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The Dynamical Ground Compliance of a Rectangular Foundation on a Viscoelastic Stratum

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Abstract

As a basic study to find the effect of ground-structure interaction on the earthquake responses and the anti-seismic safety of structural systems, it is important to describe the dynamic properties of the sub-soil ground under the foundations of such structures.

Based on the wave propagation theory, this paper deals with the "Dynamical Ground Compliance (*D.G.C.*)" of a model of foundation-ground systems more realistic than those used in any previous studies on foundation vibrations, namely, a rectangular foundation resting on a viscoelastic stratum over a rigid half-space for the cases where the vertical, horizontal, or rotational harmonic excitations act on the foundation. Defined in this paper as the transfer function describing the ratio between the dynamic complex displacement of a massless foundation and a harmonic exciting force, *D.G.C.* is a collective representation of the dynamic characteristics of the sub-soil ground under a foundation. Its limiting expression in zero frequency results in the "Statical Ground Compliance", which means the inverse of the statical ground stiffness. It is assumed that the stratum is composed of a three-dimensional, homogeneous, isotropic, Voigt solid, and lies in welded contact with the supporting rigid half-space.

The analytical expressions of *D.G.C.* as well as the displacements at the ground surface are obtained through the multiple Fourier transformation technique in the double integral representation including an improper infinite integral. The poles of the integrand are the roots of the frequency equations connected with Rayleigh and Love waves. The detailed discussions on the properties of these free waves are also developed in relation to the attenuation and the resonance phenomena of the viscoelastic stratum. Some numerical calculations of *D.G.C.* as well as the equivalent coefficients of a Voigt model evaluated according to *D.G.C.* are given to clarify the effects of various parameters related to the frequency of excitation, the constants of the viscoelastic stratum, and the shape and size of the foundation.

Compared with the results of similar problems obtained in the authors' preceding works for a perfectly elastic half-space and stratum, the present study has led to several principal conclusions on: the effects of energy attenuation owing to mechanisms of both wave radiation and internal dissipation, the comparison between the cases of a stratum and a half-space especially as to the resonance and energy attenuation phenomena, and the condition under which a stratum can be practically treated as a half-space.

1. Introduction

Numerous studies have recently been made on the effect of soil-structure interaction on the anti-seismic safety of structural systems. In most of these analytical studies, the wave propagation theory under the assumption of elastic or viscoelastic media has been applied for clarifying the dynamic properties of the sub-soil ground neighboring the structural foundations as well as the mechanism of seismic wave propagation.

Starting from the well-known work of E. Reissner[1)2)], a large number of studies on foundation vibrations have been made by various investigators on the basis of the wave propagation theory. T.Y. Sung[3)], P.M. Quinlan[4)], I. Toriumi[5)], R.N. Arnold et al.[6)], G.N. Bycroft[7)], and H. Tajimi[8)] have investigated the harmonic vibrations of a rigid circular foundation resting on an elastic half-space. The case of a rigid non-circular foundation, that is, the rectangular foundation prevalent in actual structures, has been analyzed by H. Tajimi[8)], J. Elorduy et al.[9)], and W.T. Thomson and T. Kobori[10)11)]. R.N. Arnold, G.N. Bycroft and G.B. Warburton[6)7)12)] studied the vibrations of a rigid circular foundation on different kinds of soil condition, in which an elastic stratum is assumed to be welded with a supporting rigid half-space in the case of torsional vibration and to permit of a horizontal free-slide at the bottom in the vertical and rotational cases, respectively. T.M. Lee[13)] studied the case of a viscoelastic half-space having complex Lamé's constants, while his main interest lay in the behavior of far field surface vibrations as a means to establish the foundation vibration technique for in-situ determination of the dynamic properties of sub-soil ground.

Following Thomson and Kobori's work, the authors have already carried out a series of studies on the forced-vibration problems of a rigid rectangular foundation resting on an elastic half-space and an elastic stratum lying over a rigid half-space[14)-18)]. In these studies, "Dynamical Ground Compliance", that is, the transfer function prescribing the relation between the foundation displacement and a disturbing force, is introduced in order to represent the dynamic properties of a massless-foundation-ground system.

In all of the above-mentioned studies except reference[9)], the unknown stress distribution beneath a foundation is assumed appropriately in order to avoid the mathematical difficulties involved in the so-called "mixed boundary value problem". Beginning with E. Reissner and H.F. Sagoci's studies[19)20)], a number of attempts to solve precisely the foundation vibration problem as a mixed boundary value problem have been made recently. In addition, several analytical methods to obtain an approximate solution have been presented by earthquake engineers[9)21)].

Although a rigid circular foundation on a perfectly elastic half-space has been adopted as a typical foundation-ground model in most of the previous studies, the actual sub-soil ground may be reasonably regarded as a stratified medium rather than a half-space. The presence of soil layers is important in wave propagation problems because of resonance phenomena arising from the reflection and refraction of traveling waves at the boundaries. Moreover, the nature of the internal energy dissipation may always be observed in the actual soil medium. Therefore it seems realistic, in the present stage of theoretical approach, to express the soil ground as a viscoelastic medium. Waves may thus be attenuated owing to internal dissipation together with geometrical wave radiation. Furthermore, most actual structures have rectangular foundations rather than circular ones.

To consider the dynamic properties of a more realistic ground model, namely, a stratified medium having internal dissipation, the foundation vibrations on a three-dimensional, homogeneous, isotropic, viscoelastic stratum over a rigid half-space are investigated in this paper, in which authors concentrate mainly upon the derivation of the analytical and numerical expressions of the "Dynamical Ground Compliance" of a rigid rectangular foundation. The viscoelastic stratum is assumed to be welded to the supporting half-space and composed of a Voigt solid, a model adopted frequently in seismic wave propagation problems. In relation to the resonance and wave attenuation phenomena, a detailed consideration of the properties of Rayleigh and Love's free waves is presented as well. Comparing the results with previous solutions in the case of a perfectly elastic medium and especially paying attention to the resonance and wave attenuation effect, the analysis is made from the viewpoint to discuss: what influence the dissipative attenuations have on the dynamic characteristics of the system; what differences do exist between the two cases of a stratum and a half-space; and under what conditions a stratum can or cannot be practically treated as a half-space.

2. Basic Equations and General Solutions

For a three-dimensional, homogeneous, isotropic, Voigt solid, the displacement vector, $\{u, v, w\}$, satisfies the following equation of motion in Cartesian coordinates (x, y, z);

$$\left[(\lambda+\mu)+(\lambda'+\mu')\frac{\partial}{\partial t}\right]\left\{\frac{\partial \Delta}{\partial x}, \frac{\partial \Delta}{\partial y}, \frac{\partial \Delta}{\partial z}\right\} + \left[\left(\mu+\mu'\frac{\partial}{\partial t}\right)\nabla^2-\rho\frac{\partial^2}{\partial t^2}\right]\{u, v, w\}=0 \tag{1}$$

where

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\text{dilatation} \tag{2a}$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\text{Laplacian operator} \tag{2b}$$

λ and μ=Lamé's constants; λ' and μ'=viscosity constants corresponding to λ and μ; ρ=density; and t=time.

By eliminating the displacement components u, v, and w, the wave equation for the dilatation Δ is obtained as

$$\left[\left(1+\frac{\lambda'+2\mu'}{\lambda+2\mu}\cdot\frac{\partial}{\partial t}\right)\nabla^2-\frac{1}{c_1^2}\cdot\frac{\partial^2}{\partial t^2}\right]\Delta=0 \tag{3}$$

where

$$c_1=\sqrt{\frac{\lambda+2\mu}{\rho}} \tag{4}$$

indicates the dilatational wave velocity in a perfectly elastic solid.

For convenience of analysis of the boundary value problem in the following sections, the triple Fourier transform of Δ with respect to the coordinates x and y, and time t is introduced.

$$\mathfrak{F}^3[\Delta(x, y, z, t)]=\bar{\Delta}(\beta, \gamma, z, \omega)$$
$$=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(x, y, z, t)e^{-i(\beta x+\gamma y+\omega t)}dx\,dy\,dt \tag{5}$$

Hence, the inverse transform is given by the equation

$$\mathfrak{F}^{-3}[\tilde{\Delta}(\beta, \gamma, z, \omega)] = \Delta(x, y, z, t)$$
$$= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\Delta}(\beta, \gamma, z, \omega) e^{+i(x\beta+y\gamma+t\omega)} d\beta\, d\gamma\, d\omega \tag{6}$$

By making use of the theorem of the Fourier transform[22)]

$$\mathfrak{F}^{3}\left[\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right\}\Delta\right] = \{i\beta, i\gamma, i\omega\}\tilde{\Delta} \tag{7}$$

the triple Fourier transform of Eq. (3) becomes

$$\left[\frac{d^2}{dz^2} - \{(\beta^2+\gamma^2) - h^2 g_1\}\right]\tilde{\Delta} = 0 \tag{8}$$

The solution of the above equation is expressed by

$$\tilde{\Delta} = A_1 e^{-\alpha_1 z} + A_2 e^{+\alpha_1 z} \tag{9}$$

where

$$\alpha_1^2 = \beta^2 + \gamma^2 - h^2 g_1, \qquad h^2 = \frac{\omega^2}{c_1^2}, \qquad g_1 = \left[1 + i\omega\frac{\lambda' + 2\mu'}{\lambda + 2\mu}\right]^{-1} \tag{10}$$

Let the triple Fourier transform of the displacement vector $\{u, v, w\}$ be denoted as

$$\mathfrak{F}^{3}[\{u, v, w\}] = \{\tilde{u}, \tilde{v}, \tilde{w}\} \tag{11}$$

Substitution of Eqs. (9) and (11) into the triple Fourier transform of Eq. (1) yields

$$\left[\frac{d^2}{dz^2} - \alpha_2^2\right]\{\tilde{u}, \tilde{v}, \tilde{w}\} = (\kappa^2 g_2 - h^2 g_1)\{-i\beta X_1(z), -i\gamma X_1(z), \alpha_1 X_2(z)\} \tag{12}$$

where

$$\alpha_2^2 = \beta^2 + \gamma^2 - \kappa^2 g_2, \qquad \kappa^2 = \frac{\omega^2}{c_2^2}, \qquad g_2 = \left[1 + i\omega\frac{\mu'}{\mu}\right]^{-1} \tag{13a}$$

$$X_1(z) = \frac{A_1 e^{-\alpha_1 z} + A_2 e^{+\alpha_1 z}}{h^2 g_1}, \qquad X_2(z) = \frac{A_1 e^{-\alpha_1 z} - A_2 e^{+\alpha_1 z}}{h^2 g_1} \tag{13b}$$

$$c_2 = \sqrt{\frac{\mu}{\rho}} \tag{13c}$$

and c_2 indicates the distortional wave velocity in a perfectly elastic solid.

The general solutions of $\{u, v, w\}$ for Eq. (12) are then

$$\{\tilde{u}, \tilde{v}, \tilde{w}\} = \{-i\beta X_1(z), -i\gamma X_1(z), \alpha_1 X_2(z)\}$$
$$+ \{B_1, C_1, D_1\} e^{-\alpha_2 z} + \{B_2, C_2, D_2\} e^{+\alpha_2 z} \tag{14}$$

By the substitution of Eqs. (9) and (14) into the triple Fourier transform of Eq. (2a), the following identity is obtained;

$$(-i\beta B_1-i\gamma C_1+\alpha_2 D_1)e^{-\alpha_2 z}-(i\beta B_2+i\gamma C_2+\alpha_2 D_2)e^{+\alpha_2 z}=0 \qquad (15)$$

The quantities D_1 and D_2 can then be expressed in terms of B_1, B_2, C_1, and C_2, that is,

$$D_1=\frac{i}{\alpha_2}(\beta B_1+\gamma C_1), \qquad D_2=-\frac{i}{\alpha_2}(\beta B_2+\gamma C_2) \qquad (16)$$

The independent quantities A_1, A_2, B_1, B_2, C_1, and C_2 in the preceding equations are the arbitrary constants to be determined from the boundary conditions.

With the aid of Eqs. (14) and (16), the general solutions for the triple Fourier transforms of the stress components $\{\tau_{xz}, \tau_{yz}, \sigma_z\}$ are obtained as follows;

$$\{\tilde{\tau}_{xz}, \tilde{\tau}_{yz}, \tilde{\sigma}_z\}=\mathfrak{F}^3\left[\left\{\frac{\mu}{g_2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right), \frac{\mu}{g_2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),\right.\right.$$
$$\left.\left.\left(\frac{\lambda+2\mu}{g_1}-2\frac{\mu}{g_2}\right)\Delta+2\frac{\mu}{g_2}\cdot\frac{\partial w}{\partial z}\right\}\right]$$
$$=\left\{\frac{\mu}{g_2}\left(i\beta\tilde{w}+\frac{d\tilde{u}}{dz}\right), \frac{\mu}{g_2}\left(i\gamma\tilde{w}+\frac{d\tilde{v}}{dz}\right), \left(\frac{\lambda+2\mu}{g_1}-2\frac{\mu}{g_2}\right)\tilde{\Delta}+2\frac{\mu}{g_2}\cdot\frac{d\tilde{w}}{dz}\right\}$$
$$=-\frac{\mu}{g_2}\left[\{-2i\alpha_1\beta X_2(z), -2i\alpha_1\gamma X_2(z), (2\alpha_2^2+\kappa^2 g_2)X_1(z)\}\right.$$
$$\left.+\{Y_{11}, Y_{21}, 2\alpha_2 D_1\}e^{-\alpha_2 z}-\{Y_{12}, Y_{22}, 2\alpha_2 D_2\}e^{+\alpha_2 z}\right] \qquad (17)$$

where

$$Y_{1j}=\frac{(\beta^2+\alpha_2^2)B_j+\beta\gamma C_j}{\alpha_2}, \qquad Y_{2j}=\frac{\beta\gamma B_j+(\gamma^2+\alpha_2^2)C_j}{\alpha_2} \qquad [j=1 \text{ and } 2] \quad (18)$$

The general solutions for a Voigt solid obtained in this section are easily extended to those for the general class of linear viscoelastic solids, when the quantities g_1 and g_2 are replaced by the somewhat general expressions $g_1(i\omega)$ and $g_2(i\omega)$, respectively. They may now be considered as operators representing the frequency characteristics of a viscoelastic solid. In particular, when substituting $\lambda'=\mu'=0$, *i. e.*, $g_1=g_2=1$, in the foregoing equations, they result in the solutions for a perfectly elastic solid.

3. Boundary Conditions

In the problem schematically shown in Fig. 1, the following three kinds of exciting force are considered:

$$P_l Q(t) \qquad [l=V, H, \text{ and } R]$$

where $Q(t)$ is a time factor; P_l is the amplitude of an exciting force; and V, H, and R are subscripts indicating vertical, horizontal, and rotational excitations, respectively. Each exciting force acts on a rigid, rectangular, massless foundation of dimensions

$2b \times 2c$ resting on the surface of a three-dimensional, homogeneous, isotropic, Voigt-type viscoelastic stratum which has the constant thickness H lying over a rigid half-space.

Boundary conditions at the upper surface of the stratum ($z=0$) — Strictly speaking, the boundary conditions at the upper surface of the stratum have to be expressed by the combination of the displacements in the loading area beneath the rigid foundation and the stresses outside it. Since such treatment of the problem generally results in a so-called "mixed boundary value problem" involving simultaneous dual integral equations, it seems difficult to obtain the analytical solutions of the dynamic problem. Hence, we adopt the same treatment as in most of the previous studies in which all of the boundary conditions are expressed in stresses by assuming appropriate stress distributions beneath the foundation.

