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Relation between Wave Characteristics of Cnoidal Wave Theory Derived by Laitone and by Chappelear

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Abstract

This paper presents the relation between wave characteristics of the second order approximate solution of the cnoidal wave theory derived by Laitone and by Chappelear.

If the expansion parameters L_0 and L_3 in the Chappelear theory are expanded in a series of the ratio of wave height to water depth and the expressions for wave characteristics of the second order approximate solution of the cnoidal wave theory by Chappelear are rewritten in a series form to the second order of the ratio, the expressions for wave characteristics of the cnoidal wave theory derived by Chappelear agree exactly with the ones by Laitone, which are converted from the depth below the wave trough to the mean water depth. The limiting area between these theories for practical application is proposed, based on numerical comparison.

In addition, some wave characteristics such as wave energy, energy flux in the cnoidal waves and so on are calculated.

1. Introduction

In recent years, the various higher order solutions of finite amplitude waves based on the perturbation method have been extended with the progress of wave theories. For example, systematic deviations of the cnoidal wave theory, which is a nonlinear shallow water wave theory, have been made by Keller[1], Laitone[2], and Chappelear[3] respectively. All are based on a perturbation expansion method, which suitably stretches the vertical dimension, as developed by Friedrichs[4]. Keller confirmed the results of Korteveg de Vries[5] which is a primary intuitive theory. The second approximation of the cnoidal wave theory was derived primarily by Laitone and in succession the third one by Chappelear. However, the mutual relation between them is not made clear, since expressions for these solutions are different from each other.

In this paper, the analytical relation between wave characteristics of the second approximate solution of the cnoidal wave theory by Laitone and by Chappelear is presented and the limiting area between the theories for practical use is proposed from the comparison of the numerical results on the expansion parameters. In addition, some wave characteristics such as wave energy, energy flux in the conidal waves and so on are calculated.

As already pointed out by Stokes[6], the physical definition is necessary to determine the wave celerity in the extension to the higher order approximate solution of finite amplitude wave theory. The Stokes second definition of wave celerity is used by Laitone and the first definition by Chappelear, although recently

Tsuchiya, one of the authors and Yasuda[7] calculated a new cnoidal wave theory, making use of the Gardner-Morikawa transform, in which the deficiency is overcome. As the cnoidal wave theory by the same definition must be used in the comparison, the cnoidal wave theory recalculated by the authors[8], using the second definition, is utilized in place of the one by Chappelear.

2. Second Order Approximate Solution of Cnoidal Wave Theory

If the coordinate system x, z as shown in Fig. 1 is taken, the wave characteristics by the second approximate solution of the conidal wave theory derived by Laitone[9], which is converted from the depth below the wave trough h_t to the mean water depth h and from the coordinate system taken at the depth of the wave trough to the one at the mean water depth are given by the following equations respectively; for the wave celerity c and the wave length L, they are written as

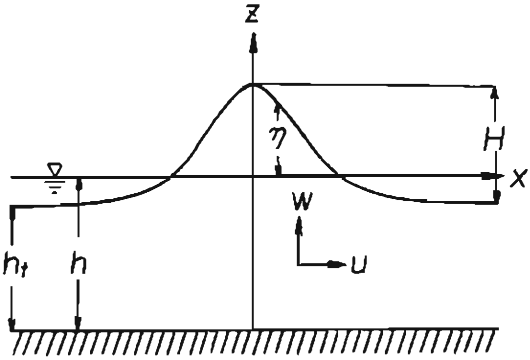


Fig. 1 Definition sketch of coordinate system

$$\frac{c}{\sqrt{gh}}=1+\left(M_1-\frac{N_1}{2}\right)\left(\frac{H}{h}\right)+\left(M_2+\frac{M_1N_1}{2}-\frac{N_1^2}{2}-\frac{N_2}{2}\right)\left(\frac{H}{h}\right)^2 \tag{1}$$

$$\frac{L}{h}=\frac{4\kappa K}{\sqrt{3(H/h)}}\left\{1+\left(\frac{7\kappa^2-2}{8\kappa^2}-\frac{3}{2}N_1\right)\left(\frac{H}{h}\right)\right\} \tag{2}$$

in which H is the wave height, g the acceleration of gravity, κ the modulus of the Jacobian elliptic function and K the complete elliptic integral of the first kind and M_1, M_2, N_1 and N_2 are given as

$$\left.\begin{aligned}&M_1=\frac{1}{\kappa^2}\left(\frac{1}{2}-\frac{E}{K}\right), \quad M_2=\frac{1}{\kappa^4}\left\{-\frac{1}{40}(\kappa^4+14\kappa^2-9)+\frac{E}{K}\left(\frac{E}{K}+\frac{3}{4}\kappa^2-1\right)\right\}\\&N_1=\frac{1}{\kappa^2}\left(\kappa^2-1+\frac{E}{K}\right), \quad N_2=\frac{1}{4\kappa^4}\left\{2(1-\kappa^2)-(2-\kappa^2)\frac{E}{K}\right\}\end{aligned}\right\} \tag{3}$$

in which E is the complete elliptic integral of the second kind.

The water surface displacement η and the horizontal and vertical water particle velocities u, w are expressed respectively as

$$\frac{\eta}{h}=(\mathrm{cn}^2-N_1)\left(\frac{H}{h}\right)+\left(-\frac{3}{4}\mathrm{cn}^2+\frac{3}{4}\mathrm{cn}^4-N_2\right)\left(\frac{H}{h}\right)^2 \tag{4}$$

$$\left.\begin{aligned}\frac{u}{\sqrt{gh}}=&\left(-1+\frac{1}{2\kappa^2}+M_1+\mathrm{cn}^2\right)\left(\frac{H}{h}\right)+\left[-\frac{N_1}{2}\left(1-\frac{1}{2\kappa^2}-\mathrm{cn}^2\right)+M_2+\frac{M_1N_1}{2}\right.\\&+\frac{21\kappa^4-6\kappa^2-9}{40\kappa^4}-\left[\frac{7\kappa^2-2}{4\kappa^2}+\frac{3}{2}\left(2-\frac{1}{\kappa^2}\right)\left\{2\frac{z}{h}+\left(\frac{z}{h}\right)^2\right\}\right]\mathrm{cn}^2\\&\left.+\left[\frac{5}{4}+\frac{9}{4}\left\{2\frac{z}{h}+\left(\frac{z}{h}\right)^2\right\}\right]\mathrm{cn}^4-\frac{3}{4}\left(\frac{1}{\kappa^2}-1\right)\left\{2\frac{z}{h}+\left(\frac{z}{h}\right)^2\right\}\right]\left(\frac{H}{h}\right)^2\end{aligned}\right\}\quad(5)$$

