Kyoto University
Economic Review

MEMOIRS OF
THE DEPARTMENT OF ECONOMICS
IN
THE IMPERIAL UNIVERSITY OF KYOTO

VOLUME XII
1937

PUBLISHED BY THE DEPARTMENT
OF ECONOMICS IN
THE IMPERIAL UNIVERSITY OF KYOTO

KYOTO UNIVERSITY

ECONOMIC REVIEW

MEMOIRS OF THE DEPARTMENT OF
ECONOMICS IN THE IMPERIAL
UNIVERSITY OF KYOTO

VOLUME XII (*July 1937*) NUMBER 1

DETERMINATION OF THE RATE OF INTEREST

By YASUMA TAKATA

1. DETERMINATION OF THE RATE OF INTEREST—ASSUMPTION OF A FIXED PERIOD OF PRODUCTION

As what height is the rate of interest fixed? The answer to this question is that the interaction of the various conditions which act upon it determines its height. Needless to say, it is influenced by all circumstances connected with production and exchange. Particularly closely is it concerned with the productivity of capital in society and the total quantity of capital. The rate of interest cannot, indeed, be fixed independently of these factors.

Let us assume, to begin with, that the quantity of capital is given and that no accumlation (or consumption) of capital takes place, so that interest is entirely consumed. Let it further be assumed that the period of production for all goods is represented by a constant, say, one year. Let us consider the conditions which operate to determine the rate of interest under such circumstances.

Now, let us suppose that there are m number of kinds of consumptive goods, $x, y, \ldots\ldots o \ldots\ldots$, and n number of kinds of productive goods, $a, b, \ldots\ldots$. Let the subjects be represented by from $1, 2, \ldots\ldots$ to θ. As φ_{1x} is taken to show the marginal degree of utility in regard to the first subject 1, $\varphi_{1x}(x_1)$ represents the marginal degree of the utility of x, when the quantity possessed by the subject 1 is x_1. The prices of the various goods, $x, y, \ldots\ldots a, b, \ldots\ldots$ are represented by $p_x, p_y, \ldots\ldots p_a, p_b, \ldots\ldots$. Further, let the quantity possessed by each subject (1, for instance) at the beginning be x_{10}, $y_{10}, \ldots\ldots$ and the quantity possessed at the time of equilibrium be $x_1, y_1, \ldots\ldots$. Let us also suppose that $a_x, a_y, \ldots\ldots b_x, b_y, \ldots\ldots$ represent the coefficients of production, that is, the average quantity of $a, b \ldots\ldots$, required for the production of one unit of x and the average quantity of $a, b, \ldots\ldots$, required for the production of one unit of y. The quantity of capital possessed by each at the beginning is $k_{10}, k_{20}, \ldots\ldots$ while i represents the rate of interest. Of the goods as products, let o, for instance, embody the price unit (numéraire).

(A) For a certain subject, say, 1, the weighted marginal degrees of utility of all kinds of goods, viz., the quotients obtained by dividing the marginal degree of each goods by its price, are equal to each other.

(B) The quantity which each subject receives or delivers, that is, the money quantity which is receives and the money quantity which it pays are equal. Let us here add another hypothesis, namely, that, in a static state, neither fresh accumulation nor fresh consumption of capital takes place. In other words, interest only is consumed, principal being preserved intact.

(C) The quantities of all productive goods necessary for the production of one unit of one product, that is, the coefficients of production of all these productive goods concerned with the product multiplied by their respective prices embody the cost of production per unit. The aggregate of the principal and interest of this cost of production is equal to the price of the product.

Number of equations

$$(A)\begin{cases} \frac{1}{p_x}\varphi_{1x}(x_1)=\frac{1}{p_y}\varphi_{1y}(y_1)=\ldots\ldots \\ \qquad =\frac{1}{p_a}\varphi_{1a}(a_1)=\frac{1}{p_b}\varphi_{1b}(b_1)=\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \qquad \theta\,(m+n-1)$$

$$(B)\begin{cases} p_x(x_{10}-x_1)+p_y(y_{10}-y_1)+\ldots\ldots \\ \qquad +p_a(a_{10}-a_1)+\ldots\ldots+i\,k_{10}=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \qquad \theta$$

$$(C)\begin{cases} (p_a a_x+p_b b_x+\ldots\ldots)(1+i)=p_x \\ p_a a_y+b_b b_y+\ldots\ldots)(1+i)=p_y \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \qquad m$$

Next, the circumstances connected with production must be considered. In doing so, let us make use of the following signs with the meanings attached to them. We shall take x, y,...... to signify the respective quantity of product produced by an enterprise. As, in a static state, production will be on the same scale in all enterprises in consequence of competition, the quantity of the product in all enterprises which produce x will be equal. The same is true of y. Let $\bar{a}_x$, $\bar{b}_x$,...... represent the quantity of a, the quantity of b,...... respectively, required in an enterprise for producing x. Then, the various conditions such as are mentioned below must rule. These may be called the conditions of production, if those already mentioned can be termed the conditions of price.