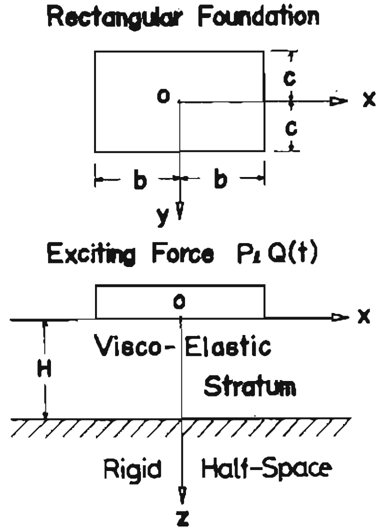


Fig. 1. Geometry of the foundation-ground system considered.

Corresponding to the type of exciting force, they are given as follows:

(1) *Vertical excitation in the z direction*

$$\sigma_z = \begin{cases} 0 & \text{(in the domain } \bar{D}) \\ -q_{0V}\cdot Q(t) & \text{(in the domain } D), \end{cases} \qquad \tau_{xz}=\tau_{yz}=0 \tag{19}$$

(2) *Horizontal excitation in the x direction*

$$\tau_{xz} = \begin{cases} 0 & \text{(in the domain } \bar{D}) \\ -q_{0H}\cdot Q(t) & \text{(in the domain } D), \end{cases} \qquad \sigma_z=\tau_{yz}=0 \tag{20}$$

(3) *Rotational excitation about the y-axis*

$$\sigma_z = \begin{cases} 0 & \text{(in the domain } \bar{D}) \\ -q_{0R}\dfrac{x}{b}\cdot Q(t) & \text{(in the domain } D), \end{cases} \qquad \tau_{xz}=\tau_{yz}=0 \tag{21}$$

where

$$D=\{|x|\leqq b \cap |y|\leqq c\} \tag{22a}$$

$$\bar{D}=\{|x|>b \cup |y|>c\} \tag{22b}$$

The triple Fourier transforms of the above Eqs. (19) to (21) are:

(1) *Vertical excitation*

$$\tilde{\sigma}_z = -\frac{4q_{0V}bc}{2\pi} S(\beta, \gamma)\bar{Q}(\omega), \qquad \tilde{\tau}_{xz} = \tilde{\tau}_{yz} = 0 \tag{23}$$

(2) *Horizontal excitation*

$$\tilde{\tau}_{xz} = -\frac{4q_{0H}bc}{2\pi} S(\beta, \gamma)\bar{Q}(\omega), \qquad \tilde{\sigma}_z = \tilde{\tau}_{yz} = 0 \tag{24}$$

(3) *Rotational excitation*

$$\tilde{\sigma}_z = i\frac{4q_{0R}bc}{2\pi} \cdot \frac{S(\beta, \gamma)}{\sin \beta b} N(\beta b)\bar{Q}(\omega), \qquad \tilde{\tau}_{xz} = \tilde{\tau}_{yz} = 0 \tag{25}$$

where

$$S(\beta, \gamma) = \frac{\sin \beta b \cdot \sin \gamma c}{\beta b \cdot \gamma c} \tag{26a}$$

$$N(\beta b) = \frac{\sin \beta b}{\beta b} - \cos \beta b \tag{26b}$$

$$\bar{Q}(\omega) = \mathfrak{F}_t[Q(t)] = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} Q(t) e^{-i\omega t} dt \tag{26c}$$

The amplitude of the exciting force for each case thus becomes

$$P_V = -4bcq_{0V}, \qquad P_H = -4bcq_{0H}, \qquad M_R = -\frac{4}{3} b^2 c q_{0R} \tag{27}$$

Boundary conditions at the surface between the stratum and the rigid half-space ($z=H$)—Here is considered the case of a welded contact at the boundary surface between the stratum and its supporting rigid half-space, namely,

$$u = v = w = 0 \tag{28}$$

Since three of the arbitrary constants can now be eliminated by the above conditions, A_2, B_2, and C_2 are expressed in terms of A_1, B_1, and C_1.

4. Frequency Equations

If all of the stress components given by by Eq. (17) vanish at the surface of a stratum, $z=0$, three equations for the independent arbitrary constants, A_1, B_1, and C_1, are derived. In order that these equations may have nontrivial solutions, the determinant of the coefficients must vanish. After some tedious manipulation, the two equations

$$\begin{aligned} F(\beta, \gamma) \equiv\ & 4\alpha_1\alpha_2(\beta^2+\gamma^2)(2\alpha_2^2+\kappa^2 g_2)\operatorname{cosech} \alpha_1 H \cdot \operatorname{cosech} \alpha_2 H \\ & -\alpha_1\alpha_2\{4(\beta^2+\gamma^2)^2+(2\alpha_2^2+\kappa^2 g_2)^2\}\coth \alpha_1 H \cdot \coth \alpha_2 H \\ & +(\beta^2+\gamma^2)\{4\alpha_1^2\alpha_2^2+(2\alpha_2^2+\kappa^2 g_2)^2\} = 0 \end{aligned} \tag{29}$$

and

$$L(\beta,\gamma)\equiv\coth\alpha_2 H=0 \tag{30}$$

are obtained. These are respectively the frequency equations for the Rayleigh and Love waves in a viscoelastic stratum over a rigid half-space. If the thickness of the stratum H tends to infinity in Eq. (29), it reduces to the frequency equation for the Rayleigh waves in a viscoelastic half-space;

$$F_0(\beta,\gamma)\equiv\{2(\beta^2+\gamma^2)-\kappa^2 g_2\}^2-4\alpha_1\alpha_2(\beta^2+\gamma^2)=0 \tag{31}$$

By the introduction of the two kinds of variable transform

$$\zeta^2=\beta^2+\gamma^2 \tag{32a}$$

$$\xi=\frac{\zeta}{\kappa}=c_2\frac{\zeta}{\omega}=\frac{c_2}{c_p} \tag{32b}$$

where $\zeta=$ the wave number and $c_p=\omega/\zeta=$ the phase velocity, and by using the non-dimensional quantities

$$n^2=\frac{h^2}{\kappa^2}=\frac{c_2^2}{c_1^2}=\frac{1-2\nu}{2(1-\nu)} \tag{33a}$$

$$a_1=\kappa H=\frac{\omega}{c_2}H \tag{33b}$$

$$\eta_{H2}=\frac{c_2}{H}\cdot\frac{\mu'}{\mu},\qquad \eta_{H1}=\frac{c_2}{H}\cdot\frac{\lambda'+2\mu'}{\lambda+2\mu}=n^2\left(\frac{\lambda'}{\mu'}+2\right)\eta_{H2} \tag{33c}$$

where $\nu=$Poisson's ratio, Eqs. (29) to (31) are transformed into the following non-dimensional forms;

$$\begin{aligned}F(\xi)\equiv&[4\xi^2(2\xi^2-g_2)\operatorname{cosech}(\sqrt{\xi^2-n^2g_1}a_1)\cdot\operatorname{cosech}(\sqrt{\xi^2-g_2}a_1)\\&-\{4\xi^4+(2\xi^2-g_2)^2\}\coth(\sqrt{\xi^2-n^2g_1}a_1)\cdot\coth(\sqrt{\xi^2-g_2}a_1)]\\&\cdot\sqrt{\xi^2-n^2g_1}\sqrt{\xi^2-g_2}+\xi^2\{4(\xi^2-n^2g_1)(\xi^2-g_2)+(2\xi^2-g_2)^2\}=0\end{aligned} \tag{34}$$

$$L(\xi)\equiv\coth(\sqrt{\xi^2-g_2}a_1)=0 \tag{35}$$

$$F_0(\xi)\equiv(2\xi^2-g_2)^2-4\xi^2\sqrt{\xi^2-n^2g_1}\sqrt{\xi^2-g_2}=0 \tag{36}$$

where

$$g_1=\frac{1}{1+i\eta_{H1}a_1},\qquad g_2=\frac{1}{1+i\eta_{H2}a_1} \tag{37}$$

Eqs. (34) and (35) generally have an infinite number of complex roots whereas Eq.

(36) has a finite number of complex roots. Let these roots be denoted by

$$\xi_{ok}=\boldsymbol{R}(\xi_{ok})+i\boldsymbol{I}(\xi_{ok}) \qquad [k=1,\ 2,\ \cdots] \tag{38}$$

where $\boldsymbol{R}$ and $\boldsymbol{I}$ represent respectively the real and the imaginary parts of a complex number. The real and imaginary parts of each root relate to the phase velocity c_p of the Rayleigh or the Love waves of a certain mode and the attenuation constant of the amplitude of the free waves, respectively. Eq. (36) for a half-space has only one complex root in the range $-\pi/2<\arg(\xi_{ok})\leqq\pi/2$. For the case of a perfectly elastic medium in which $\eta_{H1}=\eta_{H2}=0$, all of the above frequency equations have a finite number of real roots.

Since the frequency equations for a perfectly elastic medium are real-valued functions, their roots can easily be found by examining the sign of the value of the equation with increase of the argument ξ. For a viscoelastic medium, on the other hand, it becomes difficult to find the complex roots, because the frequency equations are transcendental complex-valued equations involving parabolic functions. Their complex-valued roots are calculated here by the Newton-Raphson method in which the real roots obtained for a perfectly elastic medium are used as initial values.

From the roots of the frequency equations the following quantities can be evaluated;

attenuation constant	$\alpha=\boldsymbol{I}(\zeta_{ok})$	$\left[\dfrac{\alpha}{\omega}c_2=\boldsymbol{I}(\xi_{ok})\right]$	(39a)
wave number	$\boldsymbol{R}(\zeta_{ok})$	$[\boldsymbol{R}(\zeta_{ok})H=\boldsymbol{R}(\xi_{ok})a_1]$	(39b)
Hamiltonian	$\omega=\omega(\boldsymbol{R}(\zeta_{ok}))$	$\left[\omega\dfrac{H}{c_2}=a_1(\boldsymbol{R}(\zeta_{ok})H)\right]$	(39c)
phase velocity	$c_p=\dfrac{\omega}{\boldsymbol{R}(\zeta_{ok})}$	$\left[\dfrac{c_p}{c_2}=\dfrac{1}{\boldsymbol{R}(\xi_{ok})}\right]$	(39d)
group velocity	$c_g=\dfrac{d\omega}{d\boldsymbol{R}(\zeta_{ok})}$	$\left[\dfrac{c_g}{c_2}=\dfrac{da_1}{d[\boldsymbol{R}(\zeta_{ok})H]}\right]$	(39e)

where

$$\boldsymbol{R}(\zeta_{ok})=\frac{\omega}{c_2}\boldsymbol{R}(\xi_{ok}), \qquad \boldsymbol{I}(\zeta_{ok})=\frac{\omega}{c_2}\boldsymbol{I}(\xi_{ok}) \tag{40}$$

and the quantities in brackets are the corresponding non-dimensional representations.

Results for the case of Poisson's ratio $\nu=1/4$ (i.e., $n^2=1/3$) and $\lambda'/\mu'=1$, meaning that the viscosity constant for the dilatational waves agrees with that for the distortional waves, are shown in Figs. 2–4. Figs. 2a, 3a, and 4a indicate the solutions for the first five modes of Rayleigh waves and Figs. 2b, 3b, and 4b show those for the first three modes of Love waves, respectively. Inspection of the frequency equations suggests that there are always a couple of complex roots, $\pm\xi_{ok}$, whose arguments differ by π from each other. Only the roots that satisfy the condition, $-\pi<\arg(\xi_{ok})\leqq 0$, are shown in these figures.

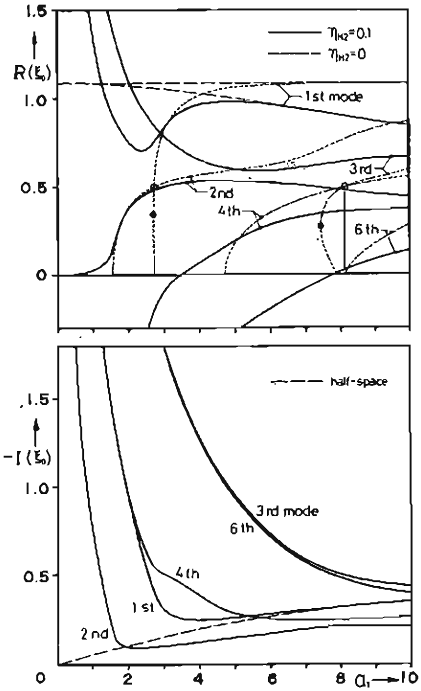
The real and imaginary parts of complex roots are separately shown in Fig. 2 as a function of the non-dimensional circular frequency a_1, which is defined as the ratio of the product of the circular frequency ω and the thickness of the stratum H to the velocity of the distortional waves c_2 in a perfectly elastic medium. The dotted and dashed lines in these figures indicate the solutions for a perfectly elastic stratum and those for a half-space, respectively.

Figs. 3 and 4 show respectively the non-dimensional phase velocity c_p/c_2 and the non-dimensional group velocity c_g/c_2 versus the product of the thickness of the stratum H and the wave number $\boldsymbol{R}(\zeta_{ok})$ which will be abbreviated to ζ in this section. The dotted lines indicate the case of a perfectly elastic stratum. The dashed lines in Fig. 3a mean that the curves are transferred from the third quadrant to the first and in Fig. 4a from the second to the first, respectively.

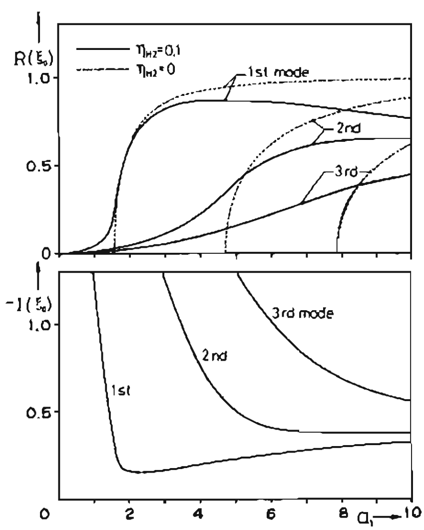
From these figures the following remarks on the properties of free waves are obtained:

Rayleigh waves—Only one mode of propagation is possible for the Rayleigh waves in a half-space and, especially in a perfectly elastic medium, their phase velocity is determined independently of the wave number ζ. For both a perfectly elastic and a viscoelastic stratum, on the other hand, various modes of Rayleigh waves may be propagated and all of these modes are "dispersive", that is, their velocities depend on the wave number.

In a perfectly elastic stratum, the number of real roots increases, that is, the number of the mode of propagation increases with the frequency parameter $a_1=\omega H/c_2$. It is noticed that no Rayleigh waves appear in the range $0\leqq a_1<a_{1cr}$, in which $a_{1cr}=\pi/2$ corresponds to the fundamental natural frequency $\omega_{cr}=\pi\sqrt{\mu/\rho}/2H$ of the lateral vibration of a rod with the same length H and similar end conditions (that is, one end

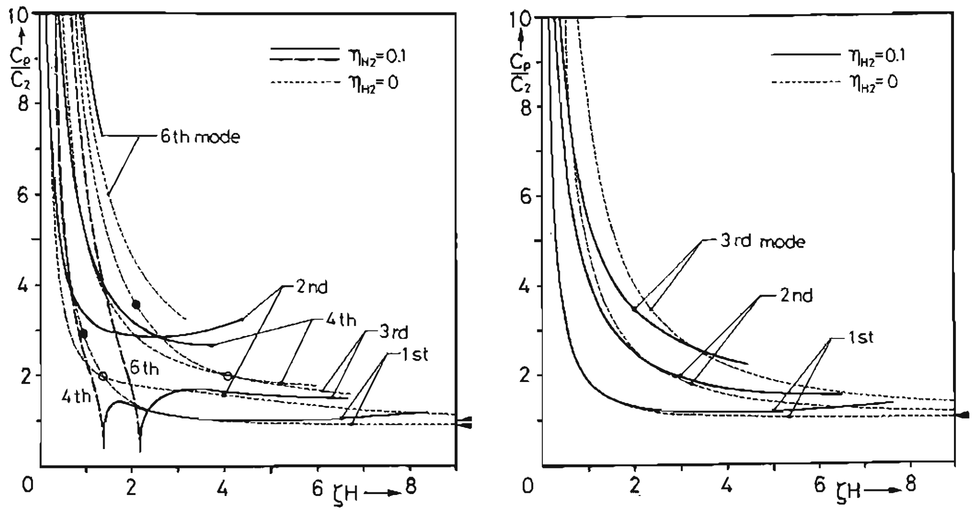


(a) Rayleigh waves.



