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TECHNICAL COMPLEMENTARITY IN LINEAR PROGRAMMING

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I. Introduction

Economists of the Neo-Classical School usually start their analysis of production from a smooth isoquant. We do not, however, begin our analysis of production with such a smooth curve. We will begin with input-coefficients. (*Gentani* in mengenmässige Rechnung) Thus it enables us to carry the analysis of production one step back from isoquant, which is otherwise the starting point. In other words, it enables us to derive isoquant from more elementary concepts, input-coefficients. And it also enables us to handle a number of special problems that have hitherto eluded our analysis.

In this paper we utilized the method of LP. (Linear Programming) We cannot answer the question whether our approach is superior to another. Sometimes the assumption of divisibility or additivity will be superior to the smooth isoquant of the Neo-Classical analysis, but in other cases it will be inferior to it. With this method of approach we will consider the following problems. First of all, we will derive isoquant from input-coefficients. After this is done we will consider the nature of isoquant and that of the faimily of isoquants. But we will leave the problem of prices outside of the scope of this paper, because by this way, we can analyze the nature of thchnology better.

II. Input-Coefficient

Consider a company which produces one kind of output (automobile) with the use of two factors. (labor and machine) This company may be able to produce one unit of output with one unit operation of a new type of machine. When this company does not utilize this machine, it will be

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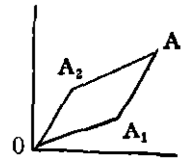
required to employ 2 units of labor to get the same unit of output. When we measure the labor units on the horisontal axis and the operation hours of the machine on the vertical axis, the combinations of two inputs–labor and machine–will be shown by points on either axis. When this company produces one unit of output using 3/2 units of labor and 1/4 units of machine in combination, then this combination of inputs can be expressed by a point A in the plane, but not on the axis. Each of these points represents a combination of inputs, which are required to produce one unit of output, i. e., it represents input-coefficient.

Last combination of input coefficients shown by A means the following; in order to produce one unit of automobile, it is necessary to input labor and machine in the ratio of 6 : 1. This ratio is assumed to be constant so long as we use this method of production. We can show this ratio by a ray through this point. By a ray we mean a half line from the origin, and a ray which goes through A will be denoted as (A). If there is another method of production, there will be another ray correspondingly.

In the above statement we assumed that, given one method of production, the ratio of the units of labor and machine will remain constant, even if the units of output vary. But even if this assumption is fullfilled, we cannot say whether the combination of inputs which is required for the output of the first unit of production is equal to the inputs which is required for the second unit of production. If the second unit of production requires the smaller units of input than that of the first unit of production, we usually say that there are economies of large scale production. In this case we can save input by large scale production. On the contrary, if it is smaller, there are diseconomies due to large scale production. However we assume for simplicity that there are no economies nor diseconomies due to large scale production. In other words, we assume constant return to scale. This assumption is called the postulate of divisivility in LP. (This is not equal to the assumption that the units of output can vary continuously, which we assume in LP.)

In the above statement, we plotted a point representing input combination in the input-plane. We can think this point represents also unit level of output. In this case, if we plot points showing 1, 2, 3,... units of output on this ray, then the interval of these points have equal distances between them. These points show that, when the units of inputs are doubled, tripled, etc., then the resulting output will be doubled, tripled, etc., correspondingly. When the law of diminishing return to scale holds then these intervals become shorter, and when the law of increasing return to scale holds, then they will become longer as the scale of production expands.

In this way we can regard this point as showing the activity level of this production method.



Next, let us consider the case where our company has two methods of production, and it can use both of these two, without changing their respective input-coefficients. To explain this case, consider a company producing (or assembling) automobiles. In one method of production, we require 2 units of labor and 1 unit of machine operation. This may be expressed by a point $A_1 = [2\ \ 1]'$. (Prime means the transpose of a vector.) Another method requires 1 unit and 1.5 units of labor and machine respectively. $A_2 = [1\ \ 1.5]'$. If this company uses these two methods simultanously, 3 units of labor and 2.5 units of machine $A = [3\ \ 2.5]'$, then it will be able to get 2 units of automobiles. This can be thought of as using two methods of production successively, i. e., first when we input $[2\ \ 1]'$ by the method I then we shall move to the point A_1, *after* that when we input $[1\ \ 1.5]'$ by the method II then (because AA_1 is parallel and equal to $0A_2$) we will move from A_1 to A. This can be eqpressed as

$$A_1 + A_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 1+1.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2.5 \end{bmatrix} = A.$$

This relation means that the elements of A are equal to the sum of the corresponding elements of A_1 and A_2.

In order to be able to get this relation, the following postulate must be fulfilled. That is, in order to produce one unit of automobile from 0, we need inputs by $[1\ \ 1.5]'$. But, starting from A_1, can we produce the same units of automobiles or can we get more, when we use the same units $[1\ \ 1.5]'$ of inputs? In the real world, it may happen that this same input $[1\ \ 1.5]'$ on the second step may produce more than one unit of automobile by the help of the former production. In other words, it may produce more automobiles that mere arithmetical addition. But, here, we ignore this possibility, and we simply assume that when we use input $[1\ \ 1.5]'$ from A_1, we can produce exactly the same one unit of automobile. Our assumption that these are no other possibilities than mere arithmetical addition is called the postulate of additivity in LP.

