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ON THE "AVERAGE PERIOD" OF J. R. HICKS

By Sempei SAWA*

I Preface

This article is intended to confirm by mathematical analysis the theory of the "Average Period" which J. R. Hicks states mostly verbally and rather concisely in Chapter 14 of his *Value and Capital*, and to examine it more closely. For it seems possible that different assumptions lead to different conclusions so that this theory is not necessarily applicable to all cases in general.

I am very grateful to Mr. Shinryo Taniyama for his *Structure and Character of Life Insurance*[1], Chapter 3, in which he tried to apply the theory of "Average Period" to mathematics of life insurance. It needs to be added, however, that my procedure and conclusion are different in many respects from his article[2].

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1) Shinryo Taniyama, *Structure and Character of Life Insurance*, 1962, Economics Department of Osaka Prefectural University.

2) (i) Sawa's proof of the proposition seems to be more persuasive and demonstrative.
(ii) Mr. Taniyama has shown the lemma only partially.
(iii) Since Mr. Taniyama applied the theory of "Average Period" to life insurance during the process of his analysis, he cannot be said to have shown this theory as a general rule; whereas Sawa has, showing it in the beginning by mathematical analysis as a general rule, tried to apply it to life insurance afterward.

II Concept of the Average Period

Let a stream of capital values $\{M_t\}$ be denoted by M_0, M_1, M_2, ..., M_n, which are distributed into each period in the future 0, 1, 2, ..., n, beginning with the present point 0, the rate of interest by r, and the discount ratio by such an equation as $\frac{1}{1+r} = v$, the present values of these capital values can be expressed as

(1) $M_0 v^0,\ M_1 v^1,\ \cdots\cdots\cdots,\ M_n v^n$.

If $t > 0$ be correct here, M_t should be a future expected value and $M_t v^t$ a future expected value discounted to the present value. If the sum of these future expected values discounted to the present value be denoted by M, we have the following equation:

(2) $M = M_0 v^0 + M_1 v^1 + M_2 v^2 + \cdots\cdots + M_n v^n = \sum_{t=0}^{n} M_t v^t$.

When the elasticity of the sum of the stream of capital values M in regard to discount ratio v is denoted by π, it can be obtained according to the well-known definition of elasticity in general in the way,

$$\pi = \frac{v}{M}\frac{dM}{dv}$$

$$= \frac{v}{M_0v^0 + M_1v^1 + M_2v^2 + \cdots\cdots + M_nv^n}(M_1 + 2M_2v^1 + 3M_3v^2 + \cdots\cdots + nM_nv^{n-1})$$

$$= \frac{M_1 v^1 + 2M_2 v^2 + \cdots\cdots + n M_n v^n}{M_0 v^0 + M_1 v^1 + M_2 v^2 + \cdots\cdots + M_n v^n}$$

(3) $$= \frac{(0) M_0 v^0 + (1) M_1 v^1 + (2) M_2 v^2 + \cdots\cdots + (n) M_n v^n}{M_0 v^0 + M_1 v^1 + M_2 v^2 + \cdots\cdots + M_n v^n},$$

therefore

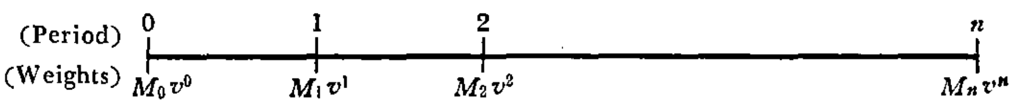
(4) $$= \frac{\sum_{t=0}^{n} t\ M_t v^t}{\sum_{t=0}^{n} M_t v^t}.$$

J. R. Hicks calls (3) or (4) the "elasticity of capital value with regard to discount ratio" and also the "Average Period", though it sounds a little strange.

He himself states that "the reader may perhaps be angry with me for appropriating the term 'Average Period' to this quantity, since he may have in his head what appears to be a very different meaning of the term". This is quite true and, to satisfy the reader, some explanation seems to be necessary as to why the "elasticity of capital values with regard to discount ratio" can be called the "Average Period", too.

In the previous statement, period t means a tract of time beginning with the present point 0. Let us take it to be the scale of a weigh-beam, and suppose that the weights marked by $M_0 v^0$, $M_1 v^1$, $M_2 v^2$, ..., $M_n v^n$ are hung at the scale marked by 0, 1, 2, ..., n respectively.

Figure 1



$$\frac{(0) M_0 v^0 + (1) M_1 v^1 + (2) M_2 v^2 + \cdots\cdots\cdots + (n) M_n v^n}{M_0 v^0 + M_1 v^1 + M_2 v^2 + \cdots\cdots\cdots + M_n v^n} = \frac{\sum_{t=o}^{n} t\, M_t\, v^t}{\sum_{t=o}^{n} M_t\, v^t}$$

In such a case the center of gravity must be the weighted arithmetic mean of these. $M_0 v^0$, $M_1 v^1$, $M_2 v^2$, ..., $M_n v^n$ are the weights to the periods 1, 2, ..., n which are the tract of time in the future off from the present point 0, and the above stated equation indicates the (weighted) average period. This is equal to (3) or (4), that is, the "elasticity of capital value with regard to discount ratio". The so-called "Average Period", therefore, rather can be said to be a *very appropriate term* as Hicks says.

III Proposition

Thus, the elasticity of the sum of the prospective stream of capital values with regard to discount ratio (rate of interest) can be understood as the "Average Period" as Hicks names it. But it is not clear only by this as to how the sum of the prospective stream of capital values is related to the discount ratio. For this reason the following proposition needs to be taken into account:

"The rise of discount ratio denoted by v (the fall of rate of interest denoted by r) leads to the extension of the average period denoted by π; on the contrary, the fall of discount ratio v (the rise of rate of interest r) leads to the shortening of it".

