Kyoto University
Economic Review

MEMOIRS OF
THE DEPARTMENT OF ECONOMICS
IN
THE IMPERIAL UNIVERSITY OF KYŌTO

VOLUME XVIII
1943

PUBLISHED BY THE DEPARTMENT
OF ECONOMICS IN
THE IMPERIAL UNIVERSITY OF KYOTO

ON THE EXTENSION OF THE CONCEPT OF A COMMODITY

A NOTE ON HICKS THEORY OF THE "GROUP OF COMMODITIES"

By HIDEO AOYAMA

INTRODUCTION

It is one of the fundamental points of view in Hicks' "*Value and Capital*" (Oxford, 1939) that when the relative prices of a group of commodities can be assumed to remain unchanged, they can be treated as a single commodity.[1] This point of view, which I will call here "*commodity-group*" *point of view* for the sake of convenience, is so frequently used in his theoretical system, that the concepts founded on this point of view, such as the demand for a group of goods or the substitution between one group of goods and another, play an important rôle in his Dynamic Economics. Nevertheless he gives us no precise information about the way to measure the quantity of a "group of commodities", about the definition of its price and about the "substitution term" in the effect of a change in the price of a commodity or commodity-group on the demand for any group of goods. The analytical exposition which he attempts in order to demonstrate that, if the prices of a set of goods change in the same proportion, that group of goods behaves just as if it were a single commodity, is the sole one which

1) Hicks gives no special name to the group of commodities which is qualified to be treated as a single commodity because of the constancy of their relative prices. For the sake of convenience, the term "*group of commodities*" (or more briefly "*commodity-group*") is devoted here to designate that set of commodities within which relative prices are assumed to be unchanged to one another.

he gives us;[2] and, what is more, in his explanation of this proposition we can find no clear definition of the amount and price of a group of commodities.

Then, precisely what means this "commodity-group" point of view? Is there any appropriate way to measure the quantity of this "group of commodities", considered as a single commodity, in spite of its being composed of physically different things? What is the price of this group of commodities? More fundamentally speaking, in what form should be represented the so-called "Fundamental Equation of Value Theory," when *the group of commodities* is concerned? These problems are what I attempt to attack in this paper. For this purpose, I will show in the first instance the precise meaning of the "commodity-group" point of view. In Section 1, I will explain it, referring to Hicks' own exposition of the law of consummer's demand under statical assumptions (verbally in Chapter II—III, and mathematically in Mathematical Appendix, pp. 307—314 of his book). Secondly, in Section 2, I will attempt to explain the economic meaning of the six rules which must be obeyed by the "substitution terms," in the application of this "commodity-group" point of view.[3] Thirdly, in Section 3, the applications of this "commodity-group" point of view to Dynamic Economics are demonstrated in connection with the dynamic problem of the consummer's planning of spending and saving. This is the problem which Hicks has verbally treated in Chapter XVIII of his book, and I will attempt here to translate this verbal exposition into a mathematical one, applying our "commodity-group" point of view precisely defined to this problem. Throughout these three sections, I will rely on Hicks' own explanations and analyses in his "*Value and Capital*," so far as posible.[4] Therefore, it is certainly

2) Cf. J. R. Hicks: *Value and Capital*, p. 312.

3) Hicks derives these six "rules of substitution terms" mathematically. See, *ibid*, pp. 309—311.

4) Relping on Hicks' own exposition, I have saved many explanations which were otherwise necessary.

questionable, how much the present work can differ from the mere mathematical translasion of Hicks' verbal explanations; but I hope, at least, that it can help somewhat to make clear one of the ideas which Hicks has employed in the construction of his theoretical system.

1

Let us consider a consummer who is making a choice of that combination of amounts bought $x_1, x_2, \cdots\cdots\cdots, x_n$ of n different commodities[1] which maximises his satisfaction of wants $u(x_1, x_2, \cdots\cdots, x_n)$ under a subsidiary condition

$$(1) \qquad M=\sum_{i=1}^{n} p_i x_i,$$

which means that this consumer spends entirely a given sum of money available for expenditure in order to buy these n commodities on the market at prices $p_1, p_2, \cdots\cdots, p_n$ which are assumed to be given to him. In order that u is at maximum, it is necessary that

$$(2) \qquad \frac{u_1}{p_1}=\frac{u_2}{p_2}= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = \frac{u_n}{p_n},$$

where u_i is written for $\frac{\partial u}{\partial x_i}$; in other words, in order that this consumer gets the "optimum combination," so to speak, of amounts bought, he must demand these commodities in accordance with the condition (2). Thus the above equations (1) and (2) define implicitly the individual demands of our consumer as functions of his income and the market prices, which we can rewrite in the following equations:

---

1) In the following, it is assumed that objects of the choice of our consummer are limited to these n different commodities. As for the designation of any one (physical) sort of commodity among these, we use, in the main, the symbol of amount brought for that sort of commodity. For instance we speak of goods x_1. But these are some cases when it is convenient to use the order in which these commodities are arranged, for this purpose. In these cases, we speak of goods No. 1, for instance, making use of the order.