III. Isoquant

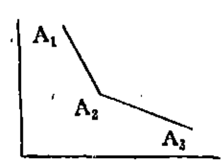
Now let us assume that our comany has three different methods of production. A_1, A_2, A_3 denote the combination of input coefficients by each method. By the three line segment connecting these three points, we can draw a triangle.

As is well known, we can express a point on the line segment $A_1\ A_3$ as

$$xA_1+(1-x)A_3 \qquad (1\leq x\leq 0)$$

where A_1, A_3 are two end points of the segment. We can think of this point on the line segment A_1A_3, as representing the required combination of inputs to produce one unit of output. When A_2 lies on the line segment A_1A_3, then we can produce one unit of automobile by two different ways, i. e., one is by A_2, another is by the appropriate combination of two methods A_1 and A_3. But both of these ways use exactly the same units of input and produce exactly the same one unit of output. Therefore, we can say that A_2 is ' logically dependent ' on A_1 and A_3. But here we exclude these possibilites. In this case. we can say that when there are three different methods of production, we have a triangle. Next, let us consider the nature of this triangle.

When we look at the points in this triangle from the view-point of output, we can think of them as having the same characteristic, that is, they all represent one unit of automobile. Each of the three extreme points and any points which lie between these points represent the common characteristic–they all show one unit of automobile. Sometimes, it is called mathematically the convex set, because it has the following properties : any point on the line segment connecting two points in the set lies in the set. In other woards, any point on the line segment connecting two points, each of which represents one unit of output shows one unit of output. But, when we look at these points from the view-point of input, then they represent a very different characteristic. For instance, when we compare any two methods of production, it is clear that the method of production, which requires larger units of input, is technically inefficient to the other one. This is also the case. when one of the quantities of the two inputs is equal in the two methods of production. In order to stress this point, we use the word ' technically efficient ' input coefficients. Hereafter, we will leave the technically inefficient methods out of our considerations and concentrate our attention only on the remaining cases, technically efficient methods. When we apply this efficiency condition, we can do as follows: first, by using the line segment A_1A_3 as separation segment, if A_2 lies on the opposite side to the origine, we call A_2 as technically inefficient one and leave it out of our considerations. Second, draw a ray through A_2 and call the intersecting point with line segment A_1A_3 as A. Then this point A represents the combination of input required to produce one unit of automobile. (by the combination of two methods A_1 and A_3) There are two points, A_1 and A_4 on a ray (A_2) which pass through A_2. But clearly, A is more inefficient than A_2. In the



following, such points will be left out of our considerations. Therefore, we are left only with the line segments A_1A_2, A_2A_3. These are the left-below boundaries of the triangle. We call this kinked line segment an isoquant. (This is supposed to be a smooth curve by the Neo-Classical School.) In this case, if we use the method A_1 which requires longer machine hours than A_2, to produce one unit of automobile, it will require shorter labor hours than A_2, i. e., the isoquant is downwards sloping. (see the above figure) When we move from $A_1=[a_{11}a_{12}]'$ to $A_2=[a_{21}a_{22}]'$, then the labor hours will be increased by $(a_{11}-a_{21})$, but machine hours will be decreased by $(a_{12}-a_{22})$. Let us call this ratio, i. e., the machine hours saved by the one unit addition of labor hour $-(a_{13}-a_{22})/(a_{11}-a_{21})$ as MRS the marginal rate of substitution of labor for machine. This represents the slope of lie segment A_1A_2 (with the negative direction of the horiz ontal axis.) It is obvious that this is positive; MRS of labor for machine is positive.

When we produce one unit of automobile by combining two methods, by a a half of A_1 and a half of A_3, the required input will be the sum of of the halves of inputs represented by A_1 and A_3. If there is another method of production between them, the required units of input will be smaller than this sum. Therefore, downward sloping isoquant is a line which may or may not have kinks. When it has a kink, it is convex to the origin.

Therefore the (absolute) value of the slope of isoquant decreases as we move to right along a isoquant. The (absolute) value of their slope represent MRS of labor for machine; therefore, we can say that MRS of labor for machine decreases as we substitute labor for machine. But when there is no (logically independent) method of production, MRS does not change.

Now, let us denote three input coefficients as

$$A_1=[a_{11}a_{12}]' \qquad A_2[a_{21}a_{22}]' \qquad A_3=[a_{31}a_{32}]'$$

The slope of A_1A_2 is greater (in absolute value) than that of A_1A_3, therefore we have

$$-\frac{a_{12}-a_{22}}{a_{11}-a_{21}} > -\frac{a_{12}-a_{32}}{a_{11}-a_{31}},$$

when the labor input is greater in A_2 than A_1, i. e., $a_{11}<a_{21}$ as in the above figure. Conversely, when $a_{11}>a_{21}$, the sign of inequality changes the direction. From these two inequalities we get

$$\begin{vmatrix} a_{31} & a_{21} \\ a_{31} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{11} \\ a_{22} & a_{12} \end{vmatrix} < \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix}$$

When the point A_2 lies on the line segment A_1A_3, this inequality sign is substituted by equality. (See the Simplex Criterion).