This will be shown below:

From (4) we have

(5) $\sum_{t=o}^{n} t M_t v^t = v \dfrac{dM}{dv}$. See (2), $M = \sum_{t=o}^{n} M_t v^t$

Denoting the left side by N, then

(6) $N = v \dfrac{dM}{dv}$,

therefore

$$(7) \qquad \pi = \frac{N}{M}.$$

Let us differentiate (7) in regard to v, and we have

$$(8) \qquad \frac{d\pi}{dv} = \frac{d}{dv}\left(\frac{N}{M}\right) = \frac{\begin{vmatrix} M & \frac{d}{dv}M \\ N & \frac{d}{dv}N \end{vmatrix}}{M^2}.$$

Whereas denominater $M^2 > 0$.

Therefore, whether $\frac{d\pi}{dv}$ becomes positive or negative depends on whether the numerator in (8) is positive or negative.

$$\text{Numerator} = \begin{vmatrix} M & \frac{d}{dv}M \\ N & \frac{d}{dv}N \end{vmatrix} = \begin{vmatrix} \sum M_t v^t & \sum t M_t v^{t-1} \\ \sum t M_t v^t & \sum t^2 M_t v^{t-1} \end{vmatrix}$$

$$= \left| \begin{pmatrix} M_0 & M_1 & M_2 & \cdots\cdots & M_n \\ 0 & 1M_1 & 2M_2 & \cdots\cdots & nM_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \\ v^2 & 2v \\ \vdots & \vdots \\ v^n & nv^{n-1} \end{pmatrix} \right|$$

$$= \sum_{i<j} \begin{vmatrix} M_i & M_j \\ iM_i & jM_j \end{vmatrix} \begin{vmatrix} v^i & i\,v^{i-1} \\ v^j & j\,v^{j-1} \end{vmatrix} \qquad \begin{pmatrix} i = 0, 1, 2, \cdots\cdots, n \\ j = 0, 1, 2, \cdots\cdots, n \end{pmatrix}$$

$$= \sum_{i<j} \begin{vmatrix} 1 & 1 \\ i & j \end{vmatrix} M_i M_j \begin{vmatrix} 1 & i \\ 1 & j \end{vmatrix} v^{i+j-1} = \sum_{i<j} \begin{vmatrix} 1 & 1 \\ i & j \end{vmatrix}^2 M_i M_j v^{i+j-1}$$

$$= \sum_{i<j} (j-i)^2 M_i M_j v^{i+j-1} > 0, \quad \text{since } M_i, M_j, v > 0.$$

Therefore it leads to

$$(9) \qquad \frac{d\pi}{dv} > 0^{3)}.$$

Thus, the proposition has been proved.

3) Unlike the previous method, it can be calculated directly:

$$\text{Numerator} = \begin{vmatrix} \sum M_t v^t & \sum t\, M_t v^{t-1} \\ \sum t M_t v^t & \sum t^2 M_t v^{t-1} \end{vmatrix} = 1^2 M_0 M_1 + 2^2 M_0 M_2 v$$
$$+ (3^2 M_0 M_3 + 1^2 M_1 M_2) v^2 + \cdots\cdots$$
$$+ \left\{(n-1)^2 M_1 M_n + (n-3)^2 M_2 M_{n-1} + (n-5)^2 M_3 M_{n-2}\right\} v^n$$
$$+ \cdots\cdots + M_{n-1} M_n v^{2n-2},$$

So that Numerator>0.

Figure 2

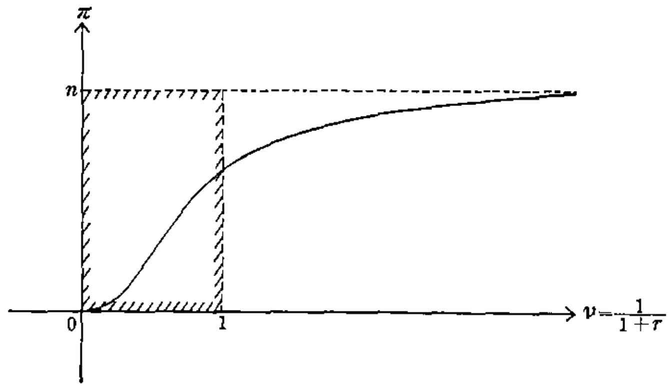
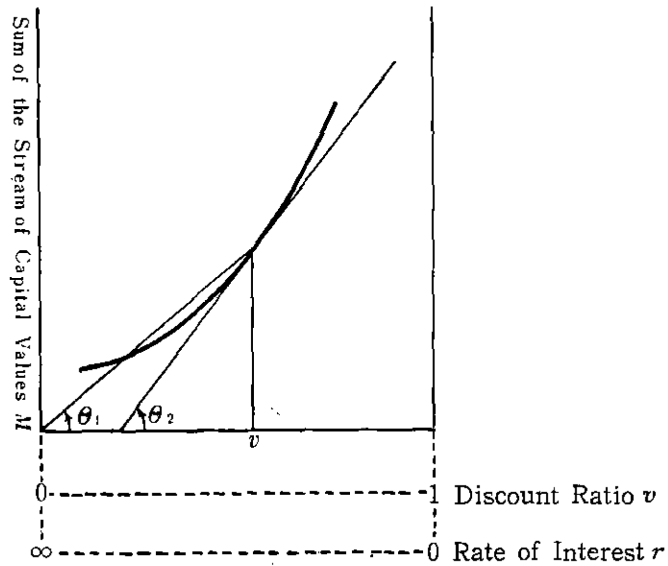


Figure 3



These figures also show the proposition in question in accordance with (4). However, only the area within the oblique lines in Figure 2 is actually available owing to the limited area of the rate of interest, that is, $0 \leqq r \leqq \infty$.