(3) $$x_i = x_i(M, p_1, p_2, \ldots\ldots\ldots\ldots\ldots\ldots, p_n).$$

Now, starting from (1) and (2), the effect on demand x_j of a change in price p_i with constant income can be shown by the "Fundamental Equation of Value Theory,"

(4) $$\frac{\partial x_j}{\partial p_i} = -x_i \frac{\partial x_j}{\partial M} + \mathbf{x}_{ij},$$

which decomposes the effect of price-change into two parts, the Income Effect and the Substitution Effect. Here $\mathbf{x}_{ij}$ denotes the same thing as this symbol in Hicks' book and it is called the "substitution term" in $\frac{\partial x_j}{\partial p_i}$.[2] As Hicks has shown,[3] this substitution term obeys six rules:

(A) $\mathbf{x}_{ij} = \mathbf{x}_{ji}$, (i.e. symmetricity of the substitution term);

(B) $\mathbf{x}_{ii} < 0$, (which means that the individual demand curve is downward sloping, so far as the Income Effect is negligible);

(C) $\sum_{j=1}^{n} p_j \mathbf{x}_{ij} = 0$,

(B′) $\sum_{i=1}^{m} \sum_{j=1}^{m} p_i p_j \mathbf{x}_{ij} < 0$, (for all values of m less than n);

(D) $\sum_{j \neq i} p_j \mathbf{x}_{ij} > 0$, (where $\sum_{j \neq i}$ means that the summation is taken over all values of j except i); and

(D′) $\sum_{i=1}^{m} \sum_{j=m+1}^{n} p_i p_j \mathbf{x}_{ij} > 0$, $(m < n)$.

In order to clarify the contents of the "commodity-group" point of view, let us suppose that prices $p_1, p_2, \cdots\cdots$, p_m change in the same proportion and therefore m com-

2) On the "Fundamental Equation of Value Theory," see Hicks: *ibid*, pp. 308—309.

3) Hicks: *ibid*, pp. 310—311.

modities $x_1, x_2, \ldots\ldots, x_m$ can be lumped together into one "commodity-group," as stated above. If we choose a particular constant set of relative prices, $w_1, w_2, \ldots\ldots, w_m$ of the commodities in this group, then, owing to the assumption of constancy of relative prices of the group of commodities, we have

$$(5) \qquad p_i=\gamma w_i, \quad (i=1, 2, \ldots\ldots, m),$$

where γ is the proportional factor and therefore is regarded as variable. Now we define the price of the group of commodities in question by this proportional factor γ and its amount by the weighted sum

$$(6) \qquad G=w_1x_1+w_2x_2+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +w_mx_m,$$

so that the budget equation (1) becomes

$$(7) \qquad M=\gamma G+p_{m+1}x_{m+1}+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots +p_nx_n.$$

Further, we will use this G as the designation of the group of commodities in question also.

This is our definition of amount and price of a group of commodities. Then we can show the proposition that if we put

$$(8a) \qquad \mathbf{x}_{Gj}=w_1\mathbf{x}_{1j}+w_2\mathbf{x}_{2j}+ \ldots\ldots\ldots\ldots\ldots\ldots +w_m\mathbf{x}_{mj},$$

$$(8b) \qquad \mathbf{x}_{iG}=w_1\mathbf{x}_{1j}+w_2\mathbf{x}_{i2}+ \ldots\ldots\ldots\ldots\ldots\ldots +w_m\mathbf{x}_{im},$$ [4)]

then we have

$$(9) \qquad \frac{\partial x_j}{\partial \gamma}=-G\frac{\partial x_j}{\partial M}+\mathbf{x}_{Gj}, \quad (j=1, 2, \ldots\ldots n,)$$

$$(10) \qquad \frac{\partial G}{\partial p_i}=-x_i\frac{\partial G}{\partial M}+\mathbf{x}_{iG}, \quad (i=1, 2, \ldots\ldots n,)$$

which is nothing else than the fundamental equations concerning the group of commodities. In fact, we obtain (9) in

4) In consequence of the symmetricity (i.e. rule (A)) of the substitution term, $\mathbf{x}_{Gi}=\mathbf{x}_{iG}$.

the following way:

$$\begin{aligned}\frac{\partial x_j}{\partial \gamma} &= \sum_{i=1}^{m} \frac{\partial x_j}{\partial p_i}\frac{dp_i}{d\gamma} \\ &= \sum_{i=1}^{m}\left[w_i\left(-x_i\frac{\partial x_j}{\partial M}+\mathbf{x}_{ij}\right)\right], \quad \text{(owing to (4) and (5))} \\ &= \sum_{i=1}^{m}(-w_i x_i)\frac{\partial x_j}{\partial M}+\sum_{i=1}^{m} w_i \mathbf{x}_{ij} \\ &= -G\frac{\partial x_j}{\partial M}+\mathbf{x}_{Gj}, \quad \text{(owing to (6) and (8a)).}\end{aligned}$$

Similarly we can prove (10).