In the above, we considered MRS at A_1, increasing the units of labor.

But when we consider MRS at A_2, there is more than one value for it. We will take up this point later.

IV. Marginal Product of Labor

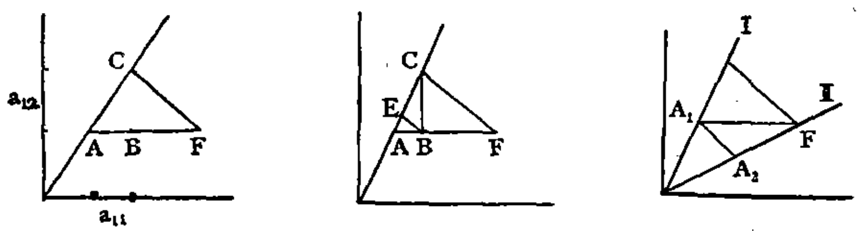
We are considering a company which produces only one kind of output, automobile, by using two kinds of input, labor and machine. And this company is here assumed to have many methods of production. When we have the following two-way table, we can tell how much output we can hope to get, it uses so much labor and so much capital. Such a table is called a production schedule.

units of capital							
	6	346	490	600	693	775	846
	5	316	448	548	632	705	775
	4	282	400	490	564	632	693
	3	245	345	423	490	548	600
	2	200	282	346	400	448	490
	1	141	200	245	282	316	346
		1	2	3	4	5	6
		units of labor					

If we want to know what output there will be when 5 units of capital and 2 units of labor are used, we count up 5 units of capital and go over 2 units of labor. The answer is 448 units of products. Similarly, we find that 3 units of capital and 6 of labor will produce 600 units of output. Thus for any combination of labor and capital, the production schedule tells us how much product we shall have. (using, of course, the best methods as decided by the technical engineer.)

This production schedule is often expressed mathematically as $P=f(L, C)$ and is called production function, where P, L, C represent product, labor and capital, respectively. In the above schedule it is implicitly assumed that there are six methods of production expressed by rays passing through the following points; (2, 3), (3, 6), (3, 4), (4, 3), (6, 3) and (6, 2). In the above we got an isoquant connecting the points of input coefficients, which is necessary to produce one unit of output. Following the same prodedure and connecting input coefficients which is required to produce two units of outputs, we get another and similar isoquant. By exactly a similar way, we can get many isoquants corresponding to 1, 2, 3,... units of outpnt. As we have postulated above, if we increase ten per cent of all inputs, starting from a combination of input of labor and capital which is required to produce ten units of automobile, we can get one extra unit of automobile. In other words, when we increase all input by k per cent, the units of output will be increased by the same proportion. In this case, when we plot points representing 1, 2, 3,.. units of output on an arbitrary ray showing a method of production, the distances between them will be the same. A ray intersects with isoquants of 1, 2, 3,... units of output at the equal distances. In this case isoquants which are cut by two neighboring rays are parallel to each other. Therefore, slopes of isoquants (MRS) at any point of one ray are equal to each other. (See the second condition of the mathematical Appendix)

Let us now consider the concept of 'MPL', the marginal product of labor. As is well known, the addition to the output caused by an extra unit of labor is called MPL. This is obtained from the above table as follows. At any point in the table, MPL can be derived by substructing the given number (representing output at that point) from the number on its right lying in the same horizontal row. Thus, when there are two units of capital and four units of labor, MPL is 48 or 448–400. Similarly we can compute MPL at any point inside the table.



Now let us take two points A and C on one ray representing one method of production, where A represents one unit of automobile, C represents two. When we move from A to C, we get one extra unit of output. At the same time we can think of it as representing the required extra input combination of labor and machine, which is equal to input coefficient. Now let us consider a following experimental case, where we produce one extra unit of automobile by adding appropriate units of labor, while keeping the operation hours of machine constant and show the resulting required combination of input by a point F on the horizontal line through A. This point F lies on the same isoquant which pass through C. If we call B the intersection of AF with the vertical line throuh C, then AB represents the units of labor and BC the units of operation hour of machine, which are required to produce one extra unit of automobile from A to C. Now we can divide the movement from A to C into two parts; one is from A to B, the other is from B to C. Then the ratio AB/AF represents the units of output, which is added by the increase in labor input by a_{11} (from A to B). Therefore MPL, i. e., the extra units of output which are obtained by the addition of labor by a_{11},

$$\frac{\text{extra units of automobile produced}}{\text{addition of labor input}}$$

