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Statistical Linearization Techniques of Hysteretic Structures to Earthquake Excitations

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Abstract

To evaluate the reliability of structures subjected to severe earthquake excitations in relation to the ultimate aseismic design method which guarantees the safety of structures in the ultimate state, the basic statistical characteristics of the response of elasto-plastic structures should be understood. At the present time there is no useful analytical method which can be used to treat problems of the response of stochastically excited hysteretic structures with strong nonlinearity. The Fokker-Planck equation approach is inapplicable to problems which involve the hysteretic nonlinearity or the non-white excitation. As an approximate analytical method, the statistical linearization technique has been widely used, since this technique overcomes the above restrictions. However, this technique requires small nonlinearity, and so may be not applicable to find statistical characteristics of the response of hysteretic structures with strong nonlinearity which is considered important in relation to the ultimate aseismic safety.

The purpose of this paper is to introduce a new statistical linearization technique that will allow as to obtain solutions to the problem of the response of hysteretic structures with strong nonlinearity to random excitation. From experimental results of the response of hysteretic structures with strong nonlinearity by means of the simulation technique, it is pointed out that the response possesses broad-band characteristics and the center of hysteresis eccentrically fluctuates. With these points as background, a new statistical linearization technique is introduced by considering the scatter of frequency and the fluctuation of the center of hysteretic oscillation. The equivalent linearization coefficients are determined by the least mean-square error method. In the present study, the average used in minimization procedure is a combination of ensemble and time averages. Though the technique developed by T. K. Caughey involves only amplitude of hysteretic oscillation as random variable, the shift of center, amplitude and frequency of hysteretic oscillation are treated as random variable. The joint probability density function of these random variables is approximately evaluated under the assumption that the response is a stationary Gaussian process.

The numerical analysis is carried out in the case of a single-degree-of-freedom structure with bilinear hysteresis to a band-limited white Gaussian excitation. From the statistics of the stationary response computed by the statistical linearization techniques and the simulation technique, it is shown that the scatter of frequency and the fluctuation of the center of hysteretic oscillation affect strongly on the displacement response of bilinear structures with severe nonlinearity, and so the statistical linearization technique is considerably improved by taking them into account. Another fact of importance is the existence of the optimum rigidity ratio which minimizes the variance of the displacement response. It is suggested that the above optimum ratio has a great significance in doing the aseismic design of the elasto-plastic structures.

1. Introduction

In the present approach to obtain a sound anti-seismic structure, a dual design concept is often used[1]. The structure is designed to resist a moderate earthquake without significant damage according to the elastic design method, and to resist a strong earthquake without extreme damage or collapse according to the elasto-plastic aseismic design method. Due to its randomness in nature, earthquake excitation has been treated as a stochastic process. Therefore, in relation to the aseismic safety of structures subjected to strong earthquake from probabilistic point of view[2)], it is considered essential to understand the basic statistical characteristics of the response of structures, particularly in the latter design method, of the response of nonlinear hysteretic structures.

At the present time there are many analytical methods which can be used to treat problems of the response of nonlinear structures to random excitations. An exact solution may be evaluated by only solving the stochastic differential equation such as the Fokker-Planck equation. Unfortunately, this Fokker-Planck equation approach has a very limited range of applicability[3)], and no solution of this equation for the response of nonlinear hysteretic system has yet been obtained. As an approximate analytical method, the statistical linearization technique has been widely used, since this technique is applicable to a nonlinear hysteretic structure subjected to a Gaussian excitation with a non-white power spectral density. This technique was developed by Booton[4)] and Caughey[5)] as a statistical extension of the method of equivalent linearization of Krylov and Bogoliubov in deterministic theory. Caughey has discussed the technique for the problem of a nonlinear hysteretic oscillator in the stationary random process[6),7)]. The application of this technique to problems of non-stationary random response of hysteretic structures has been discussed in References 8 and 9. However, the abovementioned technique requires small nonlinearity. Therefore, the technique may be not applicable to find statistical characteristics of the response of hysteretic structure with severe nonlinearity, which is considered important in relation to the ultimate aseismic safety.

From investigations of the response of hysteretic structures by means of the simulation technique, it is pointed out that as the degree of nonlinearity becomes greater, the response has the frequency character of broader band process, the fluctuation of the center of hysteresis in connection with the plastic deformation increases, and so the response increases rapidly.

With these points as background, it is the purpose of the present sutdy to introduce a new statistical linearization technique by taking into account the scatter of frequency and the fluctration of center of hysteresis. It is felt that this new technique leads both to a clearer interpretation of a tendency that the response of a hysteretic structure increases markedly, as the nonlinearity becomes more severe, and also to an improvement of the statistical linearization technique. The statistical linearization technique is based on the idea of replacing the original nonlinear sturcture by a related linear structure in such a way that the mean squared value of the difference between the two structures is minimized. In the present study, the average used in minimization procedure is a combination of ensemble and time averages. Under the assumption that the shift of the center, amplitude and frequncy of hysteretic oscillation are slowly varying random variables over any one cycle, the combination average is defined under such a law that ensemble average is taken with respect to to three random variables after time average is carried out over one cycle. The joint probability density function of the three random variables is approximately evaluated using the assumption that the response is the stationary Gaussian process.

As a numerical example, a single-degree-of-freedom structure with bilinear hysteresis subjected to a band-limited white Gaussian excitation is considered. The statistics of stationary response are computed by using the three statistical linearization techniques based on the following different analyses:

1) the one-dimensional analysis without consideration of the scatter of frequency and fluctuation of the center of hysteretic oscillation,
2) the two-dimensional analysis without consideration of the scatter of frequency of hysteretic oscillation,
3) the three-dimensional analysis with consideration of the scatter of frequency and the fluctuation of the center of hysteretic oscillation.

An indication of the relative merit of the above three techniques is proved by the experimental result obtained by means of the Monte Carlo technique.

2. Statistical Linearization Techniques

Consider a structure with nonlinear hysteresis subjected to random excitation of the earthquake type. It will be assumed that the hysteresis behaviour is a smoothly varying process and the character of hysteresis is stable. In order to obtain an approximate solution for the response of the nonlinear hysteretic structure, replacing the nonlinear hysteretic characteristics function $\varphi(\eta, \dot{\eta}; t)$ by a related linear function $\varphi_e(\eta, \dot{\eta}; t)$ which is described in terms of displacement η and velocity $\dot{\eta}$

$$\varphi_e(\eta, \dot{\eta}; t) = k_e(t)\eta + d_e(t)\dot{\eta} \tag{2.1}$$

in which $k_e(t)$ and $d_e(t)$ are the equivalent stiffness and the equivalent damping coefficient. The difference $\varphi - \varphi_e$ between the original nonlinear function and the equivalent linear function will be dependent on the choice of k_e and d_e. As η and $\dot{\eta}$ are stochastic processes, the difference is also a stochastic process. As a means of making the difference a minimum in a statistical sense, it is desirable to use the criterion that the mean squared value of the difference is a minimum. In the present study, the mean squared error function is expressed as

$$J(k_e, d_e) = \boldsymbol{E}_L[\{\varphi(\eta, \dot{\eta}; t) - \varphi_e(\eta, \dot{\eta}; t)\}^2] \tag{2.2}$$

In the above equation, the averaging operator $\boldsymbol{E}_L[\cdot]$ means a combination of time and ensemble averages, which is defined as

$$\boldsymbol{E}_L[\cdot] = \int <\cdot>_L p(\boldsymbol{x})\, d\boldsymbol{x} \tag{2.3}$$

in which $<\cdot>_L$ denotes the local time average over one cycle of oscillation and $P(\boldsymbol{x})$ is the joint probability density function of random variables $\boldsymbol{x}$ involved in $<\cdot>_L$. To minimize $J(k_e, d_e)$ requires the following conditions for k_e and d_e:

$$\frac{\partial J}{\partial k_e} = \frac{\partial J}{\partial d_e} = 0, \quad \frac{\partial^2 J}{\partial k_e^2} > 0, \quad \frac{\partial^2 J}{\partial d_e^2} > 0 \tag{2.4}$$

Then, the equivalent linearization coefficients can be obtained as follows:

$$k_e(t)=\frac{\begin{vmatrix} E_L[\varphi\eta] & E_L[\varphi\dot{\eta}] \\ E_L[\eta\dot{\eta}] & E_L[\dot{\eta}^2] \end{vmatrix}}{\begin{vmatrix} E_L[\eta^2] & E_L[\eta\dot{\eta}] \\ E_L[\eta\dot{\eta}] & E_L[\dot{\eta}^2] \end{vmatrix}}, \quad d_e(t)=\frac{\begin{vmatrix} E_L[\varphi\dot{\eta}] & E_L[\varphi\eta] \\ E_L[\eta\dot{\eta}] & E_L[\eta^2] \end{vmatrix}}{\begin{vmatrix} E_L[\eta^2] & E_L[\eta\dot{\eta}] \\ E_L[\eta\dot{\eta}] & E_L[\dot{\eta}^2] \end{vmatrix}} \tag{2.5}$$

2.1 Analysis with consideration of fluctuating hysteresis

The response of hysteretic structures subjected to random excitations has been investigated by means of the digital simulation and electronic-analog techniques in References 10), 11) and 12) and in Chap. 3. The following points are emphasized from these experimental results: As the degree of nonlinearity of the hysteretic structure becomes greater,

(1) the frequncy character of response becomes that of broader band process,
(2) the hysteresis behaviour fluctuates more obviously in a random manner and the shift of hysteresis center increases in connection with the growth of plastic drift.