Also, from definition (4) we have

$$(10) \qquad \pi = \frac{\frac{dM}{dv}}{\frac{M}{v}} = \frac{\text{marginal value}}{\text{average value}} .$$

According to this, the average perioa which is larger than 1 means $\tan\theta_1$ (average value) $<$ $\tan\theta_2$ (marginal vaue) in regard to a certain value of v.

The sum of the stream of capital values $M = f(v)$ is convex to the horizontal axis[4] in Figure 3.

$$\text{Proof:}\quad \begin{aligned} f(v) &= \sum M_t v^t \\ f'(v) &= \sum tM_t v^{t-1} > 0 \\ f''(v) &= \sum (t-1)tM_t v^{t-2} > 0 \end{aligned}$$

IV Lemma

Let us take Sequence $\{a_i\}$ for a monotone increasing sequence which is

$$(11)\qquad a_1 < a_2 < \cdots\cdots < a_n$$

and Sequence $\{w_i\}$ for a positive monotone increasing sequence which is

$$(12)\qquad w_1 < w_2 < \cdots\cdots < w_n.$$

Now, the simple arithmetic mean of (11) denoted by a can be expressed as

$$(13)\qquad a = \frac{a_1 + a_2 + \cdots\cdots + a_n}{n} = \frac{\sum_{i=1}^{n} a_i}{n}.$$

And let us denote the weighted arithmetic mean by $\hat{a}_w$ which is obtained by the multiplication of the units having the same subscript in (11) and (12), and we have

$$(14)\qquad \hat{a}_w = \frac{w_1 a_1 + w_2 a_2 + \cdots\cdots + w_n a_n}{w_1 + w_2 + \cdots\cdots + w_n} \quad \frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i}.$$

Meanwhile, consider the monotone decreasing sequence placed in the reversed order of (11) put in the form

$$(15)\qquad a_n > a_{n-1} > \cdots\cdots > a_1$$

and denote its weighted arithmetic mean by $\bar{a}_w$. The weights corresponding with a_n, a_{n-1}, ……, a_1 are w_1, w_2, ……, w_n in (12) respectively. Then we have

$$(16)\qquad \bar{a}_w = \frac{w_1 a_n + w_2 a_{n-1} + \cdots\cdots + w_n a_1}{w_1 + w_2 + \cdots\cdots + w_n} = \frac{\sum_{i=1}^{n} w_i a_{n-i+1}}{\sum_{i=1}^{n} w_i}$$

The simple arithmetic mean of (15) is equal to (13) and can be denoted by a too, especially expressed in the form

4) According to Figure 23 in Hicks's *Value and Capital*, Oxford, 1939, capital value is taken at the horizontal axis and discount ratio at the vertical axis; so that the curve is concave to the horizontal axis.

$$(17) \qquad a = \frac{a_n + a_{n-1} + \cdots\cdots\cdots + a_1}{n} = \frac{\sum_{i=1}^{n} a_{n-i+1}}{n} .$$

In the first place, let us show that $a < \hat{a}_w$ can be admitted in general[5].

When the simple arithmetic mean a takes part in (11), we can write

$$(18) \qquad a_1 < a_2 < \cdots\cdots\cdots < a_\lambda < a < a_{\lambda+1} < \cdots\cdots\cdots < a_n$$

and the weights in correspondence with (18) are

$$(19) \qquad w_1 < w_2 < \cdots\cdots\cdots < w_\lambda < w_{\lambda+1} < \cdots\cdots\cdots < w_n .$$

Hence, the following two relations are easily found.

$$(20) \qquad \sum_{i=1}^{\lambda} w_i (a_i - a) > \sum_{i=1}^{\lambda} w_\lambda (a_i - a)$$

and

$$(21) \qquad \sum_{i=\lambda+1}^{n} w_i (a_i - a) > \sum_{i=\lambda+1}^{n} w_\lambda (a_i - a) .$$

Let these two inequalities be added on each side, and then we have

$$\sum_{i=1}^{n} w_i (a_i - a) > \sum_{i=1}^{n} w_\lambda (a_i - a) ,$$

$$\sum_{i=1}^{n} w_i a_i - a \sum_{i=1}^{n} w_i > w_\lambda \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a \right) ,$$

and

$$\sum_{i=1}^{n} w_i a_i - a \sum_{i=1}^{n} w_i > w_\lambda (na - na) ; \qquad \text{See (13)}$$

therefore

$$\sum_{i=1}^{n} w_i a_i - a \sum_{i=1}^{n} w_i > 0 .$$

Let this inequality be divided by $\sum_{i=1}^{n} w_i$ (>0) on both sides, and we have

$$\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i} > a .$$

Hence

$$(22) \qquad \hat{a}_w > a . \qquad \text{See (14)}$$

In the second place, let us show that $\bar{a}_w < a$ can be admitted in general.

5) Hideo Aoyama, "On Hicks's Capital Theory of Capital", *Keizai Ronso* (Kyoto University), Vol. 56, No. 5, 1943.