(D) The quantity of the product stands in a certain definite relation to the quantity of each of the productive goods (production function).

(E) The price of the product is equal to marginal productivity less the rate of interest (Taussig's law). This point deserves some further study, but this will be deferred until we deal with the period of production regarded as variable.

(F) The coefficients of production in this case are no other than the average quantity of productive goods necessary for the production of one unit of the product (definition of the coefficients of production as the respective average quantities of productive goods necessary for the production of one unit of the product).

	Number of equations
(D) $\begin{cases} x=F_1(\bar{a}_x, \bar{b}_x, \bar{c}_x, \ldots\ldots) \\ y=F_2(\bar{a}_y, \bar{b}_y, \bar{c}_y, \ldots\ldots) \\ \ldots\ldots\ldots\ldots \end{cases}$	m
(E) $\begin{cases} \dfrac{p_a}{p_x}=\dfrac{\partial x}{\partial \bar{a}_x}\dfrac{1}{1+i}\,;\ \dfrac{p_b}{p_x}=\dfrac{\partial x}{\partial \bar{b}_x}\dfrac{1}{1+i}\,;\ldots\ldots \\ \ldots\ldots\ldots\ldots \end{cases}$	mn
(F) $\begin{cases} a_x=\dfrac{\bar{a}_x}{x}\,;\ b_x=\dfrac{\bar{b}_x}{x}\,;\ \ldots\ldots \\ \ldots\ldots\ldots\ldots \end{cases}$	mn

Next, let us consider the conditions relative to the number of enterprises. The rate of interest is related to the total quantity of capital, but the relation between production function or the quantity of the product of one enterprise and the total quantity of capital cannot be conceived without regard to the number of enterprises. As already mentioned, all enterprises for the production of goods of the the same kind are on the same scale.

(G) The quotient obtained by dividing the total quantity of a product of some kind by the product of one enterprise is equal to the number of enterprises (equality in the scale of production).

(H) The total quantity of each kind of the productive goods used by all enterprises for the production of all goods is equal to the total quantity of their supply (equilibrium of demand and supply of productive goods).

(I) The respective total quantity of the supply of each kind of the productive goods is equal to one half of the total absolute quantity received and delivered by all subjects.

(J) The respective total quantity of each product is equal to the balance of the respective total quantity possessed by all subjects in equilibrium and that possessed by them at the beginning.

	Number of equations
(G) $\begin{cases} \frac{X}{x}=\alpha_x, \quad \frac{Y}{y}=\alpha_y\,; \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$	m
(H) $\begin{cases} \alpha_x\bar{a}_x+\alpha_y\bar{a}_y+\ldots\ldots=A \\ \alpha_x\bar{b}_x+\alpha_y\bar{b}_y+\ldots\ldots=B \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$	n
(I) $\begin{cases} A=\frac{1}{2}\{\lvert a_{10}-a_1\rvert+\lvert a_{20}-a_2\rvert+\ldots\ldots\} \\ B=\frac{1}{2}\{\lvert b_{10}-b_1\rvert+\lvert b_{20}-b_2\rvert+\ldots\ldots\} \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$	n
(J) $\begin{cases} X=\sum x_1-\sum x_{10} \\ Y=\sum y_1-\sum y_{10} \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$	m

Next, two conditions are given regarding the quantity of capital.

(K) The sum total of the products obtained by multiplying the sum of the prices of the productive goods used in each enterprise by the number of enterprises is equal to the total quantity of capital, K (equality of demand and supply of capital).

(L) The total quantity of capital is equal to the aggregate of the capital possessed by all subjects at the beginning, namely, the sum total of the capital of individuals.