$$\left.\begin{aligned}\frac{w}{\sqrt{gh}}=&\left(1+\frac{z}{h}\right)\sqrt{\frac{3}{\kappa^2}\left(\frac{H}{h}\right)^3}\ \mathrm{sn\ cn\ dn}\left[1+\left[2N_1-\frac{5\kappa^2+2}{8\kappa^2}\right.\right.\\&\left.\left.-\left(1-\frac{1}{2\kappa^2}\right)\left\{2\frac{z}{h}+\left(\frac{z}{h}\right)^2\right\}-\frac{1}{2}\mathrm{cn}^2+\frac{3}{2}\left\{2\frac{z}{h}+\left(\frac{z}{h}\right)^2\right\}\mathrm{cn}^2\right]\left(\frac{H}{h}\right)\right]\end{aligned}\right\}\quad(6)$$

in which $\mathrm{sn}=\mathrm{sn}(2K(x-ct)/L)$, $\mathrm{cn}=\mathrm{cn}(2K(x-ct)/L)$ and $\mathrm{dn}=\mathrm{dn}(2K(x-ct)/L)$ are the Jacobian elliptic functions with a real period.

Also, tne wave pressure is expressed as

$$\left.\begin{aligned}\frac{p}{\rho gh}=&(\mathrm{cn}^2-N_1)\left(\frac{H}{h}\right)+\left(-\frac{3}{4}\mathrm{cn}^2+\frac{3}{4}\mathrm{cn}^4-N_2\right)\left(\frac{H}{h}\right)^2\\&-\frac{3}{4\kappa^2}\left\{2\frac{z}{h}+\left(\frac{z}{h}\right)^2\right\}\{1-\kappa^2+2(2\kappa^2-1)\mathrm{cn}^2-3\kappa^2\,\mathrm{cn}^4\}\left(\frac{H}{h}\right)^2-\frac{z}{h}\end{aligned}\right\}\quad(7)$$

in which ρ is the density of fluid.

As already pointed out by Le Méhauté[10], N_2 in the Laitone theory falls into an error in addition to the expression for the vertical water particle velocity expressed by the mean water depth.

On the other hand, the wave characteristics in the Chappelear cnoidal wave theory of the second approximation by the Stokes second definition of wave celerity are given as follows; for the wave celerity and the wave length, they are given respectively as

$$\frac{c}{\sqrt{gh}}=1+L_3+\left(1-\frac{E}{K}\right)\left\{L_0+\left(2+\kappa^2-\frac{E}{K}\right)L_0^2+5L_0L_3\right\}\quad(8)$$

$$\frac{L}{h}=\frac{4K}{\sqrt{3L_0}}\quad(9)$$

in which L_0 and L_3 are the small expansion parameters. The small parameters L_0 and L_3 can be calculated from the following equations as function of κ and H/h.

$$\frac{H}{h}=\kappa^2L_0\left\{1+\frac{1}{4}L_0(10+7\kappa^2)+6L_3\right\}\quad(10)$$

$$\left.\begin{aligned}&2L_3+L_0\left(\kappa^2+\frac{E}{K}\right)+L_0^2\left\{-\frac{1}{5}(1-6\kappa^2-9\kappa^4)+2(1+\kappa^2)\frac{E}{K}\right\}\\&+6L_0L_3\left(\kappa^2+\frac{E}{K}\right)+L_3^2=0\end{aligned}\right\}\quad(11)$$

Next, the water surface displacement and the horizontal and vertical water particle velocities are written respectively as

$$\left.\begin{aligned}\frac{\eta}{h}=&\{2L_3+L_0(1+\kappa^2)-L_0\kappa^2\mathrm{sn}^2\}+\left[L_3^2+\frac{3}{20}L_0^2(12+23\kappa^2+12\kappa^4)\right.\\&\left.+6L_0L_3(1+\kappa^2)-\left\{\frac{5}{2}L_0^2\kappa^2(1+\kappa^2)+6L_0L_3\kappa^2\right\}\mathrm{sn}^2+\frac{3}{4}L_0^2\kappa^4\mathrm{sn}^4\right]\end{aligned}\right\}\quad(12)$$

$$\left.\begin{aligned}\frac{u}{\sqrt{gh}}=&\left\{L_0\left(1-\frac{E}{K}\right)-L_0\kappa^2\mathrm{sn}^2\right\}+\left[L_0^2\left(\frac{7}{4}\kappa^2+2\right)-L_0^2(\kappa^2+3)\frac{E}{K}\right.\\&+L_0^2\left(\frac{E}{K}\right)^2+5L_0L_3\left(1-\frac{E}{K}\right)-\frac{5}{2}L_0^2\kappa^2(1+\kappa^2)\mathrm{sn}^2-5L_0L_3\kappa^2\mathrm{sn}^2\\&+\frac{5}{4}L_0^2\kappa^4\mathrm{sn}^4-\left\{2\frac{z}{h}+\left(\frac{z}{h}\right)^2\right\}\left\{-\frac{3}{4}L_0^2\kappa^2+\frac{3}{2}L_0^2\kappa^2(1+\kappa^2)\mathrm{sn}^2\right.\\&\left.\left.-\frac{9}{4}L_0^2\kappa^4\mathrm{sn}^4\right\}\right]\end{aligned}\right\}\quad(13)$$

$$\left.\begin{aligned}\frac{w}{\sqrt{gh}}=&\left(1+\frac{z}{h}\right)\frac{2\kappa^2}{\sqrt{3L_0}}\mathrm{sn\ cn\ dn}\left[\frac{3}{2}L_0^2+\left\{\frac{3}{4}\left(1+\frac{z}{h}\right)^2L_0^3(1+\kappa^2-3\kappa^2\mathrm{sn}^2)\right.\right.\\&\left.\left.+\frac{3}{2}L_0^3(1+\kappa^2)+\frac{15}{2}L_0^2L_3+3\kappa^2L_0^3\mathrm{sn}^2\right\}\right]\end{aligned}\right\}\quad(14)$$