are obtained by the unit of automobile produced AB/AF divided by the labor input required for it, a_{11}, i. e., AB/AFa_{11}. This MPL can be obtained by counting the number of intersecting points with line segment AB. Let us now call E the intersection of AC with a line parallel to CF and passing througugh B, then we get following relations,

$$AE=(AB/AF)AC, \quad EC=(BF/AF)AC.$$

When we move from A to C, the automobile will be produced by one unit ($=AC=AE+EC$). Therefore, using the above relation, we have

$$AC=(AB/AF)AC+(BF/AF)AC.$$

Here coefficient (AB/AF) shows the increase in the units of automobile due to the addition of labor by a_{11}. When we divide it by a_{11}, we get MPL which is denoted here as α. The units of labor which are required to produce one unit of automobile $AC=1$ will be denoted as $a_{11}AC$. We can say the same things about the coefficient (BE/AF), which represents the addition to automobile due to the increase in the units of machine hour by a_{12}. Then $(BF/AFa_{12})=\beta$ represents MPM (i. e., of machine) and a_{12} AC represents the units of machine which are required to produce one unit of output.

When we use the postulate that even if we extend the scale of production the input coefficients remain unchanged, we can say that, when we produce n times of automobile, the units of inputs are to be changed by n times proportionately; thus we have

$$\text{units of output}=\alpha\times\text{units of labor}+\beta\times\text{units of capital.}$$

This relation is usually derived by using the Euler's Theorem assuming differetiability. Here we derived it without reference to differentiability.

We denote the input coefficients of the method I as a_{11}, a_{21} and of the method II as a_{12}, a_{22}, then $A_1=[a_{11}\,a_{21}]'$, $A_2=[a_{12}\,a_{22}]'$ represent two methods of production each of which can produce one unit of output. When we use the machine by a_{12} in the method II (using the appropriate units of labor), then we shall be able to reach $F=[a_{31}\,a_{32}]'$, where $a_{32}=a_{11}$. Then this point F on the ray II is m times as far as that of A_2, i. e., $0F=m\,0A_2$. This means that we can get $m(=a_{12}/a_{22})$ units of automobiles. Let us use a_{12} or a_{32} as the unit of measurment of the operation hours of machine. Then A_1 represents one $(=1/a_{12})$ unit of automobile, and F represents $m(=a_{32}/a_{22}=1/a_{22})$ units of automobile. In order to produce these units of automobile, a_{11}/a_{12}, a_{21}/a_{22} units of labor will be required respectively. Therefore, when we produce $(1/a_{12})$ units of automobiles by I, we have

$$1/a_{12}=\alpha\, a_{11}/a_{12}+\beta$$

and, when we produce $(1/a_{22})$ units of automobiles by II, we have

$$1/a_{22}=\alpha\, a_{21}/a_{22}+\beta$$

Cancelling β from both of these two equations, we have

$$\alpha=\frac{a_{11}-a_{22}}{a_{21}a_{12}-a_{11}a_{22}}$$

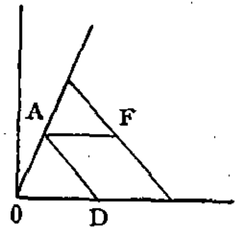
When we move from I to II, input coefficients will be changed by

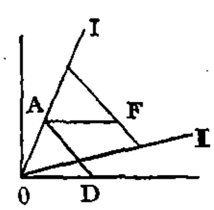
$$a_{21}=a_{11}+\triangle a_1,\quad a_{22}=a_{12}+\triangle a_2$$

then α will be expressed as

$$\alpha=1/[a_{11}-(\triangle a_1/\triangle a_2)a_{12}]$$

It is already clear that the isoquants which lie between the two rays I, II are all parallel to each other, and the slope of this isoquant is given by $-\triangle a_1/\triangle a_2$. But we must make distinction between two kinds of isoquants; 1_0 one has a kink at F and 2_0 another has not. Even if F has a kink at F, we can change it into 2_0, by properly shortening the distance of the isoquants. We shall consider the case 1_0 afterwards, and we will concentrate our attention here to the case 2_0. Now let us draw an auxiliary line with a slope of $-\triangle a_1/\triangle a_2$ from A_1 and denote the point of intersection with the horizontal axis as D. (This auxiliary line is parallel to the isoquants which lies between two rays, I and II, the latter may be identical with the horizontal axis.) The units of labor represented by D is equal to





$$a_{11}-(\triangle a_1/\triangle a_2)a_{12}.$$

And, as the length 0D is equal to AF, the units of labor represented by AF is equal to that of 0D. As ws have stated above, MPL at A (when units of labor are increased) is expressed as

AF measured by output/AF measured by input.

In the present case, the numerator is equal to 1 and the denominator to labor input expressed by 0D. Therefore, MPL at A is equal to

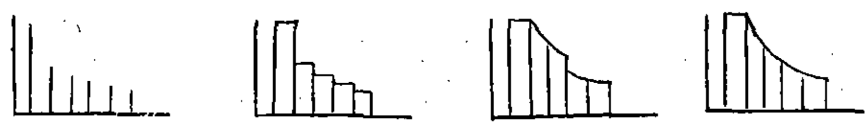
1/0D measured by input.