	Number of equations
(K) $\alpha_x(p_a\bar{a}_x+p_b\bar{b}_x+\ldots\ldots)$ $+\alpha_y(p_a\bar{a}_y+p_b\bar{b}_y+\ldots\ldots)+=K$	1
(L) $K=k_{10}+k_{20}+k_{30}+\ldots\ldots$	1

Let us now compare the number of equations with the number of unknown quantities. The total of equations is $\theta(m+n)+2mn+4m+2n+2$. Of this total, one of the equations (B) and (C) is cast aside as dependent by taking (F), (G), (H), (I), (J), (K), and (L) into consideration (this point may be expressed differently, but I shall refrain from any further discussion of the point here). The number of unknown quantities is as follows:—

$\alpha_x, \alpha_y,$	m	$\bar{a}_x, \bar{a}_y,$	mn
$x_1, y_1,$	$m\theta$	$a_x, a_y,$	mn
$a_1, b_1,$	$n\theta$	$X, Y,$	m
$p_x, p_y,$	m	$A, B,$	n
$p_a, p_b,$	n	K	1
$x, y,$	m	i	1

sum total......$\theta(m+n)+2mn+4m+2n+1$

The number of unknown quantities is equal to the num. ber of equations. The height of the rate of interest becomes unequivocally determined as other economic quantities. It must here be noted that the quantity of capital possessed by all individuals, and accordingly the total quantity of capital in society, is also given. As to the fact that the supply of productive goods is defermined by marginal utility in some sense or other, I had occasion to explain it at length in my study of power as cost.

2. DETERMINATION OF THE RATE OF INTEREST—CASES WHERE THE PERIOD OF PRODUCTION IS VARIABLE

I shall next study cases where the period of production is variable, not constant. Let it be assumed that the rate of interest (simple interest), i, is paid at regular intervals of, say, t_0, and let the value of t_0 be 1, so that all calculations may be simplified. If t_x, the period of producing x, t_y, the period of producing y, etc., are calculated on the basis of this unit, some of the equations already given will have to be re-written as follows. The introduction of these new conditions will not affect the equations (A), (B), (F), (I), (J), (K) and (L) in the least. Only the equation, (C), (D), (E), (G), (H), change their respective forms as follows according to the new situations, to be added.

Number of equations added

$$(C')\begin{cases}(p_x a_x+p_b b_x+\ldots\ldots)(1+it_x)=p_x\\(p_y a_y+p_b b_y+\ldots\ldots)(1+it_y)=p_y\\\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{cases} \qquad 0$$

$$
\text{(D}') \begin{cases} x = F(\bar{a}_x, \bar{b}_x, \ldots\ldots, t_x) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \qquad 0
$$

$$
\text{(E}') \begin{cases} \dfrac{p_a}{p_x} = \dfrac{\partial x}{\partial \bar{a}_x} \dfrac{1}{1+it_x}; \ldots\ldots, \dfrac{i}{p_x} = \dfrac{\partial x}{\partial t_x} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \qquad m
$$

$$
\text{(G}') \left\{ \frac{X}{x} t_y = \alpha_x, \frac{Y}{y} t_y = \alpha_y, \ldots\ldots\ldots\ldots\ldots\ldots \right. \qquad 0
$$

$$
\text{(H}') \begin{cases} \alpha_x \bar{a}_x \dfrac{1}{t_x} + \alpha_y \bar{a}_y \dfrac{1}{t_y} + \ldots\ldots\ldots\ldots = A \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \qquad 0
$$

As the result of this re-writing of equations, the number of equations increases by m, while the number of unknown quantities also shows an increase of m in respect of the period of producing each product. Thus, it will be seen that the re-writing of equations does not involve any change in the relation between unknown quantities and equations in point of number. The rate of interest is determined unequivocally. It has been assumed that the period in which all productive goods used in the production of each product attain maturity is idential (as, for instance, it has been assumed that the peried of maturity („Reifezeit") for a and b used in the production of x is alike t_x). Needless to say, this assumption puts actuality in too simple a form.

I have so far carried on my discusion on the assumption that there is neither consumption nor saving of capital in the possession of all subjects, but the fact that the total quantity of capital in society neither increases nor diminishes does not necessarily mean that there occurs neither consumption nor saving in the capital of all subjects. If we assume that capital is either consumed or saved, the conditions given must be somewhat altered. The assumption that the quantity of capital possessed by individuals at the beginning and that possessed after the establishment of an equilibrium are equal (this does not appear in the system of equations above given; its outcome is only shown in the equation L) must be discarded, and the quantity possessed at the beginning must be given as a known quantity and the quantity possessed after the establishment of an equilibrium dealt with as an unknown quantity. As the circumstances necessary for determing this unknown quantity, the equations A and B will take the following forms:—

	Equations	Number of equations	Additional number of equations
A′	$\frac{1}{p_x}\varphi_{1x}(x_1)=\ldots\ldots=\frac{1}{p_a}\varphi_{1a}(a_1)=\ldots\ldots$ $=\varphi_{1k}(c_1+k_1)$ ……………… …………………………………………	$\theta\,(m+n)$	θ
B′	$p_x(x_{10}-x_1)+\ldots\ldots+p_a(a_{10}-a_1)+\ldots\ldots$ $+i\,k_{10}+(k_{10}-k_1)=0$ …… …………………………………………	θ	0