In the case treated above there is only one group of commodities among the objects of preference of the consumer. Next, let us consider the case when there is another group of commodities. Suppose that this second group G' consists of l different goods $x_{m+1}, x_{m+2}, \ldots\ldots, x_{m+l}$, one particular constant set of their relative prices are $w_{m+1}, w_{m+2}, \ldots\ldots, w_{m+l}$, its amount is measured by $G' = w_{m+1}x_{m+1} + w_{m+2}x_{m+2} + \ldots\ldots + w_{m+l}x_{n+l}$ and the price of this second group of commodities is γ'. Then we can shown, in the same manner as in the above, that, when we extend the above definition (8) of the substitution term by admitting that any commodity-group (which may be different from G,) can be substituted for the commodity No. i or No. j in (8), so that

$$(10) \qquad \begin{aligned}\mathbf{x}_{GG'} &= \sum_{i=1}^{m} w_i \mathbf{x}_{iG'} = \sum_{i=1}^{m} w_i \sum_{j=m+1}^{n+l} w_j \mathbf{x}_{ij} \\ &= \sum_{i=1}^{m}\sum_{j=m+1}^{m+l} w_i w_j \mathbf{x}_{ij} = \mathbf{x}_{G'G},\end{aligned}$$

then we have

$$(12) \qquad \frac{\partial G'}{\partial \gamma} = -G\frac{\partial G'}{\partial M} + \mathbf{x}_{GG'}$$

which is the fundamental equation of this case. The proof of this proposition is done also in the same way as in the above. In fact,

$$\frac{\partial G'}{\partial \gamma}=\sum_{j=m+1}^{m+l} w_j \frac{\partial x_j}{\partial \gamma}$$

$$=\sum_{j=m+1}^{m+l} w_j\left(-G\frac{\partial x_j}{\partial M}+\mathbf{x}_{Gj}\right) \quad \text{(owing to (9))}$$

$$=-G\sum_{j=m+1} w_j \frac{\partial x_j}{\partial M}+\sum_{j=m+1}^{m+l} w_j \mathbf{x}_{Gj}$$

$$=-G\frac{\partial G'}{\partial M}+\mathbf{x}_{GG'} \qquad \text{(owing to (11)).}$$

Certainly we can treat in the same manner the case when there are many groups of commodities among the objects of preference of the consumer, we are not concerned here with such a task of generalization. It is rather important to notice the equation

$$\text{(13a)} \qquad \frac{\partial G}{\partial \gamma}=-G\frac{\partial G}{\partial M}+\mathbf{x}_{GG},$$

$$\text{(13b)} \qquad \mathbf{x}_{GG}=\sum_{i=1}^{m}\sum_{j=1}^{m} w_i w_j \mathbf{x}_{ij}$$

$$< 0, \quad \text{(by the rule (B}')\text{ of the substitution term),}$$

as a corollary obtained from the above proposition. The equation (13a) is evidently the fundamental equation which gives the effect of a change in price of a group of commodities on its demand. This effect is the only one which Hicks has mathematically analyzed with respect to the group of commodities. His result is substantially the same one as ours.[5]

Thus, when the prices of a group of commodities change in the same proportion, we can write down the fundamental equation concerning the group of commodities, simply by treating this commodity-group as a single commodity, so far as its amount, its price and its substitution term are appropriately defined. This is the exact meaning which we can give to the Hicks' fundamental proposition that " a collection

5) Cf. Hicks: *ibid*, p. 312.

of physical things can always be treated as if they were divisible into units of a single commodity so long as their relative prices can be assumed to be unchanged, in the particular problem in hand."[6)]

2

Applying the "commodity-group" point of view defined above, we can interpret the economic meaning of the six rules of substitution terms. Among these six rules, the meaning of (A) and (B) is already stated. Further, as concerns (B′), the above equations (13a, b) show that the rule (B′) is an extension of the rule (B) to the case of the group of commodities. Therefore it only remains for us to explain the economic meaning of rules (C), (D) and (D′).

Let us begin with the rule (C). For this purpose, let us suppose that *all* prices of n objects of preference rise and fall uniformly. In this case, we can lump together all objects of preference, into one "group of commodities," so that m in Section 1 coincides with n. Consequently the fundamental equation of this case becomes

$$(14)\qquad \frac{\partial x_j}{\partial \gamma} = -G\frac{\partial x_j}{\partial M} + \sum_{i=1}^{m} w_i \mathbf{x}_{ij}. \qquad (i=1,\ 2, \ldots\ldots, n)$$

With the aid of this equation (14), we can easily interpret the rule (C). The equation (14) shows that the expression $\sum_{i=1}^{n} w_i \mathbf{x}_{ij}$ is the substitution term in $\frac{\partial x_j}{\partial \gamma}$, that is to say, the substitution effect on the demand for goods x_j which is brought about by the uniform change of all prices.[1)] Now, the rule (C) asserts that this expression $\sum_{i=1}^{n} w_i \mathbf{x}_{ij}$ vanishes. (Because $\sum_{i=1}^{n} w_i x_{ij}$ is identical with $\sum_{i=1}^{n} p_i \mathbf{x}_{ij}$ except for a positive factor γ). Therefore, the rule (C) represents the evident

6) Hicks: *ibid*, p. 33.

1) We are concerned here only with the substitution effect. The demand for any goods can still change in this case owing to the income effect, which represents the first term of the right side of equation (14).

fact that the substitution effect of a uniform price-change on the demand of any goods must vanish. More accurately, it means that, in the case of universally uniform price-change, the partial substitution effects of the changes of particular prices on the demand cancel one another, so that the total substitution effect as the weighted sum of these partial effects is zero.[2)]

As regards the rules (D) and (D′), we must recall beforehand the fact that complementarity cannot appear, when the objects of preference consist of only two goods and some third goods other than these does not exist.[3)] Now I will explain the rules (D) and (D′) as the generalization of this circumstance. Let us suppose that all objects of preference are divided into two "groups of goods" in the sense defined above, to that $m+l$ in Section 1 is equal to n. Then, the rule (D′) asserts that the expression $\mathbf{x}_{GG'}$ in (11) is positive. Since it is natural to say that two commodity-groups G and G' are substitutes for one another, when $\mathbf{x}_{GG'}>0$, the rule (D′) means that, when all objects of preference are wholly divided into two commodity-groups, no complementary can arise between these two groups, what is evidently a generalization of the circumstance stated above. Now that the meaning of the rule (D′) is elucidated, it is easy to interpret the rule (D). The rule (D) is a special case of the rule (D′); that is to say, a case when one of the groups considered above is reduced to one single commodity. Therefore it means that, when we choose any one commodity from n objects of preference and lump together all things other than this commodity into a "commodity-group," then complementarity can not arise between this commodity and "other commodities" as a commodity-group.