Also, the wave pressure is given as

$$\left.\begin{aligned}\frac{p}{\rho gh}=&\{2L_3+L_0(1+\kappa^2)-L_0\kappa^2\mathrm{sn}^2\}+\left[L_3^2+\frac{3}{20}L_0^2(12\kappa^4+23\kappa^2+12)\right.\\&+6L_0L_3(1+\kappa^2)-6\left\{6L_0L_3\kappa^2+\frac{5}{2}L_0^2\kappa^2(1+\kappa^2)\right\}\mathrm{sn}^2\\&+\frac{3}{4}L_0^2\kappa^4\mathrm{sn}^4-\left\{2\frac{z}{h}+\left(\frac{z}{h}\right)^2\right\}\left\{-\frac{3}{4}L_0^2\kappa^2+\frac{3}{2}L_0^2(1+\kappa^2)\mathrm{sn}^2\right.\\&\left.\left.-\frac{9}{4}L_0^2\kappa^4\mathrm{sn}^4\right\}\right]-\frac{z}{h}\end{aligned}\right\}\quad(15)$$

3. Relation between Wave Characteristics by Both the Theories

It is assumed that the small expansion parameters L_0 and L_3 are respectively expanded into series forms of H/h as

$$L_0=a_1\left(\frac{H}{h}\right)+a_2\left(\frac{H}{h}\right)^2+\cdots\cdots \quad (16)$$

$$L_3=b_1\left(\frac{H}{h}\right)+b_2\left(\frac{H}{h}\right)^2+\cdots\cdots \quad (17)$$

in which a_i and b_i are the coefficients to be determined. Then, substituting Eqs. (16) and (17) into Eqs. (10) and (11) and truncating the series expansions to the second order of H/h, the expressions L_0 and L_3 can be given as

$$L_0=\frac{1}{\kappa^2}\left(\frac{H}{h}\right)-\frac{1}{4\kappa^4}\left(10-5\kappa^2-12\frac{E}{K}\right)\left(\frac{H}{h}\right)^2 \quad (18)$$

$$\left.\begin{aligned}L_3=&-\frac{1}{2\kappa^2}\left(\kappa^2+\frac{E}{K}\right)\left(\frac{H}{h}\right)+\frac{1}{40\kappa^4}\left\{(-6\kappa^4+26\kappa^2+4)\right.\\&\left.+5\frac{E}{K}\left(-3\kappa^2+2-\frac{E}{K}\right)\right\}\left(\frac{H}{h}\right)^2\end{aligned}\right\}\quad(19)$$

In the next, if Eqs. (18) and (19) are substituted into Eqs. (8) and (9) in order to convert the wave celerity and the wave length in the Chappelear theory into the explicit expressions of the ratio H/h, the calculation finally yields

$$\left.\begin{aligned}\frac{c}{\sqrt{gh}}=1+\frac{1}{2\kappa^2}\Big(2-\kappa^2-3\frac{E}{K}\Big)\Big(\frac{H}{h}\Big)+\frac{1}{40\kappa^4}\Big\{-16+16\kappa^2-6\kappa^4\\+5\frac{E}{K}\Big(2-\kappa^2+3\frac{E}{K}\Big)\Big\}\Big(\frac{H}{h}\Big)^2\end{aligned}\right\}\tag{20}$$

$$\frac{L}{h}=\frac{4\kappa K}{\sqrt{3(H/h)}}\Big\{1+\frac{1}{8\kappa^2}\Big(10-5\kappa^2-12\frac{E}{K}\Big)\Big(\frac{H}{h}\Big)\Big\}\tag{21}$$

Also, the water surface displacement, the horizontal and vertical water particle velocities and the the wave pressure can be expressed respectively through a similar calculation as

$$\left.\begin{aligned}\frac{\eta}{h}=\Big\{\mathrm{cn}^2-\frac{1}{\kappa^2}\Big(1-\frac{E}{K}\Big)\Big\}\Big(\frac{H}{h}\Big)+\frac{1}{4\kappa^4}\Big\{2(\kappa^2-1)+\frac{E}{K}(2-\kappa^2)\\-3\kappa^4\mathrm{cn}^2+3\kappa^4\mathrm{cn}^4\Big\}\Big(\frac{H}{h}\Big)^2\end{aligned}\right\}\tag{22}$$

$$\left.\begin{aligned}\frac{u}{\sqrt{gh}}=\Big\{\frac{1}{\kappa^2}\Big(1-\kappa^2-\frac{E}{K}\Big)+\mathrm{cn}^2\Big\}\Big(\frac{H}{h}\Big)+\Big[\frac{1}{4\kappa^4}\Big\{2(\kappa^2-1)\\+\Big(-\kappa^2+2\frac{E}{K}\Big)\frac{E}{K}-\frac{1}{4\kappa^2}\Big(5\kappa^2-2\frac{E}{K}\Big)\mathrm{cn}^2+\frac{5}{4}\mathrm{cn}^4-3\Big\{2\frac{z}{h}\\+\Big(\frac{z}{h}\Big)^2\Big\}\Big\{\frac{1}{4\kappa^2}(1-\kappa^2)+\frac{1}{2\kappa^2}(-1+2\kappa^2)\mathrm{cn}^2-\frac{3}{4}\mathrm{cn}^2\Big\}\Big]\Big(\frac{H}{h}\Big)^2\end{aligned}\right\}\tag{23}$$

$$\left.\begin{aligned}\frac{w}{\sqrt{gh}}=\Big(1+\frac{z}{h}\Big)\sqrt{\frac{3}{\kappa^2}}\Big(\frac{H}{h}\Big)^3\,\mathrm{sn\,cn\,dn}\Big[1+\frac{1}{8\kappa^2}\Big(-18+11\kappa^2+16\frac{E}{K}\\-4\kappa^2\mathrm{cn}^2\Big)+4\Big\{2\frac{z}{h}+\Big(\frac{z}{h}\Big)^2\Big\}(1-2\kappa^2+3\kappa^2\mathrm{cn}^2)\Big]\Big(\frac{H}{h}\Big)\Big]\end{aligned}\right\}\tag{24}$$

$$\left.\begin{aligned}\frac{p}{\rho gh}=\Big\{\frac{1}{\kappa^2}\Big(1-\kappa^2-\frac{E}{K}\Big)+\mathrm{cn}^2\Big\}\Big(\frac{H}{h}\Big)+\Big[\frac{1}{4\kappa^4}\Big\{2(\kappa^2-1)+(2-\kappa^2)\frac{E}{K}\\-3\kappa^4\mathrm{cn}^2+3\kappa^4\mathrm{cn}^4\Big\}-\frac{3}{4\kappa^2}\Big\{2\frac{z}{h}+\Big(\frac{z}{h}\Big)^2\Big\}\{1-\kappa^2+2(2\kappa^2-1)\mathrm{cn}^2\\-3\kappa^2\mathrm{cn}^4\}\Big]\Big(\frac{H}{h}\Big)^2-\frac{z}{h}\end{aligned}\right\}\tag{25}$$

If brief calculations are performed by substituting Eq. (3) into Eqs. (1), (2), (4), (5), (6) and (7), it is clear that the expressions for the wave characteristics in the Chappelear theory agree exactly with the ones in the Laitone theory. The second approximate solution of the cnoidal waves by Laitone corresponds to the one by Chappelear in which the expansion parameters L_0 and L_3 are expanded into the power series of H/h and the expressions for the wave characteristics are truncated to the second order of H/h. Conversely speaking, this is due to the fact that in the Laitone theory, the original expansion parameters are expanded into the power series of H/h and truncated to the second order of H/h as well as the standing long wave theory of finite amplitude derived by Shuto[11] as the nonlinear interaction problem of the cnoidal waves.