Hence, in order to apply the statistical linearization technique to problems of determining the response of hysteretic structure with large nonlinearity, consider a model of hysteretic behaviour with the above points as background as shown in Fig. 1.

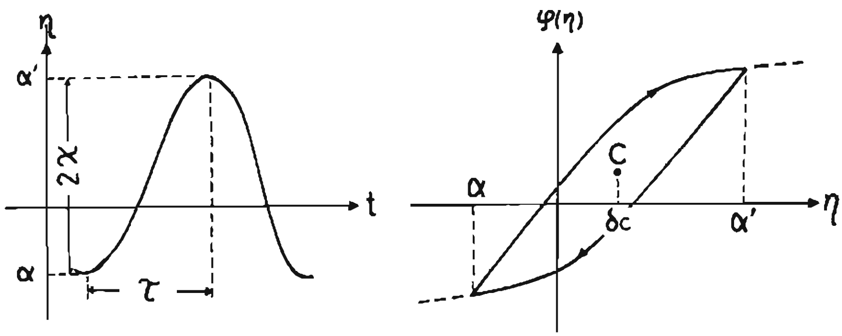


Fig. 1. Force vs displacement characteristic for a typical hysteresis.

Point C denotes the center of hysteresis which eccentrically fluctuates. $\delta_c(t)$ is the η co-ordinate of point C, $\chi(t)$ and $\omega(t)$ denote amplitude and frequency of hysteretic oscillation, respectively. The variables $\delta_c(t)$, $\chi(t)$ and $\omega(t)$ are treated as slowly varying random functions of time, and so have nearly constant values over any one cycle. Under the assumption that a structural response is smoothly varying process, the response $\eta(t)$ may be approximately a sinusoidal time function.

$$\eta(t)=\delta_c(t)-\chi(t)\cos(\omega(t)t) \tag{2.6}$$

$$\dot{\eta}(t)\approx\chi(t)\omega(t)\sin(\omega(t)t) \tag{2.7}$$

As in Fig. 1, α is the minimum, α' is the maximum and τ is the interval between the minimum and the maximum, then δ_c, χ and ω are given by

$$\delta_c = \frac{\alpha+\alpha'}{2} \equiv \upsilon_1(\alpha, \alpha'), \quad \chi = \frac{\alpha'-\alpha}{2} \equiv \upsilon_2(\alpha, \alpha') \tag{2.8}$$

$$\omega = \frac{\pi}{\tau}$$

where

$$-\infty < \alpha \leq \alpha' < \infty, \quad 0 \leq \chi < \infty$$

Using Eqs. (2.6) and (2.7), the expressions obtained after averaging η^2, $\dot{\eta}^2$ and $\eta\dot{\eta}$ with respect to time over one cycle are

$$\begin{aligned} <\eta^2>_L &= \frac{1}{2\tau}\int_0^{2\tau} \eta^2\, dt = \delta_c^2 + \frac{\chi^2}{2} \equiv B_{ke}(\delta_c, \chi) \\ <\dot{\eta}^2>_L &= \frac{1}{2\tau}\int_0^{2\tau} \dot{\eta}^2\, dt = \frac{\pi^2\chi^2}{2\tau^2} \equiv B_{de}(\chi, \tau) \\ <\eta\dot{\eta}>_L &= 0 \end{aligned} \tag{2.9}$$

Given the hysteretic characteristics function $\varphi(\eta, \dot{\eta}; t)$, the time averages $<\varphi\eta>_L$ and $<\varphi\dot{\eta}>_L$ may be expressed as

$$\begin{aligned} <\varphi\eta>_L &= \frac{1}{2\tau}\int_0^{2\tau} \varphi\eta\, dt \equiv C_{ke}(\delta_c, \chi, \tau) \\ <\varphi\dot{\eta}>_L &= \frac{1}{2\tau}\int_0^{2\tau} \varphi\dot{\eta}\, dt \equiv C_{de}(\delta_c, \chi, \tau) \end{aligned} \tag{2.10}$$

The above time averages are represented as functions of random variables δ_c, χ and τ. If $p(\delta_c, \chi, \tau; t)$ denotes the joint probability density function of δ_c, χ and τ, the equivalent linearization coefficients k_e and d_e are obtained from Eqs. (2.5), (2.9) and (2.10), as follows:

$$k_e(t) = \frac{\int_D C_{ke}(\delta_c, \chi, \tau)\, p(\delta_c, \chi, \tau; t)\, d\delta_c\, d\chi\, d\tau}{\int_D B_{ke}(\delta_c, \chi)\, p(\delta_c, \chi, \tau; t)\, d\delta_c\, d\chi\, d\tau} \tag{2.11}$$

$$d_e(t) = \frac{\int_D C_{de}(\delta_c, \chi, \tau)\, p(\delta_c, \chi, \tau; t)\, d\delta_c\, d\chi\, d\tau}{\int_D B_{de}(\chi, \tau)\, p(\delta_c, \chi, \tau; t)\, d\delta_c\, d\chi\, d\tau} \tag{2.12}$$

in which D denotes the integral area that δ_c ranges from $-\infty$ to $+\infty$, and χ and τ from 0 to ∞.

Evaluating of C_{ke}, C_{de} for bilinear hysteresis

In particular, the hysteresis considered here is the so-called bilinear hysteretic restoring force $\varphi(\eta; r)$ as shown in Fig. 2. This hysteresis is the simplest representation of idealized structures and is often used as an approximation to the yielding behaviour. Note that $\varphi(\eta; r)$ represents the non-dimensional hysteretic characteristic,

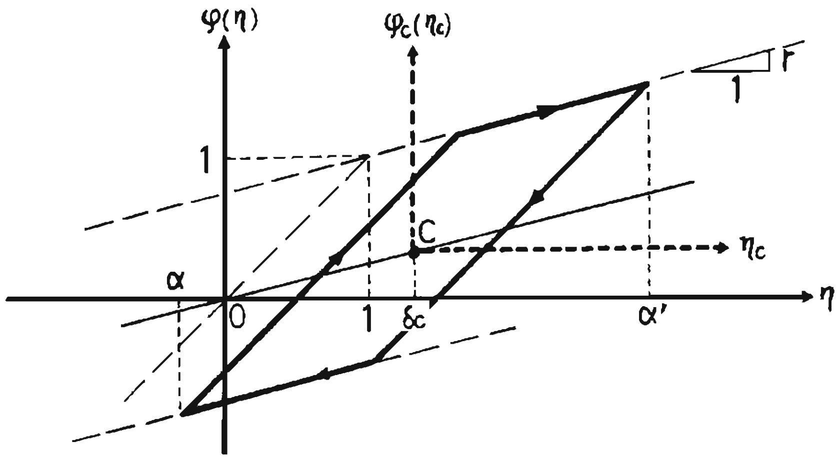


Fig. 2. Bilinear hysteretic restoring force.

r is the rigidity ratio of the second to the first branch, both the elastic limit deformation and strength are unity. Let the co-ordinates of the center of hysteresis be $(\delta_c, r\delta_c)$, the displacement η and the restoring force φ can be written from Fig. 2.

$$\begin{aligned}\eta&=\delta_c+\eta_c\\ \varphi&=r\delta_c+\varphi_c\end{aligned} \tag{2.13}$$

where

$$\eta_c=-\chi\cos(\omega t), \quad \dot{\eta}_c=\dot{\eta}=\chi\omega\sin(\omega t)$$

Then, $<\varphi\eta>_L$ and $<\varphi\dot{\eta}>_L$ become

$$\begin{aligned}<\varphi\eta>_L&=<(r\delta_c+\varphi_c)(\delta_c+\eta_c)>_L=r\delta_c^2+<\varphi_c\eta_c>_L\\ <\varphi\dot{\eta}>_L&=<(r\delta_c+\varphi_c)\dot{\eta}_c>_L=<\varphi_c\dot{\eta}_c>_L\end{aligned} \tag{2.14}$$

The final expressions of C_{ke}, C_{de} are given by

$$\begin{aligned}C_{ke}&=<\varphi\eta>_L=r\delta_c^2+\frac{\chi^2}{2}+(1-r)\left\{\frac{\chi^2}{2\pi}\cos^{-1}\left(1-\frac{2}{\chi}\right)+\frac{2-\chi}{\pi}\sqrt{\chi-1}-\frac{\chi^2}{2}\right\} && \text{for } \chi\geq 1\\ &=r\delta_c^2+\frac{\chi^2}{2} && \text{for } 0\leq\chi<1\\ C_{de}&=<\varphi\dot{\eta}>_L=\frac{2(1-r)(\chi-1)}{\tau} && \text{for } \chi\geq 1\\ &=0 && \text{for } 0\leq\chi<1\end{aligned} \tag{2.15}$$