We have elucidated the economic meaning of all six rules of substitution terms. Further it can be shown that

2) The fact that what Hicks calls the "capital value of marginal stream" and denotes by c vanishes (*cf. ibid*, p. 221.), is the result of the simple circumstance explained here.

3) Cf. Hicks: *ibid*, p. 47.

these six rules are still valid, when we consider the goods No. i or No. j in these six rules as any group of commodities.

3

In Section 1, the "commodity-group" point of view is clarified with respect to static problem. But the fruitfulness of this point of view shall be shown in its application to the dynamic problems. In this section we will consider this application. While Hicks' exposition of dynamic problems in his "Value and Capital" proceeds from the problem of the production plan of a firm (Chapter XV—XVII) to that of consumer's plan (Chapter XVIII), I will treat here only the latter problem, applying the "commodity-group" point of view to it.

In this paper, taking for granted the special devices and concepts which Hicks used in order to systematize the Dynamics in his book, we will consider directly the dynamic problem of consumer's planning. For this purpose we start from explaining the symbols which are used in the following. Supposing that the consumer makes his plan for ν weeks and $N=(\nu+1)n$, these symbols are listed as follows:

Table of Symbols

		Symbols whose second suffixes represent explicitly the distinction of "week" to which variable relate				Symbols whose suffixes run through with no regard of the distinction of "week" to which variables relate			
	Sorts of commodities / "week"	1	2	…………	n	1	2	…………	n
Demands	the current week	$\mathfrak{x}_{10}$	$\mathfrak{x}_{20}$	…………	$\mathfrak{x}_{n0}$	x_1	x_2	…………	x_n
	the first week	$\mathfrak{x}_{11}$	$\mathfrak{x}_{21}$	…………	$\mathfrak{x}_{n1}$	x_{n+1}	x_{n+2}	…………	x_{2n}
	…………	⋮	⋮	…………	⋮	⋮			
	…………			…………					
	the ν-th week	$\mathfrak{x}_{1\nu}$	$\mathfrak{x}_{2\nu}$	…………	$\mathfrak{x}_{n\nu}$	$x_{\nu n+1}$	$x_{\nu n+2}$	…………	x_N

Prices	the current week	$\mathfrak{p}_{10}\ \ \mathfrak{p}_{20}\ \cdots\cdots\ \mathfrak{p}_{n0}$	$p_1\ \ p_2\ \cdots\cdots\ p_n$
	the first week	$\mathfrak{p}_{11}\ \ \mathfrak{p}_{21}\ \cdots\cdots\ \mathfrak{p}_{1n}$	$p_{n+1}\ \ p_{n+2}\ \cdots\cdots\ p_{2n}$
	…………………	$\vdots\ \ \vdots\ \cdots\cdots\ \vdots$	$\vdots\ \ \vdots\ \cdots\cdots\ \vdots$
	…………………	$\vdots\ \ \vdots\ \cdots\cdots\ \vdots$	$\vdots\ \ \vdots\ \cdots\cdots\ \vdots$
Rate of Interest		i_t=the rate of interest per week for loans of t weeks	
Discount Ratio		$\beta_t=\dfrac{1}{(1+i_t)},\ \beta_t^t=\dfrac{1}{(1+i_t)^t}$	
Relation between Symbols		$\mathfrak{x}_{it}=x_{tn+i}$ $\beta_t^t\mathfrak{p}_{it}=p_{tn+i}$ (=disconnted value of the expected price $\mathfrak{p}_{it}$)	

Remarks.

1) The t-th week means here the week starting t weeks from now, which is called the "$(t+1)$-th week" in Hicks' book.

2) The $\mathfrak{x}_{it}$ (or x_{it}) denotes the amount of goods No. i which shall be sold in the current week (if $t=0$) or is planned to be sold in the t-th week (if $t \geqq 1$) by the consumer. Therefore it is the demand for goods No. i in the t-th week. Supplies are considered as negative demands, so that, if $\mathfrak{x}_{it}<0$, $-\mathfrak{x}_{it}$ represents the amount of goods No. i sold in the t-th week.

In this set of variables, it is evident at first sight that the current prices

$$\mathfrak{p}_{10},\ \mathfrak{p}_{20},\cdots\cdots,\ \mathfrak{p}_{n0}$$

belong to the data of the problem in hand, since they are supposed to be given owing to the assumption of the perfect competition on the market. Further, the future prices expected by the consumer in question,

$\mathfrak{p}_{11},\ \mathfrak{p}_{21},\cdots\cdots,\ \mathfrak{p}_{n1}$, (prices expected to rule during the 1st "week");

$\mathfrak{p}_{12},\ \mathfrak{p}_{22},\cdots\cdots,\ \mathfrak{p}_{n2}$, (prices expected to rule during the 2nd "week");

....

