A New Approach to Stokes Wave Theory

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Abstract

Stokes wave theories to third-order approximation have widely been employed to calculate wave properties for waves propagating over finite depths of water in most engineering applications. However, different and often inconsistent expressions of wave variables can be observed from the numerous theories available.

Examinations of the usual Stokes wave theories are on the so-called Stokes definitions of wave celerity and Bernoulli constant, as well as on the physical explanations of some theoretical problems involved. A new Stokes wave theory to third order approximation is derived by applying only necessary conditions and assumptions, without using the definitions of wave celerity. The resulting mathematical formulations for some pertinent wave variables are presented.

Comparison is made between the new third-order approximation and the usual ones derived from using the Stokes definitions of wave celerity, showing different expressions of wave celerity, horizontal water particle velocity, and mass transport velocities. It is found that the mass transport velocity exists in the Eulerian description as well as the usual Stokes drift in the Lagrangian description.

1. Introduction

A thorough understanding of wave characteristics in shallow water is essential in estimating wave forces on structures, as well as investigating various mechanisms of wave-induced transport phenomenon, such as nearshore currents and sand drift.

A finite amplitude wave theory in inviscid fluid and irrotational motion was first derived by Stokes[1] in 1847, thus bearing the name "Stokes waves". Another wave theory proposed by Korteweg de Vries[2] in 1895 has been referred to cnoidal waves. Since then, numerous theories describing wave motions have been derived to higher order of approximation for finite amplitude waves of permanent type. By the perturbation method many fruitful results have been produced by solving the governing hydrodynamic equations of wave motion. Higher order approximations have also been developed in using computers for Stokes waves by Schwartz[3] and Horikawa et al.[4], and for cnoidal waves by Fenton[5]. Approximate waves theories, applicable to water of arbitrary depth and independent of shallowness, have successfully been derived to a very high order by Cokelet[6] and Chaplin[7].

Higher order solutions of finite amplitude waves of permanent type in uniform depths of water have been derived, having results agreeing very well with experimental results, particularly when applied to wave profile and wave pressure. It is unfortunate, however, that the required exactness for other wave properties, such as water particle velocities and the resulting mass transport, has remained in doubt.

In deriving a Stokes wave theory, an additional condition, the definition of wave celerity[1] is required and yields a different expression of mass transport by waves. Besides, the inter-relationship among integral properties of periodic gravity waves of finite amplitude derived by Longuet-Higgins[8] should also be governed by the definition of wave celerity.

Table 1 Classification of usual Stokes wave theories by definition of wave celerity.

Definition of wave celerity / Wave theory	First definition	Second definition
Stokes waves	Stokes (2nd approximation) Skjelbreia (3rd approximation) Tanaka (3rd approximation) Laitone (3rd approximation) Goda & Abe (3rd approximation) Skjelbreia-Hendrickson (5th approximation) Bretscheider (5th approximation)	Tsuchiya & Yamaguchi (4th approximation) Horikawa, Nishimura & Isobe (51st approximation)
Cnoidal waves	Fenton (9th approximation)	Laitone (2nd approximation) Tsuchiya & Yamaguchi (3rd approximation) Horikawa, Nishimura & Isobe (24th approximation)

As depicted in **Table 1**, the usual wave theories for waves of finite amplitude are classified based on the kind of definition of wave celerity. The first definition of wave celerity gives

$$c=\int_0^L (c+u)dx \Big/ \int_0^L dx \quad \cdots\cdots (1)$$

while the second definition renders

$$c=\int_0^L\int_{-h}^{\eta} (c+u)dzdx \Big/ \int_0^L\int_{-h}^{\eta} dzdx \quad \cdots\cdots (2)$$

in which c is the wave celerity, u the horizontal water particle velocity, η the water surface displacement, h the water depth, and L the wave length. Given as an example, the vertical distributions of mass transport in the Stokes waves by the first and second definitions of wave celerity respectively are illustrated in **Fig.1**. In the Lagrangian description, as already described, the mass transport depends upon the definition of wave celerity, yielding quite different conclusions. It is seen from this figure that net mass transport exists in the direction of wave propagation by the first definition, but no net value results from the second definition. The vertical distribution of mass transport derived by Longuet-Higgins[9] is shown in the same figure. The effect of viscosity is taken into consideration in calculating mass transport based on the second definition of wave celerity, but no net value exists in this formulation. The

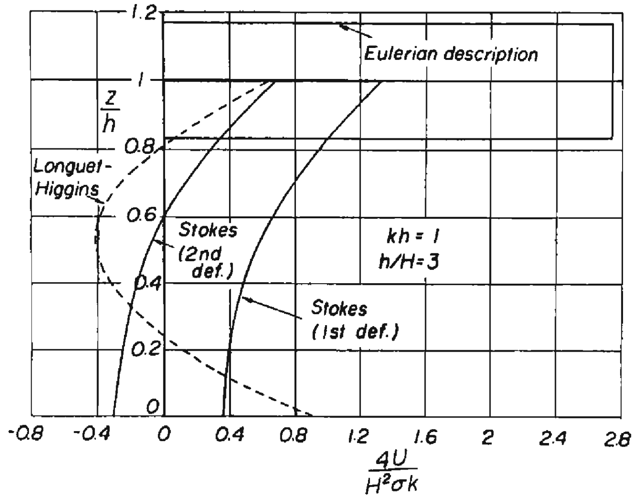

Fig. 1 Vertical distributions of mass transport velocity by usual Stokes wave theories.

results so obtained could be applied to the waves generated in a normal wave tank with end walls. But the waves may no longer be permanent type in the exact sense. Mass transport, which can be defined as the time-averaged water particle velocities, does not exist generally in the Eulerian description, but it may be obtained by the inter-relationship between the water surface displacement and water particle velocity at the free surface. To study the so-called wave-induced transport phenomena, the mass transport velocity may expectantly be formulated both in the Eulerian and Lagrangian descriptions. As mentioned previously, the expression of mass transport varies with the definition of wave celerity as shown in **Table 1**. This problem has desired further investigation.

Generally, the third-order approximation to Stokes waves has demonstrated its availability to most engineering applications. Though the solutions of Laitone[10] and of Skjelbreia[11] differ slightly from each other in form due to different treatment of the Bernoulli constant, both have not elucidated the definition of wave celerity. The discrepancy in these approximate solutions results in different expressions of water particle velocity, thus rendering the application of Stokes wave theories questionable in wave-induced transport phenomena in shallow water. Therefore, it is most certain that the required exactness in the Stokes wave theory, which has so often been applied and mathematically examined by many researchers over one century, has not yet been fully established.

In this paper, examinations on the usual Stokes wave theories are firstly made on the basis of the definition of wave celerity and Bernoulli constant. Various discussions on the theoretical problems are envisaged. A new approach to Stokes wave theory is then proposed without using the definitions of wave celerity. Comparison of this new theory with the usual approximations is illustrated, specifying also necessary conditions in deriving higher order solutions.