This is equivalent to unity divided by the units of labor which is expressed by 0D, where D is the point of intersection of horizontal axis with the above auxiliary line. When the isoquant has a kink at F, then the slope of isoquant will be changed at F, and the number of values of MPL at F is greater than one. Starting from F we can move experimentally either to right or to left along the horizontal line, i. e., we may either increase or decrease the units of labor. Corresponding to these two movements, we have two MPL at F. In the foregoing statement we considered the case of addition of the units of labor.

Thus far, we are concerned with the definition of MPL. Next, we must consider the nature of this MPL; in other words, we will consider the behavior of MPL when we change the units of labor or machine. First of these, we will consider the behavior of labor, while keeping the units of capital constant. Using the above production schedule, we can show the following nature. When we increase the units of labor successively one by one, keeping the units of capital constant, MPL will decrease monotonously. For instance, keeping the units of capital at 3, let us increase the units of labor from zero to one, two, three, . . . , then we shall have the following derived table a. This is shown by the height of the vertical line in figure b. In this figure, the height of the first vertical line shows the units of output which is obtained by an addition of one unit of labor (from zero unit of labor), keeping the units of capital at three. We can of course obtain the units of output by the first two units of labor, when we add the height of the first two lines. In the figure b, we assumed that the units of labor could not be divided smaller than the natural numbers 1, 2, 3, . . But in LP, as we indicated above, we make an assumption that the units of output and input can vary continuously. In this case, we can have the figure c for MPL.

1	245
2	101
3	77
4	67
5	58
6	52

a



b c d e

Our problem here is to compare two MPL, one is at A, another is at F, where A and F both lie on a same horizontal line. As we have seen above, we can express MPL at F (when units of labor are increased) as

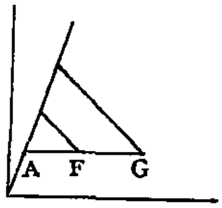
FG measured by output/FG measured by input

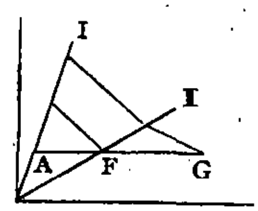
Where G represents the combination of inputs, which is required to produce one extra unit of automobile, by increasing the labor inputs from F, while keeping the units of capital constant. When we measure FG by output, we have the relation FG=1; therefore we have for MPL

1/FG measured by input.

We must distinguish here two cases. One is the case where there is no ray reyresenting production method, another is the case where there is one or more rays representing production methods, between the two rays (A) and (G), where (A), (G) denote the rays which pass through A or G, respectively. In the former case, we have AF=FG, but in the latter case, we have AF<FG. (See the following figure.) Thus the number of times of decrease of MPL is equal to the number of (logically independent) rays which lie between the two rays (A) and (G). Therefore, MPL will either

decrease or remain constant, but never increase, as the units of labor are increased.





In order to help the understanding of our argument, it will be convenient to put here one digressional remark about the differentiability of isoquant, which is usually assumed in the Neo-Classical School. In the above, in explaining the figure c, we assumed that the unit of output and input can vary continuously. But we also assumed that the number of (logically indepedent) production methods is finite following the usual custom of LP. (In the above production schedule, there are only six methods.) But if we drop this assumption and allow our company to have (logically indegendent) infinite number of production methnds, then the figure c of MPL will change into figure d, for example. But it does not necessarily follow that there are no steps on the curve of MPL. There will be, in general, some steps. (See figure d.) But if we make an additional assumption –of no steps– then we can have a smooth cusve of MPL like figure e. We must be very careful about this additional postulate. In order to see the meaning of this postulate, let us compare the two figures d and e. Crucial differences will be seen when we look at the fourth unit of labor input. In the case of figure e we have only one value of MPL. But in the case of figure d, there is some range for the values of MPL at the fourth unit of labor. In other words, consider the case where our company approaches this fourth unit of labor either from the right or from the left; i. e., either by increasing or decreasing the units of labor. When our company has a technology like figure d, there are two distinct values for MPL, but if it has a technology like e, there will be only one value for MPL. When our company has a technology like figure e, we can say that the technology satisfies the differentiability postulate. But when our company has a technology like figures b, c or d, we say that the differentiability postulate is not satisfied. As is well known, the Neo–Classical School usually postulates a technology like figure e, but LP postulates figure c. We are here following the methods of LP, so we do not assume the differentiability. Therefore we have one difficulty which we can not determine MPL uniquely. In order to ease this difficulty, we make here a supplementary rule.

When it is necessary to determine MPL uniquely, we must always measure the change in output, increasing the unit of labor, when we admit this rule, we can say that, if we want to have a unique value of MPL, we can have it at the cost of continuity in the curve of MPL. On the contrary, if we want to have a continuous (but stepped) curve of MPL, we can have it at the cost of uniqueness of MPL. When we have these remarks in mind, we can safely employ the same notations whith the ones which are commonly used in differential calculus. For instance $\partial P/\partial L$ means MPL and $\partial^2 P/\partial L^2 \leqq 0$ means the decreasing (or strictly speaking, non-increasing) MPL. (We used these notations in mathematical appendix.)