....

$\mathfrak{p}_{1\nu}, \mathfrak{p}_{2\nu}, \ldots\ldots, \mathfrak{p}_{n\nu}$, (prices expected to rule during the ν-th "week"),

can be regarded as data, since they are subjectively determined in his mind. Lastly, the rates of interest

$$i_1, i_2, \ldots\ldots\ldots, i_\nu$$

and consequently the discount ratios

$$\beta_1, \beta_2, \ldots\ldots\ldots, \beta_\nu$$

belong also to the data of the problem, either owing to the assumption of the perfect competition on the market or on account of subjective determinedness on the part of the consummer. The dynamic problem of the consumers's planning consists, therefore, in choosing such a stream of demands

$$\begin{array}{l} \mathfrak{x}_{10}, \mathfrak{x}_{20}, \ldots\ldots\ldots, \mathfrak{x}_{n0}, \\ \mathfrak{x}_{11}, \mathfrak{x}_{21}, \ldots\ldots\ldots, \mathfrak{x}_{n1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \mathfrak{x}_{1\nu}, \mathfrak{x}_{2\nu}, \ldots\ldots\ldots, \mathfrak{x}_{n\nu}, \end{array}$$

as maximises the utility

$$u\,(\mathfrak{x}_{10}, \mathfrak{x}_{20}, \ldots, \mathfrak{x}_{n0}, \mathfrak{x}_{11}, \mathfrak{x}_{21}, \ldots, \mathfrak{x}_{n1}, \ldots, \mathfrak{x}_{1\nu}, \mathfrak{x}_{2\nu}, \ldots, \mathfrak{x}_{u\nu}$$

on the foundation of knowledge of the above data. Since the task of dynamic theory consists in explaining the choice of the optimum combination of amounts bought, it is clear that the dynamic theory of the consumer's planning can be constructed in the similar way as the static theory of the consumer's choice presented in Section 1. Thus, when we regard commodities bought or sold at different dates, even when they are physically of the same sort, as different goods, it is the first step toward solving the dynamic problem parallely with the static problem. Thus the symbols without the explicit representation of the difference of "week" in the above table become useful for the present problem; in the first place, the symbols x's, by which the stream of demands for goods is represented, become to be useful.

As it can be easily conjectured by analogy from the static theory, the consumer's choice of the optimum combination must be constrained by some subsidiary conditions in this case also. The subsidiary conditions in this case must be evidently a complicated one, when we take into consideration the circumstances which are proper to dynamic problems of economics, — "securities" and its substitute "money", for instance. But the case whick Hicks treats in his book is the most simplified one, *i.e.* the case when this subsidiary condition is given in the same form of equation as the budget equation (1) in the static theory and, what is more, it is limited to it only, so that formally we need no amendment of reasoning in order to dynamize the static theory of consumer's behaviour presented in Section 1. Hicks write down this condition in the equation

(16a) $p_1x_1+p_2x_2+\dots\dots\dots+p_Nx_N=K,$

(16b) $K=-\beta_v^v C_n,$

where C_n means "the value of the securities expected to have been acquired as a result of the lending which is to take place during the period of plan" under the assumption that (α) the difference between receipts and expenditures must be made up only by changes in the consumer's holding of securities (and not by changes in his holding of money), and (β) the amount of c_n is assumed to be decided by the consumer from the beginning.[1]

So long as the subsidiary condition which restricts the consumer's choice is reduced to only one equation such as (16a), it is a matter of course that the dynamic theory of consumer's choice will be constructed in the same manner

1) Cf. Hicks: *ibid*, pp. 229—230.

as the static theory. When we let the suffix i or j change from 1 to N instead of from 1 to n and substitute K for M, then the whole static theory of consumer's choice presented in Section 1 becomes at once the dynamic theory of consumer's planning of spending and lending; consequently, the equations from (1) to (4), the six rules of substitution terms and the "commodity-group" point of view become valid in this case also. Of course the "prices" in this case are not the undiscounted ones $\mathfrak{p}$'s, but the discounted ones p's.[2]

Relying on the above explanations, we will proceed to the problem of the application of the "commodity-group" point of view to the dynamical problem of consumer's behaviour. This problem is further divided into two parts, [I] *the problem of the effect of price-change* and [II] *the problem of the effect of a change in the rates of interest.* For the former problem *the concept of the "group of physically same sort of commoditie"* at different dates

$$g_i = p_i x_i + p_{n+i} x_{n+i} + \cdots\cdots + p_{\nu n+i} x_{\nu n+i}, \quad (i=1, 2, \ldots\ldots, n) \tag{17}$$

is useful[3]; for the latter problem *the concept of the "group of contemporaneous commodities"* of any week

$$G_T = \mathfrak{p}_{1T}\mathfrak{x}_{1T} + \mathfrak{p}_{2T}\mathfrak{x}_{2T} + \cdots\cdots + \mathfrak{p}_{nT}\mathfrak{x}_{nT}, \quad (T=0, 1, 2, \ldots\ldots, \nu) \tag{18}$$

and *the "adjusted group of contemporaneous commodities"*

$$G_T' = G_T, \quad (T=0, 1, 2, \ldots\ldots, \nu-1) \tag{19a}$$

$$G_\nu' = G_\nu + C_n \tag{19b}$$

---

2) Cf. Hicks: *ibid*, p. 202—203.

