{M}(i\omega)\}} \tag{E.11}$$

From the condition (E. 5), we find

$$F(x_1)=0, \qquad A=B$$

and then

$$\Theta^*=\frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty d\lambda\left[2A\cosh\beta x_3\cos\alpha x_1\cos\alpha\lambda\right] \tag{E.12}$$

$$u_3^*=\frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty d\lambda\left[\frac{2A}{\beta}\sinh\beta x_3\cos\alpha x_1\cos\alpha\lambda\right] \tag{E.13}$$

The differential eq. (E. 2) becomes

$$\nabla^2\varphi^*=\frac{i\omega}{k}\Theta^*$$

and we obtain

$$\varphi^* = \frac{i\omega}{k}\frac{1}{(\partial^2/\partial x_1^2+\partial^2/\partial x_3^2)}\Theta^* + \frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty d\lambda\Big[(Ce^{\alpha x_3}+De^{-\alpha x_3})\cos\alpha x_1\cos\alpha\lambda\Big]$$
$$= \frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty d\lambda\Big[-\frac{i\omega}{k(\beta^2-\alpha^2)}2A\cosh\beta x_3 + (Ce^{\alpha x_3}+De^{-\alpha x_3})\Big]\cos\alpha x_1\cos\alpha\lambda$$

From conditions (E. 5) and (E. 6),

$$\alpha C - \alpha D = 0$$
$$\frac{i\omega\beta}{k(\beta^2-\alpha^2)}\cdot 2A\sinh\beta x + \alpha(Ce^{\alpha h}-De^{-\alpha h}) = 0$$

and then

$$\varphi^* = \frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty d\lambda\Big[\frac{i\omega}{k(\beta^2-\alpha^2)}2A\Big\{\cosh\beta x_3 - \frac{\beta}{\alpha}\,\frac{\sinh\beta h}{\sinh\alpha h}\cosh\alpha x_3\Big\}\cos\alpha_1 x_1\cos\alpha\lambda\Big] \qquad \text{(E.14)}$$

Carrying expressions (E. 12) and (E. 14) into condition (E. 6), we can decide the unknown constant $A(\alpha, \lambda)$.

$$-\Big\{\mathcal{L}(i\omega)+2\mathcal{M}(i\omega)\Big\}\Theta^*\Big|_{x_3=h} + \rho g\varphi^*\Big|_{x_3=h} = \rho_1 g f(x_1)$$
$$= \frac{2}{\pi}\rho_1 g\int_0^\infty d\alpha\int_0^\infty d\lambda[f(\lambda)\cos\alpha x_1\cos\alpha\lambda]$$

$$2A = -\frac{\rho_1 g}{(\mathcal{L}+2\mathcal{M})}\,\frac{\alpha\tanh\alpha h}{\beta\sinh\beta h}f(\lambda)$$

Carrying this expression into (E. 12) (E. 13) and (E. 14) and integrating these with respect to λ, we can find the solutions of this example

$$\Theta = -\frac{2H\rho_1 g}{\pi(\mathcal{L}+2\mathcal{M})}\exp i\omega t\int_0^\infty d\alpha\Big[\frac{\tanh\alpha h}{\beta\sinh\beta h}\cosh\beta x_3\cos\alpha x_1\sin\alpha a\Big] \qquad \text{(E.15)}$$

$$u_3 = -\frac{2H\rho_1 g}{\pi(\mathcal{L}+2\mathcal{M})}\exp i\omega t\int_0^\infty d\alpha\Big[\frac{\tanh\alpha h}{\beta^2\sinh\beta h}\sinh\beta x_3\cos\alpha x_1\sin\alpha a\Big] \qquad \text{(E.16)}$$

$$\varphi = \frac{2H\rho_1}{\pi\rho}\exp i\omega t\int_0^\infty d\alpha\Big[\frac{\cosh\alpha x_3}{\alpha\cosh\alpha h} - \frac{\tanh\alpha h}{\beta\sinh\beta h}\cosh\beta x_3\Big]\cos\alpha x_1\sin\alpha a \qquad \text{(E.17)}$$

Quantity u_3 will give us the time variation of the ground surface caused by the tidal oscillation of the sea level. When the rheological character of the aquifer is of Voigt's type as shown in Fig. 2, the value of $(\mathcal{L}+2\mathcal{M})$ is as follows.

$$\mathcal{L}(i\omega)+2\mathcal{M}(i\omega) = \frac{\{E_2(E_1+E_2)+(\omega\eta_2)^2\}-i\omega\eta_2 E_1}{(E_1+E_2)^2+(\omega\eta_2)^2}E_1$$

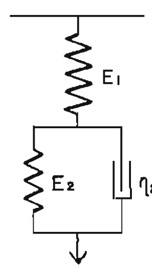

Fig. 2. Voigt's model for the aquifer.

3. Discussion

1) The gradient force f_1 of the pore pressure was explicitely taken into the equation of the skeleton motion in our treatment, while the effective stress was assumed to have no direct

action on the flow of pore water excepting the relative motion $\left(U-\sigma \frac{\partial \boldsymbol{u}}{\partial t}\right)$ between them. From the viewpoint of the principle of reaction, it may be impossible to say that the stress really exerted on soil particles would do so. In fact, it is quesionable whether there is a supporting condition in a spring model as proposed by TERZAGHI or not in the consolidation of partially permeable material. If we take up this problem, however, we are forced to solve the microscopic condition in which the soil particles and pore water support an external load. TERZAGHI's excellent success is perhaps due to dispose of the action of the effective stress on the squeezing rate of pore water to the experimental coefficient of the compressibility of the soil. Our operations $\mathcal{L}$ and $\mathcal{M}$ have quite the same meaning as in TERZAGHI's case. But our case is a saturated one and then we must seek some methods for the imcomplete saturated soil. BIOT considered this problem and introduced the physical quantities H and R which may give a clue to this research.

2) As is well known, DARCY's law is applied to the flow of a liquid through porous media when its velocity is in a certain range, but his law becomes unsuitable at a higher velocity, probably due to the fact that the frictional force is not proportional there to the velocity, even thouth the coefficient of permeability k was corrected to fit the phenomenon. When the velocity is extremly high, the soil particles could not keep their configuration and are possibly floated by the drag force of the flow. We must abandon the application of our fundamental equations at that time. One of such motions is an interesting case driven by alternating force with high frequency. In this case, the pore water cannot flow smoothly in accord with the external force and will consequently be partially compressed. BIOT[5)6)] has already investigated this problem, taking account of the compressibilities of soil particles and water. The attack on this problem may give a solution to the prevention of soft layer damage in heavy earthquakes. We shall develop our equations to be applicable to phenomenon of variable densities, refering to BIOT's theory and at the same time, considering whether the criterion is compressible or not.

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