2.2 Probability density function $p(\delta_c, \chi, \tau)$

It is assumed that the excitation is a stationary Gaussian process with zero mean value and the response of the nonlinear sturcture with a stable hysteresis is also approximately a stationary Gaussian process with zero mean value. From Eq. (2.8), the joint probability density function $p(\delta_c, \chi, \tau)$ in the stationary process can be represented as

$$p(\delta_c, \chi, \tau) = p_0(\delta_c - \chi, \delta_c + \chi, \tau) \Big/ \begin{vmatrix} \frac{\partial \upsilon_1}{\partial \alpha} & \frac{\partial \upsilon_1}{\partial \alpha'} \\ \frac{\partial \upsilon_2}{\partial \alpha} & \frac{\partial \upsilon_2}{\partial \alpha'} \end{vmatrix} = 2 p_0(\delta_c - \chi, \delta_c + \chi, \tau) \tag{2.16}$$

in which $p_0(\alpha, \alpha', \tau)$ is the joint probability density function of α, α' and τ. Futhermore, it can be expressed as

$$p_0(\alpha, \alpha', \tau) = p_0(\alpha, \alpha' | \tau) \cdot p_T(\tau) \tag{2.17}$$

in which $p_0(\alpha, \alpha' | \tau)$ is the conditional probability density function that, given a minimum of $\eta(t)$ at $t=0$ and the next maximum at $t=\tau$, then $\eta(0)=\alpha$ and $\eta(\tau)=\alpha'$, and $p_T(\tau)$ is the probability density function that, given a minimum of $\eta(t)$ at $t=0$, the next maximum occurs at $t=\tau$. Since the exact solution of probability density function $p_T(\tau)$ cannot be evaluated, an approximate expression for $p_T(\tau)$ will be evaluated by using the theory of random points developed by Stratonovich and Kuznetsov[13]. Assuming that maximums of the response process $\eta(t)$ occur at τ_j $(j=1, 2, \dots)$, after a minimum at $t=0$; i.e,

$$\dot{\eta}(0) = 0, \qquad \ddot{\eta}(0) > 0$$

$$\dot{\eta}(\tau_j) = 0, \quad \ddot{\eta}(\tau_j) < 0 \qquad \text{for } j = 1, 2, \dots$$

then the points τ_j form a system of random points, characterized by the distribution functions

$$f_1(\tau_1), f_2(\tau_1, \tau_2), \dots, f_s(\tau_1, \dots, \tau_s).$$

The distribution function $f_s(\tau_1, \dots, \tau_s)$ is given by

$$f_s(\tau_1, \dots, \tau_s) = \frac{\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \int_{0}^{\infty} |\ddot{\eta}_0| \, |\ddot{\eta}_1| \cdots |\ddot{\eta}_s| \, w(\dot{\eta}_0, \ddot{\eta}_0, \dots, \dot{\eta}_s, \ddot{\eta}_s)_{\dot{\eta}_0 = \dot{\eta}_1 = \cdots = \dot{\eta}_s = 0} \, d\ddot{\eta}_0 \cdots d\ddot{\eta}_s}{\int_0^\infty \ddot{\eta}_0 w(\dot{\eta}_0, \ddot{\eta}_0)_{\dot{\eta}_0 = 0} \, d\ddot{\eta}_0} \tag{2.18}$$

Using the distribution functions, the probability density function $p_T(\tau)$ can be obtained as follows:

$$p_T(\tau)=f_1(\tau)+\sum_{s=2}^{\infty}\frac{(-1)^{s-1}}{(s-1)!}\int_0^{\tau}\cdots\int_0^{\tau}f_s(\tau_1,\ldots,\tau_{s-1})\,d\tau_1\cdots d\tau_{s-1} \tag{2.19}$$

The above equation has been obtained by the inclusion and exclusion method as indicated by S. O. Rice[14]. The probability density function $p_T(\tau)$ can also be expressed in terms of cumulant functions

$$\int_{\tau}^{\infty}p_T(t)\,dt=\exp\left\{-\int_0^{\tau}g_1(t)\,dt+\sum_{s=2}^{\infty}\frac{(-1)^s}{s!}\int_0^{\tau}\cdots\int_0^{\tau}g_s(t_1,\ldots,t_s)\,dt_1\cdots dt_s\right\} \tag{2.20}$$

in which the cumulant functions $g_s(\tau_1,\ldots,\tau_s)$ are related to the distribution functions as follows:

$$\begin{aligned} &g_1(\tau)=f_1(\tau)\\ &g_2(\tau_1,\tau_2)=f_2(\tau_1,\tau_2)-f_1(\tau_1)\\ &g_3(\tau_1,\tau_2,\tau_3)=f_3(\tau_1,\tau_2,\tau_3)-f_1(\tau_1)f_2(\tau_2,\tau_3)-f_1(\tau_2)f_2(\tau_1,\tau_3)\\ &\qquad -f_1(\tau_3)f_2(\tau_1,\tau_2)+2f_1(\tau_1)f_1(\tau_2)f_1(\tau_3)\ \text{etc.} \end{aligned} \tag{2.21}$$

However, the calculation of higher-order terms in the series (2.20) and evaluation of multiple integral (2.18) lead to technical difficulties. Therefore, it is desirable to limit the present consideration to lower-order terms. For the sake of simplicity in computation, the system of random points is assumed to be the Possion system, that is,

$$g_s=0 \qquad \text{for } s\geq 2$$

Then, differentiating Eq. (2.20) with respect to τ gives

$$p_T(\tau)=f_1(\tau)\exp\left[-\int_0^{\tau}f_1(t)\,dt\right] \tag{2.22}$$

On the other hand, the conditional probability density function $p_0(\alpha,\alpha'|\tau)$ is given by

$$p_0(\alpha,\alpha'|\tau)=\frac{\displaystyle\int_{-\infty}^{0}\int_0^{\infty}\ddot{\eta}_0|\ddot{\eta}_\tau|\,w(\eta_0,\dot{\eta}_0,\ddot{\eta}_0,\eta_\tau,\dot{\eta}_\tau,\ddot{\eta}_\tau)_{\eta_0=\alpha,\,\eta_\tau=\alpha',\,\dot{\eta}_0=\dot{\eta}_\tau=0}\,d\ddot{\eta}_0\,d\ddot{\eta}_\tau}{\displaystyle\int_{-\infty}^{0}\int_0^{\infty}\ddot{\eta}_0|\ddot{\eta}_\tau|\,w(\dot{\eta}_0,\ddot{\eta}_0,\dot{\eta}_\tau,\ddot{\eta}_\tau)_{\dot{\eta}_0=\dot{\eta}_\tau=0}\,d\ddot{\eta}_0\,d\ddot{\eta}_\tau} \tag{2.23}$$

Substituting Eqs. (2.22) and (2.23) to Eq. (2.17), $p(\delta_c,\chi,\tau)$ is approximately obtained as

$$\begin{aligned} p(\delta_c,\chi,\tau)&=2p_0(\delta_c-\chi,\delta_c+\chi|\tau)\,f_1(\tau)\exp\left[-\int_0^{\tau}f_1(t)\,dt\right]\\ &=2\,\frac{J_3(\delta_c,\chi,\tau)}{J_1}\exp\left[-\int_0^{\tau}\frac{J_2(t)}{J_1}\,dt\right] \end{aligned} \tag{2.24}$$

in which

$$J_1=\int_0^\infty \ddot{\eta}_0 w(\dot{\eta}_0,\ \ddot{\eta}_0)_{\dot{\eta}_0=0}\,d\ddot{\eta}_0$$

$$J_2=\int_{-\infty}^0\int_0^\infty \ddot{\eta}_0|\ddot{\eta}_\tau|\,w(\dot{\eta}_0,\ \ddot{\eta}_0,\ \dot{\eta}_\tau,\ \ddot{\eta}_\tau)_{\dot{\eta}_0=\dot{\eta}_\tau=0}\,d\ddot{\eta}_0\,d\ddot{\eta}_\tau \tag{2.25}$$

$$J_3=\int_{-\infty}^0\int_0^\infty \ddot{\eta}_0|\ddot{\eta}_\tau|\,w(\eta_0,\ \dot{\eta}_0,\ \ddot{\eta}_0,\eta_\tau,\ \dot{\eta}_\tau,\ \ddot{\eta}_\tau)_{\eta_0=\alpha,\ \eta_\tau=\alpha',\ \dot{\eta}_0=\dot{\eta}_\tau=0}\,d\ddot{\eta}_0\,d\ddot{\eta}_\tau$$

Using Eq. (2.24), the probability density functions of the shift of center δ_c, the amplitude χ and the extremum α are expressed as

$$p_{\Delta c}(\delta_c)=\int_0^\infty d\tau\int_0^\infty d\chi\,p(\delta_c,\ \chi,\ \tau),\quad -\infty<\delta_c<\infty \tag{2.26}$$

$$p_X(\chi)=\int_0^\infty d\tau\int_{-\infty}^\infty d\delta_c\,p(\delta_c,\ \chi,\ \tau),\quad 0\le\chi<\infty \tag{2.27}$$

$$p_A(\alpha)=\int_0^\infty d\tau\int_0^\infty d\chi\,p(\alpha+\chi,\ \chi,\ \tau),\quad -\infty<\alpha<\infty \tag{2.28}$$