3) According to the above stated way of designating a group of commodities, the group of commodities whose quantity is measured by g_i defined by (17) should be called commodity-group g_i. Also in the following we will employ this designation, but we will call it besides "commodity-group I", using the capital letter of i which denote the sort of goods,—when the occasion demands a simple expression. Similarly, we call the "group of contemporaneous commodities" G_T simply "commodity-group" T accordiug to the order of "week" to which the "commodity-group" in question corresponds.

are useful.[4]

[*I*] *The effect of price-change.*

In the first place we consider the case when the (current or expected) price of a commodity No. j at a week T changes isolatedly and all other prices (, inclusive of rates of interest,) are constant. In this case, the discounted prices of all commodities except the one in which price-change occurs is constant according to the assumption. Hence the ratio of discounted prices of the same sort of goods

$$(20) \qquad p_i : p_{n+i} : \cdots\cdots\cdots : p_{\nu n+i}$$

is constant in reference to any sort of goods excepting the case when $i=j$. Thus we can lump together any one sort of goods at different weeks into one "commodity-group" with the same exception as before. Further, employing the discounted prices p's as weights, the quantity of these "commodity-groups" can be measured by g_i defined by (17). Applying the "commodity-group" point of view stated in Section 1 to this case, we obtain at once

$$(21) \qquad \frac{\partial g_i}{\partial p_{Tn+j}} = -x_{Tn+j}\frac{\partial g_i}{\partial K} + \mathbf{x}_{I,\,Tn+j}, \qquad (i=1,\ 2,\cdots\cdots,\ j-1,\ j+1,\cdots\cdots,\ n),$$

4) Owing to the definition of g_{it} stated above, G_T means the difference between "expenditure" and "receipts" in the T week, therefore the "borrowing" (or negative "lending") in this week. Thus, what Hicks calls the "stream of lendings" and the "adjusted stream of lendings" can be represented respectively by the sequence

$$-G_0,\ -G_1,\ -G_2,\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots -G_\nu.$$

and

$$-G_0',\ -G_1',\ -G_2',\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots -G_\nu'.$$

We shall use, in the following, the fact that the present value of this adjusted stream of lendings is zero, and this fact can be easily ascertained from the above definitions. In fact, we obtain from (16a, b)

$$\beta_0{}^0G_0+\beta_1{}^1G_1+\beta_2{}^2G_2+\cdots\cdots\cdots\cdots+\beta_\nu{}^\nu(G_\nu+c_n)=0,$$

hence, according to (19a, b)

$$\beta_0{}^0G_0'+\beta_1{}'G_1'+\beta_2{}^2G_2'+\cdots\cdots\cdots\cdots+\beta_\nu{}^\nu G_\nu'=0.$$

(cf. Hicks: *ibid*, pp. 229—230.)

where I in $\mathbf{x}_{I, T_{n+j}}$ means the commodity-group g_i as already remarked and the expression $\mathbf{x}_{I, T_{n+j}}$ has the contents which is defined in conformity with the equation (8).[5]

Next, we proceed to the case when the elasticity of price-expectation is unity. This assumption of unity elasticity of expectation means that when the current price $\mathfrak{p}_{j0}$ of the good No. j, for instance, rises by 10%, then all future prices of that sort of goods expected by the individual

$$\mathfrak{p}_{j1},\ \mathfrak{p}_{j2},\ldots\ldots\ldots,\ \mathfrak{p}_{j\nu}$$

rises by 10%. Thus, under this assumption the prices of goods No. j in different "weeks" change in the same proportion; consequently, we can lump together this sort of commodities at different "weeks" into one commodity-group g_j or J, whose amounts we measure by

$$g_j = p_j x_j + p_{n+j} x_{n+j} + \cdots\cdots + p_{\nu n+j} x_{\nu n+j}$$

in conformity with the definition (17) and whose price we denote by γ. It is a matter of course that we can lump together the other sorts of commodities into groups. Then we obtain

$$\text{(22a)} \qquad \frac{\partial g_i}{\partial \gamma} = -g_j \frac{\partial g_i}{\partial K} + \mathbf{x}_{I, T}, \quad (i=1,\ 2,\ldots\ldots,\ n)$$

$$\text{(22b)} \qquad \frac{\partial x_{tn+i}}{\partial \gamma} = -g_j \frac{\partial x_{tn+i}}{\partial K} + \mathbf{x}_{J,\ tn+i}, \quad (i,\ j=1,\ 2,\ldots\ldots,\ n)$$

in consequence of the direct application of our "commodity-group" point of view.[6]

---

5) In the following, we use frequently substitution terms concerning commodity-groups, such as $\mathbf{x}_{I,\ T_{n+j}}$ in the above. It must be remembered that the definition of these substitution terms are always given by equation (8).

6) In the special case when $i=j$ we have

$$\mathbf{x}_{J,\ tn+j} = \sum_{t=0}^{\nu} p_{Tn+j} \mathbf{x}_{Tn+j,\ tn+j}$$

according to the definition (8). Hicks discussed this case at full length. (cf. *ibid*, pp. 207—212.) In my opinion, what he calls "substitution over time" and "conplementarity over time," relates to the sign of the expression $\mathbf{x}_{Tn+j,\ tn+j}$.