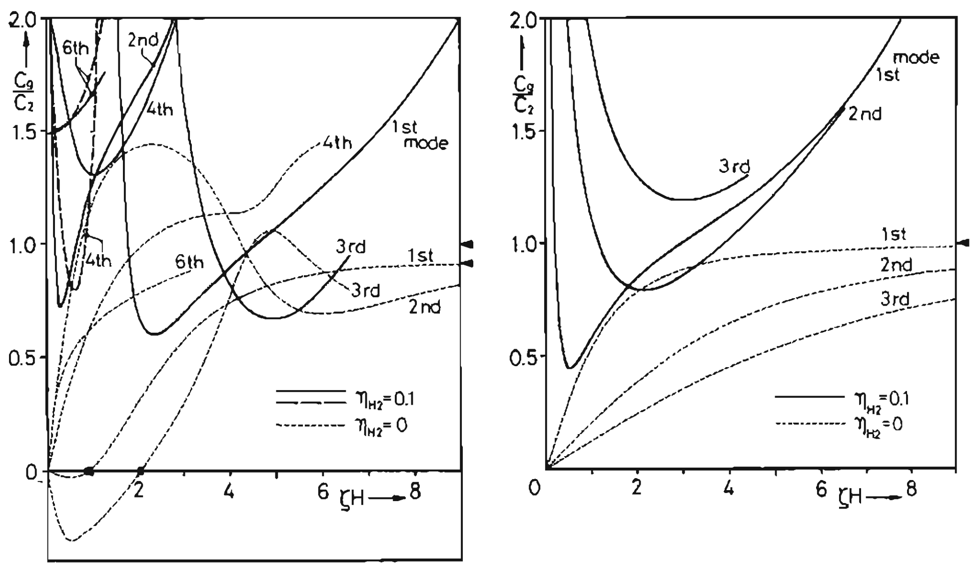
(b) Love waves.

Fig. 2. Roots of the frequency equation of a stratum. ($\nu=1/4$ and $\lambda'/\mu'=1$.)



(a) Rayleigh waves. (b) Love waves.

Fig. 3. Phase velocity of free waves. ($\nu=1/4$ and $\lambda'/\mu'=1$.)



(a) Rayleigh waves. (b) Love waves.

Fig. 4. Group velocity of free waves. ($\nu=1/4$ and $\lambda'/\mu'=1$.)

is fixed and the other is free) as the stratum. As a_1 or ζH increases, the phase and the group velocity of the 1st mode approach the Rayleigh wave velocity in a half-space and those of other modes all approach the distortional wave velocity.

In a viscoelastic stratum, on the other hand, all modes of propagation of Rayleigh waves are possible in the whole range of the frequency parameter a_1. The real parts of the complex roots $\boldsymbol{R}(\xi_{ok})$ are separated into two groups. As a_1 dicreases, the real part approaches zero in one group and diverges to infinity in the other; the phase velocity diverges in the former group and vanishes in the latter. Since the imaginary part $\boldsymbol{I}(\xi_{ok})$ or the attenuation constant $\alpha=\boldsymbol{I}(\zeta_{ok})$ of each mode increases rapidly with the decrease of a_1 in the lower frequency range of the associated critical frequency below which the corresponding mode of Rayleigh waves in a perfectly elastic stratum does not appear, it is suggested that the Rayleigh waves of the mode may have little effect on the dynamic properties of a stratum in such a lower frequency range.

As for the relation among the modes of Rayleigh waves in a viscoelastic stratum, it is indicated that the phase velocity of the 1st (or 3rd) mode approaches the velocity with opposite sign of the 4th (or 6th) mode, and the attenuation constant of the former mode also approaches that of the latter, respectively as the parameter a_1 decreases.

The phase velocity for all modes of Rayleigh waves in a perfectly elastic stratum is a monotonically decreasing function of ζH in the range $0<\zeta H<\infty$. On the other hand, for a viscoelastic stratum, the waves of even modes appear in the whole range of ζH and most of them show a similar tendency to those for a perfectly elastic stratum. For the waves of odd modes, however, there exists the range $0<\zeta H<(\zeta H)_{cr}$ in which the waves of the mode do not appear.

In the case of perfectly elastic stratum there are two kinds of double root, namely:

(*a*) *A double root connected with some single mode at which the group velocity vanishes.*
(The mark "●" in the figures indicates these points.)

(*b*) *A double root connected with two distinct modes at which the phase velocities for the two modes coincide with each other.*
(The mark "○" in the figures indicates these points.)

It is found that the phase velocity at the double root of type (*b*) is just twice the distortional wave velocity in a perfectly elastic medium and also that the value of the parameter a_1 at which the double root of the type (*b*) occurs agrees with the value of a_1 at which ξ_{ok} vanishes in one of the two distinct modes or in another possible mode. In Fig. 2a, for instance, the parameter a_1 at which the double root occurs associated with the 1st and 2nd modes (or the 3rd and 4th modes) agrees with a_1 at which ξ_{ok} vanishes in the 1st (or 6th) mode. It is noted, moreover, that these values of a_1 correspond to the natural frequencies, $a_1=(2m+1)\pi/2n$ [m=integer], of the longitudinal vibration of a rod having a length H which equals the thickness of the stratum and end conditions of the fixed-free type which are similar to the boundary conditions of the stratum. The points at which dispersion curves cross the abscissa in Fig. 2a show the condition $\xi_{ok}=0$ that means the zero wave number or the infinite wavelength. Furthermore, the frequencies evaluated from the enumerably infinite number of values a_1's which satisfy the equation $\xi_{ok}=0$ coincide with the natural frequencies of the above-mentioned one-dimensional rod for the transverse and longitudinal vibrations and also correspond to the resonant frequencies of *Case* ① that will be described in section 8.

Love waves — Any modes of wave propagation of this type are impossible in a half-space. It is found, however, that various modes of Love waves, all of which are dispersive, may propagate in both a perfectly elastic and a viscoelastic stratum.

In a perfectly elastic stratum, the real roots ξ_{ok} for Love waves have a tendency as regards the frequency parameter a_1 similar to that for Rayleigh waves, and no waves of this type appear in the range $0 \leqq a_1 < a_{1cr}$, where a_{1cr} is common with the previously-mentioned critical frequency for Rayleigh waves. The phase velocity, which is a monotonically decreasing function of a_1 or ζH, is always larger than the distortional wave velocity. On the contrary, the group velocity, which is a monotonically increasing function of a_1 or ζH, is always smaller than the distortional wave velocity, to which both the phase and the group velocities approach as the parameter a_1 or ζH increases.

In a viscoelastic stratum, on the other hand, an enumerably infinite number of modes of propagation of Love waves is possible in the whole range of a_1 and as the parameter a_1 decreases, the real part of every mode vanishes, that is, the phase velocity diverges to infinity. No complex root appears in the first and third quadrants in the complex ξ-plane for these waves different from that for Rayleigh waves. The behavior of the imaginary part with a_1 is similar to that of Rayleigh waves; the attenuation constant of each mode increases rapidly in the lower frequency range of the relevant critical frequency below which Love waves of the corresponding mode do not appear in a perfectly elastic stratum. In this case, in contrast to the case of Rayleigh waves, there does not exist the range of ζH in which the propagation of free waves of this type is impossible.

The frequencies, evaluated from the enumerably infinite number of values a_1's which satisfy the equation $\xi_{ok}=0$ for a perfectly elastic stratum, coincide with the natural frequencies of lateral vibration of the one-dimensional rod mentioned previously in the case of Rayleigh waves. For waves of the Love type, there are no such double roots as have appeared in the case of Rayleigh waves.

5. Solutions for Displacements at the Ground Surface

Three equations for the arbitrary constants, A_1, B_1, and C_1, are obtained by substituting the Fourier transforms of the boundary conditions given by Eqs. (23) to (25) into the solutions of the transformed stress components, Eq. (17), and by setting $z=0$. Since all of the eight unknown constants contained in the transformed displacement components, Eq. (14), are then determined from these equations, the solutions of the displacement components at the surface of the ground can now be obtained by performing the inverse Fourier transform. The solutions for the displacements at the ground surface corresponding to each type of exciting force given by Eqs. (19) to (21) are expressed in the following integral representations:

(1) *Vertical excitation*

$$u=\mathfrak{F}^{-3}\left[-\frac{i\beta}{F(\beta,\gamma)}W(\beta,\gamma)\left\{\frac{4bcq_{0V}}{2\pi\mu}S(\beta,\gamma)\bar{Q}(\omega)\right\}\right]$$

$$=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}i\frac{P_V g_2}{2\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{F(\beta,\gamma)}W(\beta,\gamma)S(\beta,\gamma)\bar{Q}(\omega)$$

$$\cdot e^{+i(x\beta+y\gamma+t\omega)}d\beta\, d\gamma\, d\omega$$

$$= -\frac{P_V g_2}{\pi^2 \mu}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\beta}{F(\beta,\gamma)}W(\beta,\gamma)S(\beta,\gamma)$$

$$\cdot\sin x\beta\cdot\cos y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \qquad (41a)$$

$$v = -\frac{P_V g_2}{\pi^2 \mu}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\gamma}{F(\beta,\gamma)}W(\beta,\gamma)S(\beta,\gamma)$$

$$\cdot\cos x\beta\cdot\sin y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \qquad (41b)$$

$$w = \frac{P_V g_2}{\pi^2 \mu}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\alpha_1\kappa^2}{F(\beta,\gamma)}T_V(\beta,\gamma)S(\beta,\gamma)$$

$$\cdot\cos x\beta\cdot\cos y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \qquad (41c)$$

where

$$W(\beta,\gamma) = \{(2\alpha_2^2+\kappa^2 g_2)(\beta^2+\gamma^2)+2\alpha_1^2\alpha_2^2\} + \alpha_1\alpha_2\{(2\alpha_2^2+\kappa^2 g_2)+2(\beta^2+\gamma^2)\}$$

$$\cdot(\mathrm{cosech}\,\alpha_1 H\cdot\mathrm{cosch}\,\alpha_2 H - \coth\alpha_1 H\cdot\coth\alpha_2 H) \qquad (42a)$$

$$T_V(\beta,\gamma) = (\beta^2+\gamma^2)\coth\alpha_1 H - \alpha_1\alpha_2\coth\alpha_2 H \qquad (42b)$$

(2) *Horizontal excitation*

$$u = \frac{P_H g_2}{\pi^2 \mu}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\left[\frac{\kappa^2 g_2\alpha_2^2\beta^2}{F(\beta,\gamma)}T_H(\beta,\gamma) - \frac{\gamma^2}{L(\beta,\gamma)}\right]\frac{S(\beta,\gamma)}{\alpha_2(\beta^2+\gamma^2)}$$

$$\cdot\cos x\beta\cdot\cos y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \qquad (43a)$$

$$v = -\frac{P_H g_2}{\pi^2 \mu}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\beta\gamma\left[\frac{\kappa^2 g_2\alpha_2^2}{F(\beta,\gamma)}T_H(\beta,\gamma) - \frac{1}{L(\beta,\gamma)}\right]\frac{S(\beta,\gamma)}{\alpha_2(\beta^2+\gamma^2)}$$

$$\cdot\sin x\beta\cdot\sin y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \qquad (43b)$$

$$w = \frac{P_H g_2}{\pi^2 \mu}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\beta}{2}\left[\frac{\kappa^2 g_2}{F(\beta,\gamma)}G(\beta,\gamma)+1\right]\frac{S(\beta,\gamma)}{\alpha_2(\beta^2+\gamma^2)}$$

$$\cdot\sin x\beta\cdot\cos y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \qquad (43c)$$

where

$$T_H(\beta,\gamma) = (\beta^2+\gamma^2)\coth\alpha_2 H - \alpha_1\alpha_2\coth\alpha_1 H \qquad (44a)$$

$$G(\beta,\gamma) = (2\alpha_2^2+\kappa^2 g_2)\{(\beta^2+\gamma^2) - \alpha_1\alpha_2\coth\alpha_1 H\cdot\coth\alpha_2 H\}$$

$$+ 2\alpha_1\alpha_2(\beta^2+\gamma^2)\,\mathrm{cosech}\,\alpha_1 H\cdot\mathrm{cosech}\,\alpha_2 H \qquad (44b)$$

(3) *Rotational excitation*

$$u=-\frac{3M_R g_2}{\pi^2\mu b}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\beta}{F(\beta,\ \gamma)}W(\beta,\ \gamma)\frac{S(\beta,\ \gamma)}{\sin\beta b}N(\beta b)$$
$$\cdot\cos x\beta\cdot\cos y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \tag{45a}$$

$$v=\frac{3M_R g_2}{\pi^2\mu b}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\gamma}{F(\beta,\ \gamma)}W(\beta,\ \gamma)\frac{S(\beta,\ \gamma)}{\sin\beta b}N(\beta b)$$
$$\cdot\sin x\beta\cdot\sin y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \tag{45b}$$

$$w=\frac{3M_R g_2}{\pi^2\mu b}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\alpha_1\kappa^2}{F(\beta,\ \gamma)}T_V(\beta,\ \gamma)\frac{S(\beta,\ \gamma)}{\sin\beta b}N(\beta b)$$
$$\cdot\sin x\beta\cdot\cos y\gamma\cdot\bar{Q}(\omega)e^{+it\omega}d\beta d\gamma d\omega \tag{45c}$$

The above solutions reduce to those for a viscoelastic half-space if the thickness of stratum H tends to infinity.

6. Definition of Dynamical Ground Compliance and Its Analytical Expressions

The solutions for Fourier-transformed displacement components at the surface of the ground may be represented by the general expression

$$\bar{\boldsymbol{U}}(\omega)=P_l\bar{Q}(\omega)\cdot\bar{\boldsymbol{J}}(\omega) \tag{46a}$$

or its inverse expression

$$\boldsymbol{U}(t)=P_lQ(t)*\boldsymbol{J}(t)=P_l\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}Q(\tau)\boldsymbol{J}(t-\tau)d\tau$$
$$=P_l\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{t}Q(\tau)\boldsymbol{J}(t-\tau)d\tau \tag{46b}$$

where

$$\bar{\boldsymbol{U}}(\omega)=\{\tilde{u},\ \tilde{v},\ \tilde{w}\}=\mathfrak{F}_t[\{u,\ v,\ w\}]=\mathfrak{F}_t[\boldsymbol{U}(t)] \tag{47a}$$

$$\bar{\boldsymbol{J}}(\omega)=\mathfrak{F}_t[\boldsymbol{J}(t)]\quad[l=V,\ H,\ \text{and } R=\text{type of exciting force}] \tag{47b}$$

and $\mathfrak{F}_t$=operator indicating the Fourier transform with respect to time t.

Since the function $\bar{\boldsymbol{J}}(\omega)$ is defined as the output-input ratio, $\bar{\boldsymbol{U}}(\omega)/P_l\bar{Q}(\omega)$, in the domain of the Fourier transform with respect to time t, it means the complex transfer function of displacement components to an exciting force associated with the dynamic problem of a massless-foundation-ground system considered in this paper.

If the time factor

$$P_lQ(t)=\delta(t)[=\text{Dirac's delta-function}] \tag{48}$$

in substituted into Eq. (46b), the equation

$$U(t)=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\delta(\tau)\cdot J(t-\tau)\,d\tau=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}J(t)\quad t\geqq 0,\qquad U(t)=0\quad t<0 \tag{49}$$

is then derived to represent the impulsive response of this system.