Apart from the practical applicability for the prediction of the wave characteristics, it may therefore be concluded that the solution by Chappelear is more exact than the one by Laitone, as far as mathematical formulation is concerned.

4. Numerical Computation and Consideration

In this section, the computed results for some wave characteristics based on both the original mathematical solutions are compared to each other and the limiting condition for practical use is considered

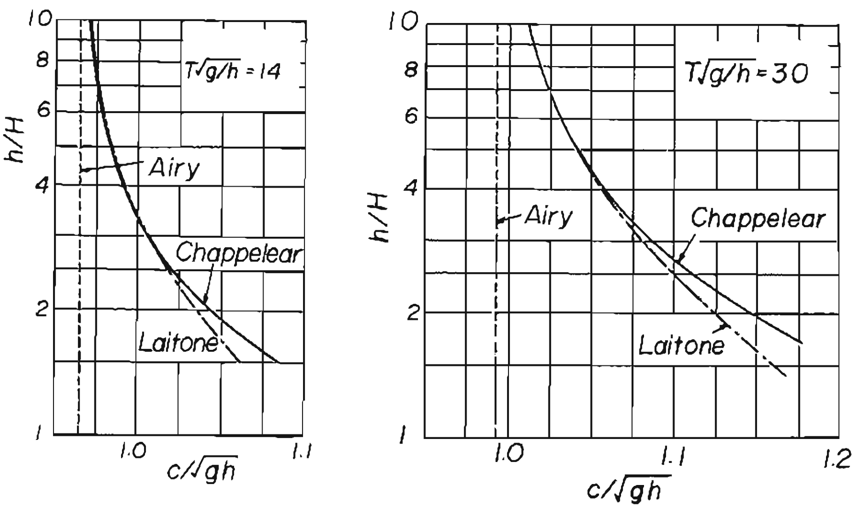


Fig. 2 Comparison between numerical results for wave celerity by both theories

Fig. 2 shows one of the comparisons for the wave celerity, in which T is the wave period. The solid line, the broken line and the one-dotted chain line in the figure indicate respectively the theoretical curve by the Chappelear theory, by the Airy wave theory and by the Laitone theory. The theoretical curves by both the

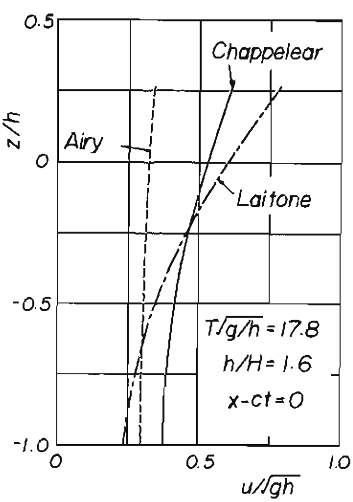


Fig. 3 Comparison between numerical results for vertical distribution of horizontal water particle velocity of wave crest phase based on both theories

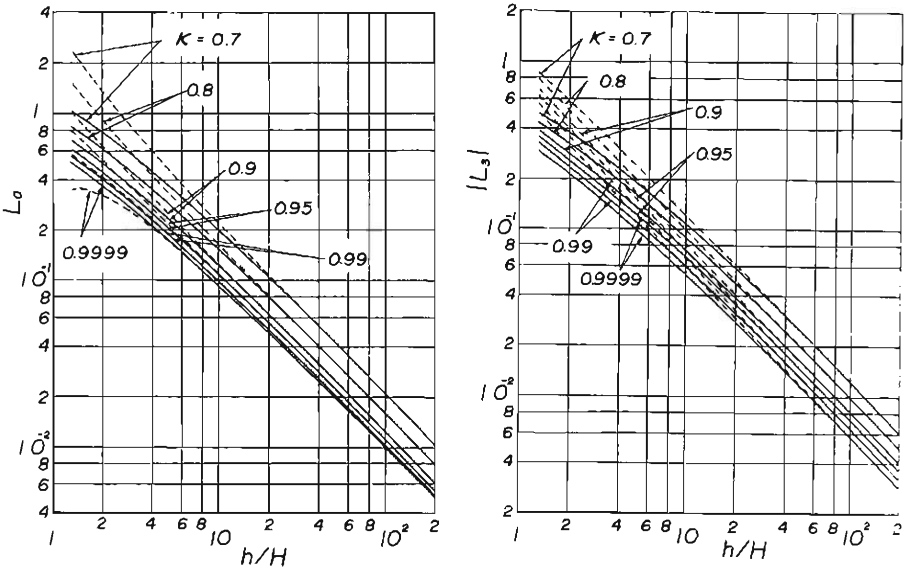


Fig. 4 Comparison between numerical results for expansion parameters in Chappelear's theory.

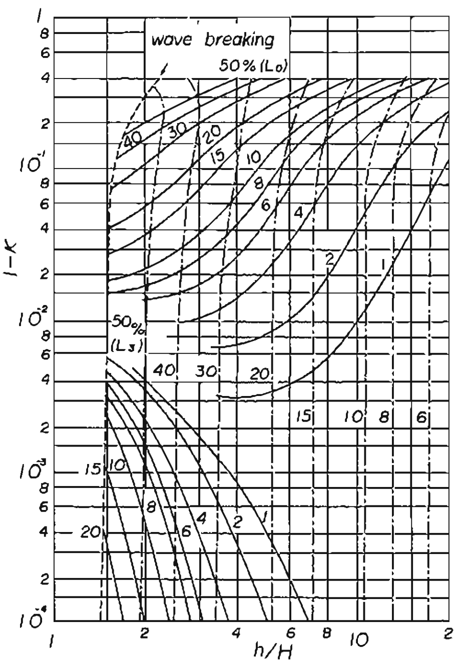


Fig. 5 Comparison of degree of deviation between expansion parameters computed exactly and those approximately in Chappelear's theory

cnoidal wave theories agree with each other in the case of larger value of h/H, while the result by the Chappelear theory becomes slightly larger than the one by the Laitone theory, as the value of h/H decreases.