Using the same method as in (18), we write (15) as follows:

(23) $a_n > a_{n-1} > \cdots\cdots\cdots > a_{n-\lambda+1} > a > a_{n-\lambda} > \cdots\cdots\cdots > a_1 .$

When the weights in correspondence with this are placed in the order of (19), the following two relations are easily found:

(24) $\sum_{i=1}^{\lambda} w_i \left(a_{n-i+1} - a\right) < \sum_{i=1}^{\lambda} w_\lambda \left(a_{n-i+1} - a\right),$

(25) $\sum_{i=\lambda+1}^{n} w_i \left(a_{n-i+1} - a\right) < \sum_{i=\lambda+1}^{n} w_\lambda \left(a_{n-\lambda+1} - a\right).$

Let these two inequalities be added on each side, and then we have

$$\sum_{i=1}^{n} w_i \left(a_{n-i+1} - a\right) < \sum_{i=1}^{n} w_\lambda \left(a_{n-i+1} - a\right),$$

$$\sum_{i=1}^{n} w_i a_{n-i+1} - a\sum_{i=1}^{n} w_i \quad < w_\lambda \left(\sum_{i=1}^{n} a_{n-i+1} - \sum_{i=1}^{n} a\right),$$

$$\sum_{i=1}^{n} w_i a_{n-i+1} - a \sum_{i=1}^{n} w_i < w_\lambda \left(na - na\right); \qquad \text{See (17)}$$

therefore

$$\sum_{i=1}^{n} w_i a_{n-i+1} - a \sum_{i=1}^{n} w_i < 0 .$$

Let this inequality be divided by $\sum_{i=1}^{n} w_i \left(> 0\right)$ on both sides, and we have

$$\frac{\sum_{i=1}^{n} w_i a_{n-i+1}}{\sum_{i=1}^{n} w_i} < a .$$

Hence

(26) $\check{a}_w < a .$ See (16)

Therefore, we obtain from (22) and (26) the following inequality:

(27) $\hat{a}_w > a > \check{a}_w$

This is the lemma which we have searched for and which is very useful for explaining the theory of the "Average Period". This will be discussed in the next chapter.

V Theory of the "Average Period"

Let us suppose that receipts and payments compose such streams of capital values over n periods as

(28) Receipts: R_1, R_2, ... , R_n

and

(29) Payments: S_1, S_2, ... , S_n.

We obtain the present values of these streams under an assumption[6] that R_i and S_i take part in the streams of capital values at the end of each period. Then we have

(30) $R_1 v^1$, $R_2 v^2$, ... , $R_n v^n$

and

(31) $S_1 v^1$, $S_2 v^2$, ... , $S_n v^n$.

Let the sums of the present values of (30) and (31) be denoted by R and S respectively, then

(32) $$R = R_1 v^1 + R_2 v^2 + \cdots\cdots + R_n v^n = \sum_{t=1}^{n} R_t v_t$$

and

(33) $$S = S_1 v^1 + S_2 v^2 + \cdots\cdots + S_n v^n = \sum_{t=1}^{n} S_t v^t .$$

Now supposing that $R = S$ be required for the planning for equating receipts and payments all over n periods, that is,

(34) $$\sum_{t=1}^{n} R_t v^t = \sum_{t=1}^{n} S_t v^t .$$

In other words, what we have to do in this research is to find what the change of discount ratio (or rate of interest) should mean under the condition of (34). We will represent (34) by k [7],

(35) $$\sum_{t=1}^{n} R_t v^t = \sum_{t=1}^{n} S_t v^t = k . \qquad k > 0$$

The average period of (32), (33) is denoted by π_r, π_s respectively, in accordance with the symbol of (4). (From now on, $\sum_{t=1}^{n}$ will be simply represented by $\sum$.) Thus we have

(36) $$\pi_r = \frac{\sum t R_t v^t}{\sum R_t v^t}$$

and

(37) $$\pi_s = \frac{\sum t S_t v^t}{\sum S_t v^t}$$

6) Pay special attention to this assumpton; see Chapter VI.

7) By assuming R_t and S_t to be given and v (therefore r) to be unknown and solving (34), we can search for v or r that satisfy the planning for equating receipts and payments in question. But this is not our present subject.

We subtract (37) from (36) on each side. Then we have

$$\pi_r - \pi_s = \frac{\sum t R_t v^t}{\sum R_t v^t} - \frac{\sum t S_t v^t}{\sum S_t v^t},$$

therefore
$$= \frac{\sum t R_t v^t}{k} - \frac{\sum t S_t v^t}{k}; \qquad \text{See (35)}$$

$$= \frac{1}{k} \sum t (R_t v^t - S_t v^t); \tag{38}$$

transforming (38) into

$$\pi_r - \pi_s = \frac{1}{k} \left\{ \sum t \cdot \frac{\sum t (R_t v^t - S_t v^t)}{\sum t} \right\}. \tag{39}$$

Now taking the stream of residuals which are the subtractions of the present values of payments (31) from those of receipts (30) in each period:

$$(R_1 v^1 - S_1 v^1),\ (R_2 v^2 - S_2 v^2),\ \cdots\cdots\cdots,\ (R_n v^n - S_n v^n) \tag{40}$$

And as a method of consideration, let us make a distinction between (40) which results in a monotone increasing stream and that which causes a monotone decreasing stream. Although the practical, especially *ex post*, stream of capital values does not always reveal either a monotone increasing or a monotone decreasing process, it is often possible that either one of them is produced in the system of the *ex ante* program of capital.