[*II*] *The effect of interest-change.*

In the case of interest-change, all prices are assumed to be unchanged. Consequently, with respect to each week,

$$p_{Tn+1} : p_{Tn+2} : \cdots\cdots : p_{(T+1)n} = \mathfrak{p}_{1T} : \mathfrak{p}_{2T} : \cdots\cdots : \mathfrak{p}_{nT}$$

is constant; thus it is possible to consider the "group of contemporaneous commodities" G_T or T defined by (18) of which undiscounted prices $\mathfrak{p}_{1T}, \mathfrak{p}_{2T}, \cdots\cdots, \mathfrak{p}_{nT}$ are used as weights and the discount ratio β_T^T is considered as the price of the commodity-group G_T owing to the relation

$$\beta_T^T \mathfrak{p}_{it} = p_{Tn+t}$$

stated above.[7]

Now that the construction of the group of commodities in this case has been explained, it remains for us only to apply the "commodity-group" point of view defined above and to write down the fundamental equations of this case, in order to see the effect of interest-change. Let us suppose, in the first instance, the case when only the rate of interest i_t per week for loans of t weeks changes and all other rates of interest are constant. Applying the commodity-group point of view, we obtain, for $t=1, 2, \cdots\cdots, \nu-1$,

$$\text{(24a)} \qquad \frac{\partial x_{Tn+i}}{\partial \beta_t} = \frac{\partial x_{Tn+i}}{\partial \beta_t^t} \frac{d\beta_t^t}{d\beta_t} = -t\beta_t^{t-1} G_t \frac{\partial x_{Tn+i}}{\partial K} + t\beta_t^{t-1} \mathbf{x}_{t,\, Tn+i}$$

$$\text{(25a)} \qquad \frac{\partial G_T}{\partial \beta_t} = \frac{\partial G_T}{\partial \beta_t^t} \frac{d\beta_t^t}{d\beta_t} = -t\beta_t^{t-1} G_t \frac{\partial G_T}{\partial K} + t\beta_t^{t-1} \mathbf{x}_{t,\, T}$$

$$(i=1, 2, \cdots\cdots, n;\ T=0, 1, 2, \cdots\cdots, \nu)$$

and, for $t=\nu$,

$$\text{(24b)} \qquad \frac{\partial x_{Tn+i}}{\partial \beta_t} = -t\beta_t^{t-1} (G_\nu + C_\nu) \frac{\partial x_{Tn+i}}{\partial K} + t\beta_t^{t-1} \mathbf{x}_{t,\, Tn+i},$$

$$\text{(25b)} \qquad \frac{\partial G_T}{\partial \beta_t} = -t\beta_t^{t-1} (G_\nu + C_\nu) \frac{\partial G_T}{\partial K} + t\beta_t^{t-1} \mathbf{x}_{t,\, T}$$

---

7) Cf. Hicks: *ibid*, p. 215.

taking into consideration that, according to (16b),

$$\frac{\partial K}{\partial \beta_\nu}=-\nu\beta_\nu^{\nu-1}C_n.$$

Employing the concept of the "adjusted group of contemporaneous commodities" defined above, we can rewrite these results in more convenient from as follow:

$$(24)\qquad \frac{\partial x_{T+nt}}{\partial \beta_t}=-t\beta_t^{t-1}G_t'\frac{\partial x_{Tn+t}}{\partial K}+\mathbf{x}_{t,\,Tn+t}$$

$$(25)\qquad \frac{\partial G_T}{\partial \beta_t}=-t\beta_t^{t-1}G_T'\frac{\partial G_T}{\partial K}+\mathbf{x}_{t,\,T}.$$

$$(t=1,2,\dots,\nu\,;\ i=1,2,\dots,n\,;\ T=0,1,2,\dots,\nu)$$

Next, we proceed on the case when all rates of interest change uniformly and consider the change of the expenditure G_T of any week caused by this uniform change of interest-rates. Let us suppose that $d\beta_t=\theta\beta_t$, θ to be the uniform rate of change of rates of interest.[8]

Then the increment δG_T of the expenditure G_T in the T week is given by

$$(26)\qquad \delta G_T=\sum_{t=0}^{\nu}\frac{\partial G}{\partial \beta_t}d\beta_t=\sum_{t=0}\theta\beta_t\left[-t\beta_t^{t-1}G_T'\frac{\partial G_t}{\partial K}+t\beta_t^{t-1}\mathbf{x}_{t,\,T}\right].$$

Especially, when

$$i_1=i_2=\dots\dots\dots=i_\nu=i$$

and consequently

$$\beta_1=\beta_2=\dots\dots\dots=\beta_\nu=\beta,$$

we obtain

$$(27)\qquad \delta G_T=-\theta\frac{\partial G_T}{\partial K}\sum_{t=0}^{\nu}t\beta^tG_t'+\theta\sum_{t=0}^{\nu}t\beta^t\mathbf{x}_{t,\,T}$$

8) In the following it is not necessary to take care of the special circumstance that $d\beta_0=0$.