2. Governing Equations

In order to study the so-called "Stokes waves" rigorously, the variables of this wave system contained in the governing equations and other essential conditions should receive a fuller examination. This can equally be applied to all assumptions used in deriving the wave theory. Stokes waves can be regarded as a kind of nonlinear dispersive waves with weak nonlinearity having the full dispersion relation to the lowest order given by

$$c=\{(g/k)\tanh kh\}^{1/2}=\sqrt{gh}\ \{1-(2\pi h/L)^2/6 + (29/360)(2\pi h/L)^4-(55/3024)(2\pi h/L)^6 \ldots +\} \qquad \text{(3)}$$

as well as being periodic progressive waves of permanent type over uniform depths of water. Upon examining Eq.(3), it is found in the first order that a basic term relating to $\sqrt{gh}$ alone is introduced into the dispersion relationship, representing a fundamental component of wave celerity of Stokes waves, besides the corrections in terms of shallowness h/L, available to high-order expansion. While nonlinear corrections associated with wave steepness at each higher order approximation have not yet been included. This is a remarkable feature of the Stokes waves.

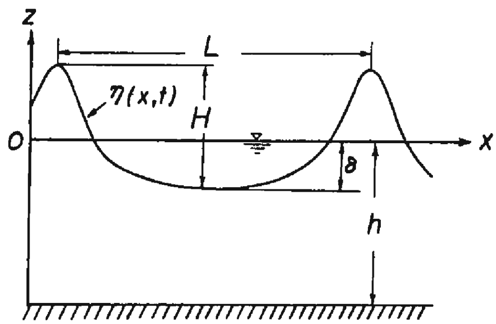

Fig. 2 Co-ordinate system used.

Considering the velocity potential ϕ for irrotational wave motion, and taking the Cartesian co-ordinate system as shown in **Fig.2**, the governing equation given by Laplace's equation renders

$$\nabla^2\phi=0 \qquad \text{(4)}$$

The dynamic and kinematic boundary conditions at the free water surface are given respectively by

$$\phi_t+(\phi_x^2+\phi_z^2)/2+gz|_{z=\eta}=0 \qquad \text{(5)}$$

$$\eta_t+\eta_x\phi_x-\phi_z|_{z=\eta}=0 \qquad \text{(6)}$$

in which η is the water surface elevation, g the acceleration due to gravity, x and z the Cartesian co-ordinates and t the time. In Eq.(5) the Bernoulli constant Q is included in the first term ϕ_t. The bottom boundary condition is given as

$$\phi_z|_{z=-h}=0 \quad \cdots\cdots (7)$$

Additional conditions for the periodic, progressive waves of permanent type can be specified as follows,

$$\eta(x, t)=\eta(x-ct) \quad \cdots\cdots (8)$$

$$\eta(x, t)=\eta(x+2\pi/k, t)=\eta(x, t+2\pi/\sigma) \quad \cdots\cdots (9)$$

$$\nabla\phi(x, z, t)=\nabla\phi(x+2\pi/k, z, t)=\nabla\phi(x, z, t+2\pi/\sigma) \quad \cdots\cdots (10)$$

Eqs.(9) and (10) describe the periodicity in wave profile and water particle velocity.

In addition, the continuity equation of the profile is given as

$$\int_0^{2\sigma/k}\eta(x, t)nx=\int_0^{2\pi/\sigma}\eta(x, t)nt=0 \quad \cdots\cdots (11)$$

in which k is the wave number and σ the wave frequency (i.e. $2\pi/T$). Part or all of the conditions above have also been employed in deriving various progressive waves of permanent type other than the Stokes waves, but are specified only as obvious assumptions.

Since solutions of the linearized equations specifying the free water surface boundary conditions also satisfy Eq.(4), the following perturbation series with respect to a small parameter ε are assumed, giving

$$\phi=\sum_{n=1}^{\infty}\varepsilon^n\phi_n \quad \cdots\cdots (12)$$

$$\eta=\sum_{n=1}^{\infty}\varepsilon^n\eta_n \quad \cdots\cdots (13)$$

to the unknown functions ϕ and η, in order to perturb the nonlinear terms contained in Eqs.(5) and (6).

3. Usual Stokes Wave Theories

Although numerous expressions have been developed for Stokes wave theories since Stokes in 1847, including very high order solutions and computer applications, the methods of derivation are not unified and some basic problems remain. To explain this, besides the clearness of the Stokes definition of wave celerity, differences exist in treating the time-dependent term associated with the Bernoulli constant and in the perturbed solution of the velocity potential. **Table 2** summarizes the major difierences in various Stokes wave theories to third order approximation, in which λ is the small parameter $\pi H/L$ and H the wave height.

It is beneficial to discuss in length the physical significance of the Bernoulli constant, as also revealed in **Table 2**. From the theoretical ground of hydrodynamics, the Bernoulli constant may well be included in the expression of the velocity potential, as pointed out by Stoker[12], and other researchers. However, Skjelbreia[11] has treated the Bernoulli constant Q alone without relating it to the velocity potential, hence resulting in an approximate solution to Q by perturbation. The reason why the constant Q has been so perturbed then requires a justifiable explanation, as often is the case in the field of applied mathematics. The unknown functions of ϕ and η may legitimately be perturbed, because the governing equations are nonlinear with

Table 2 Classification of approaches applied to usual Stokes wave theories.

Author	Bernoulli constant $(k^2/\lambda)Q$	Time-dependent term $(k/\lambda)^2(\phi/\sigma_0{}^2t)$	Distance between still water level and x-axis	Averaged horizontal water particle velocity	Difinition of wave celerity	Periodic terms in ϕ_3	Periodic terms in η_3
Skjelbreia	$-\frac{1}{2\sinh 2kh}$	0	0	0	1	$\sin 3(kx-\sigma t)$	$\cos 3(kx-\sigma t)$
Skjelbreia & Hendrickson	$-\frac{1}{2\sinh 2kh}$	0	0	0	1	$\sin(kx-\sigma t)$, $\sin 3(kx-\sigma t)$	$\cos 3(kx-\sigma t)$
Goda & Abe	0	$-\frac{1}{4}(\coth^2 kh-1)$	0	0	1	$\sin 3(kx-\sigma t)$	$\cos(kx-\sigma t)$, $\cos 3(kx-\sigma t)$
Laitone	0	0	Finite value (corrected)	0	1	$\sin 3(kx-\sigma t)$	$\cos(kx-\sigma t)$, $\cos 3(kx-\sigma t)$
Tsuchiya & Yamaguchi	$-\left(\frac{1}{2\sinh 2kh}+\frac{1}{2kh}\right)$	$\frac{\coth kh}{2kh}$	0	$\frac{\coth kh}{2kh}$	2	$\sin(kx-\sigma t)$, $\sin 3(kx-\sigma t)$	$\cos 3(kx-\sigma t)$