Case I in which (40) brings about a monotone increasing stream:

$$(R_1 v^1 - S_1 v^1) < (R_2 v^2 - S_2 v^2) < \cdots\cdots < (R_n v^n - S_n v^n). \tag{41}$$

The arithmetic mean of (41) weighted by a monotone increasing sequence which is

$$t: \qquad 1 < 2 < \cdots\cdots < n \tag{42}$$

is quite of the same characteristic as (14). The only difference is that the weight w_i in (14) is replaced by t in this case. Let us, therefore, denote it by $\hat{a}_w$, and we have

$$\hat{a}_w = \frac{1 (R_1 v^1 - S_1 v^1) + 2 (R_2 v^2 - S_2 v^2) + \cdots\cdots + n(R_n v^n - S_n V^n)}{1 + 2 + \cdots\cdots + n} \tag{43}$$

$$= \frac{\sum t (R_t v^t - S_t v^t)}{\sum t}.$$

Meanwhile taking the simple arithmetic mean of (41),

$$a = \frac{(R_1 v^1 - S_1 v^1) + (R_2 v^2 - S_2 v^2) + \cdots\cdots + (R_n v^n - S_n v^n)}{n}$$

$$= \frac{\sum R_t v^t - \sum S_t v^t}{n} = 0. \qquad \text{See (34)} \tag{44}$$

According to Lemma (27), we have

$$\frac{\sum t (R_t v^t - \sum S_t v^t)}{\sum t} > 0. \tag{45}$$

Applying (45) to (39), the inequality below is led to because of $k>0$ and $\sum t>0$:

(46) $\quad \pi_r - \pi_s > 0$.

Case II in which (40) brings about a monotone decreasing stream:

(47) $\quad (R_1 v^1 - S_1 v^1) > (R_2 v^2 - S_2 v^2) > \cdots\cdots\cdots\cdots > (R_n v^n - S_n v^n)$.

The arithmetic mean of (47) weighted by (42) is quite of the same characteristic as (16), which is denoted by $\bar{a}_w$.
Then in the same way as Case I, we will have

(48) $$\bar{a}_w = \frac{\sum t\,(R_t v^t - S_t v^t)}{\sum t} .$$

Applying (48) to Lemma (27),

(49) $$\frac{\sum t\,(R_t v^t - S_t v^t)}{\sum t} < 0 .$$ See (44)

Applying (49) to (39), we have the inequality below for the same reason as in Case I:

(50) $\quad \pi_r - \pi_s < 0$

The above-stated can briefly be put into a table below:

(51)

(Case)	(40)	(46) (50)
(I)	Monotone Increasing Stream	$\pi_r - \pi_s > 0$
(II)	Monotone Decreasing Stream	$\pi_r - \pi_s < 0$

Next let us obtain from (30) and (31) the stream of present values of residuals denoted by G_t, that is,

(52) $\quad G_1 = (R_1 v^1 - S_1 v^1)$, $G_2 = (R_2 v^2 - S_2 v^2)$, $\cdots\cdots\cdots\cdots$, $G_n = (R_n v^n - S_n v^n)$

and represent the sum of these by G, then

(53) $\quad G = \sum Gt = \sum (R_t v^t - S_t v^t)$.

Differentiating G in regard to v, we have

(54) $$dG = \left\{ \sum t R_t v^{t-1} - \sum t S_t v^{t-1} \right\} dv .$$

We assume $\dfrac{dv}{v} = \theta$,

(55) $\quad dv = \theta v$.

Let (55) be substituted in (54), and we obtain

$$dG = \left\{ \sum t R_t v^{t-1} - \sum t S_t v^{t-1} \right\} \theta v$$

therefore

$$(56) \qquad = \theta \left\{ \sum t R_t v^t - \sum t S_t v^t \right\}.$$

Transforming (56) slightly into

$$(57) \qquad dG = \theta \left\{ \sum R_t v^t \frac{\sum t R_t v^t}{\sum R_t v^t} - \sum S_t v^t \frac{\sum t S_t v^t}{\sum S_t v^t} \right\}.$$

Then using (35), (36), and (37), we will come to the following conclusion:

$$(58) \qquad dG = \theta k \left\{ \pi_r - \pi_s \right\}.$$

In this case, since $k > 0$, we can find the following relations:

(59)

	Rate of Interest r	Discount Ratio v	Rate of Change of the Discount Ratio θ	Marginal Surplus dG	Balance of Payments
$\pi_r - \pi_s > 0$ (Planning to be a Borrower)	Down	Up	Positive	Positive	Better
	Up	Down	Negative	Negative	Worse
$\pi_r - \pi_r < 0$ (Planning to be a Lender)	Down	Up	Positive	Negative	Worse
	Up	Down	Negative	Positive	Better

Some explanation about (59) seems to be useful.

That the average period of the stream of receipts is larger than that of the stream of payments, that is, $\pi_r - \pi_s > 0$, indicates a planning that the receipts in each period are relatively little at present or in the near future and relatively large in the far future; whereas the payments in each period are relatively large at present or in the near future and relatively little in the far future. The deficit caused by the greater payment over the receipts in each period at present or in the near future are to be made amends for by the surplus which will be brought about by the greater receipts over the payments in each period in the far future. This is the "planning to be a borrower" as Hicks names it.

That the average period of the stream of payments is larger than that of the stream of receipts, that is, $\pi_r - \pi_s < 0$, indicates a planning that the payments in each period are relatively little at present or in the near future and relatively large in the far future; whereas the receipts in each period are relatively large at present or in the near future and relatively little in the far future. The surplus caused by the greater receipts over the payments in each period at present or in the near future is to be provided so as to make amends for the deficit which will be brought about by the greater payments over the receipts in each period in the far future. This is the "planning to be a lender" as Hicks names it.

According to (59), which is our conclusion, the fall (rise) of rate of interest improves (deteriorates) the balance of payments in the planning to be a borrower; the fall (rise) of rate of interest deteriorates (improves) it in the planning to be a lender.

A remark on $dG \gtrless 0$: in short, this indicates a positive or negative sign of the quantity of change (which is the approximate value) of the sum of residuals G which is caused by a small change of v (or r) out of a certain balance of payments over n periods. In the above, we have substituted (35) in (57) to lead (58). If (35) be substituted in (53), G should be equal to 0, that is, $G=0$; in other words, the balance of payments over n periods is to be attained. This means that v_1 obtained from solving (35) is the root of $G=0$. dG is the approximate value of the quantity of change of G in response to the change of v out of $G=0$ in this root.