Under the assumption that $\eta(t)$, $\dot{\eta}(t)$ and $\ddot{\eta}(t)$ are the stationary Gaussian processes with zero mean values, the joint probability density function $w(\eta_0,\ \dot{\eta}_0,\ \ddot{\eta}_0,\ \eta_\tau,\ \dot{\eta}_\tau,\ \ddot{\eta}_\tau)$ is the multi-dimensional normal density function and is expressed as

$$w(\eta_0,\ \dot{\eta}_0,\ \ddot{\eta}_0,\ \eta_\tau,\ \dot{\eta}_\tau,\ \ddot{\eta}_\tau)=\frac{1}{(2\pi)^3|\boldsymbol{K}|}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{6}s_{ij}x_ix_j\right] \tag{2.29}$$

in which $|\boldsymbol{K}|$ is the determinant of the covariance matrix $[\boldsymbol{K}]$, S_{ij} is the element in the ith row and jth column of the inverse of the matrix $[\boldsymbol{K}]$ and $(x_1,\ x_2,\ \dots,\ x_6)$ denotes $(\eta_0,\ \dot{\eta}_0,\ \ddot{\eta}_0,\ \eta_\tau,\ \dot{\eta}_\tau,\ \ddot{\eta}_\tau)$. The covariance matrix $[\boldsymbol{K}]$ is

$$[\boldsymbol{K}]=\begin{bmatrix} K_{11}\cdots K_{16}\\ \cdots\quad\cdots\\ \cdots\quad\cdots\\ \cdots\quad\cdots\\ \cdots\quad\cdots\\ K_{61}\cdots K_{66}\end{bmatrix}=\begin{bmatrix} R(0) & 0 & R^{(2)}(0) & R(\tau) & R^{(1)}(\tau) & R^{(2)}(\tau)\\ 0 & -R^{(2)}(0) & 0 & -R^{(1)}(\tau) & -R^{(2)}(\tau) & -R^{(3)}(\tau)\\ R^{(2)}(0) & 0 & R^{(4)}(0) & R^{(2)}(\tau) & R^{(3)}(\tau) & R^{(4)}(\tau)\\ R(\tau) & -R^{(1)}(\tau) & R^{(2)}(\tau) & R(0) & 0 & R^{(2)}(0)\\ R^{(1)}(\tau) & -R^{(2)}(\tau) & R^{(3)}(\tau) & 0 & -R^{(2)}(0) & 0\\ R^{(2)}(\tau) & -R^{(3)}(\tau) & R^{(4)}(\tau) & R^{(2)}(0) & 0 & R^{(4)}(0)\end{bmatrix} \tag{2.30}$$

in which

$$K_{ij}=E[x_ix_j],\quad i,\ j=1,\dots,\ 6$$

and $R(\tau)$ denotes the correlation function, that is,

$$R(\tau)=E[\eta(t)\eta(t+\tau)]$$

$$R^{(i)}(\tau)=\frac{d^i}{d\tau^i}\,R(\tau),\quad i=1,\ldots,4 \tag{2.31}$$

The expression obtained after integration on α by making use of Eq. (2.24) is

$$\int_{-\infty}^{\infty}\delta_c^n p(\delta_c,\,x,\,\tau)\,dx=A(\tau)\exp(-Bx^2)\begin{Bmatrix}U_0(x)\\ 0\\ \dfrac{1}{2a_0}\,U_0(x)+\dfrac{a_1^2}{4a_0^2a^2}U_2(x)\end{Bmatrix}\qquad\text{for }n=\begin{Bmatrix}0\\1\\2\end{Bmatrix} \tag{2.32}$$

in which

$$A=\frac{2}{(2\pi)^3J_1|\boldsymbol{K}|^{1/2}a_2^2}\sqrt{\frac{\pi}{a_0}}\,\exp\left(-\int_0^{\tau}\frac{J_2(t)}{J_1}\,dt\right)$$

$$B=s_{11}-s_{14},\ a_0=s_{11}+s_{14},\ a_1=s_{13}+s_{16},\ a_2=\frac{1}{4}\left(2s_{33}-\frac{a_1^2}{a_0}\right)$$

$$J_1=\frac{1}{2\pi}\sqrt{\frac{k_{\ddot{\eta}\ddot{\eta}}}{k_{\dot{\eta}\dot{\eta}}}}=\frac{1}{2\pi}\sqrt{-\frac{R^{(4)}(0)}{R^{(2)}(0)}}$$

$$J_2=\frac{M_3^{3/2}}{4\pi^2(M_1^2-M_2^2)}\,[1+H\cot^{-1}(-H)]$$

$$M_1=R^{(4)}(0)\{R^{(2)^2}(0)-R^{(2)^2}(\tau)\}+R^{(2)}(0)R^{(3)^2}(\tau)$$

$$M_2=-R^{(4)}(\tau)\{R^{(2)^2}(0)-R^{(2)^2}(\tau)\}-R^{(2)}(\tau)R^{(3)^2}(\tau)$$

$$M_3=\{R^{(2)}(0)R^{(4)}(0)-R^{(2)}(\tau)R^{(4)}(\tau)+R^{(3)^2}(\tau)\}^2$$
$$\qquad-\{R^{(2)}(0)R^{(4)}(\tau)-R^{(2)}(\tau)R^{(4)}(0)\}^2$$

$$H=\frac{M_2}{\sqrt{M_1^2-M_2^2}}$$

$$U_0(x)=\frac{1}{2}\int_0^{\pi/4}\frac{\sin 2\theta}{(1+c_1\sin 2\theta)^2}\left[(1+\xi^2)+\sqrt{\pi}\,\xi e^{\xi^2}\left(\frac{3}{2}+\xi^2\right)\{1+erf(\xi)\}\right]d\theta$$

$$U_2(x)=\frac{1}{2}\int_0^{\pi/4}\frac{(\cos\theta-\sin\theta)^2\sin 2\theta}{(1+c_1\sin 2\theta)^3}\left[2+\frac{9}{2}\xi^2+\xi^4\right.$$
$$\left.+\sqrt{\pi}\,\xi e^{\xi^2}\left(\frac{15}{4}+5\xi^2+\xi^4\right)\{1+erf(\xi)\}\right]d\theta$$

$$\xi=\frac{c_2(\sin\theta+\cos\theta)}{\sqrt{1+c_1\sin 2\theta}}\,x$$

$$c_1=\frac{\dfrac{a_1^2}{a_0}-2s_{36}}{2s_{33}-\dfrac{a_1^2}{a_0}},\qquad c_2=\frac{s_{13}-s_{16}}{\sqrt{2s_{33}-\dfrac{a_1^2}{a_0}}}$$

$$erf(\xi)=\frac{2}{\sqrt{\pi}}\int_0^{\xi}e^{-z^2}dz$$

Using Eq. (2.23), the first and second order moments for the amplitude χ and the extremum α are obtained as

$$E[\chi]=\int_0^\infty\int_0^\infty A\chi U_0(\chi)\exp(-B\chi_2)\,d\chi\,d\tau \tag{2.33}$$

$$E[\alpha]=-E[\chi] \tag{2.34}$$

$$E[\chi^2]=\int_0^\infty\int_0^\infty A\chi^2U_0(\chi)\exp(-B\chi^2)\,d\chi\,d\tau \tag{2.35}$$

$$E[\alpha^2]=\int_0^\infty\int_0^\infty A\left\{\left(\frac{1}{2a_0}+\chi^2\right)U_0(\chi)+\frac{a_1^2}{4a_0^2a_2}U_2(\chi)\right\}\exp(-B\chi^2)\,d\chi\,d\tau \tag{2.36}$$

$$E[\alpha\chi]=-E[\chi^2] \tag{2.37}$$

In particular, the mean and variance of the shift of the center of hysteresis are given by

$$\begin{aligned}E[\delta_c]&=E[\alpha]+E[\chi]=0\\ K_{\delta_c\delta_c}&=E[\alpha^2]-E[\chi^2]\end{aligned} \tag{2.38}$$

As above mentioned, the new technique of statistical linearization is introduced by taking into consideration the scatter of frequency and the fluctuation of the center of hysteretic oscillation. It may be proper to call such a technique a statistical linearization technique based on the three-dimensional analysis, since three variables δ_c, χ and τ in this analysis are treated as random variables. To investigate the analytical technique, two different techniques are introduced.

First, if only τ among three variables is a deterministic parameter, the probability density $p(\delta_c, \chi, \tau)$ is reduced to

$$\begin{aligned}p(\delta_c,\chi,\tau)&=2p_0(\delta_c-\chi,\ \delta_c+\chi\,|\,\tau)\cdot\delta(\tau-\tau^*)\\&=2\,\frac{J_3(\delta_c,\chi,\tau)}{J_2(\tau)}\cdot\delta(\tau-\tau^*)\end{aligned} \tag{2.39}$$

in which $\delta(\cdot)$ is Dirac's delta function and

$$\tau^*=\frac{1}{N_p} \tag{2.40}$$

where N_p is the expected number of extremum per unit time and is given by

$$N_p=\int_{-\infty}^{\infty}|\ddot{\eta}|\,w(\dot{\eta},\ddot{\eta})_{\dot{\eta}=0}\,d\ddot{\eta}=\frac{1}{\pi}\sqrt{\frac{K_{\ddot{\eta}\ddot{\eta}}}{K_{\dot{\eta}\dot{\eta}}}} \tag{2.41}$$

This technique based on the probability density $p(\delta_c, \chi, \tau)$ of Eq. (2.39) instead of Eq. (2.24) may be called a technique based on the two-dimensional analysis in correspondence with the three-dimensional analysis.