respect to these two quantities, and more over, the perturbation is necessary to linearize the equations and to evaluate successively the nonlinear effect at each order of approximation. On the other hand, the Bernoulli constant Q is contained within the governing equation simply as a linear quantity, hence the constant Q can be determined without being perturbed. As a matter of fact, Stokes did not perturb the Bernoulli constant. It appears therefore that the reason why the Bernoulli constant was perturbed by Skjelbreia[10] can not clearly be understood. In order to try explaining the reason, it is guessed that the additional constant is needed to eliminate some existing constant terms resulting from solving the free water surface boundary conditions. As the velocity potential is expressed in terms of the harmonic function in sinusoidal terms and the coefficients are then decided by using Eqs. (5) and (6), a constant term arises from the squared terms contained in Eq. (5) at least at the second order approximation and higher. And, consequently, a constant referred to the Bernoulli constant is required to eliminate this term. Laitone[9] assumed the relations $\eta(x,t)=\eta(x-ct)$ and $\phi(x,z,t)=\phi(x-ct,z)$, but didn't take the Bernoulli constant and the time-dependent term associated with ϕ into consideration. So the constant term resulting from solving Eqs. (5) and (6) could not properly be managed and was included in η. As a result, the continuity equation of the wave profile represented by Eq. (11) is not satisfied and the mean water level disagrees with the still water level. Including the Bernoulli constant Q within the velocity potential ϕ, Goda et al.[13] assumed a zero value for Q so that the time-dependent term pertained to ϕ. The latter method has been judged as adequate, yielding the final expressions of velocity potential at each order of approximation which are composed of the harmonic functions in sinusoidal forms and a linear time-dependent function. It can be inferred implicitly that the first definition of wave celerity was used in the derivations, as no use had been made of the component of averaged horizontal water particle velocity. Following Skjelbreia's treatment of Bernoulli constant in the governing equation, Tsuchiya & Yamaguchi[14] have included a nonperiodic term into the velocity potential and adopted a second definition of wave celerity. The averaged water-particle velocity distribution through the depth for the horizontal component was accordingly defined in the Eulerian description, rendering the Stokes drift in the reverse direction of wave propagation. It is felt that the above treatment is not complete, and a question remained on the point that the unknown Bernoulli constant is left in the governing equation, while the time dependent term was deliberately absorbed in ϕ.

Whitham[15] assumed the Bernoulli constant a zero value and let $\eta(x,t)=\eta(x-ct)$ and $\phi(x,z,t)=\phi(x-ct,z)$, although the solution was not given in **Table 2.** Whitham has stated that the mean value ϕ_t should not become zero, while ϕ must at least have a time-dependent term in its solution (i.e. terms proportional to t or x are acceptable in ϕ). He has further proposed that a term proportional to x can be normalized to zero, as it represents a nonzero mean in the horizontal velocity. However, the resulting velocity potential has failed in satisfying the relation of $\phi(x,z,t)=\phi(x-ct,z)$ and is inconsistent with the initial assumption with respect to ϕ.

Secondly, there are differences between the derived expressions of perturbation solutions, as shown in **Table 2.** In the third order approximation calculated by Skjelbreia[11], both ϕ_3 and η_3 include only the third harmonics. While, in a fifth order solution of Skjelbreia[10] and fourth order by Tsuchiya & Yamaguchi[14], ϕ_3 has a first order component as well as a third harmonic, and η_3 contains only a third harmonic. In the solution of Laitone[9] and Goda et al.[13], a third order harmonic is observed in the expressions of ϕ_3 and η_3, an additional term in linear order is also seen in η_3. The differences briefly mentioned above might be resulted from different approaches used in derivation. In all the examples above, the relations of $\phi(x,z,t) =\phi(x-ct,z)$ and $\eta(x,t)=\eta(x-ct)$ have fundamentally been assumed. Hence, upon the variable transformation $x-ct=X$ and the elimination of η from Eqs.(5) and (6), it yields a combined free water surface boundary condition with respect to ϕ,

$$c^2\phi_{tt}+g\phi_z=\frac{1}{2}c(\phi_x{}^2+\phi_z{}^2)+g\eta_x\phi_x|_{z=\eta} \quad \cdots\cdots(14)$$

Eq.(14) can successively be solved by a perturbation method and its basic solution, which represents the small amplitude waves, can easily be obtained at the lowest order. At the second order and higher, a particular solution which reflects the correction term at each order of approximation to the basic solution should be derived together with the complementary solution. As far as the progressive waves of permanent type are concerned, it is reasonable that the desired solution be not a complementary solution, similar to the basic one, but a particular solution representing nonlinear correction. Functions which satisfy the governing equations together with the ϕ function so obtained may be used to determine the solution of η. Thus, as η is derived directly from ϕ at each order of approximation, the nature of a particular solution has inherently included in the solution of η. Accordingly, it might be said that the approximate solutions of Laitone and Goda et al. are appropriate, but the solutions of Skjelbreia[10] and Tsuchiya & Yamaguchi[14] are inadequate, as the latter cases contain a complementary solution of the first order component.

It may, therefore, be concluded that the lack of unity in various Stokes wave theories examined previously has resulted from diverse opinions on what the wave motion is and how the solution should be derived.

4. Perturbation Method

A third-order approximation to Stokes waves is derived herein by using the conditions and assumptions presented already, referring also to the consideration mentioned previously. First, in order to eliminate the secular term resulting from deriving the approximate solution, as done by Stokes, a change of variable for the time t is introduced as

$$\tau=\sigma t \quad \cdots\cdots(15)$$

in which σ is the angular frequency which is assumed to be expanded as a power series with respect to the small parameter ε as

$$\sigma=\sigma_0+\varepsilon\sigma_1+\varepsilon^2\sigma_2+ \quad \cdots\cdots(16)$$

In addition, the velocity potential ϕ at the free surface $z=\eta$ may be expressed in terms of the Taylor expansion at $z=0$ as

$$\phi(x,\eta,t)=\sum_{n=1}^{\infty}\varepsilon^n\sum_{m=1}^{\infty}\frac{\eta^m}{m!}\left(\frac{\partial^m\phi_n}{\partial z^m}\right)\Bigg|_{z=0} \quad \cdots\cdots(17)$$

Upon introducing Eqs.(12), (13), (15), (16) and (17) into Eqs.(4), (5), (6) and (7) and collecting terms of each order in ε yields the necessary equations to each order of approximation for constant at $z=0$. The terms in $O(\varepsilon)$ are given by

$$\left.\begin{aligned}&\nabla^2\phi_1=0, \quad g\eta_1+\sigma_0\phi_{1\tau}|_{z=0}=0\\&\phi_{1z}-\sigma_0\eta_{1\tau}|_{z=0}=0, \quad \phi_{1z}|_{z=-h}=0\end{aligned}\right\} \quad \cdots\cdots(18)$$

and those in $O(\varepsilon^2)$ are

$$\left.\begin{aligned}&\nabla^2\phi_2=0\\&g\eta_2+\sigma_0\phi_{2\tau}=-\sigma_1\phi_{1\tau}-\sigma_0\eta_1\phi_{1\tau z}-(\phi_{1x}^2+\phi_{1z}^2)/2|_{z=0}\\&\phi_{2z}-\sigma_0\eta_{2\tau}=\sigma_1\eta_{1\tau}+\eta_{1x}\phi_{1x}-\eta_1\phi_{1zz}|_{z=0}\\&\phi_z|_{2z=-h}=0\end{aligned}\right\} \quad \cdots\cdots(19)$$