In this section, we considered the behavior of MPL, changing the units of labor input. In the following section we will again consider the bahavior of the same MPL, but by changing the units of capital input instead of labor input. This is so-called the problem of technical complementarity.

V. Technical Complementarity

Let us again return to our production schedule. As we have said above, we derived MPL for three units of capital (third row of the following table), increasing one unit of labor step by step. By a similar way, we can compute MPL for each row of the following table. In this way we can derive the following table showing MPL, increasing one unit of lobor step by step.

units of machine							
	6	346	144	110	94	82	71
	5	316	132	100	84	73	70
	4	282	118	90	74	68	61
	3	245	101	77	67	58	52
	2	200	82	64	54	48	42
	1	141	59	45	37	34	30
		0	1	2	3	4	5
		units of lrbor					

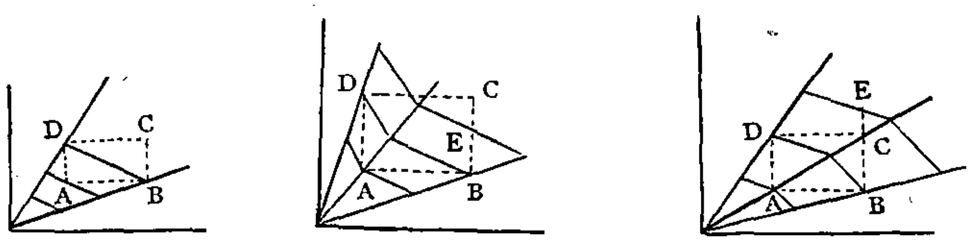
We can find very interesting natures in this table. When we look at each (horizontal) row, we can find that MPL is decreasing as we move to the right: as we increase the units of labor. Similarly, when we look at each (vertical) column from bottom to top, we can find that MPL is increasing; it increases as we increase the units of capital. In this section, we will consider this latter nature.

In the real world, it is very interesting to ask whether there is a difference between the following two cases. The first case is this. Our comapny starts to produce 10 units output by using the method I from zero point. Another case is this. Our company is already producing 100 units of output by the method II, and begins to produce extra 10 units of output by another method I. Of course, we cannot answer this question generally. But as mentioned above we simply postulated the additivity; there is no difference between these two cases. In other words, it makes no difference whether our company produces 10 units 1_0 by method I and uses no other methods or 2_0 by method I *after* it produces some units of output by the other method, II. This means that even if our comapny uses two different methods simultanously, they do not help nor check each other. When we look at this nature superficially, it seems to us that there is no possibility of technical complementaroity. But when we look it more closely, we can find that there is the possibility of technical complementarity. Moreover, we can show that there cannot arise the case of technical substitution. We will take up this problem in this section.

First of all, we will attempt to show the complementary relation on the two dimentional diagram, and after that we will derive the complementary relation from the divisibility postulate without using the help of prices. On this point Professor Hicks made the following misleading comment;

"It is a very curious consequence of our new definiion that the indifference diagram... proves to be of little direct use for that particular problem... the problem of related goods cannot be treated on two dimentions to represent the two related goods and *money* (the necessary background). This means that the theory is most conveniently presented either in algebra..., or, as here, in ordinary words." p. 45.

Our company is producing automobiles by using two kinds of input labor and machine. We denote by the point A the combination of inputs which are required to produce two units of automobiles. Starting from this point, let us experimentally, move to B where we produce one extra unit of automobile by adding appropriate labor, while keeping the units of capital constant. By exactly the same way, let us move from A to D where we produce one more extra unit of automobile by adding appropriate capital instead of labor and keeping the units of labor constant. Clearly, both B and D lie on the same isoquant which represents three units of automobile.



a b c

Now let us here construct an auxiliary rectangle which has AB and AD as its two neighboring sides and call the fourth vertex as C. If our company has a technology like figure a, the isoquant which represents the four units of outomobile will pass through the point C. This point C represents a combination of inputs which are required to produce four units of automobile adding one extra unit of automobile from D. In the present caė the units of labor which are required to move from A to B are equal to that which is required to move from D to C, where both movements represent, respectively, the one unit additon of output.

But we have a quite different situation if the technology of our company is shown by the figure b. In the former case there was no (technically independent) method of production between two rays (B) and (D). But in the present case there is one. Here we assume that it is shown by a ray (A) which passes through the point A. Therefore, isoquants have kinks on this ray (A). When we construct an auxiliary rectangle as before, the fourth vertex C does not lie on the isoquant which represents the four units of automobile. In the cases of the technology shown by the figure b, the point C lies on the outside (the opposide to the origin, using an isoquant which passes through B and D as the separating line.) In this case, the units of labor which are required to produce one extra unit of automobile from A are greater than that from D. That is, in order to produce one extra unit of automobile, it is required to use labor input by AB or DC when we start from A, while it is only DE when we start from D, i. e., in the latter case where the operational hours of the machine are longer than in the former case, labor input will be saved by EC.