On the other hand, if the time factor

$$Q(t)=e^{i\omega t} \tag{50}$$

with its Fourier transform

$$\bar{Q}(\omega')=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}e^{i(\omega-\omega')t}\,dt=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}2\pi\delta(\omega-\omega')=\left(\frac{1}{2\pi}\right)^{-\frac{1}{2}}\delta(\omega-\omega') \tag{51}$$

is substituted into Eq. (46a), the inverse Fourier transform of $\bar{U}(\omega')$ becomes

$$U(t)=P_l\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\right)^{-\frac{1}{2}}\delta(\omega-\omega')\bar{J}(\omega')e^{+it\omega'}\,d\omega'=P_l e^{i\omega t}\bar{J}(\omega) \tag{52}$$

That is to say, it is found that the function $\bar{J}(\omega)$ is expressed also by the ratio of the displacement components $\bar{U}(t)$ to the complex harmonic exciting force $P_l e^{i\omega t}$.

The complex frequency response of a representative displacement of a foundation in the exciting direction, which can be evaluated from the function $\bar{J}(\omega)$, is now defined as the "Dynamical Ground Compliance (*D.G.C.*)" of a rectangular foundation on a viscoelastic medium. As previously mentioned, this is the force-displacement transfer function with the inverse dimension of stiffness and represents comprehensively the dynamic characteristics of a massless-foundation-ground system. If this quantity is determined in the whole frequency range, the representative displacement of a foundation subjected to an arbitrary transient exciting force $P_l Q(t)$ can be evaluated from Eq. (46b).

For a representative displacement of a foundation, the following displacements of the ground in the loading area can be adopted;

(1) *Vertical excitation* $\qquad w_0=w|_{x=y=z=0}$ (53a)

(2) *Horizontal excitation* $\qquad u_0=u|_{x=y=z=0}$ (53b)

(3) *Rotational excitation* $\qquad \phi=w|_{x=b,\,y=z=0}/b$ (53c)

That is to say, the displacement in the exciting direction at the center of the foundation is chosen in the cases of vertical and horizontal excitations, and the average rotational angle defined by the ratio between the vertical displacement at the center of a side of the foundation area parallel to the rotational axis and the distance from the axis to the center of the side is selected in the case of rotational excitation.

The analytical expressions of *D.G.C.* are derived from the solutions of the displacement components, Eqs. (41c), (43a), and (45c), as follows;

(1) *Vertical excitation*

$$\frac{\mathfrak{F}_t[w_0]}{\mathfrak{F}_t[P_V Q(t)]}=\frac{w_0}{P_V e^{i\omega t}}=\frac{g_2}{\pi^2\mu}\int_0^{\infty}\int_0^{\infty}\frac{\alpha_1\kappa^2}{F(\beta,\gamma)}T_V(\beta,\gamma)S(\beta,\gamma)\,d\beta\,d\gamma \tag{54}$$

(2) *Horizontal excitation*

$$\frac{\mathfrak{F}_t[u_0]}{\mathfrak{F}_t[P_H Q(t)]}=\frac{u_0}{P_H e^{i\omega t}}=\frac{g_2}{\pi^2\mu}\int_0^\infty\int_0^\infty\left[\frac{\kappa^2 g_2\alpha_2^2\beta^2}{F(\beta,\gamma)}T_H(\beta,\gamma)-\frac{\gamma^2}{L(\beta,\gamma)}\right]\cdot\frac{S(\beta,\gamma)}{\alpha_2(\beta^2+\gamma^2)}d\beta\,d\gamma \tag{55}$$

(3) *Rotational excitation*

$$\frac{\mathfrak{F}_t[\phi]}{\mathfrak{F}_t[M_R Q(t)]}=\frac{\phi}{M_R e^{i\omega t}}=\frac{3g_2}{\pi^2\mu b^2}\int_0^\infty\int_0^\infty\frac{\alpha_1\kappa^2}{F(\beta,\gamma)}T_V(\beta,\gamma)S(\beta,\gamma)N(\beta b)\,d\beta\,d\gamma \tag{56}$$

Although all of the above equations include a double infinite integral, one of the infinite integrals can be reduced to a finite one by the following transformation of the integration variables.

$$\beta=\zeta\cos\theta,\qquad \gamma=\zeta\sin\theta \tag{57}$$

In addition, the transformation

$$\xi=\frac{\zeta}{\kappa} \tag{58}$$

and the non-dimensional quantities

$$a_0=\kappa b=\frac{\omega}{c_2}b=a_1\frac{b}{H} \tag{59a}$$

$$\eta_2=\frac{c_2}{b}\cdot\frac{\mu'}{\mu}=\eta_{H2}\frac{H}{b},\qquad \eta_1=\frac{c_2}{b}\cdot\frac{\lambda'+2\mu'}{\lambda+2\mu}=n^2\left(\frac{\lambda'}{\mu'}+2\right)\eta_2=\eta_{H1}\frac{H}{b} \tag{59b}$$

$$g_1=\frac{1}{1+i\eta_1 a_0},\qquad g_2=\frac{1}{1+i\eta_2 a_0} \tag{59c}$$

are introduced. The analytical expressions of $D.G.C.$ are then written in a non-dimensional form as follows;

(1) *Vertical excitation*

$$\begin{aligned}\frac{\mathfrak{F}_t[w_0]}{\mathfrak{F}_t[P_V Q(t)]}\cdot b\mu&=\frac{a_0 g_2^2}{\pi^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{\xi\sqrt{\xi^2-n^2 g_1}}{F(\xi)}T_V(\xi)S(a_0\xi,\theta)\,d\theta\,d\xi\\&=\frac{a_0 g_2^2}{\pi^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{\xi\sqrt{\xi^2-n^2 g_1}}{F_V(\xi)}S(a_0\xi,\theta)\,d\theta\,d\xi\end{aligned} \tag{60}$$

(2) *Horizontal excitation*

$$\frac{\mathfrak{F}_t[u_0]}{\mathfrak{F}_t[P_H Q(t)]}\cdot b\mu=\frac{a_0 g_2^2}{\pi^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\xi\left[\frac{\sqrt{\xi^2-g_2}}{F(\xi)}T_H(\xi)\cos^2\theta-\frac{\sin^2\theta}{g_2\sqrt{\xi^2-g_2}L(\xi)}\right]\cdot S(a_0\xi,\theta)\,d\theta\,d\xi$$

$$=\frac{a_0 g_2^2}{\pi^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\xi\left[\frac{\sqrt{\xi^2-g_2}}{F_H(\xi)}\cos^2\theta-\frac{\sin^2\theta}{g_2\sqrt{\xi^2-g_2}L(\xi)}\right]S(a_0\xi,\theta)\,d\theta\,d\xi \tag{61}$$

(3) *Rotational excitation*

$$\frac{\mathfrak{F}_t[\phi]}{\mathfrak{F}_t[M_R Q(t)]}\cdot\frac{b^3\mu}{3}=\frac{a_0 g_2^2}{\pi^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{\xi\sqrt{\xi^2-n^2 g_1}}{F(\xi)}T_V(\xi)S(a_0\xi,\theta)N(a_0\xi,\theta)\,d\theta\,d\xi$$

$$=\frac{a_0 g_2^2}{\pi^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{\xi\sqrt{\xi^2-n^2 g_1}}{F_V(\xi)}S(a_0\xi,\theta)N(a_0\xi,\theta)\,d\theta\,d\xi \tag{62}$$

where

$$F_V(\xi)=\{(2\xi^2-g_2)^2\coth(\sqrt{\xi^2-n^2 g_1}a_1)-4\xi^2\sqrt{\xi^2-n^2 g_1}\sqrt{\xi^2-g_2}\coth(\sqrt{\xi^2-g_2}a_1)\}$$
$$-\frac{\xi^2}{T_V(\xi)}\{2\sqrt{\xi^2-n^2 g_1}\sqrt{\xi^2-g_2}\,\mathrm{cosech}(\sqrt{\xi^2-g_2}a_1)$$
$$-(2\xi^2-g_2)\,\mathrm{cosech}(\sqrt{\xi^2-n^2 g_1}a_1)\}^2 \tag{63a}$$

$$F_H(\xi)=\{(2\xi^2-g_2)^2\coth(\sqrt{\xi^2-g_2}a_1)-4\xi^2\sqrt{\xi^2-n^2 g_1}\sqrt{\xi^2-g_2}\coth(\sqrt{\xi^2-n^2 g_1}a_1)\}$$
$$-\frac{\xi^2}{T_H(\xi)}\{2\sqrt{\xi^2-n^2 g_1}\sqrt{\xi^2-g_2}\,\mathrm{cosech}(\sqrt{\xi^2-n^2 g_1}a_1)$$
$$-(2\xi^2-g_2)\,\mathrm{cosech}(\sqrt{\xi^2-g_2}a_1)\}^2 \tag{63b}$$

$$T_V(\xi)=\xi^2\coth(\sqrt{\xi^2-n^2 g_1}a_1)-\sqrt{\xi^2-n^2 g_1}\sqrt{\xi^2-g_2}\coth(\sqrt{\xi^2-g_2}a_1) \tag{63c}$$

$$T_H(\xi)=\xi^2\coth(\sqrt{\xi^2-g_2}a_1)-\sqrt{\xi^2-n^2 g_1}\sqrt{\xi^2-g_2}\coth(\sqrt{\xi^2-n^2 g_1}a_1) \tag{63d}$$

$$S(a_0\xi,\theta)=\frac{\sin(a_0\xi\cos\theta)}{a_0\xi\cos\theta}\cdot\frac{\sin\left(\frac{c}{b}a_0\xi\sin\theta\right)}{\frac{c}{b}a_0\xi\sin\theta} \tag{63e}$$

$$N(a_0\xi,\theta)=\frac{\sin(a_0\xi\cos\theta)}{a_0\xi\cos\theta}-\cos(a_0\xi\cos\theta) \tag{63f}$$

If the thickness H tends to infinity, Eqs. (63) and (34) become

$$F_V(\xi)=F_H(\xi)=F_0(\xi) \tag{64a}$$

$$T_V(\xi)=T_H(\xi)\equiv T(\xi)(=\xi^2-\sqrt{\xi^2-n^2 g_1}\sqrt{\xi^2-g_2}) \tag{64b}$$

$$F(\xi)=F_0(\xi)\cdot T(\xi) \tag{64c}$$

The substitution of Eq. (64) into Eqs. (60) to (62) results in the analytical expressions of $D.G.C.$ for a viscoelastic half-space.

7. Definition of Statical Ground Compliance and Its Analytical Expressions

The limiting representations of the analytical expressions of $D.G.C.$ when the frequency of the exciting force ω tends to zero give, in particular, the "Statical Ground Compliance ($S.G.C.$)". This is defined as the ratio of the static displacement of a foundation in the forcing direction to a static load. Hence, its inverse is the static stiffness of the foundation-ground system. The final expressions of $D.G.C.$ given by Eqs. (60) to (62) are not available in calculating their limiting expressions, because the frequency ω is included in the denominator of the integration variable, $\xi=\zeta/\kappa=\zeta\cdot c_2/\omega$. However, if L. Hospital's theorem is applied to the limiting calculation of the integral representations with respect to the variables defined by Eq. (57) when the frequency parameter, $\kappa=\omega/c_2$, tends to zero, the analytical expressions of $S.G.C.$ can be evaluated as follows;

(1) *Vertical excitation*

$$f_{SV}=\frac{w_0}{P_V}\cdot b\mu=\frac{1}{2\pi^2\frac{H}{b}}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{(1+n^2)\sinh\zeta'\cdot\cosh\zeta'-(1-n^2)\zeta'}{1+(1-n^2)\{(1+n^2)\sinh^2\zeta'+(1-n^2)\zeta'^2\}}\cdot S\left(\zeta'\frac{b}{H},\ \theta\right)d\theta\, d\zeta' \tag{65}$$

(2) *Horizontal excitation*

$$f_{SH}=\frac{u_0}{P_H}\cdot b\mu=\frac{1}{2\pi^2\frac{H}{b}}\int_0^\infty\int_0^{\frac{\pi}{2}}\left[\frac{(1+n^2)\sinh\zeta'\cdot\cosh\zeta'+(1-n^2)\zeta'}{1+(1-n^2)\{(1+n^2)\sinh^2\zeta'+(1-n^2)\zeta'^2\}}\cdot\cos^2\theta-2\frac{\sin^2\theta}{\coth\zeta'}\right]S\left(\zeta'\frac{b}{H},\ \theta\right)d\theta\, d\zeta' \tag{66}$$

(3) *Rotational excitation*

$$f_{SR}=\frac{\phi}{M_R}\cdot\frac{b^3\mu}{3}=\frac{1}{2\pi^2\frac{H}{b}}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{(1+n^2)\sinh\zeta'\cdot\cosh\zeta'-(1-n^2)\zeta'}{1+(1-n^2)\{(1+n^2)\sinh^2\zeta'+(1-n^2)\zeta'^2\}}\cdot S\left(\zeta'\frac{b}{H},\ \theta\right)N\left(\zeta'\frac{b}{H},\ \theta\right)d\theta\, d\zeta' \tag{67}$$

where

$$\zeta'=\zeta H \tag{68}$$

If the thickness H tends to infinity in the expressions of $S.G.C.$ whose integration variable is ζ, the limiting representations reduce to the analytical expressions of $S.G.C.$ for a half-space.

8. Resonance Phenomena

Perfectly elastic medium — Resonance phenomena do not appear in the case of a half-space. On the other hand, for the case of a stratum over a rigid half-space, waves are reflected perfectly at both of the boundary surfaces of the stratum. Therefore, if a certain relation is satisfied between the depth of the stratum H and the frequency of the exciting force ω, the resonance phenomena that the amplitude of displacements diverges may appear when a harmonic exciting force is acting on the foundation. These resonance phenomena can be separated into the following three cases:

Case ①; the integral representing D.G.C. diverges in the neighborhood of $\xi=0$, that is, the resonant frequency agrees with the natural frequency of a one-dimensional rod having the same length as the depth of stratum.

(1) *Vertical excitation*

Supposing the variable ξ to be an infinitesimal positive number, and if the relation

$$[\coth(\sqrt{\xi^2-n^2}a_1)]_{\xi=0}=\coth(ina_1)=-i\cot(na_1)=0 \tag{69}$$

i.e.,

$$a_1=\frac{2m+1}{2n}\pi \qquad \text{or} \qquad \omega=\frac{2m+1}{2H}\pi\sqrt{\frac{1}{\rho}\cdot\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}} \tag{70}$$

where m=integer and E=Young's modulus of stratum, is satisfied, an approximate expression

$$[\coth(\sqrt{\xi^2-n^2}a_1)]_{a_1=\frac{2m+1}{2n}\pi}$$
$$\simeq -i\cot\left[\frac{2m+1}{2}\pi\left(1-\frac{\xi^2}{2n^2}\right)\right]\simeq -i\frac{2m+1}{2}\pi\cdot\frac{\xi^2}{2n^2} \tag{71}$$

can be obtained. It is found, using Eq. (71), that both of the functions $F(\xi)$ [Eq. (34)] and $F_V(\xi)$ [Eq. (63a)] are of the order of ξ^{-2}. Now let ε_1 and ε ($\varepsilon_1>\varepsilon$) be infinitesimal positive numbers and J_V be Cauchy's principal value of the integral with respect to ξ in the double integral representation of $D.G.C.$ given by Eq. (60), respectively. Then, the integral J_V is expressed by

$$J_V=\boldsymbol{P}\int_0^\infty\frac{\xi\sqrt{\xi^2-n^2}}{F_V(\xi)}S(a_0\xi,\ \theta)d\xi$$
$$=\frac{\frac{c}{b}\left(\frac{2m+1}{2}\pi\frac{b}{H}\right)^2\cot\frac{2m+1}{2n}\pi}{(4n^2+1)+\frac{2m+1}{4n}\pi\cot\frac{2m+1}{2n}\pi}\lim_{\varepsilon\to 0}\left[\int_\varepsilon^{\varepsilon_1}\frac{d\xi}{\xi}\right]+\boldsymbol{P}\int_{\varepsilon_1}^\infty\frac{\xi\sqrt{\xi^2-n^2}}{F_V(\xi)}S(a_0\xi,\ \theta)d\xi \tag{72}$$

The first term of the above equation obviously diverges. So the resonance phenomenon may occur when Eq. (70) is satisfied. The resonant frequencies of this case coincide with the natural frequencies of longitudinal vibration of a one-dimensional rod one end of which is fixed while the other is free. This equivalent rod has the longitudinal rigidity, $(1-\nu)E/(1+\nu)(1-2\nu)$, and the same length H as the thickness of the stratum.