The governing equations in $O(\varepsilon^3)$ are as follows,

$$\left.\begin{aligned}&\nabla^2\phi_3=0\\&g\eta_3+\sigma_0\phi_{3\tau}=-\sigma_1\eta_1\phi_{1\tau z}-\sigma_0\eta_2\phi_{1\tau z}-\sigma_0\eta_1\phi_{2\tau z}-\sigma_0\eta_1^2\phi_{1\tau zz}/2\\&\qquad-\eta_1(\phi_{1x}\phi_{1xz}+\phi_{1z}\phi_{1zz})-\phi_{1x}\phi_{2x}-\phi_{1z}\phi_{2z}|_{z=0}\\&\phi_{3z}-\sigma_0\eta_{3\tau}=\sigma_1\eta_{2\tau}+\sigma_2\eta_{1\tau}-\eta_1\phi_{1zz}-\eta_1^2\phi_{1zzz}/2\\&\qquad-\eta_1\phi_{2zz}+\eta_1\eta_{1x}\phi_{1xz}+\eta_{1x}\phi_{2x}+\eta_{2x}\phi_{1x}|_{z=0}\\&\phi_{3z}|_{z=-h}=0\end{aligned}\right\} \quad \cdots\cdots(20)$$

A general solution for ϕ_r which satisfies both the Laplace equation, Eq.(4) and bottom boundary condition, Eq.(7), is assumed:

$$\phi_r=\sum_{n=0}^{\infty}\{A_n^{(r)}(\tau)\sin nkx+B_n^{(r)}(\tau)\cos nkx\}\cosh nk(h+z)+C^{(r)}x \quad \cdots\cdots(21)$$

in which the subscript r indicates the order of approximation, $A_n^{(r)}(\tau)$ is a periodic function of time with the condition of Eq.(10), $B_n^{(r)}(\tau)$ is composed of periodic and non-periodic functions, and $C^{(r)}$ represents the existence of a uniform flow component in the wave field.

(1) First order solution

From Eq.(18), the combined free water surface boundary condition of ϕ_1 after eliminating η is obtained as

$$\sigma_0^2\phi_{1\tau\tau}+g\phi_{1z}|_{z=0}=0 \quad \cdots\cdots(22)$$

Substituting Eq.(21) with $r=1$ into Eq.(22) yields the following ordinary differential equation with respect to $A_n^{(1)}$ and $B_n^{(1)}$,

$$\begin{aligned}\sum_{n=0}^{\infty}[&\{ngkA_n^{(1)}\sinh nkh+\sigma_0^2A_{n\tau\tau}^{(1)}\cosh nkh\}\sin nkx\\&+\{ngkB_n^{(1)}\sinh nkh+\sigma_0^2B_{n\tau\tau}^{(1)}\cosh nkh\}\cos nkx]=0 \quad \cdots\cdots(23)\end{aligned}$$

in which both $A_n^{(1)}$ and $B_n^{(1)}$ with $n\geqq1$ satisfy the condition of periodicity derived from Eq.(10),

$$\left.\begin{aligned}A_n^{(1)}(\tau+2\pi)=A_n^{(1)}(\tau)\\B_n^{(1)}(\tau+2\pi)=B_n^{(1)}(\tau)\end{aligned}\right\} \quad \cdots\cdots(24)$$

When n equals zero, $B_{0\tau\tau}{}^{(1)}=0$, and

$$B_0{}^{(1)}=\alpha_0\tau+\beta_0 \qquad (25)$$

When n equals 1,

$$\frac{A_{1\tau\tau}{}^{(1)}}{A_1{}^{(1)}}=\frac{B_{1\tau\tau}{}^{(1)}}{B_1{}^{(1)}}=-\frac{gk}{\sigma_0{}^2}\tanh kh=-\lambda_1{}^2 \qquad (26)$$

The next solutions can be derived as

$$\left.\begin{aligned} A_1{}^{(0)}&=P_1\cos\lambda_1\tau+Q_1\sin\lambda_1\tau \\ B_1{}^{(0)}&=R_1\cos\lambda_1\tau+S_1\sin\lambda_1\tau \end{aligned}\right\} \qquad (27)$$

in which P_1, Q_1, R_1 and S_1 are integral constants. Applying Eq.(24) into Eq.(26) yields the dispersion relation,

$$\lambda_1=1,\quad \sigma_0{}^2=gk\tanh kh \qquad (28)$$

For $n\geqq 2$, an equation identical to Eq.(26) is rendered,

$$\frac{A_{n\tau\tau}{}^{(1)}}{A_n{}^{(1)}}=\frac{B_{n\tau\tau}{}^{(1)}}{B_n{}^{(1)}}=-\frac{n\tanh nkh}{\tanh kh}=-\lambda_n{}^2\neq -1 \qquad (29)$$

No suitable solution to Eq.(29) can be found for satisfying Eq.(24), for $A_n{}^{(1)}$ and $B_n{}^{(1)}$ when $\lambda_n\neq 1$. However, $A_n{}^{(1)}$ and $B_n{}^{(1)}$ which satisfy Eq.(29) in the case of $n\geqq 2$ are obtained as

$$A_n{}^{(1)}=0,\quad B_n{}^{(1)}=0 \qquad (30)$$

To derive the expression of η_1 corresponding to ϕ_1 in this case, it may be assumed that

$$\eta_1=\sum a_n{}^{(1)}\cos n(kx-\tau) \qquad (31)$$

Inserting Eq.(31) into the second equation of Eq.(18) and determining the unknown constants yields

$$\left.\begin{aligned} &\alpha_0=0,\quad P_1=ga_1{}^{(1)}/\sigma_0\cosh kh \\ &Q_1=R_1=0,\quad a_n{}^{(1)}=0\ (n\geqq 2),\quad S_1=P_1 \end{aligned}\right\} \qquad (32)$$

Hence, the solutions of ϕ_1 and η_1 which satisfy the necessary boundary conditions can be easily written as

$$\phi_1=C^{(1)}x+\frac{\sigma_0 a_1{}^{(1)}}{k}\,\frac{\cosh k(h+z)}{\sinh kh}\sin(kx-\tau) \qquad (33)$$

$$\eta_1=a_1{}^{(1)}\cos(kx-\tau) \qquad (34)$$

(2) Second order solution

In the same manner as for Eq.(22) the combined free surface boundary condition for ϕ_2 is given from Eq.(19) by

$$\begin{aligned} \sigma_0{}^2\phi_{2\tau\tau}+g\phi_{2z}&=g\sigma_1\eta_{1\tau}-\sigma_0\sigma_1\phi_{\tau\tau}-\sigma_0{}^2\eta_{1\tau}\phi_{1\tau z} \\ &-\sigma_0{}^2\eta_1\phi_{1\tau\tau z}-\sigma_0\phi_{1x}\phi_{1x\tau}-\sigma_0\phi_{1z}\phi_{1z\tau}-g\eta_1\phi_{1zz}+g\eta_{1x}\phi_{1x} \end{aligned} \qquad (35)$$