Secondly, as treated in References 6) and 7), it is assumed that the response is contained within a narrow band frequncy and the amplitude of this process forms the Rayleigh distribution. Then, both the scatter of frequency and the fluctuation of the center of hysteretic oscillation are neglected. Therefore, the probability density function $p(\delta_c, \chi, \tau)$ reduces to

$$p(\delta_c, \chi, \tau) = p_x(\chi)\cdot\delta(\delta_c)\cdot\delta(\tau-\bar{\tau}) \tag{2.42}$$

in which

$$\bar{\tau} = \frac{\pi}{\tilde{\omega}}, \quad \tilde{\omega} = \pi N_0 \tag{2.43}$$

$$N_0 = \int_{-\infty}^{\infty} |\dot{\eta}|\, w(\eta, \dot{\eta})_{\eta=0}\, d\dot{\eta} \tag{2,44}$$

where $\tilde{\omega}$ is the mean frequency and N_0 is the expected number of zero crossing per unit time. $p_X(\chi)$ is the probability density function of peak amplitude and is given by

$$p_X(\chi) = \frac{\chi}{K_{\eta\eta}} \exp\left(-\frac{\chi^2}{2K_{\eta\eta}}\right) \tag{2,45}$$

Since this analysis treats only the amplitude χ as a random variable, the technique is called that based on the one-dimensional analysis.

In general, numerical work needed to evaluate the probability density function $p(\delta_c, \chi, \tau; t)$ involving time-dependent covariances of $\eta(t)$, $\dot{\eta}(t)$ and $\ddot{\eta}(t)$ will become significantly heavy if the response process considered is non-stationary. However the evaluation of $p(\delta_c, \chi, \tau; t)$ in the case of the technique based on the one-dimensional analysis does not seem extremely difficalt. In the case of non-stationary random process, $p(\delta_c, \chi, \tau; t)$ is expressed as

$$p(\delta_c, \chi, \tau; t) = p_X(\chi; t)\cdot\delta(\delta_c)\cdot\delta(\tau-\bar{\tau}(t)) \tag{2.46}$$

in which $p_X(\chi; t)$ and $\bar{\tau}(t)$ are given as a function of the covariances $K_{\eta\eta}(t)$, $K_{\eta\dot{\eta}}(t)$ and $K_{\dot{\eta}\dot{\eta}}(t)$ in the following equations[8]:

$$p_X(\chi; t) = \exp\left(-\frac{\chi^2}{2K_{\eta\eta}(t)}\right)\left[\frac{\chi}{K_{\eta\eta}(t)} \exp\left(-\frac{\rho^2\chi^2}{2(1-\rho^2)K_{\eta\eta}(t)}\right)\right.$$
$$\left. + \rho\sqrt{\frac{\pi}{2(1-\rho^2)K_{\eta\eta}(t)}}\left(\frac{\chi^2}{K_{\eta\eta}(t)}-1\right)\mathrm{erf}\left(\frac{\rho\chi}{\sqrt{2(1-\rho^2)K_{\eta\eta}(t)}}\right)\right] \tag{2.47}$$

and

$$\bar{\tau}(t) = \pi/\tilde{\omega}(t) = \pi\sqrt{\frac{K_{\eta\eta}(t)}{(1-\rho^2)K_{\dot{\eta}\dot{\eta}}(t)}} \tag{2.48}$$

where

$$\rho = K_{\eta\dot{\eta}} / \sqrt{K_{\eta\eta}(t)\, K_{\dot{\eta}\dot{\eta}}(t)} \tag{2.49}$$

3. Numerical Example

As an example of the application of the present techniques to problems of structural response, consider a single-degree-of-freedom structure with bilinear hysteresis subjected to an excitation $f(t)$. The dimensionless equation of motion associated with the ductility ratio η is expressed by

$$\frac{d^2}{dt^2}\eta + 2h\frac{d}{dt}\eta + \varphi(\eta; r) = -f(t) \tag{3.1}$$

and

$$\eta = \frac{x}{\Delta}, \quad t = \Omega_0 T$$

in which x is relative displacement response, T is time, Δ is the elastic limit deformation, h is the critical damping ratio, Ω_0 is the natural frequency of the structure, t is nondimensional time, $\varphi(\eta; r)$ is the bilinear hysteretic characteristic as shown in Fig. 2 and r is the rigidity ratio of the second to the first branch. For the present investigation, the excitation $f(t)$ is taken to be a stationary random function with a power spectral density $S_f(\omega)$, a Gaussian probability distribution and zero mean value. The statistical linearization techniques in the previous chapter are applicable to the Gaussian excitation of which frequency characteristics is an arbitrary type such as actual earthquake excitation. In order to investigate the approximate analytical techniques and to understand the applicability of these techniques to problems of the response of hysteretic structure, it is desirable that the frequency characteristic of the excitation is simple such as white. However, Eq. (2.24) involves the covariance function of the second derivative $\ddot{\eta}(t)$ of the response, which does not exist in the case that the frequency characteristic is white. Therefore, let the excitation $f(t)$ be a band-limited white Gaussian process with the power spectral density represented in the following form:

$$S_f(\omega) = \begin{cases} \dfrac{S_0}{2\pi} & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where ω_c is a cut off frequancy and S_0 is a parameter associated with the intensity of excitation.

3.1 Monte Carlo Solutions

In order to understand the statistical characteristics of the response of hysteretic structure subjected to a random excitation and to check the accuracy of the analytical values obtained through the statistical linearization techniques, a numerical simulation based on the Monte Carlo technique has been carried out on a digital computer.

Numerical procedure

The artificial earthquake has been simulated according to M. Shinozuka and C.-M. Jan[16]. A stationary band-limited white Gaussian noise with zero mean and the power spectral density given by Eq. (3.2) is generated by following equation:

$$f(t)=\left(\frac{2S_0}{\pi}\Delta\omega\right)^{1/2}\sum_{i=1}^{N}\cos(\omega_i' t+\phi_i) \tag{3.3}$$

and

$$\omega_i'=\omega_i+\delta\omega$$

$$\omega_i=\left(i-\frac{1}{2}\right)\Delta\omega \qquad \text{for } i=1, 2,\ldots, N \tag{3.4}$$

$$\Delta\omega=\frac{\omega_c}{N}$$

where ϕ_i is the independent random phase uniformly distributed between 0 and 2π, $\delta\omega$ is a small random frequancy introduced to avoid the periodicity of the simulated process and is uniformly distributed between $-\Delta\omega'/2$ and $\Delta\omega'/2$ with $\Delta\omega'\ll\Delta\omega$. The response of the structure subjected to the above excitation was obtained by numerical integration of Eq. (3.1) using the Runge-Kutta method. For each combination of r and S_0, an ensemble of 400 time-histories of response was generated. Initial values of η, $\dot{\eta}$ and $\ddot{\eta}$ were zero for each sample. This procedure was carried out till the variance built up from zero toward its stationary level. The set of parameters used is $h=0.01$, $\omega_c=2$, $N=50$ and $\Delta\omega'=\frac{\Delta\omega}{20}$. Fig. 3 shows the variances of the transient response calculated as an ensemble average at each time by the following

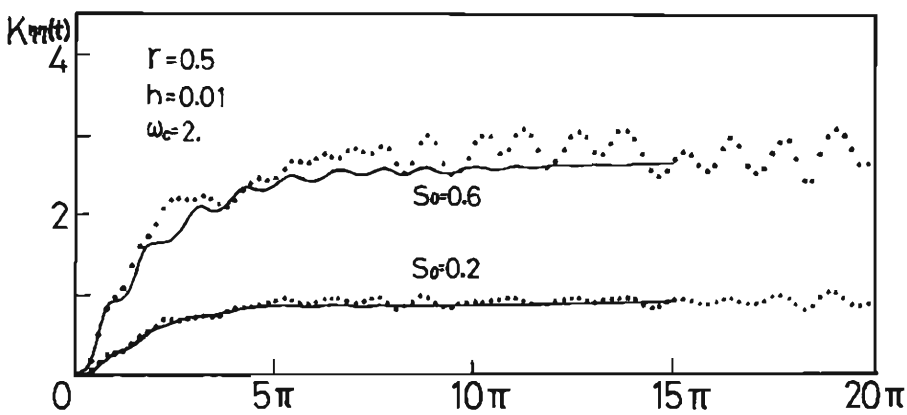


Fig. 3(a) Moderately nonlinear (r=0.5)

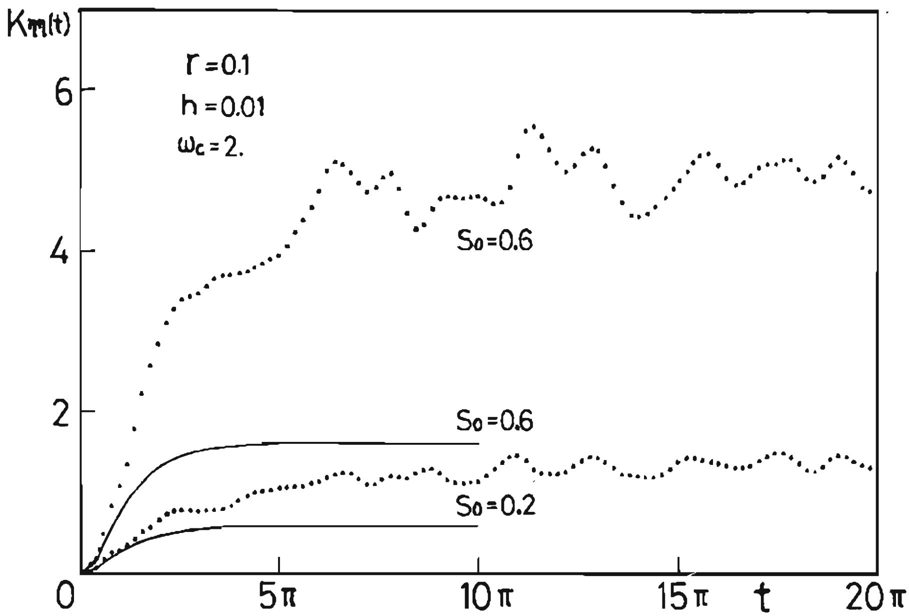


Fig. 3(b) Severely nonlinear (r=0.1)