In the above equations, c_1, c_2,…… represent the anticipated income of each subject in the next period; k_{10}, k_{20},…… the quantity of capital possessed by each subject at the beginning; and k_1, k_2,…… that possessed by each subject when an equailibrium is established. It is assumed that transactions in all goods—productive goods and products—take place at the end of the period. It is therefore assumed that the interest income of each subject is the product of the rate of interest multiplied by k_{10} representing the quantity of capital possessed at the beginning. Further, φ_k represents the function of the present utility of the future money income, and it is determined by many conditions, such as the future commodity prices, state of desire and interest rate and the rate of underestimation of the future satisfaction of desire. In some cases, k_1, k_2,…… denote positive quantities, but in other cases they denote negative quantities. Where they are positive, capital is loaned, while when ther are negative, it is borrowed. What household economy borrows is intended for consumption; it embodies the demand for capital. The sum total of k_1, k_2,…… alone is, therefore, supplied for production purposes. Next, if the size of $k_{10}-k_1$, $k_{20}-k_2$,…… shows a positive quantity, it means that capital is consumed to that extent in the period concerned. On the other hand, if it shows a negative quantity, capital is saved to that extent. Let us now study how the consumptive demand and the productive demand for capital act on each other in this connection.

Needless to say, the size of k_1, k_2,...... is influenced by the rate of interest, but the interest rate that affects it is the interest rate in the next period, not that in the present period. What is influenced by the interest rate in the present period must be k_{10}, k_{20},......, which are determined at the end of the previous period. In anticipation of the interest rate in the present period, the consumption and accumulation of capital and accordingly the consumptive demand for capital are determined at the end of the previous period. Of k_{10}, k_{20},......, all those which bear the minus sign represent loans for consumption. Those which bear the plus sign represent the quantity available for meeting the consumptive or productive demand. If the aggregate of the latter is K_0 and the aggregate of the former K', K or the quantity available for meeting the productive demand may be shown in the following equation:—

$$|K_0|-|K'|=k_{10}+k_{20}+\ldots\ldots\ldots\ldots\ldots=K.$$

What are simply expressed as φ_{1k}, φ_{2k},...... in A′ are, as a matter of fact, of a fairly complex structure. In effect, φ_{1k},...... contain the rate of interest as an independent variable. Therefore, although the condition given in the present instance, viz., that neither increase nor decrease takes place in productive capital in society, may appear to make the number of the equations more than the number of the unknown quantities, it must be noted that the rate of interest in the next period is, in reality, included as a new unknown quantity. In other words, because the interest rate anticipated for the next period is of a certain height, the quantity of productive capital in the present period can remain unchanged.

3. ON THE SCOPE OF PRODUCTION

Let us now consider the scope of production and the size of capital in an enterprise. Let it be assumed, to begin with, that the period of production is fixed. Then, the size of all productive goods used by each enterprise during the period, that is, the size of $\bar{a}_x$, $\bar{b}_x$,...... in regard to the production of x in the aforementioned case, will be determined at a point where the respective marginal productivity discounted, is equal to the respective price of each of the productive goods. The addition of productive goods ceases at this point, so to speak. To study the point further, it will be seen that this discounted marginal productivity must also be equal to the discounted average productivity. The scope of production will be expanded as far as this applies to all productive goods which are concerned with the production of x.

The above-mentioned example assumes a most simple case. It presuppose a case where all productive goods, $\bar{a}_x$, $\bar{b}_x$,......, are in use from the beginning of the period of production. Even where the period of production or the absolute length of production from its beginning to its end is fixed, if these goods are gradually applied, one on the top of another, the manner of their addition will be diverse. The so-called function of the employment of services will take a variety of forms. Then, the condition relative to productivity, already mentioned, will apply to any portion of productive goods also, no matter at what point of time during the period of production they may be invested. It must be mentioned, however, that this assumes the continuity of the productive function, that is, the unfailing occurrence of changes in products in response to any small changes which take place either in the quantity of productive goods or in the period in which these goods are employed.

Now, let us proceed with our study by assuming the most simple case of the investment of productive goods, in which, as already mentioned, all productive goods for use are invested at the beginning of the period of production. Let us suppose that the period of production. which has so far been assumed to be fixed, is variable. There are two points deserving consideration in this connection.