Applying ϕ_2 from Eq.(21) at $r=2$ together with ϕ_1 and η_1 from Eqs.(33) and (34) into Eq.(35), an ordinary differential equation in terms of $A_n{}^{(2)}$ and $B_n{}^{(2)}$ is arrived at

$$\begin{aligned} &\sum_{n=0}^{\infty}\{ngkA_n{}^{(2)}\sinh nkh+\sigma_0{}^2A_{n\tau\tau}\cosh nkh\}\sin nkx \\ &\quad+\{ngkB_n{}^{(2)}\sinh nkh+\sigma_0{}^2B_{n\tau\tau}{}^{(2)}\cosh nkh\}\cos nkx \\ &\quad=2ga_1{}^{(1)}(\sigma_1-kc)(\sin kx\cos\tau-\cos kx\sin\tau) \\ &\quad+(3/2)(a_1{}^{(1)})^2\sigma_0{}^3(1-\coth^2 hk)(\sin 2kx\cos 2\tau-\cos 2kx\sin 2\tau) \end{aligned} \qquad (36)$$

For $n=0$, the relations of $B_{\tau\tau}^{(2)}=0$ and

$$B_0^{(2)}=\alpha_1\tau+\beta_1 \qquad (37)$$

are produced similar to Eq. (25) previously. When $n=1$, the use of Eq. (28) results in

$$\begin{Bmatrix} A_{1\tau\tau}^{(2)} \\ B_{1\tau\tau}^{(2)} \end{Bmatrix} + \begin{Bmatrix} A_1^{(2)} \\ B_1^{(2)} \end{Bmatrix} = \frac{ga_1^{(1)}}{\sigma_0^2}(\sigma_1-kC^{(1)})(1+\mathrm{sech}\, kh)\begin{Bmatrix} \cos\tau \\ -\sin\tau \end{Bmatrix} \qquad (38)$$

As explained earlier, ϕ_2 should not contain a complementary solution, hence, the secular terms included in the expressions of $A_1^{(2)}$ and $B_1^{(2)}$,

$$\begin{Bmatrix} A_1^{(2)} \\ B_1^{(2)} \end{Bmatrix} = \frac{ga_1^{(1)}}{\sigma_0^2}(\sigma_1-kC^{(1)})(1+\mathrm{sech}\, kh)\tau\begin{Bmatrix} \sin\tau \\ \cos\tau \end{Bmatrix} \qquad (39)$$

must vanish in order to satisfy the condition of permanent waves. Upon employing the following relationship,

$$\sigma_1=kC^{(1)} \qquad (40)$$

it is found that

$$A_1^{(2)}=B_1^{(2)}=0 \qquad (41)$$

Thus, the secular terms τ have successfully been eliminated. For $n=2$, the following equation is found,

$$\begin{Bmatrix} A_{2\tau\tau}^{(2)} \\ A_{2\tau\tau}^{(2)} \end{Bmatrix} + \frac{n\tanh nkh}{\tanh kh}\begin{Bmatrix} A_2^{(2)} \\ B_2^{(2)} \end{Bmatrix} = \gamma_2\begin{Bmatrix} \cos\tau \\ -\sin\tau \end{Bmatrix} \qquad (42)$$

in which

$$\gamma_2=(3/2)\sigma_0(a_1^{(1)})^2\,\mathrm{sech}\, 2kh(1-\coth^2 kh) \qquad (43)$$

Expressions of $A_2^{(2)}$ and $B_2^{(2)}$ which satisfy Eq. (10) are deduced from Eq. (42), rendering

$$\begin{Bmatrix} A_2^{(2)} \\ B_2^{(2)} \end{Bmatrix} = -\frac{3}{8}\sigma_0(a_1^{(0)})^2\frac{1}{\sinh^4 kh}\begin{Bmatrix} \cos 2\tau \\ -\sin 2\tau \end{Bmatrix} \qquad (44)$$

Another differential equation for $n\geqq 3$ is given as,

$$\begin{Bmatrix} A_{n\tau\tau}^{(2)} \\ B_{n\tau\tau}^{(2)} \end{Bmatrix} + \frac{n\tanh nkh}{\tanh kh}\begin{Bmatrix} A_n^{(2)} \\ B_n^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (45)$$

The solutions of Eq. (45) which satisfy Eq. (10) are

$$A_n^{(2)}=B_n^{(2)}=0 \qquad (46)$$

Taking η_2 which corresponds to ϕ_2 as

$$\eta_2=\sum_{n=1}^{\infty}a_n^{(2)}\cos n(kx-\tau) \qquad (47)$$

and substituting this into the second equation of Eq. (20) yields the following relationships,

$$\left.\begin{aligned} &\alpha_1=-(C_1^{(1)})^2/2\sigma_0-\sigma_0(a_1^{(1)})^2/4\sinh^2 kh, \quad a_1^{(2)}=0, \\ &a_2^{(2)}=\frac{k(a_1^{(1)})^2}{4}\frac{\coth kh}{\sinh^2 kh}(\cosh 2kh+2), \quad a_n^{(2)}=0 \ (n\geqq 3) \end{aligned}\right\} \qquad (48)$$

Therefore, the complete solutions for the second order approximation can now be written as

$$\begin{aligned} \phi_2=&-\left\{\frac{(C^{(1)})^2}{2\sigma_0}+\frac{\sigma_0(a_1^{(1)})^2}{4\sinh^2 kh}\right\}\tau+C^{(2)}x \\ &+\frac{3}{8}\sigma_0(a_1^{(1)})^2\frac{\cosh 2k(h+z)}{\sinh^4 kh}\sin 2(kx-\tau) \end{aligned} \qquad (49)$$

$$\eta_2=\frac{k(a_1^{(1)})^2}{4}\frac{\coth kh}{\sinh^4 kh}(\cosh 2kh+2)\cos 2(kx-\tau) \qquad (50)$$

(3) Third order solution

Upon eliminating η_3 from the second and the third equations in Eq.(20), the combined free water surface boundary equation is obtained. Inserting Eqs.(33), (40), (34), (49) and (50) for ϕ_1, η_1, σ_1, ϕ_2 and η_2 into its right hand side, a governing equation is given as

$$\sigma_0^2\phi_{3\tau\tau}+g\phi_{3z}=-\left\{\frac{1}{8}k\sigma_0^3(a_1^{(1)})^3\frac{\coth kh}{\sinh^4 kh}(8\cosh^4 kh\right.$$
$$\left.-8\cosh^2 kh+9)-2ga_1^{(1)}(\sigma_2-C^{(2)}k)\right\}\sin(kx-\tau)$$
$$-\frac{3}{2}k\sigma_0^2C^{(1)}(a_1^{(1)})^2\frac{\cosh 2kh}{\sinh^4 kh}\sin 2(kx-\tau)$$
$$+\frac{3}{8}k\sigma_0^3(a_1^{(1)})^3\frac{\coth kh}{\sinh^4 kh}(4\cosh^2 kh-13)\sin 3(kx-\tau) \quad\cdots\cdots(51)$$

Inserting ϕ_r of Eq.(21) with $r=3$ into the left hand side of Eq.(51) and processing it in the same way as for the second-order approximation, the results are $B_{0\tau\tau}^{(3)}=0$ and

$$B_0^{(3)}=\alpha_2\tau+\beta_2 \quad\cdots\cdots(52)$$

for $n=0$.