Fig. 3. Transient variance of displacement response of structures with bilinear hysteresis. —— : 1-dimensional analysis. ●●● : Simulation.

equation:

$$K_{\eta\eta}(t)=<(\eta(t)-<\eta(t)>)^2> \tag{3.5}$$

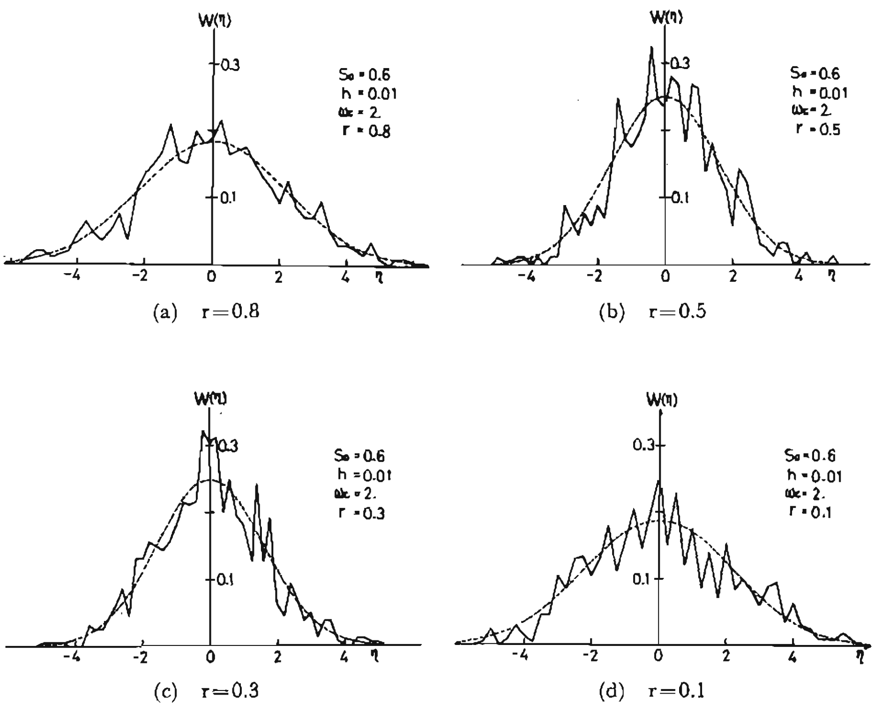
in which $<\cdot>$ denotes an ensemble average over the ensemble. As in these figures, the variance $K_{\eta\eta}(t)$ fluctuates with time, even if the variance amounts to its stationary level. Therefore the variance of the stationary response was newly calculated in this study by the following method of average to eliminate the time variation of the variance $K_{\eta\eta}(t)$. This average was the time average of $K_{\eta\eta}(t)$ over the interval, of 5 times the natural period, in which the response was considered as stationary. This stationary variance is denoted herein by $K_{\eta\eta}$. The variances $K_{\dot{\eta}\dot{\eta}}$ and $K_{\ddot{\eta}\ddot{\eta}}$ of the stationary velocity and acceleration response were estimated by using the similar averaging method.

Experimental results

In Figs. 6(a) and 9, experimental results of $K_{\eta\eta}$ are plotted for the parameters of $S_0=0.2$, 0.6 and 1.0, and $r=0.1$, 0.3, 0.5 and 0.8. Note that the structure becomes linear when $r\rightarrow1$ and the perfect elasto-plastic hysteretic structure when $r\rightarrow0$. Therefore, these figures give an indication of how the degree of nonlinearity affects the structural response. From these figures, it can be seen that the variance of displacement response for the same intensity of excitation is strongly influenced by the degree of nonlinearity, and is minimized when the rigidity ratio is 0.3. Figs. 6(b) and (c) show experimental results of $K_{\dot{\eta}\dot{\eta}}$ and $K_{\ddot{\eta}\ddot{\eta}}$ for $S_0=0.6$ and $r=0.1$, 0.3, 0.5 and 0.8.

The variances of velocity and acceleration response are sensitive to the rigidity ratio r, but they decrease monotonically with decreasing r.

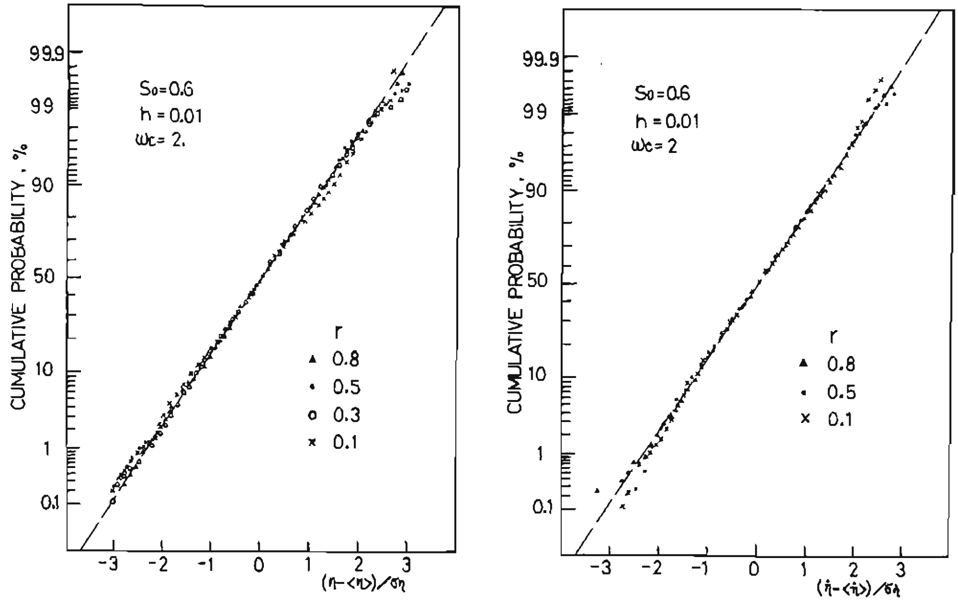
To investigate the assumption used in the previous chapter that the response forms the Gaussian distribution, the probability density function of the displacement response in the stationary situation obtained by the simulation is plotted in Fig. 4. In these figures, the dashed line represents the normal density function with the mean and the variance obtained by the simulation. Further more, Figs. 5(a) and (b) show how the probability distributions of the displacement and velocity response are influenced by the regidity ratio, respectively. In these figures, the scales are chosen so that the Gaussian distribution plots as a straight line. From Fig. 5(a), it seems that the experimental distribution of the displacement response is on the whole similar to the Gaussian distribution. However, for the severe nonlinearity ($r=0.1$), the experimental distribution diverges increasingly from the Gaussian distribution as the displacement level $\left|\frac{\eta-<\eta>}{\sigma_\eta}\right|$, normalized by the standard deviation σ_η of η, becomes larger. The effect of nonlinearity is less significant for the distribution of the velocity response than for that of the displacement response. From these results, the assumption of normality is proper for the response of the structure with small to moderate nonlinearity. On the other hand, for severe nonlinearity, the assumption of normality is, strictly speaking, invalid, and so this fact may have some effects on the results obtained through the statistical linearization techniques.



(a) r=0.8 (b) r=0.5

(c) r=0.3 (d) r=0.1

Fig. 4. Probability density of displacement response for various rigidity ratios r. ——: Simulation. -------: Gaussian



(a) Displacement response (b) Velocity response

Fig. 5. Probability distribution of response.

3.2 Approximate analytical solutions

By making use of the equivalent stiffness coefficient k_e and the equivalent damping coefficient d_e, the original nonlinear equation (3.1) can be replaced by a related linear equation

$$\frac{d^2}{dt^2}\eta + 2h_{eq}\omega_{eq}\frac{d}{dt}\eta + \omega_{eq}^2 = -f(t) \tag{3.6}$$

in which

$$\omega_{eq} = \sqrt{k_e} \qquad h_{eq} = \frac{2h + d_e}{2\sqrt{k_e}} \tag{3.7}$$

The statistics of the stationary response governed by Eq. (3.6) are readily obtained. The correlation function of the response η is given by

$$R(\tau) = E[\eta(t)\eta(t+\tau)] = \int_{-\infty}^{\infty} S_f(\omega)\,|H(\omega)|^2 \cos\omega\tau\, d\omega$$

$$= \frac{S_0}{2\pi}\int_{-\omega_c}^{\omega_c} \frac{\cos\omega\tau}{(\omega_{eq}^2 - \omega^2)^2 + 4h_{eq}^2\omega_{eq}^2\omega^2}\, d\omega \tag{3.8}$$

In particular, when $\tau=0$, the covariances of the displacement, velocity and acceleration response are expressed as follows:

$$
\begin{aligned}
&K_{\eta\eta} = R(0) = \frac{S_0}{4h_{eq}\omega_{eq}^3} I_0 & & K_{\eta\dot{\eta}} = -R^{(1)}(0) = 0 \\
&K_{\dot{\eta}\dot{\eta}} = -R^{(2)}(0) = \frac{S_0}{4h_{eq}\omega_{eq}} I_2 & & K_{\dot{\eta}\ddot{\eta}} = R^{(3)}(0) = 0 \\
&K_{\ddot{\eta}\ddot{\eta}} = R^{(4)}(0) = \frac{S_0\omega_{eq}}{4h_{eq}} I_4 & &
\end{aligned}
\tag{3.9}
$$

in which

$$
\begin{aligned}
&I_j = \frac{4h_{eq}}{\pi}\int_0^q \frac{u^j}{(1-u^2)^2 + 4h_{eq}^2 u^2}\, du, \quad j = 0,\, 2,\, 4 \\
&q = \frac{\omega_c}{\omega_{eq}}
\end{aligned}
\tag{3.10}
$$