From (53), we have

(60) $\quad G = \sum (R_t - S_t)\, v^t \,;$

In this form, G is the function of v, that is,

(61) $\quad G = G(v)$

Let one of the roots of $G=0$ be denoted by v_1 and substituted in $\dfrac{dv}{v} = \theta$, and it becomes $\theta_1 = \dfrac{dv}{v_1}$. So (58) can be written in the form

(62) $$dG \Big|_{v=v_1} = \frac{1}{v_1} k \left\{ \pi_r - \pi_s \right\} dv$$

(62) indicates the approximate value of the quantity of change of G in response to the change of v out of $v = v_i$. Both π_r and π_s on the right side of (62), however, are the values with respect to $v = v_1$.

VI Advanced Inquiry into the Theory of the "Average Period"

In the former chapters we have proved by mathematical analysis and revealed the utility of the theory of the "Average Period" which Hicks developed —— though only verbally and rather simply —— in his excellent work, *Value and Capital.* We, furthermore, would like to examine this theory a little more, because it seems possible that it might not necessarily be applicable to all cases depending upon the kind and nature of an assumption presupposed.

We have assumed that, in (28) and (29), receipts and payments take part in the streams of capital values respectively at the end of each period. Now we assume, in the first place[8], that the receipts take part in the stream of capital values at the beginning and the payments at the end of each

8) In the latter part of this chapter, it will be assumed that "receipts take part in the stream of capital values at the end, and payments at the beginning of each period".

period. (In the mathematics of life insurance such an assumption is rather common.) In this case the streams of present values of receipts and payments over n periods would become

$$(63) \qquad R_1 v^0, \quad R_2 v^1, \cdots\cdots\cdots\cdots, R_n v^{n-1}$$

and

$$(64) \qquad S_1 v^1, \quad S_2 v^2, \cdots\cdots\cdots\cdots, S_n v^n .$$

Denoting the sums of these streams of present values by R and S respectively, thus

$$(65) \qquad R = R_1 v^0 + R_2 v^1 + \cdots\cdots + R_n v^{n-1} = \sum R_t v^{t-1}$$

and

$$(66) \qquad S = S_1 v^1 + S_2 v^2 + \cdots\cdots + S_n v^n = \sum S_t v^t .$$

For the planning of equating receipts and payments over all periods, $R = S$ is required. Let us assume $R = S = k$, which can be expressed as

$$(67) \qquad \sum R_t v^{t-1} = \sum S_t v^t = k .$$

Then, let us search for the elasticities of the capital values of R and S with regard to the discount ratio, that is, the average periods, π_r and π_s:

$$\pi_r = \frac{v}{R} \frac{dR}{dv}$$

$$= \frac{v}{R_1 v^0 + R_2 v^1 + \cdots\cdots + R_n v^{n-1}} \left[R_2 + 2R_3 v^1 + \cdots\cdots + (n-1) R_n v^{n-2} \right]$$

$$= \frac{1R_2 v^1 + 2R_3 v^2 + \cdots\cdots + (n-1) R_n v^{n-1}}{R_1 v^0 + R_2 v^1 + \cdots\cdots + R_n v^{n-1}}$$

$$= \frac{(1) R_1 v^0 + (2) R_2 v^1 + \cdots\cdots + (n) R_n v^{n-1}}{R_1 v^0 + R_2 v^1 + \cdots\cdots + R_n v^{n-1}} - \frac{R_1 v^0 + R_2 v^1 + \cdots\cdots + R_n v^{n-1}}{R_1 v^0 + R_2 v^1 + \cdots\cdots + R_n v^{n-1}}$$

therefore

$$(68) \qquad = \frac{\sum t R_t v^{t-1}}{\sum R_t v^{t-1}} - 1 .$$

In the same way

$$\pi_s = \frac{v}{S} \frac{dS}{dv} ,$$

therefore

$$(69) \qquad = \frac{\sum t S_t v^t}{\sum S_t v^t} . \qquad \text{See (37)}$$

Subtracting (69) from (68) on each side:

$$\pi_r - \pi_s = \frac{\sum t R_t v^{t-1}}{\sum R_t v^{t-1}} - \frac{\sum t S_t v^t}{\sum S_t v^t} - 1$$

$$= \frac{1}{k} \sum t (R_t v^{t-1} - S_t v^t) - 1 \qquad \text{See (67)}$$

transforming into

$$(70) \qquad = \frac{1}{k} \left\{ \sum t \frac{\sum t (R_t v^{t-1} - S_t v^t)}{\sum t} \right\} - 1$$

Next, let us also take from (63) and (64) the stream of residuals in each period, and we have

$$(71) \qquad (R_1 v^0 - S_1 v^1),\ (R_2 v^1 - S_2 v^2),\ \cdots\cdots\cdots,\ (R_n v^{n-1} - S_n v^n).$$

The arithmetic mean of these residuals weighted by

$$(72) \qquad t: \quad 1, 2, \cdots\cdots\cdots\cdots, n$$

becomes

$$(73) \qquad \hat{a}_w, \bar{a}_w = \frac{\sum t (R_t v^{t-1} - S_t v^t)}{\sum t}$$

When (71) is a monotone increasing stream, (73) becomes $\hat{a}_w$; When (71) is a monotone decreasing stream, (73) becomes $\bar{a}_w$. And the simple arithmetic mean a of (71) becomes

$$a = \frac{1}{n} \left\{ (R_1 v^0 - S_1 v^1) + (R_2 v^1 - S_2 v^2) + \cdots\cdots\cdots + (R_n v^{n-1} - S_n v^n) \right\}$$

$$= \frac{1}{n} \left\{ \sum R_t v^{t-1} - \sum S_t v^t \right\},$$

therefore

$$(74) \qquad = 0; \qquad \text{See (67)}$$

Applying Lemma (27) to (70), we can find the following relations:

$\pi_r - \pi_s$

	(71)	(73)	(70)
(75)	Monotone Increasing Stream	Positive	Impossible to Judge Positive or Negative
	Monotone Decreasing Stream	Negative	Negative

This leads to a different result from that of (51) in the previous chapter which was reached under the assumption that both receipts and payments take part in each stream of capital values at the end of each period.