Now, for $n=1$, it is found that

$$\begin{Bmatrix}A_{1\tau\tau}^{(3)}\\B_{1\tau\tau}^{(3)}\end{Bmatrix}+\begin{Bmatrix}A_1^{(3)}\\B_1^{(3)}\end{Bmatrix}=-\left\{\frac{1}{8}k\sigma_0^3(a_1^{(1)})^3\frac{\coth kh}{\sinh^4 kh}(8\cosh^4 kh\right.$$
$$\left.-8\cosh^2 kh+9)-2ga_1^{(1)}(\sigma_2-C^{(2)}k)\right\}\begin{Bmatrix}\cos\tau\\-\sin\tau\end{Bmatrix} \quad\cdots\cdots(53)$$

Excluding the complementary solution to ϕ_3 and the secular terms, the following relations are determined,

$$\sigma_2=C^{(2)}k+\frac{1}{16}k^2\sigma_0(a_1^{(1)})^2\frac{1}{\sinh^4 kh}(8\cosh^4 kh-8\cosh^2 kh+9) \quad\cdots\cdots(54)$$

$$A_1^{(3)}=B_1^{(3)}=0 \quad\cdots\cdots(55)$$

For $n\geqq 2$, it is observed that

$$\begin{Bmatrix}A_2^{(3)}\\A_2^{(3)}\end{Bmatrix}=\frac{3}{8}kC^{(1)}(a_1^{(1)})^2\frac{2\cosh^2 kh-1}{\sinh^6 kh}\begin{Bmatrix}\cos 2\tau\\-\sin 2\tau\end{Bmatrix} \quad\cdots\cdots(56)$$

$$\begin{Bmatrix}A_3^{(3)}\\B_3^{(3)}\end{Bmatrix}=\frac{1}{64}k\sigma_0(a_1^{(1)})^3\frac{1}{\sinh^7 kh}(13-4\cosh^2 kh)\begin{Bmatrix}\cos 3\tau\\-\sin 3\tau\end{Bmatrix} \quad\cdots\cdots(57)$$

$$A_n^{(3)}=B_n^{(3)}=0 \quad (n\geqq 4) \quad\cdots\cdots(58)$$

Similarly, it is assumed that

$$\eta_3=\sum_{n=1}^{\infty}a_n^{(3)}\cos n(kx-\tau) \quad\cdots\cdots(59)$$

To determine all unknown constants, the substitution of Eq.(59) into Eq.(21) yields

$$\alpha_2=-C^{(2)}C^{(1)}/\sigma_0 \quad\cdots\cdots(60)$$

$$a_1^{(3)}=\frac{1}{16}k^2(a_1^{(1)})^3\frac{1}{\sinh^4 kh}(2\cosh^4 kh+10\cosh^2 kh-9), \quad\cdots\cdots(61)$$

$$a_2^{(3)}=\frac{3}{4\sigma_0}C^{(1)}k^2(a_1^{(1)})^2\frac{\cosh 2kh}{\sinh^6 kh}(3\cosh^2 kh-1)\tanh kh, \quad\cdots\cdots(62)$$

and

$$a_3^{(3)}=\frac{3}{64}k^2(a_1^{(1)})^3\frac{1}{\sinh^6 kh}(8\cosh^6 kh+1), \quad a_n^{(3)}=0 \quad (n\geqq 4) \quad\cdots\cdots(63)$$

Finally, ϕ_3 and η_3 can be expressed as

$$\phi_3=-\frac{C^{(1)}C^{(2)}}{\sigma_0}\tau+C^{(3)}x+\frac{\sigma_0 k}{64}(a_1^{(0)})^3\frac{13-4\cosh^2 kh}{\sinh^7 kh}\cosh 3k(h+z)\sin 3(kx-\tau) \quad \cdots\cdots(64)$$

$$\eta_3=a_1^{(3)}\cos(kx-\tau)+a_2^{(3)}\cos 2(kx-\tau)+a_3^{(3)}\cos 3(kx-\tau) \quad \cdots\cdots(65)$$

(4) Determining the unknown constants

As mentioned above, the third order approximation to Stokes waves was derived on the basis of some clear assumptions. So far, however, the unknown constants of $C^{(1)}$, $C^{(2)}$, $C^{(3)}$ and $a_1^{(1)}$ await to be determined. First of all, $a_1^{(1)}$ can be found directly from the equation describing the maximum wave height H of the Stokes waves to the third order,

$$\begin{aligned}H&=\eta_{max}-\eta_{min}\\&=2\varepsilon a_1^{(1)}+\frac{1}{32}k^2\varepsilon^3(a_1^{(1)})^3\frac{1}{\sinh^6 kh}(32\cosh^6 kh+32\cosh^4 kh-76\cosh^2 kh+39)\end{aligned} \quad \cdots\cdots(66)$$

Only the unique real solution of Eq.(63) will be used for $a_1^{(1)}$. Upon applying the first definition of wave celerity (see Skjelbreia and Goda et al.), $C^{(r)}$ can be calculated, having

$$C^{(r)}=0 \quad (r=1, 2, 3) \quad \cdots\cdots(67)$$

If the second definition of wave celerity is used (see Tsuchiya & Yamaguchi[14]), values of $C^{(r)}$, showing different forms when compared with Eq.(64), are given as

$$\left.\begin{aligned}&C^{(2)}=C^{(3)}=0,\\&C^{(2)}=-\frac{1}{2h}\sigma_0\varepsilon^2(a_1^{(1)})^2\coth kh<0\end{aligned}\right\} \quad \cdots\cdots(68)$$

However, there is no reason why $C^{(r)}$ must be zero. And, it is not clear how the uniform current is opposite the direction of wave propagation in the wave field where geometrical restriction is imposed at both ends of a wave flume. An explanation is herein presented. The governing equations consisting of ϕ and η both closely related to each other would propose a complete agreement in phase relationship between that of ϕ and η through the following assumption,

$$\left.\begin{aligned}&\eta(x, t)=\eta(x-ct),\\&\phi(x, z, t)=\phi(x-ct, z)\end{aligned}\right\} \quad \cdots\cdots(69)$$

The above assumption is inevitable, for determining the unknown constant $C^{(r)}$ while employing no additional condition physically. Laitone and Whitham have deduced an approximation by using Eq.(69). And two theories[16,17] of finite amplitude waves in shallow water, cnoidal wave theory and Quasi-Stokes wave theory, have already been developed by the authors based on this assumption. Deciding $C^{(r)}$ through Eq.(69) yields

$$\left.\begin{aligned}&C^{(1)}=C^{(3)}=0,\\&C^{(2)}=\frac{1}{4}k\sigma_0\varepsilon^2(a_1^{(1)})^2\frac{1}{\sinh^2 kh}\end{aligned}\right\} \quad \cdots\cdots(70)$$

This suggests that a time-averaged horizontal water particle velocity accompanying the wave motion exists in the direction of wave propagation, thus, a mass transport velocity can be defined also in the Eulerian description. As a result, it is evident that mass transport phenomena depend upon the nonlinearity of velocity field within the waves itself, in addition to the so-called Stokes drift, usually a Lagrangian formulation.