The final expressions of I_j are given by

$$
\begin{aligned}
&I_0 = \frac{1}{\pi}\tan^{-1}\frac{2h_{eq}q}{1-q^2} + \frac{h_{eq}}{2\pi\sqrt{1-h_{eq}^2}} \log\frac{1+q^2+2q\sqrt{1-h_{eq}^2}}{1+q^2-2q\sqrt{1-h_{eq}^2}} \\
&I_2 = \frac{1}{\pi}\tan^{-1}\frac{2h_{eq}q}{1-q^2} + \frac{h_{eq}}{2\pi\sqrt{1-h_{eq}^2}} \log\frac{1+q^2-2q\sqrt{1-h_{eq}^2}}{1+q^2+2q\sqrt{1-h_{eq}^2}} \\
&I_4 = \frac{4h_{eq}q}{\pi} + 2(1-2h_{eq}^2)I_2 - I_0
\end{aligned}
\tag{3.11}
$$

Numerical procedure

The estimates for the stationary response are computed for bilinear hysteretic structures by using the three statistical linearization techniques based on the following three different analyses:

(1) one-dimensional analysis without consideration of the distributions of center and frequency of hysteretic oscillation,

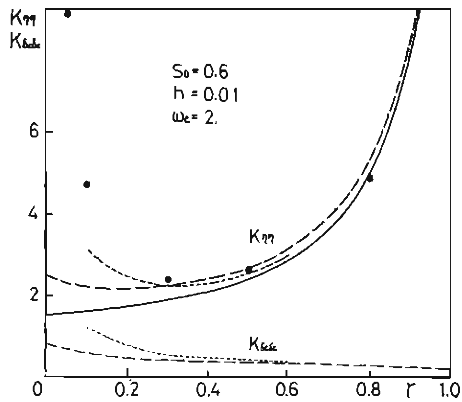
(2) two-dimensional analysis with consideration of the distributions of center and amplitude of hysteretic oscillation,

(3) three-dimensional analysis with consideration of the distributions of center, amplitude and frequency of hysteretic oscilltion.

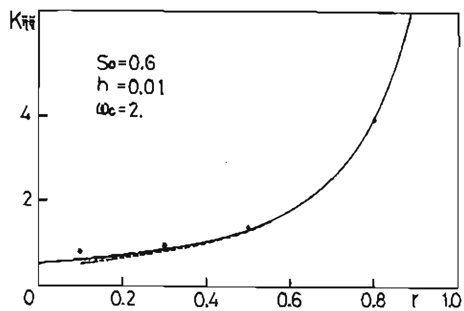
In order to find the basic statistics of the nonlinear response by using the proceeding statistical linearization techniques, Eqs. (2.11), (2.12) and (3.9) must be solved by using of Eqs. (2.9), (2.15), (3.7), (3.11) and (2.24), for the three-dimensional analysis. Note that instead of Eq. (2.4), Eq. (2.39) is used for the two-dimensional analysis, and Eq. (2.42) is used for the one-dimensional analysis. These are nonlinear algebaric equations and it is difficult to solve them directly. However, they can, in general, be solved by the following iteration scheme on the digital computer. Assuming a set of values for k_e and d_e, then the covariance matrix can be solved easily, and by using these results, a new set of values for k_e and d_e is introduced. This procedure can be repeated until the required accuracy is obtained.

Numerical results

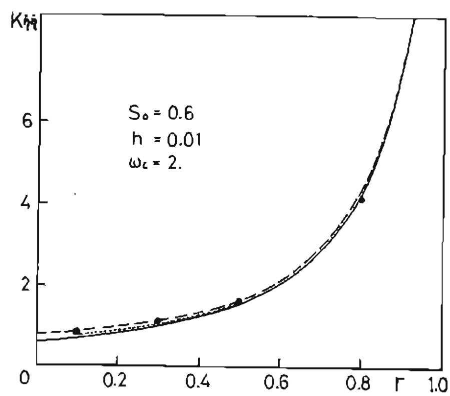
Figs. 6(a), (b) and (c) show variances of the displacement, velocity and accelera-



(a) Displacement response and center of hysteresis



(b) Velocity response



(c) Acceleration response

Fig. 6. Variance of stationary response of structures with bilinear hysteresis. ——: 1-dimensional analysis. — — —: 2-dimensional analysis. ·······: 3-dimensional analysis. ●: simulation

tion response as a function of the rigidity ratio r for the set of parameters, $S_0=0.6$, $h=0.01$. Remember that the rigidity ratio r indicates the degree of nonlinearity for the bilinear hysteretic sturcture. From these figures it can be seen that the degree of nonlinearity affects storngly the over-all response. In particular, it is interesting that the variance $K_{\eta\eta}$ of the displacement response is strongly influenced by the degree of nonlinearity and the difference of the analytical technique as indicated in Fig. 6(a). For the structure with small to moderate nonlinearity ($r>0.5$) all analytical estimates agree rather well with the simulated results obtained in the previous section. For the strong nonlinearity, however, the effect of the different analytical techniques on predicting response is clearly apparent. Whereas the simulated result indicates that there is a noticeable tendency for the variance $K_{\eta\eta}$ to increase rapidly with decreasing r when the rigidity ratio is less than 0.3, the one-dimensional estimate decreases monotonically and the two-dimensional estimate increases slightly with decreasing r. The three-dimensional estimate gives considerably a better prediction than other analytical estimates and shows the above tendency. However, this estimate also becomes qualitatively much less satisfactory as the rigidity ratio decreases below about 0.3.

In Fig. 6(a), the variance $K\delta_c\delta_c$ of the shift δ_c of the center of hysteresis is also plotted as a function of r. It is found that the variance $K\delta_c\delta_c$, in general, increases with decreasing r, and the estimate for $K\delta_c\delta_c$ computed from Eq. (2.38) based on the three-dimensional analysis increases rapidly when $r<0.3$, and is always greater than one based on the two-dimensional analysis. It is noted here that the variance $K\delta_c\delta_c$ based on the one-dimensional analysis does not exist from its definition indicated as Eq. (2.42). Therefore, the effect of the fluctuating center of hysteresis is evident in the fact that the enlargement of displacement response with decreasing r when $r<0.3$.

On the other hand, variances $K_{\dot{\eta}\dot{\eta}}$ and $K_{\ddot{\eta}\ddot{\eta}}$ of the velocity and acceleration response decrease monotonically with decreasing r as shown in Figs. 6(b) and (c). From these figures, it is found that the discrepancy between analytical and simulated results, and the effect of the different analytical techniques are much less significant than that of the displacement response.

Fig. 7 shows how the probability density function $P_T(\tau)$, for the interval τ between

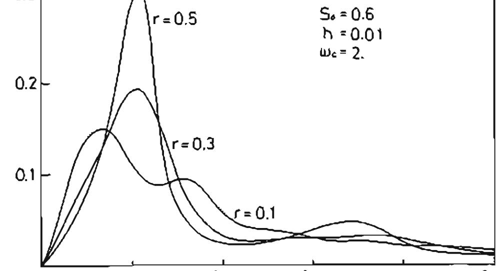
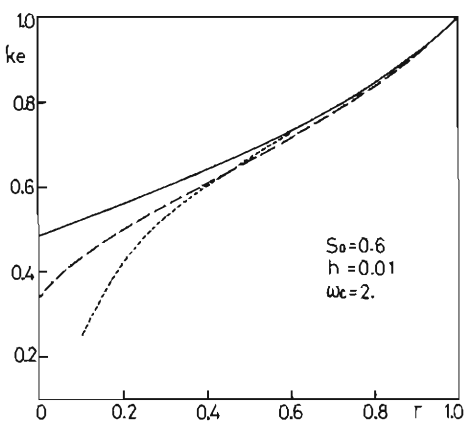


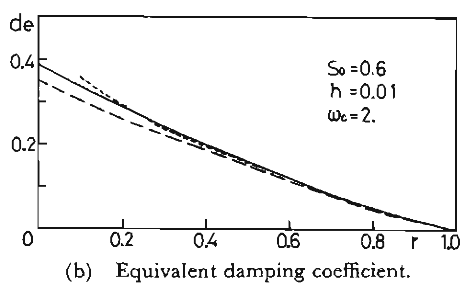
Fig. 7. Probability density function of interval τ between successive extremums of response.

zero crossing of $\dot{\eta}(t)$, evaluated approximately by Eq. (2.22), is influenced by the rigidity ratio r. For the moderate nonlinearity ($r=0.5$), the probability density has a remarkable peak and the scatter of τ is small. Therefore, the response retains narrow-band characteristics, even with the introducing of yielding. For the severe nonlinearity ($r=0.1$), there is no distinguished peak and the scatter of τ is large. This indicates that the response does not possess narrow-band characteristics. By comparing the result of $K_{\eta\eta}$ based on the two-dimensional analysis with that based on the three-dimensional analysis, in Fig. 6(a), it is suggested that the scatter of τ as well as the fluctuating center of hysteresis has greatly significant effect on predicting the displacement response of hysteretic structures with severe nonlinearity.