As the final case, let us consider the following imaginary case, where our company's technology is shown by figure c. In this cuse, as in the second case, there is a (logically independent) production method which is shown by a ray (A), between (B) and (D). But contrary to the second case, the point C lies on the inner side (the same side with the origin using the isoquant, which passes through B and D, as the separating line). In this case, in order to produce one extra unit of automobile, it is required to use labor by AB or DC when we start to expand from A, while the labor input will be DE when we start from D, i. e., in this case where the units of operational hours of machine are greater than former, the units of labor inputs will be greater by CE.

When we look at the difference between two points A and D, we can find that there is only one difference, i. e., the operational hours of the machine are longer in D than in A. In the figure b, one extra unit of labor can produce more automobiles when the operational hours are longer. In this case we can say that the machine helps the productivity of labor. We call such relation between machine and labor as techincal complementarity. This is the definition of technical complementarity given by Professor Hicks. He says; "The two lactors will be complementary, we must remember, if an increase in the employment of *A* (with *B* constant), and consequent increase in the output of *X*, moves the marginal rate of transformations of B into X in favour of B; that is to say, raises the marginal product of B . . ." p. 95.

Contrary to this case, when the machine operation checks the productivity of labor, as in the figure c, we call the relation between them as technical substitution. And, when machine operation does neither help nor check the productivity of labor, we call the relation as technical indenendence. See figure a. Then we can say that technical complementary relation corresponds to the convexity (see figure b) and substitution to concavity (see figure c) to the origin of the isoquant. In the second section of this paper we excluded the techincally inefficient methods of production, so our isoquant can never have the shape of concavity to the origin. So we are left with technically complementary or independent relations.

Professor Hicks stated as follows, " Now consider what happens in those special conditions of production, when the contribution of the fixed ' productive opportunity ' of the enterprise vanishes, so that costs do not rise with increasing output; and in which no economies of large scale are present either, so that costs do not fall with increasing output, and the situation is just consistent with perfect competition. Costs (both average and marginal) are constant; the surplus is zero; when each factor is paid a price per unit equal to its marginal product, the total product is exactly exhausted. Since marginal product is constant, the increase in product due to a simultanous product of the two factors taken together) must be constant. . . Therefore the factors A and B must be compementary. Thus, if the fixed productive opportunity does nothing to limit the scale of production, the two factors must be complementary." pp. 94-95.

This is the Professor Hicks' derivation of complemeetary relations from the postulate of constant return to scale. But he used the help of prices in his derivation. We derived the same conclusion without using the help of prices.

If we drop the postulate of divisivility and admit the possibility of diseconomies of large scale production, it is possible to have the case of technical independence or substitution, as Professor Hicks says; " . . . if the fixed productive opportunity' does nothing to limit the scale of production, the two factors must be complementary. As soon as it does something to limit expansion, the two factors are not inddeed, necessarily complementary." p. 95.

VI. Mathematical Appendix

In the last section, we considered the case where there were only three commodities, one output and two inputs. Here, we can easily generalize the above argument. But in order to do this it is necessary to make one additional assumtion. This assumption is pointed out by Professor Hicks as follows: " My test would deped on what happened to the marginal product of B (B constant) if the supplies of other factors (C & c.) were not kept constant. but varied in such a way as to leave their marginal products unchanged." p. 94. footnote 2.

Then all of the conditions for complementary relation can be formulaten as follows. As our company produces one kind of output x_0 employing n kinds of inputs $x_1, \ldots, x_n$, we express thechnical production function of it as

$$x_0+R(x_1, \ldots, x_n)=0$$

Then labor x_1 is complementary with capital x_2 if MPL increases when the operational hours increases ($\delta x_2>0$) $\delta R_1/\delta x_2>0$, while keeping the following terms constant;

1 production function
2 the units of labor x_1
3 MP of x_k $(k=3, \ldots, n)$

By solving the following simultanous equations;

$$\frac{dx_0}{dx_a}+R_2+R_3\frac{dx_3}{dx_1}+\cdots\cdots+R_n\frac{dx_n}{dx_1}=0$$

$$R_{10}\frac{dx_0}{dx_2}+R_{12}+R_{13}\frac{dx_3}{dx_2}+\cdots\cdots+R_{1n}\frac{dx_n}{dx_2}=\frac{\delta R_1}{\delta x_2}$$

$$R_{20}\frac{dx_0}{dx_2}+R_{22}+R_{23}\frac{dx_3}{dx_2}+\cdots\cdots+R_{2n}\frac{dx_n}{dx_2}=\frac{\delta R_2}{\delta x_2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R_{n0}\frac{dx_0}{dx_2}+R_{n2}+R_{n3}\frac{dx_3}{dx_2}+\cdots\cdots+R_{nn}\frac{dx_n}{dx_2}=9$$