Since all the unknown constants have been decided, expressions for both ϕ and η to third order approximation can finally be written respectively as

$$\frac{\phi}{ac_0}=\frac{\cosh k(h+z)}{\sinh kh}\sin(kx-\tau)+\frac{3ka}{8}\left\{\frac{\cosh 2k(h+z)}{\sinh^4 kh}\sin 2(kx-\tau)+\frac{2}{3\sinh^2 kh}(kx-\tau)\right\}+\frac{(ka)^2(13-4\cosh^2 kh)}{64\sinh^7 kh}\cosh 3k(h+x)\sin 3(kx-\tau) \quad \text{(71)}$$

$$\frac{\eta}{a}=\left\{1+\frac{(ka)^2}{16\sinh^4 kh}(2\cosh^4 kh+10\cosh^2 kh-9)\right\}\cos(kx-\tau)+\frac{ka\coth kh}{4\sinh^2 kh}(\cosh 2kh+2)\cos 2(kx-\tau)+\frac{3(ka)^2}{64\sinh^6 kh}(8\cosh^6 kh+1)\cos 3(kx-\tau) \quad \text{(72)}$$

in which a is the wave amplitude to first order being related to $a_1^{(1)}$ by

$$a=\varepsilon a_1^{(1)} \quad \text{(73)}$$

The wave celerity to first order, c_0 is expressed as

$$c_0^2=(g/k)\tanh kh \quad \text{(74)}$$

5. Mathematical Expressions of Stokes Waves

Having derived the solutions for Stokes waves to each order of approximation, mathematical formulations for some pertinent wave variables can now be written to a third-order. These include the wave profile, wave celerity, water particle velocities, wave pressure, mass transport velocities in the Eulerian and Lagrangian descriptions respectively, and momentum flux. Appropriate expressions of some of these wave variables are also derived on the basis of the first and second definition of wave celerity, and compared with the mathematical formulations.

The wave profile η to third-order is expressed as

$$\frac{\eta}{a}=\left\{1+\frac{1}{16}(ka)^2\frac{1}{\sinh^4 kh}(2\cosh^4 kh+10\cosh^2 kh-9)\right\}\cos k(x-ct)+\frac{1}{4}ka\frac{\coth kh}{\sinh^2 kh}(\cosh 2kh+2)\cos 2k(x-ct)+\frac{3}{64}(ka)^2\frac{1}{\sinh^6 kh}(8\cosh^4 kh+1)\cos 3k(x-ct) \quad \text{(75)}$$

in which the wave celerity is given as

$$c=\left(\frac{g}{k}\tanh kh\right)^{1/2}\left\{1+\frac{1}{16}(ka)^2\frac{1}{\sinh^4 kh}(8\cosh^4 kh-4\cosh^2 kh+5)\right. \quad \text{(76)}$$

Eq.(75) is seen to be different from the solution of the usual Stokes wave theories. Moreover, the relationships between the wave celerity from Eq.(75) and that by the first and second definition are presented respectively as

$$c_{\mathrm{I}}=c-\frac{c_0(ka)^2}{4\sinh^2 kh} \qquad \text{(77)}$$

$$c_{\mathrm{II}}=c-\frac{c_0(ka)^2}{4\sinh^2 kh}\left(1+\frac{\sinh 2kh}{kh}\right) \qquad \text{(78)}$$

in which the subscripts I and II stand for the first and the second definition respectively.

The water particle velocities in the horizontal and vertical directions can be derived respectively as

$$\begin{aligned}\frac{u}{c_0}=&ka\frac{\cosh k(h+z)}{\sinh kh}\cos(kx-ct)\\&+\frac{3}{4}(ka)^2\left\{\frac{\cosh 2k(h+x)}{\sinh^4 kh}\cos 2(kx-ct)+\frac{1}{3\sinh^2 kh}\right\}\\&+\frac{3}{64}(ka)^3\frac{13-4\cosh^2 kh}{\sinh^7 kh}\cosh 3k(h+z)\cos 3(kx-ct)\end{aligned} \qquad \text{(79)}$$

$$\begin{aligned}\frac{w}{c_0}=&ka\frac{\sinh k(h+z)}{\sinh kh}\sin(kx-ct)\\&+\frac{3}{4}(ka)^2\frac{\sinh 2k(h+z)}{\sinh^4 kh}\sin 2(kx-ct)\\&+\frac{3}{64}(ka)^3\frac{13-4\cosh^2 kh}{\sinh^7 kh}\sin 3(kx-ct)\end{aligned} \qquad \text{(80)}$$

With regard to the horizontal water particle velocity u, a difference exists between the above expression and the one by the Stokes definitions of wave celerity. The expressions of the horizontal water particle velocities derived from the first and second definition of wave celerity can also be related to u/c_0 from Eq.(79) as

$$\frac{u_{\mathrm{I}}}{c_0}=\frac{u}{c_0}-\frac{(ka)^2}{4\sinh^2 kh} \qquad \text{(81)}$$

$$\frac{u_{\mathrm{II}}}{c_0}=\frac{u}{c_0}-\frac{(ka)^2}{4\sinh^2 kh}\left(1+\frac{\sinh 2kh}{kh}\right) \qquad \text{(82)}$$

The pressure in the wave motion is obtained as

$$\begin{aligned}\frac{p}{\rho gh}=&-\frac{z}{h}+\frac{\tanh kh}{kh}\Bigg[\frac{(ka)^2}{4\sinh^2 kh}\left\{\frac{\sigma}{\sigma_0}-\cosh 2k(h+z)\right\}\\&+\left\{\frac{\sigma}{\sigma_0}ka\frac{\cosh k(h+z)}{\sinh kh}-\frac{1}{16}(ka)^3\frac{1}{\sinh^5 kh}(2\sinh^2 kh\cosh k(h+z)\right.\\&\left.+3\cosh k(h+z))\right\}\cos(kx-\sigma t)-\frac{(ka)^2}{4\sinh^4 kh}\left\{\sinh^2 kh\right.\\&\left.-3\left(\frac{\sigma}{\sigma_0}\right)\sinh^2 kh\cosh 2k(h+z)\right\}\cos 2(kx-\sigma t)\\&-\frac{3}{64}(ka)^3\frac{1}{\sinh^7 kh}\left\{4\sinh^2 kh\cosh k(h+z)\right.\\&\left.-\frac{\sigma}{\sigma_0}(13-4\cosh^2 kh)\cosh 3k(h+z)\right\}\cos 3(kx-\sigma t)\Bigg]\end{aligned} \qquad \text{(83)}$$