Figs. 8(a) and (b) show the equivalent linearization coefficients k_e and d_e as a function of r, respectively. As the rigidity ratio r deceases, the equivalent stiffness k_e decreases by reason of the softning nature of the hysteretic structure, and the difference among the results obtained with use of the three different analytical techniques increases. Particularly, the three-dimensional estimate for k_e decreases rapidly when the rigidity ratio r is less than 0.3. By contrast, the equivalent damping d_e, associated with the hystertic energy dissipation due to yielding, increases with decreasing r as shown in Fig. 8(b). The difference among the three estimates for the equivalent damping is less than for the equivalent stiffness. It is well known



(a) Equivalent stiffness coefficient.



(b) Equivalent damping coefficient.

Fig. 8. Equivalent linearization coefficients. ——: 1-dimensional analysis. -----: 2-dimensional analysis. ·······: 3-dimensional analysis.

that the softning nature of the nonlinear sturcture tends to increase the displacement response, while the damping due to the energy dissipation tends to decrease the response. For the structure with small to moderate nonlinearity, the damping effect dominates, and so the displacement response is decreased as in Fig. 6(a). Contrary to this, from the results based on the three-dimensional analysis, it is found that for the structure with severe nonlinearity, the displacement response is increased by the reason that the effect of the softning nature dominates even though the equivalent damping coefficient increases. On the other hand, the velocity and accerelation responses are affected by only the damping effect.

Fig. 9 shows the results of the variance $K_{\eta\eta}$ obtained by the simulation technique and the two statistical linearization techniques based on the one-dimensional analysis and the three-dimensional analysis for $S_0=0.2$, 0.6, 1.0. This figure gives an indication about the range of the applicability of the statistical linearization techniques to the prediction of stationary random response of bilinear structures. Increasing the intensity of excitation as well as decreasing the rigidity ratio increases the degree of nonlinearity of structures, and so increases the discrepancy among the analytical results and the simulated result. Furthermore, from the result by the three-dimensional analysis or the simulation in this figure, it is clearly evident that the variance of the displacement response of the structure for the same intensity of excitation has a minimum. The optimum rigidity ratio which gives the minimum variance is about 1/3. It is considered that this optimum value depends on the intensity S_0 of excitation and the critical damping ratio h of the structure. When the ratio

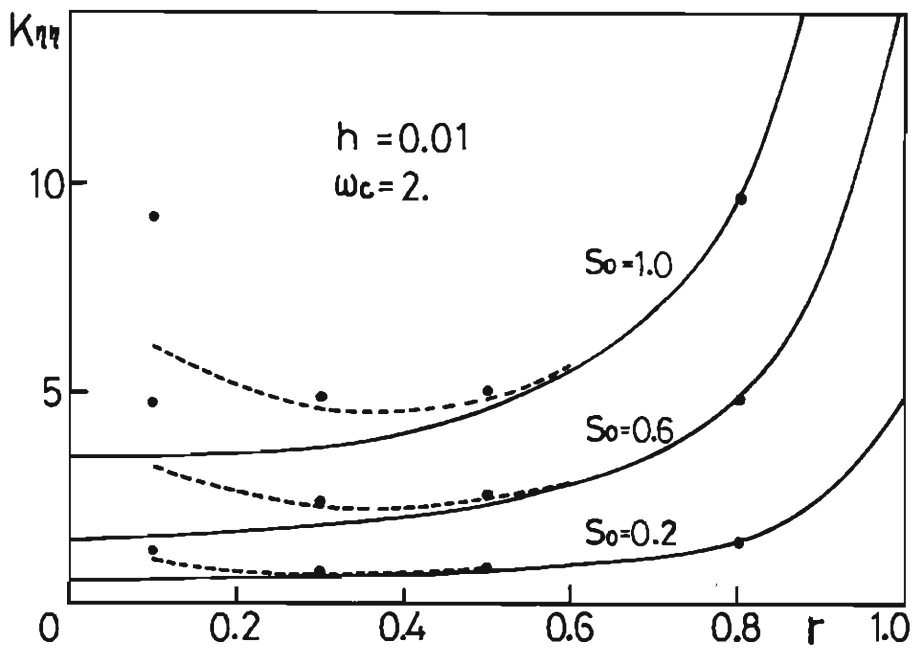


Fig. 9. Effect of rigidity ratio r on variance of displacement response.
——: 1-dimensional analysis. – – – –: 3-dimensional analysis.
●: Simulation.

r is less than the optimum value, the characters of the response differ markedly from that of structure with small to moderate nonlinearity, for example, the variance of the center of hysteresis, the probability density function $P_T(\tau)$ and the probability distribution of the displacement response, as described before.

The statistical linearization technique based on the three-dimensional analysis proposed in this paper is applicable to the prediction of stationary random response of bilinear structure when the rigidity ratio is greater than the above optimum value. The technique based on the one-dimensional analysis is applicable for small to moderate nonlinearity ($r>0.5$), as recognized by other investigations[10)-13)].

The technique based on the three-dimensional analysis gives underestimated results for the variance of the displacement response in comparison with the simulated results when the rigidity ratio is much less than the optimum value. This major discrepancy between the two results for the hysteretic structure with severe nonlinearity may be due to the following factors:

(1) The response of the structure possesses the broad-band characteristics, and so is not a smoothly varying process. Hence, the continuous behaviour of hysteresis cannot be treated discretely as the closed hysteresis over one cycle.

(2) The probability distribution of the response is not Gaussian distribution.

(3) The probability density function $P_T(\tau)$ given by equation (2.22) is drastically apporximate and gives an unsatisfactory estimate for large τ.

As a example of the application to non-stationary random process, also shown in Fig. 3 are values of the transient variance $K_{\eta\eta}(t)$ evaluated by using the step-by-step linearization technique based on the one-dimensional analysis[8),9)]. It appeares that the analytical results agree rather well with the exprimental results for moderately nonlinear system. Contrary to this, the discrepancy between two results is quite apparent for severely nonlinear system. Thus the technique based on the one-dimensional analysis appears to be very powerful to anyalze non-stationary response as well as stationary response only when r is greater than 0.5.

4. Conclusions

In order to understand the statistical characteristics of the responses of hysteretic structures subjected to severe earthquake excitations, a new statistical linearization technique is introduced according to the facts that the response of a hysteretic structure with severe nonlinearity is not, in general, a narrow band process and hysteretic behaviour obviously fluctuates eccentrically in a random manner. The basic statistics of the stationary response of a single-degree-of-freedom structure with bilinear hysteresis subjected to a band-limited white Gaussian excitation with zero mean value are numerically evaluated by the statistical linearization techniques based on the three different analyses; i.e., 1) the one-dimensional analysis, 2) the two-dimensional analysis and 3) the three-dimensional analysis. To investigate the range of applicability of these techniques, a numerical simulation has been carried out.

Though the general features of the random response of nonlinear hysteretic structures, which are varied depending on the characteristics of hysteresis and excitations, cannot be derived from this limited analysis, some of the significant finds can be summarized as follows:

(1) The response of the bilinear hysteretic structure is, in general, quite sentisive

to the rigidity ratio of the second to the first branch of hysteresis. Particularly, the experimental result by the simulation technique shows a noticeable tendency that as the rigidity ratio decreases, the variance of the displacement response of the structure for the same intensity of excitation decreases gradually, and increases rapidly after it has a minimum at a certain rigidity ratio. Apparently this tendency is also recognized by the statistical linearization technique based on the three-dimensional analysis. It is considered that the optimum rigidity ratio which minimizes the variance depends on the intensity of excitation and the damping ratio of structure. The optimum value is nearly 1/3 in the range of parameters used in this study. Other techniques do not explain the tendency. In particular, the variance estimated by the technique based on the one-dimensional analysis decreases monotonically with decreasing rigidity ratio.

(2) The variance for the shift of center of hysteresis evaluated by the three-dimensional analysis is always greater than one by the two-dimensional analysis, increases gradually with decreasing rigidity ratio, and rapidly increases when the rigidity ratio is less than the optimum value. The probability density function of the interval between two adjacent extremes of displacement response is apporximately evaluated, and indicates that the scatter of frequency of hysteretic oscillation is noticeably large for severely nonlinear structures. Therefore, the scatter of frequency and the fluctuation of the center of hysteretic oscillation affect strongly on the displacement response of the hysteretic structure with severe nonlinearity.

(3) From the error survey made with the aid of a numerical simulation based on the Monte Carlo method, it can be said that the statistical linearization technique based on the three-dimensional analysis is applicable to the prediction of stationary response of structures with bilinear hysteresis when the rigidity ratio is greater than the optimum value.

From the above-mentioned remarks, it is evident that the investigation on the scatter of frequency and the fluctuation of center of hysteretic oscillation are necesasry to make clear the response of hysteretic structures with severe nonlinearity. It can be said that the statistical linearization technique is considerably improved by taking into account the distributions of them if not satisfactory. The large displacement response caused by the shift of the center of hysteresis may be a serious problem on the aseismic safety of hysteretic structures with severe nonlinearity in relation to the instantaneous failure due to large deformation and the low-cycle fatigue phenomenon due to repeated deformation. It is suggested that the existence of the optimum rigidity ratio which minimizes the variance of the displacement response may have a great significance in doing the aseismic design of structures.

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