In the second place, unlike the previous assumption let us assume that receipts take part in the stream of capital values at the end, and payments at the beginning of each period. In this case each stream of present values of receipts and payments over n periods will become respectively

$$(76) \qquad R_1 v^1 , \quad R_2 v^2 , \ \cdots\cdots\cdots\cdots , \quad R_n v^n$$

and

$$(77) \qquad S_1 v^0 , \quad S_2 v^1 , \ \cdots\cdots\cdots\cdots , \quad S_n v^{n-1} .$$

Suppose that $R = S = k$ be required for the planning of equating receipts and payments all over n periods, that is,

$$(78) \qquad \sum R_t v^t = \sum S_t v^{t-1} = k .$$

Let us search for π_r and π_s, the average periods of R and S: in the same way as before,

$$(79) \qquad \pi_r = \frac{\sum t R_t v^t}{\sum R_t v^t}$$

and

$$(80) \qquad \pi_s = \frac{\sum t S_t v^{t-1}}{\sum S_t v^{t-1}} - 1 .$$

Subtracting (80) from (79) on each side, then

$$(81) \qquad \pi_r - \pi_s = \frac{1}{k} \left\{ \sum t \cdot \frac{\sum t (R_t v^t - S_t v^{t-1})}{\sum t} \right\} + 1 .$$

Taking the stream of residuals in each period from (76) and (77) which becomes

$$(82) \qquad (R_1 v^1 - S_1 v^0) , \quad (R_2 v^2 - S_2 v^1) , \ \cdots\cdots\cdots\cdots , \quad (R_n v^n - S_n v^{n-1}) ,$$

we have an arithmetic mean weighted by (72) as

$$(83) \qquad \hat{a}_w , \ \bar{a}_w = \frac{\sum t (R_t v^t - S_t v^{t-1})}{\sum t} .$$

When (82) is a monotone increasing stream, (83) becomes $\hat{a}_w$; when (82) is a monotone decreasing stream, (83) becomes $\bar{a}_w$. And the simple arithmetic mean a becomes

$$\begin{aligned} a &= \frac{1}{n} \left\{ (R_1 v^1 - S_1 v^0) + (R_2 v^2 - S_2 v^1) + \cdots\cdots + (R_n v^n - S_n v^{n-1}) \right\} \\ &= \frac{1}{n} \left\{ \sum R_t v^t - \sum S_t v^{t-1} \right\}, \end{aligned}$$

therefore

$$(84) \qquad = 0 . \qquad\qquad \text{See (78)}$$

Applying Lemma (27) to (81), we can find the following relations:

$\pi_r - \pi_s$

(85)

(82)	(83)	(81)
Monotone Increasing Stream	Positive	Positive
Monotone Decreasing Stream	Negative	Impossible to Judge Positive or Negative

This leads to a different result from that of (51) in the previous chapter in which both receipts and payments are assumed to take part in each stream of capital values at the end of each period.

The result of (51), (75), and (85) which are cases derived from different assumptions is as follows.

$\pi_r - \pi_s$

(86)

Assumptions		Streams of residuals of present values	
		M. I. S.	M. D. S.
(A)	Both receipts and payments taking part in the streams at the end of each period	Positive	Negative
(B)	Receipts taking part in the stream at the beginning, and payments at the end of each period	Impossible to Judge Positive or Negative	Positive
(C)	Receipts taking part in the stream at the end, payments at the beginning of each period	Positive	Impossible to Judge Positive or Negative

In short, $\pi_r - \pi_s$ is always settled as either positive or negative under Assumption (A); whereas there can be some cases in which it is impossible to judge positive or negative under Assumptions (B) and (C). In other words, in the cases in which the stream of residuals of present values indicates monotone increase under Assumption (B) and cases which it has a monotone decrease under Assumption (C), $\pi_r - \pi_s$ cannot be judged either positive or negative. In such cases, therefore, it is impossible to judge whether the balance of payments in (59) improves or deteriorates.

In the last place, we will take, as a concrete case, the management of life insurance into account, applying the theory which we have treated so far.

In the case of life insurance, insurance premiums which are receipts for the insurance company are received at the beginning of each period;

while insurance amounts which are payments for the company are paid at any time when the insured die. So the time when insurance amounts are paid is, on an average, to be supposed at the middle of each respective period.

We will explain, as an example, the whole life insurance system, the most representative of all the life-insurance contract systems. The stream of present values of receipts and payments is

$$(87)\qquad \text{Receipts:}\qquad R_1 v^0,\ R_2 v^1,\ \cdots\cdots\cdots,\ R_n v^{n-1}$$

and

$$(88)\qquad \text{Payments:}\qquad S_1 v^{\frac{1}{2}},\ S_2 v^{1+\frac{1}{2}},\ \cdots\cdots\cdots,\ S_n v^{(n-1)+\frac{1}{2}}.$$

(In the mathematics of life insurance, it is often assumed that insurance amounts be paid at the end of each period for the purpose of simple calculation. In this case, (88) is to be quite the same as (64). But we assume here that they be paid by the company at the middle of each period for the purpose of strict calculation.)