First, it may be asked how far the peried of production can be extended. The answer is that it is extended to a point where productivity ascribable to the marginal unit of prolongation in time of production is equal to the rate of interest for this marginal period. To express it in different words, it is extended to a point where the marginal productivity, as viewed from the angle of the duration of the service of capital, becomes equal to the rate of interest. Needless to say, the fundamental principle that the size of marginal quantity is equal to the average quantity ought to operate in this case also. The average productivity in the period of production, that is, the ratio of gross productivity minus the prices of all productive goods combined to the service of capital (the product of the capital quantity multiplied by the period of production) must be equal to this marginal productivity. According to this method of explanation, if the rate of interest is given, the period of production is extended until the productivity of the marginal period, or, to be more exact, the marginal productivity of the service of capital in its dimension of time becames equal to the rate of interest. This also applies to the case where the size of the service of capital is considered in its dimension of capital quantity. If asked how far the service of capital will be increased in quantity, the right answer will be that it will be increased until gross productivity minus the prices of all productive goods used, that is, the marginal productivity of capital in its relation to quantity becomes equal to the rate of interest. In short, the service of capital has two directions or dimensions. One dimension refers to quantity and the other dimension to the duration of time. In either dimension, the service of capital will be extended until its marginal productivity becomes equal to the rate of interest. In this presentation of the case, nothing is set forth beyond that, given the rate of interest, productivity will conform to it. But, as is always the case with general equilibrium, if all enterprises, not individual enterprises only, are considered, it will easily be inferred that the rate of interest is determined only in accord with the marginal productivity of capital.

Secondly, the process of production become synchronised, so to speak. In any enterprise, no matter how long its period of production, the process of production goes on incessantly and the product is being turned out constantly. Accordingly, all stages along the line of the period of production, or, in other words, in the process of production, keep on operating, each as the present stage of production. That is to say, all stages operate simultaneously at the same point of time. Productive goods are accordingly being applied constantly at their proper stages and the product manufactured with them is being turned out every moment. It is for this reason that it is said that there is no need to "wait", from the application of productive goods to the acquisition of the product, barring the case of the commencement of production, that is, the initiation of the enterprise. Under such circumstances, the scale of production in an enterprise is, to some extent, the expression of the length of production period. That is to say productive goods of a quantity large enough to cover the entire area of the so-called productive trapezoid must be applied and accordingly capital large enough to pay for these goods must be invested. In other words, all productive goods, the application of which is required in order to maintain the process of production without interruption throughout the period of production, must be used and these must operate simultaneously at the same point of time. Such being the case, the scale of an enterprise depends at once on the quantity of the flow of products, that is, the quantity of products during an unit of time and accordingly the quantity of the productive goods necessary for such production and on the length of the production period or the duration of time from their application to their maturity into the product. The scale of the enterprise is determined, roughly expressed, by the quantity of the product multiplied by the period of production. As regards capital, the amount required is equivalent to the total of the sum of the prices of productive goods at all stages of the productive trapezoid and interest thereon since its investment. Therefore, other things being equal, the period of production conditions the scale of production and accordingly the amount of capital required.

To put the matter in other words, of the two dimensions of the service of capital, that of time or the period of production being projected on the present in the actuality of synchronised production, manifests itself in the dimension of the quantity of capital. Therefore, in an actual enterprise, the rate of interest, no matter from what angle it may be viewed, is equal to the marginal productivity of the quantity of capital. What was mentioned as the service of capital in the marginal period is no other than the service at the "present" time of the last unit of capital supplemented in order to extend the period of production in the enterprise, and its productivity is equal to the rate of interest. The marginal productivity of the service of capital, regarded either in terms of quantity or in terms of time, is, after all the marginal produdtivity of the present service of the entire capital of the enterprise.

Thus, we can see what is meant by the productivity of the service of capital. The fact that some substantial quantity of capital can be obtained, that is, the possibility of the investment of capital on a certain scale, renders the adoption of a certain definite method of production possible. It renders possible the collaboration of productive goods in some form. Because, as the result of this collaboration, a certain quantity of products, higher in price than the sum of the prices of the productive goods consumed, is obtained, that is, a surplus is realised, capital has productive power. After all, the productivity of the period of production is what is conceived as a result of mental analysis. Viewed from the standpoint of an actual enterprise, the above-mentioned collaboration is rendered possible by the quantity of capital of certain dimensions and a surplus is realised through this collaboration. In an enterprise in which the length of the period is reflected in the quantity of capital at the "present" moment, the service of capital has its productivity because it renders the collaboration of productive goods on a definite scale possible. As I have already stated, the productivity of the marginal unit of the quantity of capital which is actually being utilised determines the rate of interest. It must, however, be noted that although what is here termed the period of production will be comparatively easy to understand when applied to circulating capital goods only it assumes a character of great complexity when it is applied to fixed capital goods as well. The discussion of this phase of the problem I shall defer to another occasion.