The mass transport velocity $\bar{u}$ in the Eulerian description, $\bar{U}$ in the Lagrangian description, and momentum flux M have taken the following forms respectively as

$$\frac{\bar{u}}{c_0}=\frac{1}{4}(ka)^2\frac{1}{\sinh^2 kh} \qquad \text{(84)}$$

$$\frac{\bar{U}}{c_0}=\frac{1}{4}(ka)^2\frac{1}{\sinh^2 kh}+\frac{1}{4}(ka)^2\frac{\sigma_0}{\sigma}\left\{\frac{2\cosh 2k(h+z)}{\sinh^2 kh}-\frac{1}{\sinh^3 kh}\right\} \quad \cdots\cdots(85)$$

$$\frac{M}{\rho h\sqrt{gh}}=(ka)^2\left(\frac{\tanh kh}{kh}\right)^{1/2}\frac{1}{4\sinh^2 kh}\left(\frac{2}{kh}\sinh kh+1\right) \quad \cdots\cdots(86)$$

The above formulations are not identical to that by the Stokes definitions of wave celerity. The mass transport velocities and the momentum flux derived from the first definition of wave celerity are expressed respectively as

$$\bar{u}_{\mathrm{I}}/c_0=0 \quad \cdots\cdots(87)$$

$$\frac{\bar{U}_{\mathrm{I}}}{c_0}=\frac{(ka)^2}{4\sinh^2 kh}+\frac{c_0(ka)^2}{4c_{\mathrm{I}}\sinh^3 kh}\{2\sinh kh\cosh 2k(h+z)-1\} \quad \cdots\cdots(88)$$

$$\frac{M_{\mathrm{I}}}{\rho h\sqrt{gh}}=\frac{(ka)^2}{4\sinh^2 kh}\left(\frac{\tanh kh}{kh}\right)^{1/2}\left(\frac{2\sinh kh}{kh}+1\right) \quad \cdots\cdots(89)$$

And, those deduced from the second definition of wave celerity are given respectively as

$$\frac{\bar{u}_{\mathrm{II}}}{c_0}=-\frac{(ka)^2}{2kh}\coth kh \quad \cdots\cdots(90)$$

$$\frac{\bar{U}_{\mathrm{II}}}{c_0}=-\frac{(ka)^2}{2kh}\coth kh+\frac{c_0(ka)^2}{4c_{\mathrm{II}}\sinh^3 kh}\{2\sinh kh\cosh 2k(h+z)-1\} \quad \cdots(91)$$

$$M_{\mathrm{II}}=0 \quad \cdots\cdots(92)$$

Finally, the kinetic energy E_k and energy flux W which are related to the definition of wave celerity, are expressed respectively by $O(ka)^4$ as

$$E_k=\frac{\rho g a^2}{4}\left\{1+(ka)^2\left(\frac{\cosh 2kh+2}{8\sinh^4 kh}+\frac{9\cosh 2kh}{16\sinh^6 kh}+\frac{3\cosh 3kh}{4\cosh kh\sinh^4 kh}+\frac{2\sinh^2 kh+kh}{8\sinh^4 kh}\right)\right\} \quad \cdots\cdots(93)$$

$$W=\frac{1}{2}\rho g a^2 c\left[\frac{1}{2}\left(1+\frac{2kh}{\sinh 2kh}\right)+(ka)^2\left\{\frac{\cosh^2 2kh+3\cosh 2kh+2}{16\sinh^4 kh}+\frac{3}{4}\sinh^2 kh+\frac{9(2kh+\sinh 2kh)}{64\sinh^7 kh\cosh kh}+\frac{3(\cosh kh+\cosh 3kh)}{8\sinh^4 kh\cosh kh}+\frac{kh\tanh kh+\sinh^2 kh}{2\sinh^4 kh}\right\}\right] \quad \cdots\cdots(94)$$

On the other hand, the formulations for E_k and W are derived from the first (I) and second (II) definitions of wave celerity respectively as

$$(E_k)_{\mathrm{I}}=E_k-\frac{\rho g a^2}{4}(ka)^2\left\{\frac{2\sinh^2 kh+kh}{8\sinh^4 kh}\right\} \quad \cdots\cdots(95)$$

$$(E_k)_{\mathrm{II}}=E_k-\frac{\rho g a^2}{4}(ka)^2\left\{\frac{2\sinh^2 kh+kh}{8\sinh^4 kh}+\frac{(ka)^2\coth kh}{2kh}\right\} \quad \cdots\cdots(96)$$

$$(W)_{\mathrm{I}}=W-\frac{\rho g a^2 c}{2}(ka)^2\left\{\frac{kh\tanh kh+\sinh^2 kh}{2\sinh^4 kh}\right\} \quad \cdots\cdots(97)$$

$$(W)_{\mathrm{II}}=W-\frac{\rho g a^2 c}{2}(ka)^2\left\{\frac{2kh\tanh kh+\sinh^2 kh}{4\sinh^4 kh}-\frac{\coth kh}{2kh}\right\} \quad \cdots\cdots(98)$$

It is also observable that different expressions can be seen upon comparing Eqs. (95), (96), (97) and (98) with Eqs. (93) and (94) for E_k and W.

6. Conclusions

As explained already, numerous wave theories have ingeniously been developed for the so-called "Stokes Waves" to a third-order approximation or higher, by a perturbation method and even more systematically with a computer. However, the theories available have revealed a variety of final expressions to all pertinent wave variables.

To render a unified solution, it is necessary to examine rigorously all conditions (and or assumptions) from which the usual Stokes wave theories are based. These include the Stokes definitions of wave celerity and the Bernoulli constant in an equation for free water surface boundary conditions. Some questionable aspects in the current theories are discussed. Diverse formulations would have resulted from using the first and the second definition of wave celerity respectively.

A new approach is herein developed to re-derive a third-order approximation to Stokes waves, using no relationship from the Stokes definitions of wave celerity. A complete agreement in phase is found for between the velocity potential and water surface displacement. Mathematical formulations for some pertinent wave variables are derived to the third-order. Differences in these expressions can be observed when compared with the usual Stokes wave theories. A list of expressions in detail and brief discussion are included. It is of particular interest that mass transport velocity exists in the Eulerian description, in addition to the usual Stokes drift in the Lagrangian description. The magnitude of the former (the mass transport in the Eulerian description) is comparable with that of the latter. Of all the final formulations presented for wave variables, the wave celerity in the new wave theory can also be related to that under the first and the second definition respectively.

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