(2) *Horizontal excitation*

If the relation

$$[\coth(\sqrt{\xi^2-1}a_1)]_{\xi=0}=\coth(ia_1)=-i\cot a_1=0 \tag{73}$$

i.e.,

$$a_1=\frac{2m+1}{2}\pi \qquad \text{or} \qquad \omega=\frac{2m+1}{2H}\pi\sqrt{\frac{\mu}{\rho}} \tag{74}$$

is satisfied, it is also found, after similar calculations in the case of vertical excitation described above, that Cauchy's principal value of the integral representation of $D.G.C.$ given by Eq. (61) diverges, which means the resonance phenomenon of infinite amplitude. The resonant frequencies in Eq. (74) coincide also with the natural frequencies of lateral vibration of the similar equivalent rod having the shear modulus μ to the case of vertical excitation.

(3) *Rotational excitation*

Different from the above-mentioned two cases, this type of excitation exhibits no resonance phenomenon corresponding to the natural frequencies of an equivalent rod.

As for the resonant frequencies of the vertical and horizontal excitations belonging to this *Case* ①, the above discussions show that a three-dimensional stratum can be treated as an equivalent one-dimensional rod having the same length and the similar end conditions as the stratum and oscillating in the mode corresponding to the type of excitation. As mentioned in section 4, the resonant frequencies of the horizontal excitation given by Eq. (74) are the solutions of the frequency equations for both Rayleigh and Love waves with the condition $\xi_{ok}=0$ i.e., the wave number $\zeta=0$. The resonant frequencies of the vertical excitation in Eq. (70), on the other hand, are the solutions of the frequency equation of Rayleigh waves with the same condition.

Case ②; *the frequency equation, $F(\xi)=0$, has a double root in some mode, that is, the group velocity of the mode vanishes.*

Case ③; *the frequency equation, $F(\xi)=0$, has a double root associated with two distinct modes, that is, the phase velocity of the two modes agrees with each other.*

As mentioned previously and shown in Fig. 2a, the frequency equation, $F(\xi)=0$, for Rayleigh waves may have an enumerably infinite number of double roots belonging to *Case* ② or *Case* ③. They are indicated by the mark "●" (*Case* ②) and "○" (*Case* ③) in Figs. 2-4. Owing to these double roots, ξ_{ok}, at which the relations $F(\xi)|_{\xi=\xi_{ok}}=dF(\xi)/d\xi|_{\xi=\xi_{ok}}=0$ and $d^2F(\xi)/d\xi^2|_{\xi=\xi_{ok}}\neq 0$ are held, Cauchy's principal value of $D.G.C.$ diverges, that is, a resonance phenomenon having an infinite amplitude may occur. These two cases may appear in all types of excitation when certain common relations are satisfied between the depth of the stratum and the frequency of the disturbing force.

Viscoelastic medium — In this case it seems difficult to express the resonant frequencies analytically. The resonant amplitude is reduced to a finite one owing to wave dissipations in a viscoelastic medium.

9. Numerical Calculations

Method of numerical calculations

It seems impossible to derive closed-form solutions expressed in terms of the functions whose properties are well known, from the double integral representations of *D.G.C.* including an infinite improper integral. Thus, a numerical integration method must be applied in order to clarity the properties of *D.G.C.*

The outline of the numerical calculations of *D.G.C.* is as follows: Since the integration with respect to θ is common to repeated calculations of the infinite integral with respect to ξ for various frequency parameters a_0, it may be convenient to evaluate, in advance, the quantities

$$S_I(a_0\xi)=\int_0^{\frac{\pi}{2}} S(a_0\xi,\ \theta)d\theta \tag{75a}$$

$$S_{IS}(a_0\xi)=\int_0^{\frac{\pi}{2}} S(a_0\xi,\ \theta)\sin^2\theta d\theta \tag{75b}$$

$$S_{IN}(a_0\xi)=\int_0^{\frac{\pi}{2}} S(a_0\xi,\ \theta)N(a_0\xi,\ \theta)d\theta \tag{75c}$$

for an appropriate set of values of the variable $a_0\xi$, which are selected here at intervals of 0.05 from 0 to 100. With the aid of the numerical tables for the above quantities, the integration with respect to ξ is then carried out by interpolating the values of the quantities in Eq. (75) for intermediate values of $a_0\xi$ by Everett's or Stirling's formula of the 4th order. The numerical calculation of the infinite integral is carried on up to where the integrand becomes so small that the error produced by such truncation of the infinite integration is not more than 0.5 percent. In the numerical integration according to Simpson's 3/8 rule, successive subdivisions of the integration interval are made until the error becomes smaller than 0.05 percent.

Since the numerical table for the quantities given by Eq. (75) is also common to the integral representations of *S.G.C.*, the numerical integrations of *S.G.C.* are carried out by the same method as mentioned above.

Distinct differences as to the integration path on ξ may now arise between the two cases of perfectly elastic and viscoelastic media. The interpretation of the integration path will thus be given below separately for each of these two cases. Inspection of the analytical expressions of *D.G.C.* shows that results of the integration are obtained as a complex number, the real and imaginary parts of which shall be denoted by f_{1l} and f_{2l} [$l=V$, H, and R], respectively, in the following.

Perfectly elastic medium—The analytical expressions of *D.G.C.* for a perfectly elastic medium, obtained by setting $\eta_1=\eta_2=0$. i.e., $g_1=g_2=1$ in Eqs. (60) to (62), include the following singular points on the path of integration; namely,

the branch points are	$\xi=n$	and	$\xi=1$;	
the poles are	$\xi=1$,		$\xi=\xi_{ok}[k=1,2,\cdots,N]$,	and
			$\xi=\xi_{ok'}[k'=1,2,\cdots,N']$;	

where ξ_{ok} = real roots of the frequency equation, Eq. (34), for Rayleigh waves;
$\xi_{ok'}$ = real roots of the frequency equation, Eq. (35), for Love waves;

and N and N' = the total number of real roots, ξ_{ok} and $\xi_{ok'}$, respectively.

When the integration path on ξ is extended in the complex-plane, the path can be selected as a positive real axis, avoiding the above-mentioned singular points around semi-circles in the first quadrant. The integrals along these semi-circles around the singular points, $\xi=n$ and $\xi=1$, vanish when the radius of the circle tends to zero. Hence, the integrations are expressed in terms of Cauchy's principal values with respect to both poles, ξ_{ok} for Rayleigh waves and $\xi_{ok'}$ for Love waves, and the sum of the residue term which is $-\pi i$ times the value of residue at the singular point for Rayleigh or Love waves. It is noted that the residue terms for Love waves appear only in the case of horizontal excitation.

In the case of a half-space, Cauchy's principal values are complex numbers and the residue terms are pure imaginary ones. For a stratum, on the other hand, Cauchy's principal values and the sum of the residue terms give respectively the real and imaginary parts of $D.G.C.$; namely,

(1) *Vertical excitation*

$$\frac{\mathfrak{F}_t[w_0]}{\mathfrak{F}_t[P_V Q(t)]}\cdot b\mu = f_{1V} + i f_{2V} \tag{76}$$

where

$$f_{1V} = \frac{a_0}{\pi^2}\boldsymbol{P}\int_0^\infty \frac{\xi\sqrt{\xi^2-n^2}}{F_V(\xi)} S_I(a_0\xi)\,d\xi \tag{77a}$$

$$f_{2V} = -\frac{a_0}{\pi}\sum_{k=1}^{N}\left[\frac{\xi\sqrt{\xi^2-n^2}}{\dfrac{dF_V(\xi)}{d\xi}} S_I(a_0\xi)\right]_{\xi=\xi_{ok}} \tag{77b}$$

(2) *Horizontal excitation*

$$\frac{\mathfrak{F}_t[u_0]}{\mathfrak{F}_t[P_H Q(t)]}\cdot b\mu = f_{1H} + i f_{2H} \tag{78}$$

where

$$f_{1H} = \frac{a_0}{\pi^2}\boldsymbol{P}\int_0^\infty\left[\frac{\xi\sqrt{\xi^2-1}}{F_H(\xi)}\{S_I(a_0\xi)-S_{IS}(a_0\xi)\} - \frac{\xi}{\sqrt{\xi^2-1}L(\xi)}S_{IS}(a_0\xi)\right]d\xi \tag{79a}$$

$$f_{2H} = -\frac{a_0}{\pi}\sum_{k=1}^{N}\left[\frac{\xi\sqrt{\xi^2-1}}{\dfrac{dF_H(\xi)}{d\xi}}\{S_I(a_0\xi)-S_{IS}(a_0\xi)\}\right]_{\xi=\xi_{ok}} + \frac{a_0}{\pi}\sum_{k'=1}^{N'}\left[\frac{S_{IS}(a_0\xi)}{a_1}\right]_{\xi=\xi_{ok'}} \tag{79b}$$

(3) *Rotational excitation*

$$\frac{\mathfrak{F}_t[\phi]}{\mathfrak{F}_t[M_R Q(t)]}\cdot\frac{b^3\mu}{3} = f_{1R} + i f_{2R} \tag{80}$$

where

$$f_{1R}=\frac{a_0}{\pi^2}\boldsymbol{P}\int_0^\infty\frac{\xi\sqrt{\xi^2-n^2}}{F_V(\xi)}S_{IN}(a_0\xi)d\xi \tag{81a}$$

$$f_{2R}=-\frac{a_0}{\pi}\sum_{k=1}^{N}\left[\frac{\xi\sqrt{\xi^2-n^2}}{\frac{dF_V(\xi)}{d\xi}}S_{IN}(a_0\xi)\right]_{\xi=\xi_{0k}} \tag{81b}$$

In the above equations $\boldsymbol{P}$ represents Cauchy's principal value of the improper integral. In evaluating Cauchy's principal value with respect to a pole, Longman's method[23] is used; namely, separating the integrand into two parts, an even and an odd function, about the pole and choosing an appropriate symmetric range of integration, Cauchy's principal value associated with the range is calculated by only considering the contribution of the even function to the integral.

Viscoelastic medium — When some mechanism of internal energy dissipation exists in a medium, all singular points of the integrand in the integral representations of *D.G.C.* become complex-numbers. Therefore, it seems convenient, in this case, to evaluate the integral with respect to ξ directly along the positive real axis on which no singular points appear.

Numerical results

Numerical results will be shown for the case of $n^2=1/3$, i.e., Poisson's ratio of the medium, $\nu=1/4$.

For the three types of loading and for various shape ratios c/b, *S.G.C.* is shown in Fig. 5, in which the solutions

$$\bar{f}_{SV}=\frac{w_0}{P_V}\cdot\bar{b}_V\mu,\qquad \bar{f}_{SH}=\frac{u_0}{P_H}\cdot\bar{b}_H\mu,\qquad \bar{f}_{SR}=\frac{\phi}{M_R}\cdot\frac{\bar{b}_R^3\mu}{3} \tag{82}$$

are plotted versus the ratio of the thickness of stratum H to the reference half-width of the foundation $\bar{b}_i$ which is defined according to the type of excitation such that $\bar{b}_V=\bar{b}_H=\sqrt{bc}$ and $\bar{b}_R=(b^3c)^{\frac{1}{4}}$. By the use of the reference half-width these figures make it possible to find the shape effect when the cross-sectional shape of the foundation is varied while its area and its moment of inertia about the rotational axis remain constant respectively in the cases of translational and rotational excitations. The dotted lines in these figures indicate the solutions for a half-space.

For the three types of excitation *D.G.C.* of a square foundation, $c/b=1$, is shown in Figs. 6 and 7 versus the frequency parameter a_0. These figures show the case where $\lambda'/\mu'=1$ and $\nu=1/4$. This means that the non-dimensional viscosity coefficients of dilatational and distortional waves defined in Eq. (59b) are equal, namely, $\eta_1=\eta_2=\eta$. In Fig. 6 the variations of *D.G.C.* according to the ratio of stratum thickness to foundation width, H/b, are presented, and the variations according to the non-dimensional viscosity coefficient η are shown in Fig. 7.

The physical meanings of *S.G.C.* may easily be understood as the reciprocal of the static stiffness of a foundation-ground system. It seems difficult, however, to find the physical properties of *D.G.C.* directly from the present complex-valued expressions. Then, let *D.G.C.* be related to the relevant Voigt model consisting of the equivalent stiffness and viscous damping both of which are functions of frequency. Since the force-displacement transfer function of the Voigt model is expressed as $1/(\kappa_{ei}+ia_0c_{ei})$

$[l=V, H,$ and $R]$, the equivalent stiffness coefficient κ_{el} and the equivalent viscous damping coefficient c_{el} are determined from the relation

$$f_{1l}+if_{2l}=\frac{1}{\kappa_{el}+ia_0c_{el}} \tag{83a}$$

i.e.,

$$\kappa_{el}=\frac{f_{1l}}{f_{1l}^2+f_{2l}^2},\quad c_{el}=\frac{-f_{2l}}{a_0(f_{1l}^2+f_{2l}^2)} \tag{83b}$$

Results for the case where $\lambda'/\mu'=1$ and $c/b=1$ are shown in Fig. 8 for the two cases of $H/b=4$ (or $H/b=2$) and $H/b=\infty$, the latter case of which corresponds to a half-space.

The cases of a square foundation are indicated in all of the above figures of *D.G.C.* In order to find the effect of the shape of the foundation on *D.G.C.*, the equivalent coefficients of the Voigt model, $\bar{\kappa}_{el}$ and $\bar{c}_{el}$, evaluated from the following expressions of *D.G.C.* having similar form to Eq. (82) for *S.G.C.*

$$\frac{\mathfrak{F}_t[w_0]}{\mathfrak{F}_t[P_VQ(t)]}\cdot\bar{b}_V\mu=\bar{f}_{1V}+i\bar{f}_{2V} \tag{84}$$

$$\frac{\mathfrak{F}_t[u_0]}{\mathfrak{F}_t[P_HQ(t)]}\cdot\bar{b}_H\mu=\bar{f}_{1H}+i\bar{f}_{2H} \tag{85}$$

$$\frac{\mathfrak{F}_t[\phi]}{\mathfrak{F}_t[M_RQ(t)]}\cdot\frac{\bar{b}_R^3\mu}{3}=\bar{f}_{1R}+i\bar{f}_{2R} \tag{86}$$

are shown in Fig. 9, in which abscissa is the frequency parameter, $\bar{a}_0=\omega\sqrt{\rho/\mu}\cdot\bar{b}_l$, and the parameters $H/\bar{b}_l$ and $\bar{\eta}=\frac{c_2}{\bar{b}_l}\cdot\frac{\mu'}{\mu}=\frac{c_2}{\bar{b}_l}\cdot\frac{\lambda'+2\mu'}{\lambda+2\mu}$ remain constant.