The vertical distribution of the horizontal water particle velocity at the wave crest phase is compared in Fig. 3. As already pointed out by Iwagaki and Sakai[12], the Laitone theory gives the shape of the vertical distribution of the horizontal water particle velocity at the wave crest phase which increases rapidly toward the water surface from the bottom, in comparison with the Chappelear theory and the previous theories in the case of smaller value of h/H. This is caused by formulating the wave characteristics explicitly on H/h in the Laitone theory.

Fig. 4 is the comparison between the expansion parameters computed directly from Eqs. (10) and (11) in the Chappelear theory for the given values of κ and h/H and the ones from Eqs. (18) and (19) taking into account to the second order of H/h in the case in which the Chappelear theory coincides with the Laitone theory, as described already. The solid and broken lines in the figure correspond to the former and latter cases respectively. The results computed from the latter equations deviate rapidly from the ones from the former equations with decrease of h/H and moreover the tendency for L_3 is more prominent than the one for L_0.

Fig. 5 gives the degree of the mutual difference between the parameters computed exactly using Eqs. (10) and (11) and those computed approximately using Eqs. (18) and (19). In the figure, 50% (L_0) means that the degree of deviation between the parameter L_0 computed by the two methods is 50 percent and the broken line indicates the breaking criterion computed by the Chappelear theory under the assumption that the horizontal water particle velocity of the wave crest phase at the water surface becomes to be equal to the wave celerity at the wave breakig. The figure shows that the region in which the degree of deviation for both the parameters L_0 and L_3 is insignificant is limited to the higher value of h/H. It is also found that the region for L_0 is considerably wider than the region for L_3 in the same percentage.

Similarly, the ratio of the difference between the computed results on the wave characteristics by both the theories is given in Fig. 6. In the figure, η_0 is the wave crest height above the still water level at the wave crest phase, u_η, u_0 and u_b are the horizontal water particle velocities of the wave crest phase at the water surface, the mean water level and the bottom, p_0 and p_b the wave pressures of the wave crest phase at the mean water level and the bottom and w_{max} (-0.2) is the maximum vertical water particle velocity at the location of $z/h=-0.2$ respectively. It is found that although the magnitude of the errors due to the approximation mentioned above is different from each other in the computed results of the wave characteristics, the effects of the approximation on $u_\eta/\sqrt{gh}$, $u_b/\sqrt{gh}$, $p_b/\rho gh$ and $w_{max}(-0.2)/\sqrt{gh}$ are more clearly prominent than those on the other wave characteristics. Also, these effects on the ratio of h/L and $w_{max}(-0.2)/\sqrt{gh}$ are 1/2 times and 3/2 times respectively of the effect on the parameter L_0 in the case of the larger value of h/H, as would be made clear from the comparison between each equation.

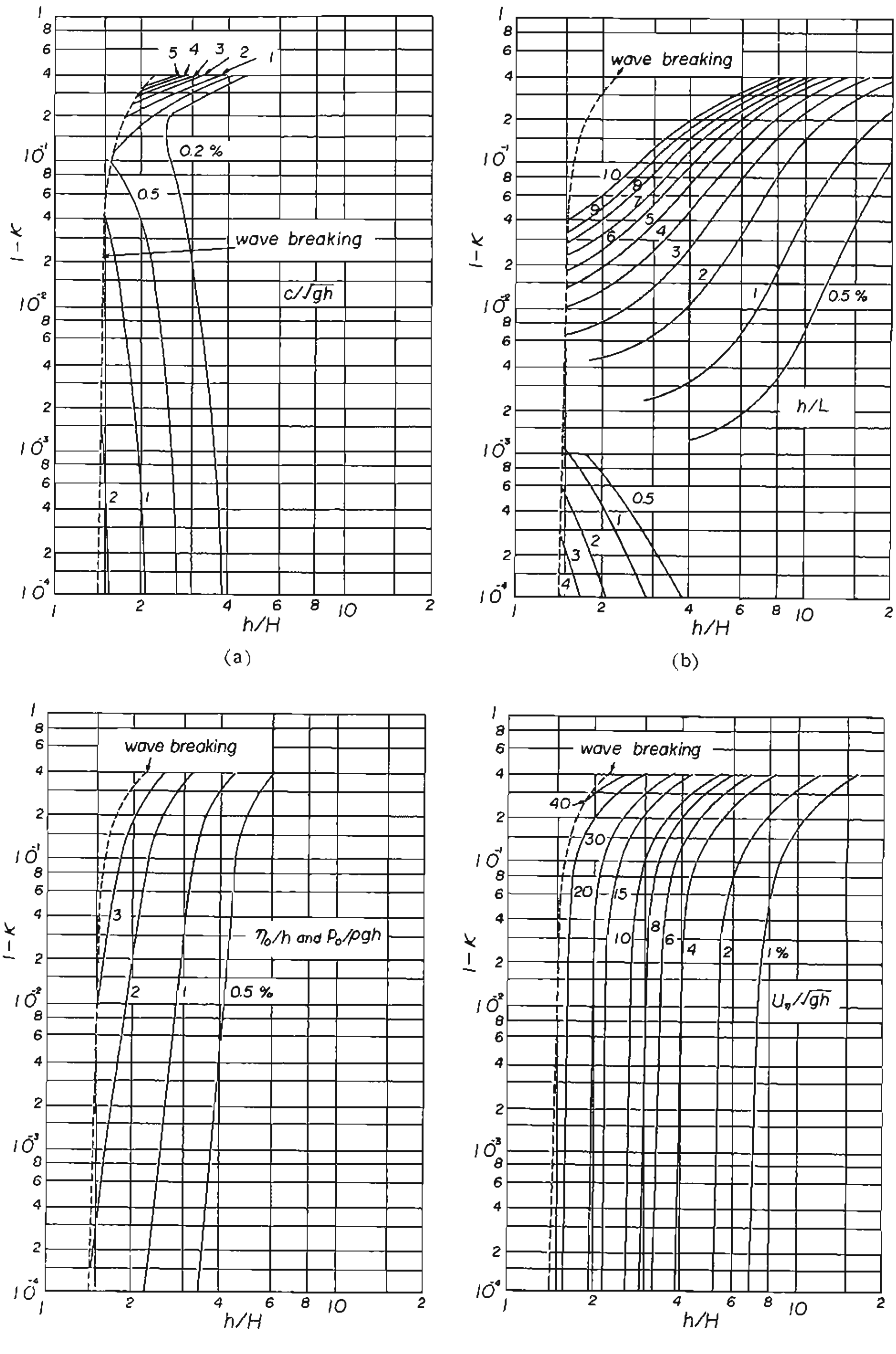
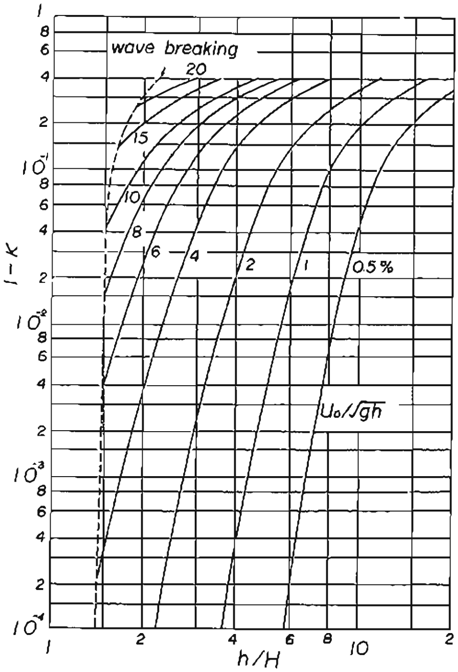