The sums of (87) and (88) denoted by R and S respectively become

$$(89)\qquad R = R_1 v^0 + R_2 v^1 + \cdots\cdots\cdots + R_n v^{n-1} = \sum R_t v^{t-1}$$

and

$$(90)\qquad S = S_1 v^{\frac{1}{2}} + S_2 v^{1+\frac{1}{2}} + \cdots\cdots + S_n v^{(n-1)+\frac{1}{2}} = \sum S_t v^{t-\frac{1}{2}}.$$

Again $R = S = k$ is required for the planning of equating receipts and payments all over n periods, and therefore

$$(91)\qquad \sum R_t v^{t-1} = \sum S_t v^{t-\frac{1}{2}} = k\ .$$

Let us search for π_r and π_s with regard to R in (89) and S in (90), and we have

$$(92)\qquad \pi_r = \frac{\sum t R_t v^{t-1}}{\sum R_t v^{t-1}} - 1 \quad : \qquad \text{this is the same as (68);}$$

and

$$(93)\qquad \pi_s = \frac{\sum t S_t v^{t-\frac{1}{2}}}{\sum S_t v^{t-\frac{1}{2}}} - \frac{1}{2}\ .$$

Subtract (93) from (92) on each side in consideration of (91), and transform into

$$(94)\qquad \pi_r - \pi_s = \frac{1}{k}\left\{\sum t \cdot \frac{\sum t\,(R_t v^{t-1} - S_t v^{t-\frac{1}{2}})}{\sum t}\right\} - \frac{1}{2}\ .$$

Furthermore we obtain the stream of residuals of present values from (87) and (88), which is

$$(95)\qquad (R_1 v^0 - S_1 v^{\frac{1}{2}}),\ (R_2 v^1 - S_2 v^{1+\frac{1}{2}}),\ \cdots\cdots,\ (R_n v^{n-1} - S_n v^{(n-1)+\frac{1}{2}}).$$

Taking the arithmetic mean weighted by

(96) $t:$ $1, 2, \cdots\cdots\cdots\cdots, n$

which becomes

(97) $$\hat{a}_w, \quad \bar{a}_w = \frac{\sum t\,(R_t v^{t-1} - S_t v^{t-\frac{1}{2}})}{\sum t}.$$

When (95) is a monotone increasing stream, (97) becomes $\hat{a}_w$; when (95) is a monotone decreasing stream, (97) becomes $\bar{a}_w$. But, in life insurance, (95) always indicates a monotone decreasing stream. The reason: in life insurance the system of paying insurance premiums, which are receipts for the insurance company, is most commonly the uniform premiums system in which the sum of premiums paid from policy-holder is constant at every period.

Therefore,

$$R_1 = R_2 = \cdots\cdots\cdots\cdots\cdots\cdots = R_n.$$

Whereas the sequence of discount ratio is decreasing;

(98) $$v^0 > v^1 > v^2 > \cdots\cdots\cdots > v^{n-1}.$$

So that the present values of premium receipts indicate a monotone decreasing stream which is

(99) $$R^1 v^0 > R_2 v^1 > \cdots\cdots\cdots > R_n v^{n-1}.$$

Against this tendency, the insurance amounts paid become steadily higher and higher as the probability of dying of the insured grows high, period to period. So they make a monotone increasing stream which is

(100) $$S_1 < S_2 < \cdots\cdots\cdots\cdots < S_n.$$

But the increasing trend of (100) is much greater than the decreasing trend of the discount ratio which is a monotone decreasing sequence, that is,

(101) $$v^{\frac{1}{2}} > v^{1+\frac{1}{2}} > \cdots\cdots\cdots\cdots > v^{(n-1)+\frac{1}{2}}.$$

So that the stream of present values of insurance amount payments becomes a monotone increasing stream such as

(102) $$S_1 v^{\frac{1}{2}} < S_2 v^{1+\frac{1}{2}} < \cdots\cdots\cdots\cdots < S_n v^{(n-1)+\frac{1}{2}}.$$

The result of this shows that the stream of residuals of present values in each period obtained from (87) and (88) which is

(103) $$(R_1 v^0 - S_1 v^{\frac{1}{2}}) > (R_2 v^1 - S_2 v^{1+\frac{1}{2}}) > \cdots\cdots > (R_n v^{n-1} - S_n v^{(n-1)+\frac{1}{2}})$$

very clearly indicates a monotone decreasing stream.

In life insurance the method of keeping the balance of payments equal is projected in such a way that the surplus of receipts brought about in the former half of all the periods (the area bounded by MPN in Figure 4) can meet the deficit brought about in the latter half (the area bounded by QPT in Figure 4). For this reason it is clear that (103) indicates a monotone decreasing sequence. According to Lemma (27), therefore, (97) can be represented by

$$(104) \quad \bar{a}_w = \frac{\sum t\,(R_t v^{t-1} - S_t v^{t-\frac{1}{2}})}{\sum t} < 0\ ;$$

substituting (104) in (94), we obtain

$$(105) \quad \pi_r - \pi_s < 0\,.$$

In terms of our conclusion which has been made clear by (86), the management of life insurance belongs to the case in which the stream of residuals of present values takes a monotone decreasing stream under Assumption (B). We, therefore, are able to see by (59) how the change of discount ratio (therefore, that of the rate of interest) influences the balance of payments of an insurance company that manages by planning for equating receipts and payments.

In conclusion, Hicks's theory of the "Average Period" is fully applicable as far as life insurance is concerned.

Figure 4