In Figs. 6–9, dotted lines represent the solutions for a half-space and fine solid lines drawn vertically indicate the position of the resonant frequencies in the case of a perfectly elastic stratum. The

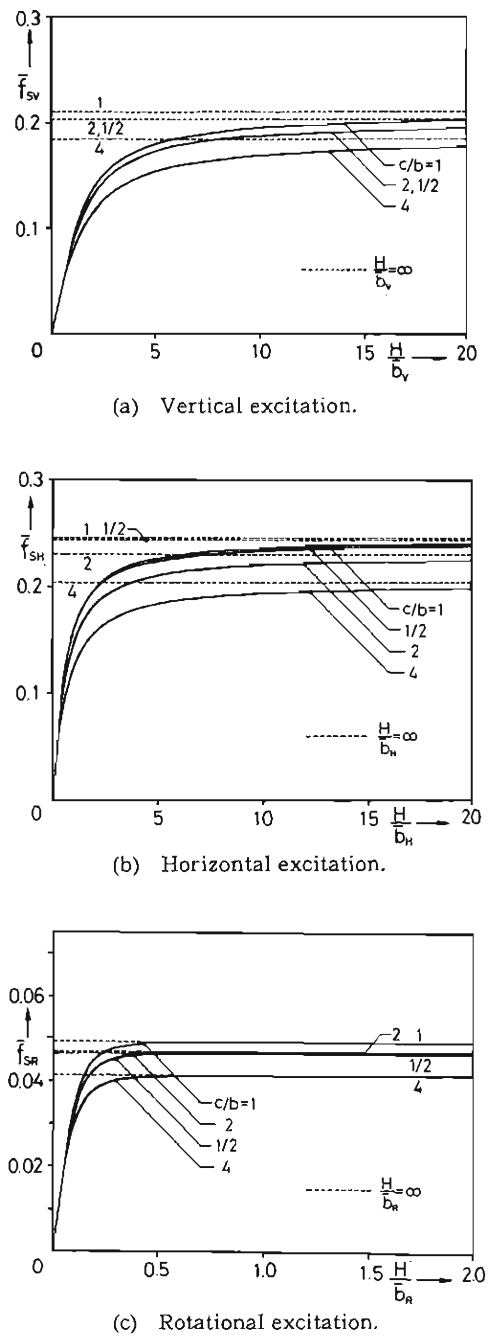
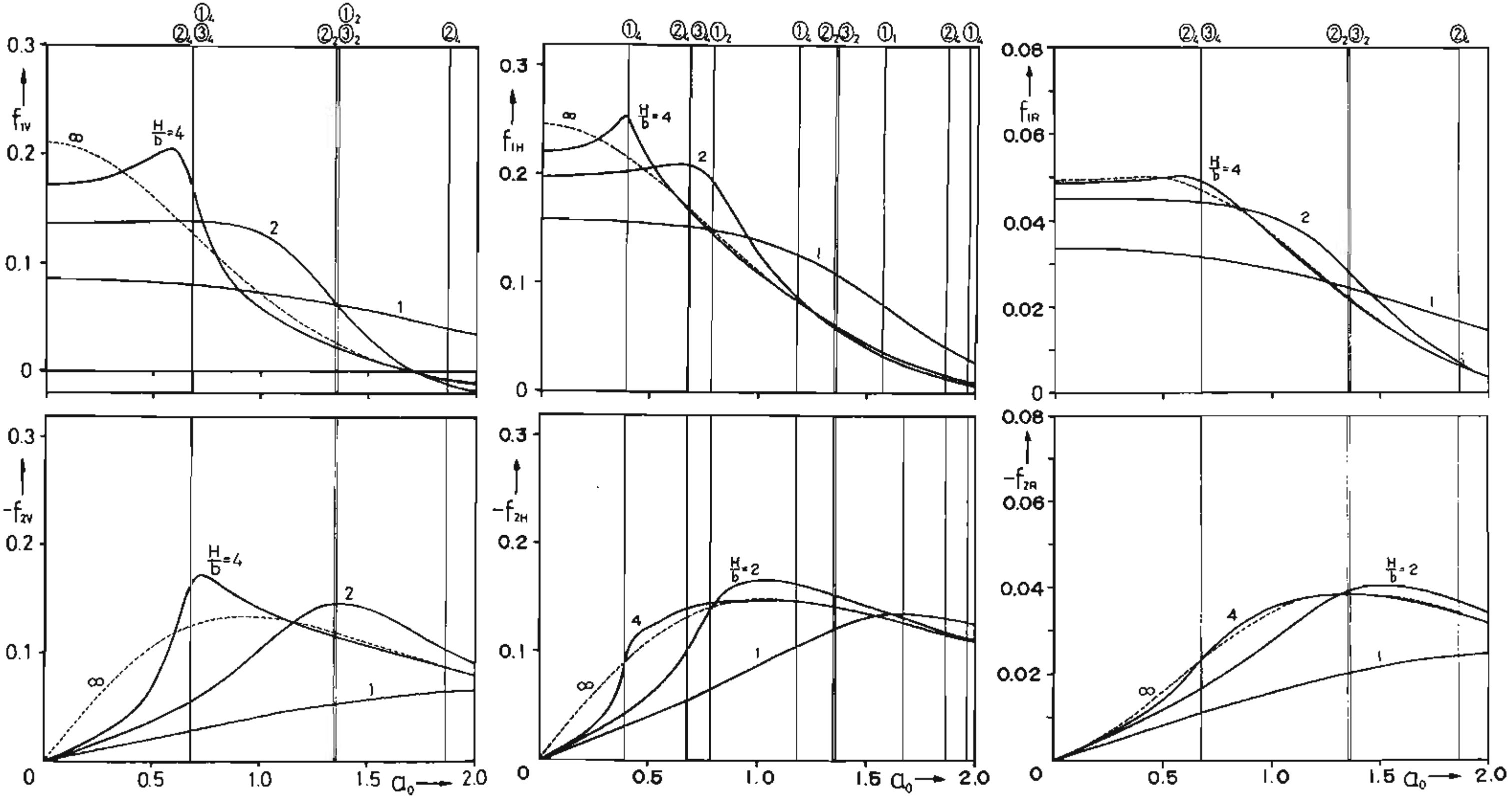
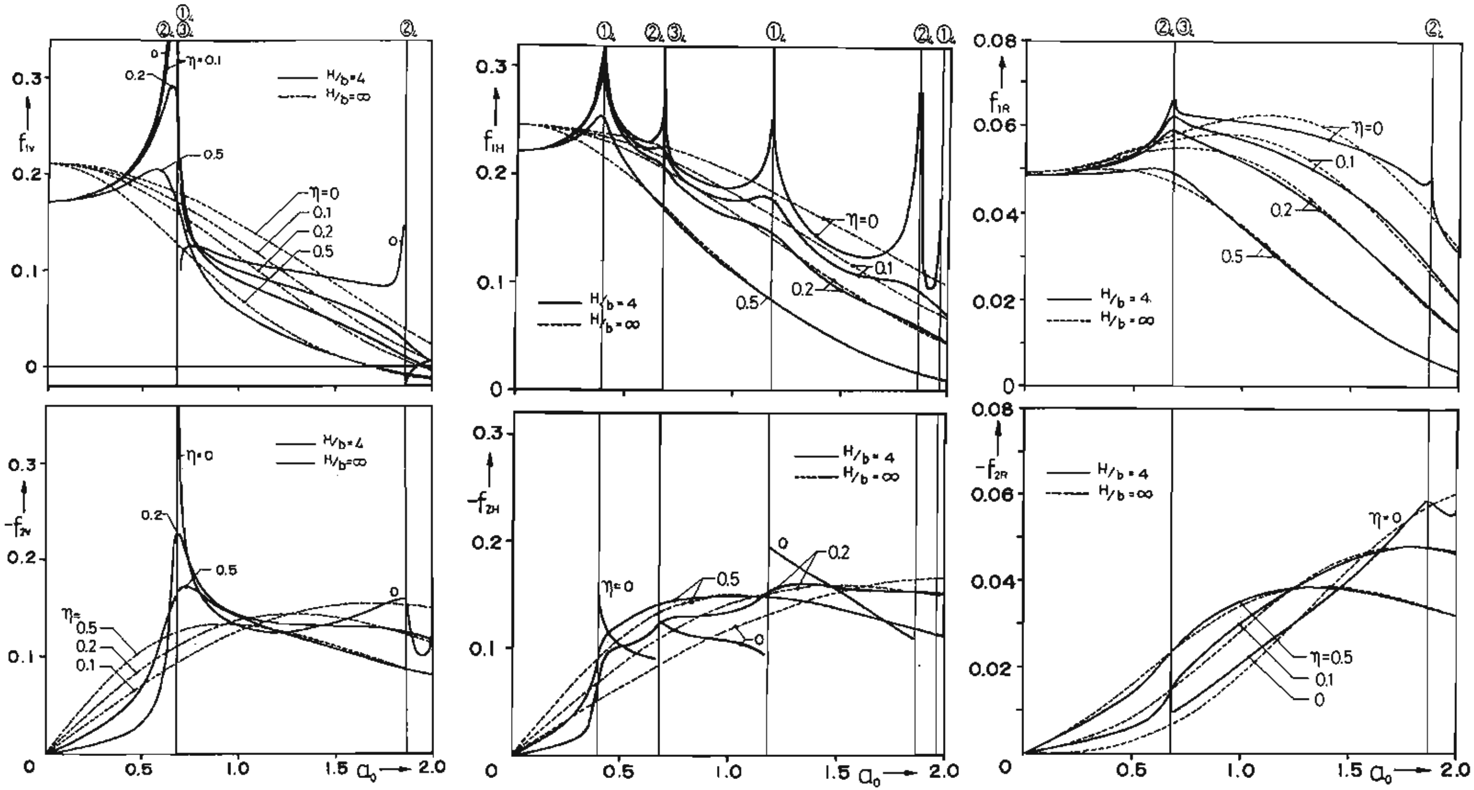


Fig. 5. Statical Ground Compliance for various shape ratios c/b. ($\nu=1/4$.)



(a) Vertical excitation.   (b) Horizontal excitation.   (c) Rotational excitation.

Fig. 6. Dynamical Ground Compliance for various thickness-width ratios H/b. ($c/b=1$, $\nu=1/4$, $\lambda'/\mu'=1$, and $\eta=0.5$.)

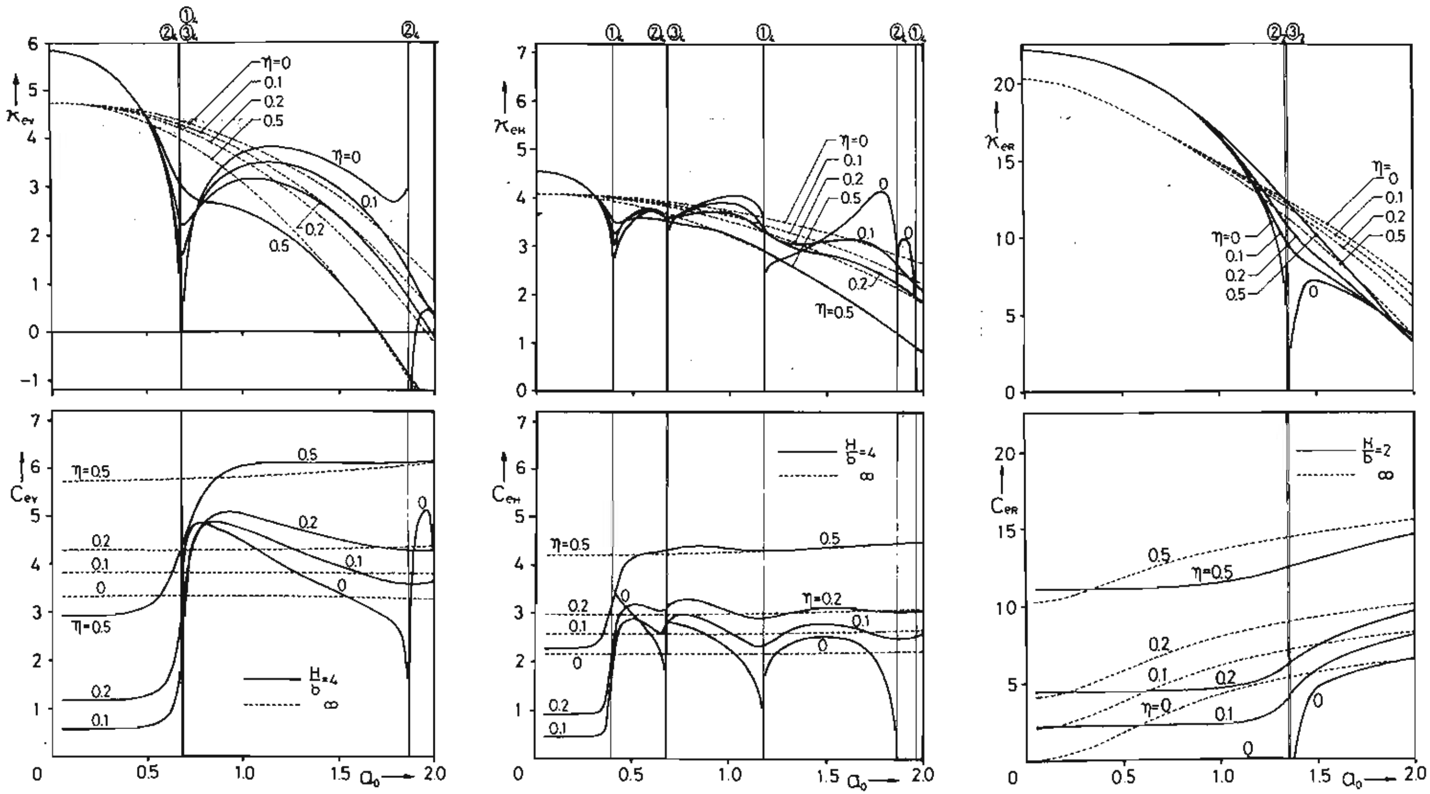


(a) Vertical excitation.

(b) Horizontal excitation.

(c) Rotational excitation.

Fig. 7. Dynamical Ground Compliance for various viscosity coefficient parameters η. ($c/b=1$, $\nu=1/4$, $\lambda'/\mu'=1$, and $H/b=4$.)

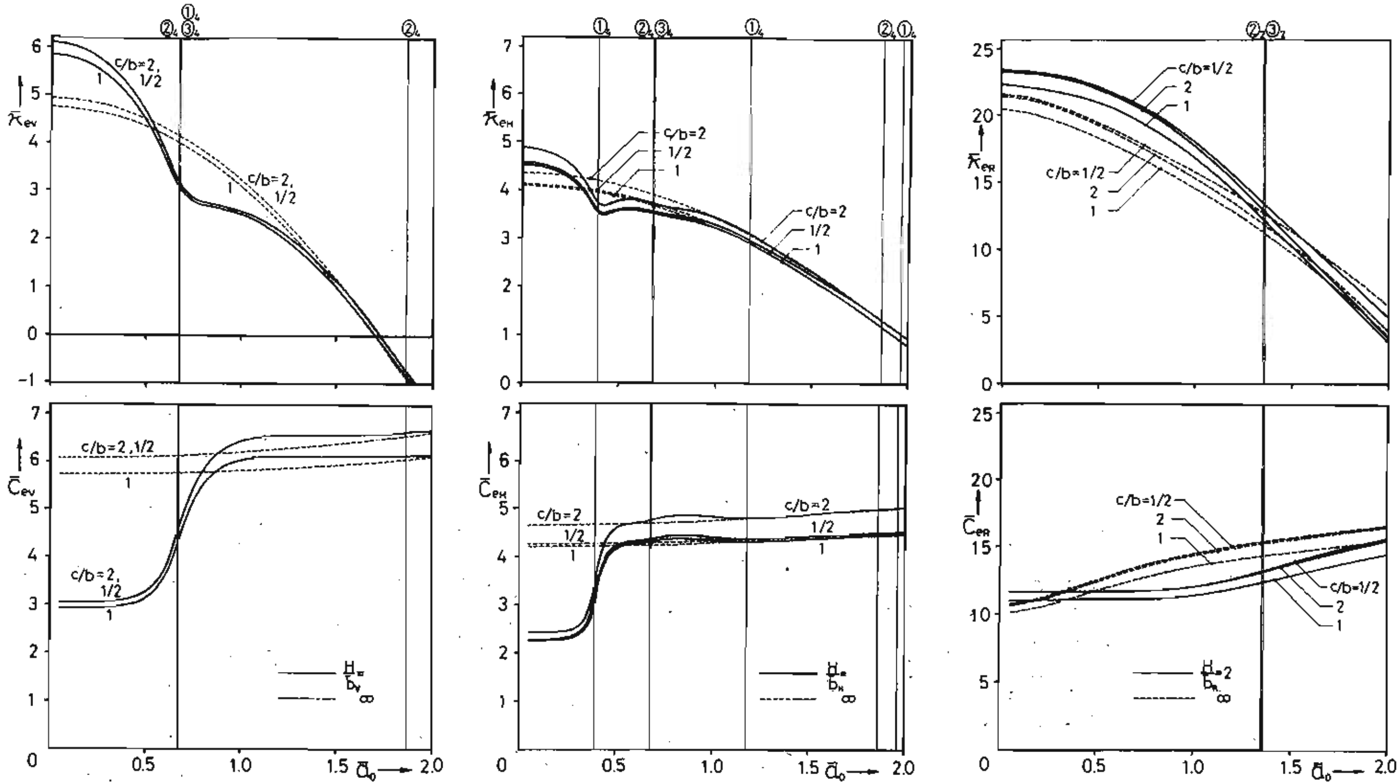


(a) Vertical excitation.