Fig. 6 Comparison of degree of deviation between computed results of wave characteristics by both theories



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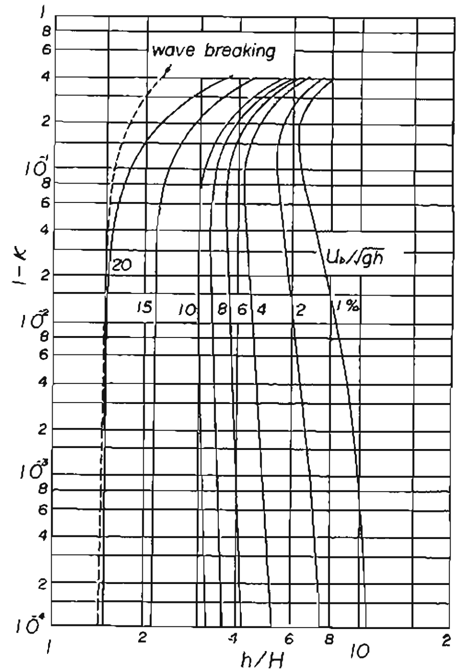

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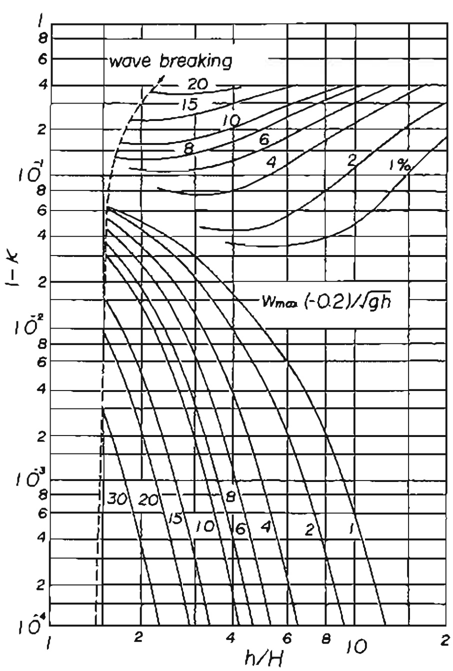

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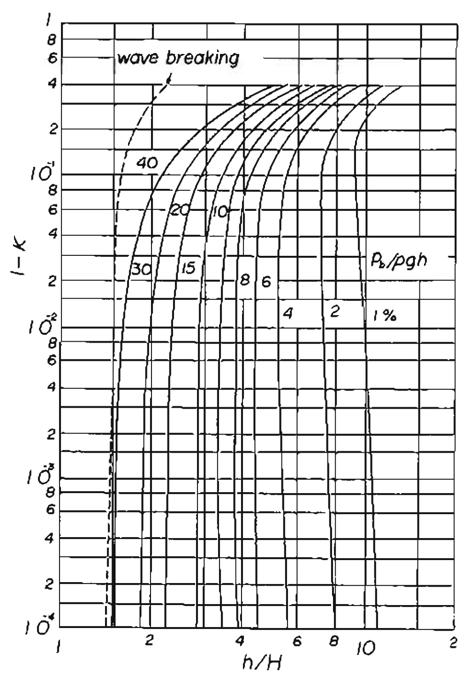

(h)

Fig. 6 Comparison of degree of deviation between computed results of wave characteristics by both theories

Since the Laitone theory is regarded as an approximation of the Chappelear theory on the expansion parameters L_0 and L_3, as mentioned previously, the limiting region of applicability of the cnoidal wave theories for the various wave characteristics has to be determined only by bounding the limiting region for the parameters L_0 and L_3.

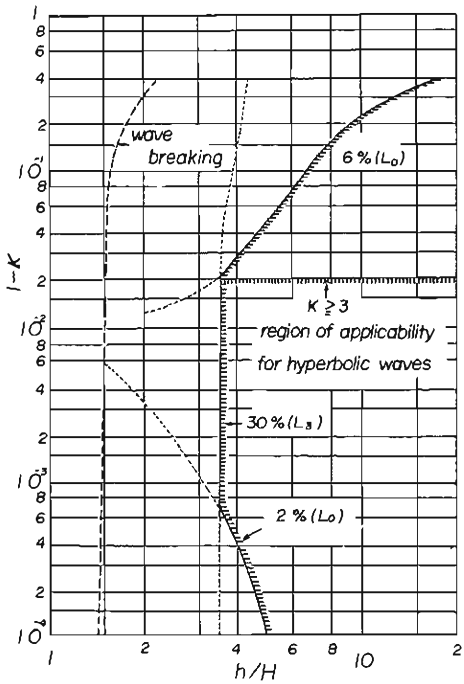

Fig. 7 Limiting region between both theories for practical use

Fig. 7 shows the limiting area of applicability for practical purposes between both the theories, in which the degree of deviation between the computed results on each wave characteristic by both the theories is roughly 20 % of the one at the breaking point except for the wave celerity in the case in which the difference is not so significant for practical computation. The limiting area is composed of the overlapped area in which the degree of difference for L_3 is smaller than 30 % and the degree for L_0 6 % in the case of the smaller value of κ and 2 % in the case of the larger value of κ. In the figure, the region corresponding to $K \geqq 3$ is the area of applicability for the Iwagaki hyperbolic wave theory[13], which is derived from the Laitone theory under the condition that $\kappa = 1$ and $E = 1$ but $K \fallingdotseq \infty$. Within the limiting area, the degree of daviation between the computed results on the wave characteristics by both the theories is less than 0.7 % for η_0/h, 6 % for $u_r/\sqrt{gh}$, 2 % for $u_0/\sqrt{gh}$, 8 % for $u_b/\sqrt{gh}$, 0.2 % for $c/\sqrt{gh}$, 3 % for h/L, 0.7 % for $p_0/\rho gh$, 8 % for $p_b/\rho gh$ and 9 % for $w_{max}(-2.0)/\sqrt{gh}$ respectively. Accordingly, the cnoidal wave theory by Laitone is only applicable in the shaded area under the above criteria and the theory by Chappelear must be used in the other region.