we have

$$\frac{\delta R_1}{\delta x_2}=-\begin{vmatrix} 1 & 0 & 0 & R_3\cdots R_n \\ R_{10} & 1 & 0 & R_{13}\cdots R_{1n} \\ R_{20} & 0 & 1 & R_{23}\cdots R_{2n} \\ R_{30} & 0 & 0 & R_{33}\cdots R_{3n} \\ \cdots & & & \cdots \\ R_{n0} & 0 & 0 & R_{n3}\cdots R_{nn} \end{vmatrix} \cdot \begin{vmatrix} 1 & R_2 & 0 & R_3\cdots R_n \\ R_{10} & R_{22} & 0 & R_{13}\cdots R_{1n} \\ R_{20} & R_{22} & 1 & R_{23}\cdots R_{2n} \\ R_{30} & R_{32} & 0 & R_{33}\cdots R_{3n} \\ \cdots & & & \cdots \\ R_{n0} & R_{n2} & 0 & R_{n3}\cdots R_{nn} \end{vmatrix}^{-1}$$

For simplicity, let us introduce the following notations

$$D=\begin{vmatrix} 1 & R_1 & R_2\cdots R_n \\ R_{10} & R_{11} & R_{12}\cdots R_{1n} \\ R_{20} & R_{21} & R_{22}\cdots R_{2n} \\ \cdots & & \cdots \\ R_{n0} & R_{n1} & R_{n2}\cdots R_{2n} \end{vmatrix}$$

D_{ij} =cofactor of R_{ij} in D

D_{ijkl}=cofactor of R_{kl} in D_{ij}

then we can express the above relation as

$$\frac{\delta R_1}{\delta x_2}=\frac{D_{1122}}{D_{21}}$$

This is the definition of related goods in terms of technology. On the other hand, the change in demand for x_1, when the price p_2 changes, is given by

$$\frac{\partial x_1}{\partial p_2}=\frac{D_{21}}{p_0 D}$$

This is the definition of related goods in terms of price change. "if we suppose... a rise in the price of the product X ... it is possible that there may be some outputs which are complementary with X so that they will be expanded with it. p. 203. From these two equations, we can form the following relation;

$$\frac{\delta R_1}{\delta x_2}\cdot\frac{\partial x_1}{\partial p_2}=\frac{1}{p_0}\ \frac{D_{11}}{D}\ \frac{D_{1122}}{D_{11}}>0$$

This shows that the two definitions of technical complementary relation, one in terms of technology, and another in terms of price change, have equal signs.

Next we will show that machine x_2 (which is chosen arbitrarily) is always technically complementary with labor x_1, when the following two condition are satisfied:

First, the output x_0 is the homogenous function of degree one with respect to all inputs, $x_1, \ldots x_n$. (divisibility postulate)

Second, MP of x_k; $-\partial x_0/\partial x_k=R_k$ is homogenous function of degree zero with respect to output x_0 and all inputs $x_1 \ldots, x_n$. (This condition is dependent to the first one. See section IV)

Fromthe first of these conditions, we have

$$dx_0+R_1dx_1+R_2dx_2+\cdots+R_ndx_n=k(x_0+R_1x_1+R_2x_2+\cdots+R_nx_n)=0$$

where $dx_i=kx_i \quad (i=1,\cdots, n)$

and from the second condition we have

$$R_{i0}dx_0+R_{i1}dx_1+R_{i2}dx_2+\cdots+R_{in}dx_n=h(R_{i0}x_0+R_{i1}x_1+R_{i2}x_2+\cdots+R_{in}x_n)=0$$

where $dx_i=hx_i\,(i=1,\cdots, n)$

using these two relations, we can show that D_{21} is equal to $-D_{22}$. By substituting the elements of the second column by the sum of all the elements of all other columns multipliying the first column by x_0/x_1 and the kth column with corresponding x_k/x_1, $(k=2\cdots n)$ and using the above relations, we have

$$D_{21}=\begin{vmatrix} 1 & R_2 & R_3\cdots R_n \\ R_{10} & R_{12} & R_{13}\cdots R_{1n} \\ R_{30} & R_{32} & R_{33}\cdots R_{3n} \\ \cdots & \cdots & \cdots \\ R_{n0} & R_{n2} & R_{33}\cdots R_{3n} \end{vmatrix} = -\begin{vmatrix} 1 & R_1 & R_3\cdots R_n \\ R_{10} & R_{11} & R_{13}\cdots R_{1n} \\ R_{30} & R_{31} & R_{33}\cdots R_{3n} \\ \cdots & \cdots & \cdots \\ R_{n0} & R_{n1} & R_{n3}\cdots R_{nn} \end{vmatrix} = -D_{22}$$

$$\therefore \frac{\delta R_1}{\delta x_2} = -\frac{D_{1122}}{D_{22}} > 0$$

This shows that labor x_1 is always complementary with machine x_2. In this mathematical appendix we proved the above relation postulating differentiability. We must pay special attention to the remark which we gave in section IV, when we do not postulate this. In this case it may happen that the values of the above determinants are zeros.

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