(b) Horizontal excitation.

(c) Rotational excitation.

Fig. 8. Equivalent coefficients of Dynamical Ground Compliance for various viscosity coefficient parameters η. ($c/b=1$, $\nu=1/4$, $\lambda'/\mu'=1$, and $H/b=4$ in Figs. 8(a) and 8(b) and 2 in Fig. 8(c), respectively.)



(a) Vertical excitation. (b) Horizontal excitation. (c) Rotational excitation.

Fig. 9. Equivalent coefficients of Dynamical Ground Compliance for various shapes of foundation. ($\nu=1/4$, $\lambda'/\mu'=1$, $\bar{\eta}=0.5$, and $H/\bar{b}_i=4$ in Figs. 9(a) and 9(b) and 2 in Fig. 9(c), respectively.)

notation Ⓘ$_k$[I=1, 2, and 3] in the upper blank of these figures means the kind of resonance phenomenon discussed in Section 4 and subscript k stand for H/b or $H/\bar{b}_l$.

10. Discussions

Statical Ground Compliance (See Fig. 5.)

S.G.C. of a stratum indicated by solid lines in Figs. 5a to 5c tends to increase rapidly in the comparatively small range of $H/\bar{b}_l$, and approaches gradually to *S.G.C.* of a half-space indicated by dotted lines as the parameter $H/\bar{b}_l$ increases. The rate of convergence of *S.G.C.* of a stratum into *S.G.C.* of a half-space with the increase of $H/\bar{b}_l$ becomes large in the order: 1 vertical, 2 horizontal, and 3 rotational loading cases. There is little difference in the convergency between the two types of translational loading in which the rate of convergence is rather small, and rapid increase continues until the thickness of the stratum H becomes about three times the reference half-width of the foundation $(bc)^{1/2}$. In the case of rotational loading the convergence is found to be extremely rapid and *S.G.C.* of both media almost coincide with each other when the thickness of the stratum H becomes more than five times the reference half-width of the foundation $(b^3c)^{1/4}$. Comparing the two cases of translational loading under an identical static force, it is found that *S.G.C.*, namely, the foundation displacement in the loading direction is always found to be larger for the horizontal case.

For all types of loading, *S.G.C.* increases almost linearly in the range where $H/\bar{b}_l$ is small, and it remains nearly constant in the range of large $H/\bar{b}_l$. By reference to the non-dimensional expressions of $\bar{f}_{sl}$ given by Eq. (82), the following relations may be approximately satisfied between *S.G.C.* with the original dimension and the cross-sectional properties of the foundation as long as the foundation has an identical shape: *S.G.C.* is almost inversely proportional to the area of the foundation for the translational loading cases and to the moment of inertia about the principal axis of the foundation for the rotational case, respectively, when the thickness of the stratum is relatively small in comparison with the reference half-width of the foundation, in other words, when the stratum is considered as a thin layer over a rigid half-space. On the contrary, when the thickness is considerably larger than the reference half-width of the foundation, namely, when the stratum may be treated approximately as a half-space, *S.G.C.* increases almost in inverse proportion to the square root of the area for the two translational cases and to the three-fourth powers of the moment of inertia for the rotational case, respectively.

As regards the effects of foundation shape on *S.G.C.*, it is found that for the translational loading cases, *S.G.C.* is maximum when the foundation is square. Comparison between the two cases in which a horizontal force acts along either of the mutually perpendicular principal axes of a foundation indicates that *S.G.C.* for a loading in the slenderer direction $(c/b>1)$ is always greater than that in the other perpendicular direction $(c/b>1)$. Both, of course, agree with each other for the vertical case. For the rotational loading, if the moment of inertia remains constant, the smaller deformation, namely, the smaller *S.G.C.* is produced as the shape becomes slenderer in either direction of the principal axes of the foundation area.

Resonance phenomena (See Figs. 6 and 7.)

One of the main aspects in the dynamic problems of a stratum different from the case of a half-space is that an enumerably infinite number of resonant frequencies appears owing to the repeated wave reflections at the upper and lower boundary surfaces of the stratum.

As mentioned in Section 8 there are three kinds of resonant frequencies in the case of a perfectly elastic stratum. The following table shows the values of the non-dimensional frequency parameter a_0 at resonant frequencies appearing within the range, $0 \leqq a_0 \leqq 2$. These frequencies are indicated by fine solid lines drawn vertically in Figs. 6–9.

Table The values of frequency parameter a_0 at resonant frequencies within $0 \leqq a_0 \leqq 2$.

<table>
<tr><th rowspan="2">the kind of resonant frequency</th><th rowspan="2">H/b</th><th colspan="3">type of excitation</th></tr>
<tr><th>vertical</th><th>horizontal</th><th>rotational</th></tr>
<tr><td rowspan="3">①</td><td>1</td><td>—</td><td>1.5708</td><td>—</td></tr>
<tr><td>2</td><td>1.3603</td><td>0.7854</td><td>—</td></tr>
<tr><td>4</td><td>0.6802</td><td>0.3927, 1.1781, 1.9635</td><td>—</td></tr>
<tr><td rowspan="3">②</td><td>1</td><td colspan="3">—</td></tr>
<tr><td>2</td><td colspan="3">1.3506</td></tr>
<tr><td>4</td><td colspan="3">0.6753 1.8637</td></tr>
<tr><td rowspan="3">③</td><td>1</td><td colspan="3">—</td></tr>
<tr><td>2</td><td colspan="3">1.3603</td></tr>
<tr><td>4</td><td colspan="3">0.6802</td></tr>
</table>

The notations in the first column of the table indicate the kind of resonant frequency such that;

①=*those which correspond to the natural frequencies of an equivalent one-dimensional rod having the same length with the thickness of stratum,*

②=*those at which the group velocity of some mode vanishes, and*

③=*those at which the phase velocities of two distinct modes coincide with each other.*

The above resonant frequencies are determined from the roots of the non-dimensional frequency equations expressed in terms of the frequency parameter, $a_1 = a_0 \cdot H/b = \omega H/c_2$, and the velocity ratio, $n = c_2/c_1$. Consequently, they are functions of only the constants of the stratum such as the thickness H, the density ρ, the shear modulus μ, and Poisson's ratio ν. They are thus independent of the shape and dimensions of the foundation. It is an interesting fact, as mentioned partly in Sections 4 and 8, that the resonant frequencies of *Case* ③ coincide with those belonging to *Case* ① associated with the vertical excitation.

As shown in Fig. 7, the resonant amplitude of *D.G.C.* of a perfectly elastic stratum ($\eta=0$) diverges at those frequencies. A viscoelastic stratum, on the other hand, has finite amplitudes at the resonant frequencies because of the internal energy dissipation in the medium. As the non-dimensional viscosity coefficient η is larger, the behavior of *D.G.C.* in the neighborhood of the resonant frequencies becomes smoother and the peak frequencies of the absolute value of *D.G.C.* i.e., the amplitude of the foundation displacement, are gradually shifted into the lower frequency range.

Attenuation mechanism (See Figs. 7 and 8.)

In general, whether the medium is perfectly elastic or viscoelastic, *D.G.C.* is expressed as a complex number and its imaginary part is concerned with some energy attenuations.

Perfectly elastic medium — Even though the medium has no attenuative nature owing to internal dissipation, it shows an apparent attenuation caused by wave radiations, that is, waves are radiating away from the exciting source at the surface into the medium. In this case, *D.G.C.* is expressed as the sum of Cauchy's principal values of the improper integral, which are complex or real numbers, and the residue terms around the poles related to Rayleigh and Love waves, which are pure imaginary numbers. This summation of the residue terms means that the contribution of Rayleigh's or Love's free waves with definite amplitudes may appear in the solution of *D.G.C.*, because these poles give the components with the wave number of such free waves. It is suggested by G.N. Bycroft[7] that the composed waves are the divergent waves, which radiate away exclusively from the source without including reflected waves from infinity. Therefore, the solutions obtained by this summation may have the physical meaning that the boundary conditions at the radial infinity are throughly satisfied.

The imaginary part of *D.G.C.* of a half-space consists of two parts; the imaginary part of Cauchy's principal value and the residue term around the Rayleigh pole. These parts represent respectively the downward wave radiations of bodily waves and the radial wave radiations of free waves. Energy attenuations are thus produced by these two kinds of wave radiation mechanism. In the case of a stratum, on the other hand, the imaginary part consists of only the sum of the residue terms around the finite number of poles corresponding to the free waves of possible modes. Energy attenuations in this medium are thus produced only by free waves in the radial direction, while no radiations occur downward because of perfect wave reflections at the boundary surface of the rigid half-space. It is observed from the equivalent viscous damping coefficient in the case of a perfectly elastic stratum shown in Fig. 8 that an abrupt increase of energy attenuation occurs in the immediate right side of a resonant frequency. This is because a free wave of some higher mode appears additionally when the frequency increases over the resonant frequency. For both kinds of free waves, as mentioned in Section 4, there is the frequency range, $0 \leqq a_1 < a_{1cr}$, in which the frequency equations have no real roots. In this range, any propagation of free waves is impossible in a perfectly elastic stratum and thus, the stratum exhibits no attenuations in the frequency range below the lowest resonant frequency.

Viscoelastic medium — As shown in Fig. 2, all the roots of the frequency equations are complex numbers having non-zero imaginary parts. The branch points appearing in the integrands of the analytical expressions of *D.G.C.* are also similar complex numbers. Therefore, it is convenient to evaluate the infinite integral with respect to ξ in the analytical expression of *D.G.C.* along the positive real axis having no singularities. No result is obtained in such a separated expression consisting of Cauchy's principal values and the residue terms as in the case of perfectly elastic medium. However, if the integration is made along an appropriate path in the complex ξ-plane, a formal separation into those two parts may be possible. The infinite integral in the analytical expressions of *D.G.C.* means the continuous summation of the effects of every wave number included. Although the imaginary part of *D.G.C.* appears only in a certain definite range of the integration variable, $\xi = c_2\zeta/\omega$, and at the poles on the positive real axis for a perfectly elastic medium, it always appears in a viscoelastic medium in the whole range of wave numbers, because the dissipative attenuation may arise for waves of whatever wave numbers. Moreover, the residue terms at the complex-valued poles are in general complex numbers. Thus, in contrast to the case of a perfectly elastic medium, a physical separation of the imaginary part of *D.G.C.* of a viscoelastic medium into the radiative and dissipative attenuations will not be allowed, even though its formal separation be made such as the imaginary parts of the residue terms related to the radiation and dissipation of surface waves and of Cauchy's principal values related to those of other waves. However, in the case of a viscoelastic half-space the equivalent viscous damping coefficient may have an appearance, as shown in Fig. 8, such that the amount of energy attenuation owing to internal dissipation in the viscoelastic half-space is added to that owing to wave radiation in a perfectly elastic one. It is observed, moreover, that the former, the amount owing to internal dissipation, increases almost linearly with the non-dimensional viscosity coefficient η which is the parameter prescribing the magnitude of the dissipative attenuation. For the case of a viscoelastic stratum, such additivity of the energy attenuation seems to be invalid, because the two attenuation mechanisms of radiation and dissipation may complicatedly interact with each other.

Even in the low frequency range below the critical frequency in which no free waves appear in a perfectly elastic stratum, innumerable modes of their propagation are possible in a viscoelastic stratum. Consequently, the two kinds of attenuation always occur in the whole range of frequency. However, as shown in Fig. 2 the attenuation constant of every mode is extremely large in this low frequency range, so that all of the free waves may attenuate immediately as they propagate and the dissipative attenuation may greatly predominate over the radiative one. In addition to the resonance phenomena, it is one of the noticeable characteristics of a stratum different from the half-space that the energy attenuation caused by wave radiation is extremely small or does not exist at all in the lower range below the lowest resonant frequency.

In general, the two kinds of free waves, namely, Rayleigh and Love waves, may propagate in both a perfectly elastic and a viscoelastic stratum. It may be readily understood from the assumption of stress distributions produced by an exciting force beneath the foundation that both free waves appear in the case of horizontal excitation while only Rayleigh waves are caused in the cases of vertical and rotational excitations. The occurrence of Love waves in addition to Rayleigh waves may be pointed out as a characteristic of the case of horizontal excitation.

The effects of dissipative attenuation in a medium of Voigt solid increase remarkably with the frequency, because the imaginary parts of complex elastic moduli are proportional to the product of the non-dimensional viscosity coefficient η and the frequency parameter a_0. Therefore, as shown in Fig. 7, the peaks of *D.G.C.* near the resonances become gradually smoother as the frequency is higher, and the distinguished peak of the fundamental resonance can be particularly observed. It is considered in the range where ηa_0 is large that the waves produced at the source on the surface of a stratum may almost attenuate before they reach the boundary of a supporting rigid half-space and also that the reflected waves from the lower boundary may scarcely affect the behavior of the foundation at which *D.G.C.* is defined. Hence, the solutions of a viscoelastic stratum have a close resemblance to those of a half-space in the range of large ηa_0. Such resemblance in *D.G.C.* is more remarkable in the case of a horizontal excitation than in the cases of vertical and rotational excitations. This may be interpreted by the following two reasons:

(1) In the case of a horizontal excitation, Love waves are caused together with Rayleigh waves. For the other two types of excitation, however, only Rayleigh waves are produced, and they contribute considerably to *D.G.C.* of these cases. Since the attenuation constants of the first several modes of Love waves are surely larger than those of the corresponding modes of Rayleigh waves, the components of Love waves, which are inherent in a stratified medium, may attenuate more rapidly than those of Rayleigh waves and the latter wave components become distinguished when Voigt-type internal dissipation exist in a stratum. Therefore, as regards the contribution of free waves to the solutions of a stratum, it approaches the case of a half-space more closely for a horizontal excitation than for the other two types of excitation.

(2) Horizontal excitation is liable to produce distortional waves, and the other two cases to produce dilatational waves. The phase velocity or the wavelength of the latter waves is always larger than that of the former as long as the same frequency is concerned. Moreover, the difference in their wavelengths becomes larger with the increase of ηa_0. In the range where ηa_0 is large, waves of a shorter wavelength are thus distinguished for the case of a horizontal excitation in which the distortional wave components may have the main role. On account of the wavelength effect of the bodily waves mentioned above, a supporting rigid half-space may have less effect on *D.G.C.* in this case of excitation, particularly in the range of large ηa_0.

Effects of the ratio of stratum thickness to foundation width (See Fig. 6.)