Also, the hyperbolic wave theory by Iwagaki must be used in the region of κ more than 0.98 ($K \fallingdotseq 3$) of the shaded area.

According to the above considerations, the errors due to the approximation should be previously estimated from Fig. 6 when applying the cnoidal wave theory by Laitone and the hyperbolic wave theory for the calculation of each wave characteristic.

5. Some Calculations on Wave Characteristics

According to Whitham[14], the density and flux of mass, momentum and energy in water waves are defined respectively as

(a) Mass: $$P_0=\int_{-h}^{\eta}\rho dz \quad (26) \qquad Q_0=\int_{-h}^{\eta}\rho u dz \quad (27)$$

(b) Momentum: $$P_1=\int_{-h}^{\eta}\rho u dz \quad (28) \qquad Q_1=\int_{-h}^{\eta}(p+\rho u^2)dz \quad (29)$$

(c) Energy: $$P_2=\int_{-h}^{\eta}\left\{\frac{1}{2}\rho(u^2+w^2)+\rho gz\right\}dz \quad (30)$$

$$Q_2=\int_{-h}^{\eta}\left\{\frac{1}{2}\rho(u^2+w^2)+p+\rho gz\right\}u dz \quad (31)$$

in which P_i and Q_i indicate the density and flux respectively.

The mean values of these quantities on a wave length are calculated using the second approximate solution of the cnoidal wave theory by Chappelear and by Laitone.

If the expressions for the wave characteristics in the Chappelear theory are substituted into Eqs. (26), (27), (28) and (29), the calculation for the density and flux of mass and momentum yields

$$\bar{P}_0=\rho h \quad (32) \qquad \bar{Q}_0=\bar{P}_1=0 \quad (33)$$

$$\left.\begin{aligned}\bar{Q}_1=\frac{1}{2}\rho gh^2+\rho gh^2\Big[&2L_3+L_0\left(\kappa^2+\frac{E}{K}\right)+3L_3^2+8L_0L_3\left(\kappa^2+\frac{E}{K}\right)\\&+\frac{L_0^2}{10}\left\{-7+17\kappa^2+23\kappa^4+10\left(2\kappa^4+4-\frac{E}{K}\right)\frac{E}{K}\right\}\Big]\end{aligned}\right\} \quad (34)$$

which are correct to the second order of the expansion parameters. The flux of mass and the density of momentum in waves calculated using the wave theory derived by the Stokes second definition of wave celerity vanish, as would be expected from the definition.

In the next, the kinetic energy can be given from the first term of Eq. (30) as

$$\left.\begin{aligned}\bar{P}_{2k}=\frac{1}{2}\rho gh^2\Big[&\frac{L_0^2}{3}\left\{\kappa^2-1+2(2-\kappa^2)\frac{E}{K}-3\left(\frac{E}{K}\right)^2\right\}+4L_0^2L_3\Big\{\kappa^2-1\\&+2(2-\kappa^2)\frac{E}{K}-3\left(\frac{E}{K}\right)^2\Big\}+\frac{L_0^3}{15}\Big\{17\kappa^4+9\kappa^2-26+(-34\kappa^4+19\kappa^2\\&+101)\frac{E}{K}-45(\kappa^2+2)\left(\frac{E}{K}\right)^2+15\left(\frac{E}{K}\right)^3\Big\}\Big]\end{aligned}\right\} \quad (35)$$

If the potential energy is measured from the mean water level, it can be written as

$$
\left.\begin{aligned}
\bar{P}_{2p}=\frac{1}{2}\rho gh^2\Big[&4L_3^2+4L_0L_3\Big(\kappa^2+\frac{E}{K}\Big)+\frac{L_0^2}{3}\Big\{3\kappa^4+\kappa^2-1+4(\kappa^2+1)\frac{E}{K}\Big\}\\
&+4L_3^3+\frac{L_0^2L_3}{5}\Big\{96\kappa^4+44\kappa^2-24+120(\kappa^2+1)\frac{E}{K}\Big\}+26L_0L_3^2\\
&\times\Big(\kappa^2+\frac{E}{K}\Big)+\frac{L_0^3}{15}\Big\{54\kappa^6+55\kappa^4-3\kappa^2-22+(76\kappa^4+134\kappa^2+76)\frac{E}{K}\Big\}\Big]
\end{aligned}\right\} \tag{36}
$$

Using Eq. (11), the kinetic energy can be proved to be identical with the potential energy to the first approximation of the cnoidal wave theory. The energy partition to the second approximation, however is not in the same ratio.

Accordingly, the density of wave energy is expressed as a sum of Eqs. (35) and (36) by the following equation.

$$
\left.\begin{aligned}
\bar{P}_2=\frac{1}{2}\rho gh^2\Big[&4L_3^2+4L_0L_3\Big(\kappa^2+\frac{E}{K}\Big)+\frac{L_0^2}{3}\Big\{3\kappa^4+2\kappa^2-2+2(\kappa^2+4)\frac{E}{K}\\
&-3\Big(\frac{E}{K}\Big)^2\Big\}+4L_3^3+26L_0L_3^2\Big(\kappa^2+\frac{E}{K}\Big)+\frac{L_0^2L_3}{5}\Big\{96\kappa^4+64\kappa^2-44\\
&+40(2\kappa^2+5)\frac{E}{K}-60\Big(\frac{E}{K}\Big)^2\Big\}+\frac{L_0^3}{15}\Big\{54\kappa^6+72\kappa^4+6\kappa^2-48\\
&+(42\kappa^4+153\kappa^2+177)\frac{E}{K}-45(\kappa^2+2)\Big(\frac{E}{K}\Big)^2+15\Big(\frac{E}{K}\Big)^3\Big\}\Big]
\end{aligned}\right\} \tag{37}
$$

The energy flux in the Chappelear theory is calculated from Eq. (31). The result is

$$
\left.\begin{aligned}
\bar{Q}_2=\rho gh^2\sqrt{gh}\Big[&\frac{L_0^2}{3}\Big\{\kappa^2-1+2(2-\kappa^2)\frac{E}{K}-3\Big(\frac{E}{K}\Big)^2\Big\}+\frac{13}{3}L_0^2L_3\Big\{\kappa^2-1\\
&-2(\kappa^2-2)\frac{E}{K}-3\Big(\frac{E}{K}\Big)^2\Big\}+\frac{L_0^3}{15}\Big\{18\kappa^4+11\kappa^2-29+(-36\kappa^4+6\kappa^2\\
&+124)\frac{E}{K}-5(7\kappa^2+25)\Big(\frac{E}{K}\Big)^2+30\Big(\frac{E}{K}\Big)^3\Big\}\Big]
\end{aligned}\right\} \tag{38}
$$

Therefore, the transfer velocity of wave energy c_g, which is one of the definitions for the group velocity of waves, is expressed as the ratio of the wave energy flux to the wave energy density, although an explicit expression is difficult.