4. ON THE MULTILINEAL STRUCTURE OF PRODUCTION

What I have so far stated may be applicable to the so-called lineal structure of production as well, but my remarks do not necessarily assume such a structure of production. The problem was analysed on the assumption that some of the productive goods may well be intermediate products. But the points are left unclarified as to the process by which means of production are produced and the circumstances under which they carry on self-reproduction, as they are actually doing. It is very imperfect as illustrative of the multilineal structure of production. How will the rate of interest be determined under such a structure of production, that is, how far will the mechanism for the determination of the interest rate in the market, such as has already been described, be altered in such circumstances? All that is needed in order that the rate of interest may be unequivocally determined is simply to introduce some new unknown quantities and new conditions of the same number, namely equations.

Now, let us introduce l number of the intermediate products, q, r, s,......, and let it be assumed that they are used for self-reproduction. Then, the conditions will have to be altered as follows:—

(A) condition will remain unchanged. Nor will there be any change in (B) condition.

(C) condition will be altered thus. The factors forming the cost of production will have a new addition of the product of the coefficients of production of all capital goods multiplied by their prices.

In (D) condition, the quantities of products, x, y...... q, r,......, will represent not only the functions of the original productive goods, a, b,......, but those of the capital goods.

In (E) condition, the proposition that the sum of the prices of all productive goods is equal to discounted productivity will hold good not only in regard to the original productive goods but about capital goods also.

(F) condition, which indicates the relation between the average coefficient of production and the quantity of products must apply to capital goods also.

(G) The relation between the quantity of social products and the number of enterprises must exist in regard to capital goods as well as in respect of consumer's enjoyable goods.

(H) condition that the sum of the quantities of one kind of productive goods used in all kinds of industry is equal to the entire quantity of supply must rule in regard to capital goods also.

(K) The total quantity of capital is equal to the sum of the total quantity of the prices of original productive goods and the total quantity of capital goods.

(I), (J), and (L) conditions will suffer no change. The changed content are shown in the following equations:—

		Number of equations	Number of equations added
(C)	$(p_a a_x + p_b b_x + \ldots p_q q_x + p_r r_x + \ldots)(1+i) = p_x$		
	....	$m+l$	l
	$(p_a a_q + p_b b_q + \ldots p_q q_q + p_r r_q + \ldots)(1+i) = p_q$		
	....		

(D) $\begin{cases} x=F_x(\bar{a}_x,\ \bar{b}_x,\ldots\ldots\bar{q}_x,\ \bar{r}_x\ldots\ldots) \\ \ldots\ldots\ldots\ldots \\ q=F_q(\bar{a}_q,\ \bar{b}_q,\ \ldots\ldots q_q,\ r_q\ldots\ldots) \\ \ldots\ldots\ldots\ldots \end{cases}$	$m+l$	1
(E) $\begin{cases} \frac{p_a}{p_x}=\frac{\partial x}{\partial \bar{a}_x}\frac{1}{1+i};\ \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ 1=\frac{\partial x}{\partial q_q}\frac{1}{1+i};\ \frac{p_r}{p_q}=\frac{\partial q}{\partial r_a}\frac{1}{1+i};\ldots\ldots \\ \ldots\ldots\ldots\ldots \end{cases}$	$m(n+l)$ $+l(n+1)$	$ml+nl$ $+ll$
(F) $\begin{cases} a_x=\frac{\bar{a}_x}{x},\ b_x=\frac{\bar{b}_x}{x},\ldots\ldots,\ q_x=\frac{\bar{q}_x}{x},\ r_x=\frac{\bar{r}_x}{x}, \\ \ldots\ldots\ldots\ldots \\ a_q=\frac{\bar{a}_q}{q},\ b_q=\frac{\bar{b}_q}{q},\ldots\ldots,\ q_q=\frac{\bar{q}_q}{q},\ r_q=\frac{\bar{r}_q}{q}, \\ \ldots\ldots\ldots\ldots \end{cases}$	$m(n+l)$ $+l(n+l)$	$ml+nl$ $+ll$
(G) $\left\{\frac{X}{x}=\alpha_x,\ \frac{Y}{y}=\alpha_y,\ldots\ldots,\ \frac{Q}{q}=\alpha_q,\ \frac{R}{r}=\alpha_r,\ldots\ldots\right.$	$m+l$	l
(H) $\begin{cases} \alpha_x\bar{a}_x+\alpha_y\bar{a}_y+\ldots\ldots+\alpha_q\bar{a}_q+\alpha_r\bar{a}_r+\ldots\ldots=A \\ \ldots\ldots\ldots\ldots \\ \alpha_x\bar{q}_x+\alpha_y\bar{q}_y+\ldots\ldots+\alpha_q\bar{q}_q+\alpha_r\bar{q}_r+\ldots\ldots=Q \\ \ldots\ldots\ldots\ldots \end{cases}$		
(K) $\begin{cases} \alpha_x(p_a\bar{a}_x+p_b\bar{b}_x+\ldots\ldots)+\ldots\ldots \\ \quad +\alpha_q(p_a\bar{a}_q+p_b\bar{b}_q+\ldots\ldots)+\ldots\ldots=K \end{cases}$	l	0