As the thickness of the stratum is relatively large in comparison with the width of the foundation, the reflected waves from the boundary of a supporting half-space have less effect on the behavior of *D.G.C.*, which is defined at a point in a surface exciting area, because of both the phenomena of wave radiation and of internal dissipation. Hence, in the case where the ratio of the stratum thickness H to the half-width of foundation b is large, *D.G.C.* of a stratum shows a closer resemblance to the case of a half-space. It is found that the rapid convergence of *D.G.C.* of a stratum to *D.G.C.* of a half-space with the increase of the ratio H/b is extremely conspicuous in the case of rotational excitation. In the two cases of translational excitation the effect of a supporting rigid half-space on *D.G.C.* is considered to be accumulative, since the assumed stress distribution beneath the foundation takes the same sign at the same instance. For the case of rotational excitation, on the other hand, the assumed distribution is proportional to the distance from the rotational axis, and both tensile and compressional exciting stresses are equally applied on the surface area at the same instance. Since such surface excitations may have some cancelling effect, the dynamic responses of a stratum may be reduced rapidly with the increase of the depth from the surface. This may be the reason why the lesser effect of a supporting rigid half-space appears on *D.G.C.* and the rapid convergence of *D.G.C.* to the case of a half-space is observed in almost the whole frequency range except for the neighborhoods of the lower resonant frequencies in the case of rotational excitation. However, it is noticed that in the range where ηa_0 is large a strong resemblance in *D.G.C.* is observed between the two kinds of medium, particularly in the case of horizontal excitation, because the effect of internal dissipation on the attenuation of free waves as well as on the wavelength of bodily waves appears more significantly for this type of excitation as mentioned previously.

Equivalent coefficients of Voigt model (See Fig. 8.)

Halfspace (indicated by dotted lines in figures.) — The equivalent stiffness coefficients κ_{el} for the three types of excitation are monotonously decreasing functions of the frequency parameter a_0. Although little difference in κ_{el} according to the change of the non-dimensional viscosity coefficient η is seen in the range where a_0 is small, the difference becomes more noticeable as the parameter a_0 increases and the coefficients κ_{el} become smaller with the increase of the product ηa_0 representing the magnitude of dissipative attenuation. In the case of rotational excitation, the equivalent stiffness coefficient is little affected by the parameter η in a considerably wider frequency range than in the other types of excitation.

The equivalent viscous damping coefficients c_{el}, on the other hand, show considerably different characteristics between the cases of translational and of rotational excitations. The coefficients c_{el} in the vertical and horizontal cases are nearly constant over the frequency parameter range considered, that is, in these two cases the viscous damping coefficients are almost independent of the frequency. In the case of rotational excitation, however, c_{el} is an increasing function of the frequency parameter a_0 at least in the range considered. Strictly, the coefficients c_{el} for both two translational cases show a slowly increasing tendency in the range where a_0 is large. Regardless of the type of excitation and of the value of the frequency, the increase of the parameter η yields an increase of c_{el} approximately proportional to η in addition to the value of c_{el} of a perfectly elastic medium caused only by wave radiation. It is noted that in the case of rotational excitation the equivalent viscous damping coefficient of a perfectly elastic medium is inconsiderable for a small a_0, and the energy attenuation owing to wave radiation is insignificant as compared with the dissipative attenuation due to the viscoelasticity of the medium, especially in a low frequency range.

Stratum (indicated by solid lines in figures.) — As for the equivalent coefficients, κ_{el} and c_{el}, a similar tendency as has been in the discussions on *D.G.C.* can also be pointed out such that the difference between a stratum and a half-space reduces as the ratio H/b or the product ηa_0 increases. Because of the resonance phenomena inherent in a stratified medium, the local reductions of both coefficients occur in the neighborhood of the resonant frequencies. These reductions are more remarkable as the parameter η decreases, and especially for a perfectly elastic stratum both the coefficients vanish at the resonant frequencies. As the product ηa_0 increases, the variations of the coefficients with the parameter a_0 become smooth and the local reductions at the resonant frequencies gradually disappear. Also, in the range where ηa_0 is large, the equivalent coefficients of a stratum converge uniformly to those of a half-space as far as the higher frequency range above the lowest resonant frequency is concerned.

It is a noticeable characteristic common to the three types of excitation that the coefficient c_{el} is considered as nearly constant with respect to a_0 in the lower frequency range below the lowest resonant frequency where the dissipative attenuation may be predominant. Especially for the case of rotational excitation in which energy attenuation owing to wave radiation is scarcely produced even in a half-space, the equivelent viscous damping coefficient of a stratum is almost in accordance with the coefficient of a half-space in almost the whole frequency range when the ratio H/b is larger than about 4. As the parameter a_0 increases from zero, the coefficients c_{el} for all types of excitation increase rapidly in the neighborhood of the lowest resonant frequency because of the additional radiative attenuation owing to the first mode of free wave.

As for the general dynamic characteristics of the massless-foundation-ground system the equivalent coefficients show a considerable dependence on the frequency, particularly in the cases of a half-space with a large dissipative coefficient and of a stratum in general. From this aspect, the use of such a simplified model as a frequency-independent single spring-dashpot system is clearly inadequate, and the use of a higher-degree-of-freedom lumped system with approximate transfer function to *D.G.C.* of the foundation-ground system may be recommended. It is noticeable, however, that the equivalent coefficients of a Voigt model show little fluctuation with the frequency if the following two conditions are satisfied; (1) the thickness of the surface layer is much larger than the width of the foundation and a pertinent amount of dissipative attenuation exists in the medium so that the stratum under the foundation can be considered practically as a half-space; and (2) translational excitation is treated in the relatively low frequency range. In this case, a frequency-independent single spring-dashpot system may be available for a model of a massless-foundation-ground system, and the numerical values of its parameters are determined by using the equivalent stiffness and viscous damping coefficients, which are dimensionless quantities, as well as the ground constants and the shape and dimensions of the foundation.

Effects of the shape and size of the foundation (See Fig. 9.)

The shape of a foundation scarcely affects the qualitative nature of *D.G.C.* and the equivalent coefficients, and affects their quantitative properties little when the shape parameter c/b varies in the range from 1/2 to 2, while the area or the moment of inertia about the rotational axis remains constant for the cases of translational or rotational excitations, respectively. Strictly speaking, both of the two equivalent coefficients, $\bar{k}_{el}$ and $\bar{c}_{el}$, which are defined under the above-mentioned equi-area or equi-moment of inertia conditions, take minimum values when the shape ratio $c/b=1$ for all the types of excitation. It may be rather difficult to know the general characteristic of the effect of the size of a foundation on the equivalent coefficients from the results obtained herein. However, it seems probable that both coefficients with original physical dimension are approximately proportional to the square root of the area or to the three-fourth powers of the moment of inertia respectively for the case of translational or rotational excitation, when the frequency is smaller than the lowest resonant frequency and when the thickness of the stratum is considerably larger than the reference width of the foundation.

Comparison between a stratum and a half-space

In the preceding, the properties of *D.G.C.* and *S.G.C.* have been discussed from several viewpoints. In completing the discussions they are now examined from the standpoint of the comparison between a stratum and a half-space.

The noticeable aspects of the dynamic properties of a stratum different from the case of a half-space are summarized as follows:

(1) There exist several kinds of resonant frequencies which are functions of the stratum and the constants of the medium. The resonant amplitude diverges for a perfectly elastic stratum on account of the existence of a supporting rigid half-space, but remains finite for a viscoelastic stratum because of the dissipative character of the medium.

(2) In the lower frequency range below the lowest resonant frequency, the energy attenuation caused by wave radiation is extremely small in a viscoelastic stratum and does not exist at all in a perfectly elastic one.

When the ratio of the thickness of stratum H to the half-width of foundation b increases, the above-mentioned frequency range decreases and the peaks of the resonance curve become sharper and narrower. Moreover, as the viscosity coefficient parameter η is larger, the resonance curve of the stratum are more smoothed and approach to the case of a half-space in the higher frequency range above the lowest resonant frequency. Except for the case of rotational excitation, however, the resonance curve of a stratum does not converge to that of a half-space in the low frequency range in which the dissipative attenuation predominate over the radiative one. In general, when both the ratio H/b and the parameter η increase, *S.G.C.* and *D.G.C.* of a stratum converge to those of a half-space in the whole range of frequency. Since *S.G.C.* and *D.G.C.* represent respectively the static and dynamic displacement characteristics of a massless-foundation-ground system subjected to an excitation, they are considerably affected by the properties of the medium at the vicinity of the foundation. As the dissipative attenuation increases or the thickness of the stratum becomes large in comparison with the width of foundation, a rigid supporting half-space may have less influence on the characteristics of *S.G.C.* and *D.G.C.*, so that the solutions of the stratum show a closer resemblance to those of a half-space.

The distinctive features mentioned are such that, when the ratio H/b increases, the solutions of a stratum converge to those of a half-space much more rapidly for the case of rotational excitation than for the translational cases, and also that the dissipative attenuation affects the solutions more remarkably for a horizontal excitation than for the other cases; namely, while the convergence of *D.G.C.* to the half-space solution with the increase of the ratio H/b is extremely rapid for the rotational case in the lower frequency range because of the cancelling effect of the exciting force distribution and of the weak dependence on free waves, the convergence is rather rapid for the horizontal case in the higher frequency range because the dissipative attenuation increases with the frequency.

In general, it may be allowable in the range where ηa_0 is large to treat a stratum practically as a half-space notwithstanding the value of the ratio H/b. It seems, however, that a stratum cannot be treated as a half-space, especially in the low frequency range, unless the ratio H/b takes a considerably large value. From the numerical data of *S.G.C.* and *D.G.C.* of a viscoelastic medium, it is suggested that the lower bound of the ratio between the thickness of the stratum and the reference half-width of the foundation above which such practical treatment may be permissible in the whole range of frequency is about 4 for the case of rotational excitation and about 10 for the translational cases; in other words, for the validity of the practical treatment of a half-space instead of a stratum, the thickness of the stratum is to be about twice the width of an equivalent square foundation having the same moment of inertia with the rectangular foundation for the rotational excitation and about five times larger than that having the same area of the foundation for the translational excitations, respectively.

In this paper, an idealized model of a foundation-ground system, namely, a rectangular foundation on a viscoelastic stratum lying over a rigid half-space, which may have a severer condition than any other stratified medium in the sense of small radiative damping, is considered in comparing its dynamic characteristics with the case of a half-space in which a considerable amount of radiative damping may exist. It is thus supposed that the above-mentioned conclusions on *S.G.C.* and *D.G.C.* as to whether the stratum over a rigid half-space can or cannot be treated practically as a half-space may also be applicable for the case in which the supporting medium is composed of several viscoelastic layers.

11. Concluding Remarks

As a basic attempt to define the anti-seismic safety of structural systems including the sub-soil ground, the authors have made a series of theoretical studies in which the concept of the "Dynamical Ground Compliance" is introduced on the basis of the wave propagation theory in order to represent the dynamic properties of a foundation-ground system. Paying attention to the mechanism of dissipative and radiative attenuations in the ground, this paper has studied the dynamic properties of a viscoelastic stratum over a rigid half-space including a viscoelastic half-space as its limiting medium for the case in which a vertical, a horizontal or a rotational excitation is exerted on a massless rectangular foundation resting on the surface of the stratum. The stratum is assumed to be composed of a Voigt soid that has often been adopted in seismic problems as one of the most basic models of a viscoelastic continuum.

This study has led to the following principal conclusions:

(1) The energy attenuations in the viscoelastic stratum occur in two mechanisms; one of these is the dissipative attenuation owing to the viscosity of the medium and the other is the radiative attenuation owing to the wave radiation into the medium. Although the two mechanisms of energy attenuation interfere with each other in a strict sense, its total amount for a viscoelastic stratum seems in general to be the sum of the energy attenuation increasing monotonously with the viscosity coefficient and that caused by wave radiation for a perfectly elastic stratum. Especially in the case of a half-space, the amount of dissipative attenuation is almost proportional to the viscosity coefficient in a wide range of frequency.

(2) Two main aspects of the dynamic properties of a stratum different from those of a half-space are: 1. there is an enumerably infinite number of resonant frequencies to be determined from the thickness of the stratum and the constants of the medium, and the finite resonant amplitudes for a viscoelastic stratum diverge when the medium is perfectly elastic; and 2. in the lower frequency range below the lowest resonant frequency the energy attenuation owing to wave radiation scarcely appears in a viscoelastic stratum and never exists in a perfectly elastic one.

(3) Less affected by a supporting rigid half-space, the dynamic properties of a stratum show a closer resemblance to those of a half-space, as the product of the viscosity coefficient and the frequency is larger or as the thickness of the stratum is considerably larger than the width of the foundation. The convergence of *D.G.C.* of a stratum into that of a half-space is slightly rapider for the case of a horizontal excitation when the above-mentioned product increases, and also remarkably rapider for the rotational case when the ratio of the stratum thickness to the foundation width becomes larger, than for the cases of other excitations.

(4) Thus, a stratum whose thickness is much larger than the dimensions of the foundation may be treated practically as a half-space in the whole range of frequency, otherwise such treatment cannot be permissible unless the product of the viscosity coefficient and the frequency takes a certain fairly large value and also the higher

frequency range above the lowest resonant frequency is concerned.

(5) The equivalent stiffness and damping coefficients of a Voigt model evaluated from *D.G.C.* cannot be given generally as the quantities independent of the frequency when the sub-soil ground is assumed to be a perfectly elastic or a viscoelastic stratum. However, if the ground under a foundation can be considered practically as a half-space and if the translational excitation is confined in the relatively low frequency range, it is possible to determine approximately the frequency-independent coefficients of an equivalent Voigt model from the density and elastic constants of the medium and the shape and dimensions of the foundation.

The inverse of *D.G.C.* gives the displacement-force transfer function of a massless foundation-ground system, which means the complex stiffness of the system. When this transfer function is connected with the transfer function of an above-ground structural system, it may be possible in principle to perform the earthquake response analysis of a coupled ground-structure system subjected to an arbitrary transient earthquake motion. In particular, the linear harmonic responses of coupled ground-structure systems can be analyzed by making direct use of the numerical solutions of *D.G.C.* Therefore, several studies, including the authors', have been made on structural models resting on perfectly elastic half-spaces and their basic vibrational characteristics have been described in detail[24)-26)].

Even if a highly efficient digital computer is employed, immense operational time would be required for the following frontal attack method of; connecting the analytical expressions of *D.G.C.* or those of the complex stiffness, each of which is an improper double integral representation including an infinite integral, with the transfer function of an above-ground structural system; evaluating the Fourier transforms of the transient responses of the coupled ground-structure system in the frequency domain; deriving their general expressions in the time domain by performing the inverse Fourier transformation with respect to the frequency parameter; and finally, computing their numerical transient responses. However, the linear or nonlinear transient earthquake responses of a ground-structure system may be evaluated somewhat easily, if an appropriate representation of the transfer function such as a rational function approximated from the numerical solution of *D.G.C.* is simulated on the equivalent operational circuit of an analog computer or on the program of a digital computer. Such practical treatment is indeed possible, so that several studies have been made on the methods of functional approximation of the transfer function and transient response analysis of ground-structure systems and also on the earthquake response characteristics of some elasto-plastic ground-structure systems[27)28)].

Thus, *D.G.C.* studied in this paper may offer the basic data in estimating the anti-seismic safety of coupled ground-structure systems and may also have broad applicability to various dynamic problems of ground-structure systems on a viscoelastic ground.

As a first step to describe the dynamic properties of a viscoelastic stratified medium, this paper has dealt with an idealized ground model for the purpose of simplifying the treatment of the problem and also of investigating the most basic stratified medium first of all. It will be an important and interesting problem at the next step to study a more realistic ground model, such as a three-dimensional stratified medium composed of several viscoelastic layers. The results of such an investigation will be reported in a subsequent paper.

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