Hicks gives noteworthy consideration to the right side of the last equation (27). The analysis of the phenomenon of tilting of the stream caused by interest-change[9] tells us that its second term $\sum t\beta^t \mathbf{x}_{t,T}$ decreases with T. Likewise, the theory of the "average period of the stream of value" is applied to the expression $\sum_t t\beta^t G_T'$ in its first term, in order to show that the sequence $\{G_t'\}$ is increasing or decreasing (, in his expression, *crescendo* or *diminuendo*), according as $\sum_t t\beta^t G_t'$ is positive or negative.[10] Hicks explains this circumstance verbally in applying the concepts of the "average period of expenditure" and the "average period of receipts".[11]

* * *

I have elucidated Hicks' theory of consumer's planning, applying the commodity-group point of view. Hicks' theory of producer's planning can be also reproduced in the similar way. Before I put an end to the present work, I wish to add a critical remark on the theory of the so-called "average period of the stream of value."

As I take it, the fundamental proposition upon which the theory of average period is founded can be stated as follows:

---

9) Cf. Hicks: *ibid*, pp. 187—188, 232—234.

10) For the aplication of the theory of average period, it is necessary that

$$\beta_0{}^0 G_0' + \beta_1{}' G_1' + \beta_2{}^2 G_2' + \dots\dots\dots + \beta_\nu{}^\nu G_\nu' = 0.$$

But this condition follows directly from (16a, b). See above.

11) As above stated, our G_T is the difference of expenditure and receipts in the T-th week. From this fact follows that the expression $\sum_t t\beta^t G_t$ is equal to the difference of the "average period of expenditure" and the "average period of receipts" except for a positive factor. Therefore, for an individual whose "average period of expenditure" is greater than his "average period of receipts, the expression $\sum_t t\beta^t G_t'$ is positive and, according to the above reasoning, he is said to "plan to be a lender." Similarly, an individual whose "average period of expenditure" is less than his "average of receipts" is described as "planning to be a borrower." Cf. Hicks: *ibid*, pp. 187—188, 232—234.

Let us consider a finite and monotonous sequence

$$\{a_i\}=\{a_1,\ a_2,\ldots\ldots\ldots,\ a_n\},$$

whose simple arithmetical average is a. Next we make a a weighted arithmetical average A from this sequence, employing such weights w_i as

$$0\leqq w_1 < w_2 < \ldots\ldots\ldots < w_n.$$

Then, if the original sequence $\{a_i\}$ is monotonously increasing, then

$$a=\frac{a_1+a_2+\ldots\ldots+a_n}{n}<A=\frac{w_1a_1+w_2a_2+\ldots\ldots+w_na_n}{w_1+w_2+\ldots\ldots+w_n},$$

that is, the weighted average is greater than the simple average. On the contrary, if the original sequence $\{a_i\}$ is monotonously decreasing, then the weighted average is less than the simple average. Hence we obtain a criterion to judge whether a given sequence which is assumed to be monotonous is increasing or decreasing, (*crescendo* or *diminuendo*), provided that the weights employed satisfy the required conditions. A sequence is increasing or decreasing according as the weighted average is greater or less than the simple average.

Now, in the case of our problem, the stream of "discounted" expenditures

$$(28)\qquad \{\beta^t_tG_t'\}=\{\beta^0G_0',\ \beta^1G_1',\ \beta^2G_2',\ldots\ldots,\ \beta^\nu G_\nu'\}$$

is regarded as given sequence and the number t of the order of weeks as weights. Then, in consequence of the relation

$$\beta^0G_0'+\beta^1G_1'+\beta^2G_2'+\ldots\ldots+\beta^\nu G_\nu'=0,$$

the simple arithmetical average of this sequence is zero. Consequently, it depends solely upon the sign of the expression $\sum_t t\beta^tG_t'$, whether the weighted average $\frac{\sum t\beta^tG_t'}{\sum t}$ is greater or less than the simple average $\frac{\sum\beta^tG_t'}{n}$. If this expression $\sum t\beta_t G_t'$ is positive, the stream of "discounted" expenditures (28) must be *crescendo*; if, on the contrary $\sum_t t\beta^t G_t'$ is negative, the stream (28) must be *diminuendo.* Thus, we can judge the shape of the stream of discounted expenditures (28) from the sign of $\sum t\beta^t G_t'$. But what is concerned here is, however, not the shape of the stream of "discounted" expenditures, but that of the stream of "undiscounted" ones,

$$(29) \qquad \{G_t'\}=\{G_0',\ G_1',\ G_3',\ldots\ldots,\ G_\nu'\}.$$

Now, the shape of the former can differ from that of the latter. Since β is less than unity and therefore β^t decreases with t, the "undiscounted" stream (29) is necessarily increasing, when the "discounted" stream (28) is increasing, but the "undiscounted" stream can be increasing, also when the "discounted" stream is decreasing.[12] Thus, strictly speaking, it is not correct to say that the stream of "undiscounted" expenditures is decreasing and the consumer in question is "planning to be a borrower" because $\sum t\beta^t G_t'$ is negative and the average period of expenditure is less than the average period of receipts.[13]

12) This is the case, if the "undiscounted" stream is very slightly increasing.

13) Applying the lemma shown above, we can derive the effect of an upward tilt to the stream of surpluses caused by a fall in the rate of interest (explained by Hicks in his book pp. 216—217, p. 327), from the proposition that a fall in the rate of interest must increase the "average period of the plan" (demonstrated by Hicks in p. 328.). Thus we can show with this lemma that two effects of a fall in the rate of interest, that is, the upward tilt to the stream of surpluses, on the one hand, and the lengthning of average period, on the other hand, are nothing more than the two different aspects of the same thing.