The number of the equations newly added totals $2(ml+nl+ll)+4$, while the number of the unknown quantities newly brought in is as follows:—

$p_q,\ p_r\ldots\ldots l$	$q_x,\ r_x\ldots\ldots ml$	$a_q,\ a_r\ldots\ldots nl$	$q_q,\ q_r\ldots\ldots ll$
$q,\ r\ \ldots\ldots l$	$\bar{q}_x,\ \bar{r}_x\ldots\ldots ml$	$\bar{a}_q,\ \bar{a}_r\ldots\ldots nl$	$\bar{q}_q,\ \bar{q}_r\ldots\ldots ll$
$Q,\ R\ldots\ldots l$			
$\alpha_q,\ \alpha_r\ldots\ldots l$			
Total...$4l$	$+2ml$	$+2nl$	$+2ll$

It will be seen that the number of the unknown quantities newly added is equal to the number of equations newly added. As in the case previously explained, the rate of interest is unequivocally determined according to the given conditions in the new situation, that is, under the multilineal structure of production.

5. THE MULTILINEAL STRUCTURE OF PRODUCTION AND THE VARIABLE PERIOD OF PRODUCTION

So far, the period of production has been taken to be fixed, that is, constant (it has been taken to be *1*, to be more exact), but even if it is taken to be variable, the conclusion already reached need suffer no change whatever. The duration of the stay of productive goods within the process of production from their entry into it up to their transformation into a product, or the period of production, as it is here called, has hitherto been taken to be t_x in regard to x and t_y in regard to y, but now let us take it to be t_x in regard to original productive goods, $a, b, c,\ldots\ldots$, and t'_x in respect of intermediate goods, $q, r, s,\ldots\ldots$. Further, let us assume them to be different in size.

As in the previous case, capital goods do not enter the equations showing the law of equality of receipts and outlays in household economy or the law of equi-marginal utility. No new change occur in (A) and (B) conditions. With regard to the cost principle mentioned in (C), the rate of interest corresponding to the period of production for each of the productive goods must be added as an item of cost. Into condition (D) must be taken the period of producing each of the productive goods. So with (E) condition. (G) condition must be altered because of the entry of the period of production into it. The other conditions remain unaltered.

	Number of equations	Number of equations added
(C) $\begin{cases} (p_a a_x + p_b b_x + \ldots\ldots)(1+it_x) \\ \quad + (p_q q_x + p_r r_x + \ldots\ldots)(1+it'_x) = p_x \\ \ldots\ldots\ldots\ldots \\ (p_a a_q + \ldots\ldots\ldots\ldots)(1+it_q) \\ \quad + (p_q q_q + \ldots\ldots\ldots\ldots)(1+it'_q) = p_q \\ \ldots\ldots\ldots\ldots \end{cases}$	$m+l$	0
(D) $\begin{cases} x = F_x(\bar{a}_x, \bar{b}_x, \ldots\ldots, \bar{q}_x, \bar{r}_x, \ldots\ldots, t_x, t'_x) \\ \ldots\ldots\ldots\ldots \\ q = F_q(\bar{a}_q, \ldots\ldots\ldots, \bar{q}_q, \ldots\ldots\ldots, t_q, t'_q) \\ \ldots\ldots\ldots\ldots \end{cases}$	$m+l$	l
(E) $\begin{cases} \frac{p_a}{p_x} = \frac{\partial x}{\partial \bar{a}_x}\frac{1}{1+it_x}, \ldots\ldots, \frac{p_q}{p_x} \\ \quad = \frac{\partial x}{\partial \bar{q}_x}\frac{1}{1+it'_x}, \ldots\ldots\ldots\ldots, \\ \frac{\partial x}{\partial t_x} = \frac{i}{p_x}(p_a\bar{a}_x + \ldots\ldots), \ \frac{\partial x}{\partial t'_x} \\ \quad = \frac{i}{p_x}(p_q\bar{q}_x + \ldots\ldots\ldots\ldots) \\ \ldots\ldots\ldots\ldots \\ \frac{p_a}{p_q} = \frac{\partial x}{\partial \bar{q}_x}\frac{1}{1+it_q}, \ldots\ldots, \frac{p_q}{p_q} \\ \quad = \frac{\partial x}{\partial \bar{q}_q}\frac{1}{1+it'_q}, \ldots\ldots\ldots\ldots, \\ \frac{\partial x}{\partial t_q} = \frac{i}{p_x}(p_a\bar{q}_x + \ldots\ldots), \ \frac{\partial x}{\partial t'_q} \\ \quad = \frac{i}{p_x}(p_q\bar{q}_x + \ldots\ldots\ldots\ldots) \\ \ldots\ldots\ldots\ldots \end{cases}$	$m(n+l)$ $+2m+$ $l(n+1)$ $+2l$	$ml+2m$ $+nl+ll$ $+2l$
(F) $\begin{cases} a_x = \frac{\bar{a}_x}{x}, \ b_x = \frac{\bar{b}_x}{x}, \ldots\ldots, \ q_x = \frac{\bar{q}_x}{x}, \ r_x = \frac{\bar{r}_x}{r}, \\ \ldots\ldots\ldots\ldots \\ a_q = \frac{\bar{a}_q}{q}, \ b_q = \frac{\bar{b}_q}{q}, \ldots\ldots, \ q_q = \frac{\bar{q}_q}{q}, \ r_q = \frac{\bar{r}_q}{q}, \\ \ldots\ldots\ldots\ldots \end{cases}$	$m(n+1)$ $+l(n+1)$	$ml+ml$ $+ll$