On the contrary, the similar wave characteristics in the Laitone theory are calculated respectively as

$$\bar{P}_0=\rho h \tag{39}$$

$$\bar{Q}_0=\bar{P}_1=0 \tag{40}$$

$$\bar{Q}_1=\frac{1}{2}\rho gh^2+\frac{1}{2\kappa^4}\Big\{\kappa^2-1+2(2-\kappa^2)\frac{E}{K}-3\Big(\frac{E}{K}\Big)^2\Big\}\Big(\frac{H}{h}\Big)^2 \tag{41}$$

$$
\left.\begin{aligned}
\bar{P}_{2k}=\frac{1}{2}\rho gh^2\Big[&\frac{1}{3\kappa^4}\Big\{\kappa^2-1+2(2-\kappa^2)\frac{E}{K}-3\Big(\frac{E}{K}\Big)^2\Big\}\Big(\frac{H}{h}\Big)^2+\frac{1}{30\kappa^6}\Big\{-\kappa^4\\
&+3\kappa^2-2+2(\kappa^4-\kappa^2+1)\frac{E}{K}+15(\kappa^2-2)\Big(\frac{E}{K}\Big)^2+30\Big(\frac{E}{K}\Big)^3\Big\}\Big(\frac{H}{h}\Big)^3\Big]
\end{aligned}\right\} \tag{42}
$$

$$\left.\begin{aligned}\bar{P}_{2p}=\frac{1}{2}\rho g h^2\Big[&\frac{1}{3\kappa^4}\Big\{\kappa^2-1+2(2-\kappa^2)\frac{E}{K}-3\Big(\frac{E}{K}\Big)^2\Big\}\Big(\frac{H}{h}\Big)^2+\frac{1}{10\kappa^6}\\ &\times\Big\{\kappa^4-3\kappa^2+2-2(\kappa^4-6\kappa^2+6)\frac{E}{K}-5(\kappa^2-2)\Big(\frac{E}{K}\Big)^2\Big\}\Big(\frac{H}{h}\Big)^3\Big]\end{aligned}\right\} \quad (43)$$

$$\left.\begin{aligned}\bar{P}_{2}=\frac{1}{2}\rho g h^2\Big[&\frac{2}{3\kappa^4}\Big\{\kappa^2-1+2(2-\kappa^2)\frac{E}{K}-3\Big(\frac{E}{K}\Big)^2\Big\}\Big(\frac{H}{h}\Big)^2+\frac{1}{15\kappa^6}\\ &\times\Big\{\kappa^4-3\kappa^2+2+(-2\kappa^4+17\kappa^2-17)\frac{E}{K}+15\Big(\frac{E}{K}\Big)^3\Big\}\Big(\frac{H}{h}\Big)^3\Big]\end{aligned}\right\} \quad (44)$$

$$\left.\begin{aligned}\bar{Q}_2=\rho g h^2\sqrt{gh}\Big[&\frac{1}{3\kappa^4}\Big\{\kappa^2-1-2(\kappa^2-2)\frac{E}{K}-3\Big(\frac{E}{K}\Big)^2\Big\}\Big(\frac{H}{h}\Big)^2\\ &+\frac{1}{30\kappa^6}\Big\{4(-\kappa^4+3\kappa^2-2)+(8\kappa^4-53\kappa^2+53)\frac{E}{K}+60(\kappa^2-2)\Big(\frac{E}{K}\Big)^2\\ &+75\Big(\frac{E}{K}\Big)^3\Big\}\Big(\frac{H}{h}\Big)^3\Big]\end{aligned}\right\} \quad (45)$$

These equations agree exactly with the ones by the Chappelear theory in the case in which the expansion parameters L_0 and L_3 are expanded into the power series of H/h and the expressions for the wave characteristics are truncated to the second order of H/h.

Also, the energy transfer velocity can be given in an explicit form as

$$\left.\begin{aligned}c_g=\sqrt{gh}\Big[1+&\frac{(H/h)}{2\kappa^2\{\kappa^2-1+2(2-\kappa^2)(E/K)-3(E/K)^2\}}\Big\{-\kappa^4+3\kappa^2-2\\ &+2(\kappa^4-7\kappa^2+7)\Big(\frac{E}{K}\Big)+12(\kappa^2-2)\Big(\frac{E}{K}\Big)^2+12\Big(\frac{E}{K}\Big)^3\Big\}\Big]\end{aligned}\right\} \quad (46)$$

Since the preceding wave theories are derived using the perturbation method, the expressions for the horizontal water particle velocity and the other wave characteristics in the theories are expressed as the infinite power series of small parameter ε such as $u=\sum_{i=1}^{\infty}\varepsilon^i u_i$, in which u_i is the approximate solution corresponding to each order of ε. Accordingly, in the calculation of Eqs. (27), (28), (29), (30) and (31) using the truncated series solution of the second order approximation, the contribution of the higher order terms beyond the third order e.g. u_3 to the results exsists in the mathematical formulation. However, this contribution is neglected in the calculation.

6. Conclusion

The relation between wave characteristics of the cnoidal wave theory derived by Laitone and by Chappelear was established. Expanding the parameters L_0 and L_3 in the Chappelear theory into the power series of H/h and rewritting the expressions for wave characteristics in the Chappelear theory into power series of H/h to the second order, it was proved that the second order approximate solution of the cnoidal wave theory by Laitone coincides with the one by Chappelear to the second order of H/h. And based on the numerical comparison between the expansion parameters computed exactly and the ones computed approximately to the second order of H/h, the limiting region of wave characteristics in which the approximate expansion of L_0 and L_3 in the Chappelear theory on the ratio of H/h is allowable was proposed for practical use.

In addition, tbe density and flux of mass, momentum and energy were formulated using both the cnoidal wave theories of the second approximation as well as the energy tranfer velocity.

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