$$\text{(G)}\begin{cases} \dfrac{X}{x}t_x=\alpha_x,\ \dfrac{Y}{y}t_y=\alpha_y,\ \cdots\cdots\cdots\cdots \\ \dfrac{Q}{q}t_q=\alpha_q,\ \dfrac{R}{r}t_r=\alpha_r,\ \cdots\cdots\cdots\cdots \end{cases} \qquad m+1 \qquad 1$$

$$\text{(H)}\begin{cases} \alpha_x\Big\{(p_a\bar{a}_x+p_b\bar{b}_x+\cdots\cdots) \\ \qquad +(p_q\bar{q}_x+p_r\bar{r}_x+\cdots\cdots)\dfrac{t'_x}{t_x}\Big\}+ \\ \alpha_y\Big\{(p_a\bar{a}^y+p^b\bar{b}_y+\cdots\cdots) \\ \qquad +(p_q\bar{q}_x+\cdots\cdots)\dfrac{t'_y}{t_y}\Big\}+\cdots\cdots=K \end{cases} \qquad 1 \qquad 0$$

When this is compared with the case where capital goods are taken into consideration and the period of production is assumed to be uniform and fixed for all productive goods, the number of equations shows an increase of *2m+2l*, while the added number of unknown quantities is *2m+2l*, that is, t_x, $t'_x,\cdots\cdots$, t_q, $t'_q,\cdots\cdots$. Thus, as already stated, the rate of interest can be unequivocally determined.

In order to bring the whole argument nearer to actual economy, the assumption that all productive goods are invested simultaneously must be discarded, and, further, the so-called function of the employment of the service of each class of goods must be taken into consideration. Again instead of confining attention to circulating capital goods, as has been done so far, fixed capital goods must also be brought into the picture and the period of production, fixed by their period of construction and the duration of their existence, taken into the reckoning. Even if these points are taken into account, however, it will not be necessary to revise the conclusion already reached. Again, the mathematical treatment of the subject will then become too complex. Above all, as the chief point which I want to emphasise in connection with the determination of the rate of interest lies somewhere else, I shall not touch on these circumstances in the present article.

What I want to stress by the argument I have so far advanced is simply this. The rate of interest cannot find its definite level unless the total quantity of capital in society and the supply function of original factors of production are given. Even if attempts are made to find the rate of interest merely by productive function and therefore by means of the functions of productivity and of cost, it will be impossible to determine it unequivocally by this means only. The most that can be done will be to find it as the function of marginal productivity or as the relative size with regard to marginal productivity. The absolute size of this marginal productivity, and accordingly the rate of interest can only be determined by the whole mechanism of social production and by the factors determinant of it which I have already described.

If all that we want to know is the relation in which the rate of interest stands to marginal productivity, our object can be attained without regard to the whole aspect of general equilibrium and accordingly the relation with the total quantity of capital. As it is, it is impossible to attain the end without taking the supply function of capital into consideration. If I have brought the entire aspect of general equilibrium into the picture in the present article, though in a very simplified form, it is because the subject under discussion cannot otherwise be adequately explained.