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ON THE "AVERAGE PERIOD" OF J. R. HICKS

By Sempei SAWA *

I Preface

This article is intended to confirm by mathematical analysis the theory of the "Average Period" which J. R. Hicks states mostly verbally and rather concisely in Chapter 14 of his Value and Capital, and to examine it more closely. For it seems possible that different assumptions lead to different conclusions so that this theory is not necessarily applicable to all cases in general.

I am very grateful to Mr. Shinryo Taniyama for his Structure and Character of Life Insurance, Chapter 3, in which he tried to apply the theory of "Average Period" to mathematics of life insurance. It needs to be added, however, that my procedure and conclusion are different in many respects from his article.\(^1\)

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1) Shinryo Taniyama, Structure and Character of Life Insurance, 1962, Economics Department of Osaka Prefectural University.

2) (i) Sawa's proof of the proposition seems to be more persuasive and demonstrative.

(ii) Mr. Taniyama has shown the lemma only partially.

(iii) Since Mr. Taniyama applied the theory of "Average Period" to life insurance during the process of his analysis, he cannot be said to have shown this theory as a general rule; whereas Sawa has, showing it in the beginning by mathematical analysis as a general rule, tried to apply it to life insurance afterward.
II. Concept of the Average Period

Let a stream of capital values \( \{M_t\} \) be denoted by \( M_0, M_1, M_2, \ldots, M_n \), which are distributed into each period in the future 0, 1, 2, \ldots, \( n \), beginning with the present point 0, the rate of interest by \( r \), and the discount ratio by such an equation as \( \frac{1}{1+r} = v \), the present values of these capital values can be expressed as

\[
(1) \quad M_0 v^0, M_1 v^1, \ldots, M_n v^n.
\]

If \( t > 0 \) be correct here, \( M_t \) should be a future expected value and \( M_t v^t \) a future expected value discounted to the present value. If the sum of these future expected values discounted to the present value be denoted by \( M \), we have the following equation:

\[
(2) \quad M = M_0 v^0 + M_1 v^1 + M_2 v^2 + \ldots + M_n v^n = \sum_{t=0}^{n} M_t v^t.
\]

When the elasticity of the sum of the stream of capital values \( M \) in regard to discount ratio \( v \) is denoted by \( \pi \), it can be obtained according to the well-known definition of elasticity in general in the way,

\[
\pi = \frac{v}{M} \frac{dM}{dv}
\]

\[
= \frac{M_t v^t + M_1 v^1 + M_2 v^2 + \ldots + M_n v^n}{M_0 v^0 + M_1 v^1 + M_2 v^2 + \ldots + M_n v^n}(M_1 + 2M_2 v^1 + 3M_3 v^2 + \ldots + nM_n v^{n-1})
\]

\[
(3) \quad \frac{M_t v^t + M_1 v^1 + M_2 v^2 + \ldots + M_n v^n}{M_0 v^0 + M_1 v^1 + M_2 v^2 + \ldots + M_n v^n}.
\]

therefore

\[
(4) \quad \frac{\sum_{t=0}^{n} t M_t v^t}{\sum_{t=0}^{n} M_t v^t}.
\]

J. R. Hicks calls (3) or (4) the "elasticity of capital value with regard to discount ratio" and also the "Average Period", though it sounds a little strange.

He himself states that "the reader may perhaps be angry with me for appropriating the term 'Average Period' to this quantity, since he may have in his head what appears to be a very different meaning of the term". This is quite true and, to satisfy the reader, some explanation seems to be necessary as to why the "elasticity of capital values with regard to discount ratio" can be called the "Average Period", too.

In the previous statement, period \( t \) means a tract of time beginning with the present point 0. Let us take it to be the scale of a weigh-beam,
and suppose that the weights marked by $M_0 v^n, M_1 v^1, M_2 v^2, \ldots, M_n v^n$ are hung at the scale marked by 0, 1, 2, \ldots, $n$ respectively.

![Figure 1](period_weights.png)

In such a case the center of gravity must be the weighted arithmetic mean of these. $M_0 v^n, M_1 v^1, M_2 v^2, \ldots, M_n v^n$ are the weights to the periods 1, 2, \ldots, $n$ which are the tract of time in the future off from the present point 0, and the above stated equation indicates the (weighted) average period. This is equal to (3) or (4), that is, the "elasticity of capital value with regard to discount ratio". The so-called "Average Period", therefore, rather can be said to be a very appropriate term as Hicks says.

### III Proposition

Thus, the elasticity of the sum of the prospective stream of capital values with regard to discount ratio (rate of interest) can be understood as the "Average Period" as Hicks names it. But it is not clear only by this as to how the sum of the prospective stream of capital values is related to the discount ratio. For this reason the following proposition needs to be taken into account:

"The rise of discount ratio denoted by $v$ (the fall of rate of interest denoted by $r$) leads to the extension of the average period denoted by $\pi$; on the contrary, the fall of discount ratio $v$ (the rise of rate of interest $r$) leads to the shortening of it".

This will be shown below:

From (4) we have

$$\sum_{t=0}^{n} t M_t v^t = v \frac{dM}{dv},$$

See (2). $M = \sum_{t=0}^{n} M_t v^t$

Denoting the left side by $N$, then

$$N = v \frac{dM}{dv},$$
therefore

\[
(7) \quad \pi = \frac{-N}{M}.
\]

Let us differentiate (7) in regard to \( v \), and we have

\[
(8) \quad \frac{d\pi}{dv} = \frac{d}{dv} \left( \frac{-N}{M} \right) = \frac{M \frac{d}{dv} M - N \frac{d}{dv} N}{M^2}.
\]

Whereas denominator \( M^2 > 0 \).

Therefore, whether \( \frac{d\pi}{dv} \) becomes positive or negative depends on whether the numerator in (8) is positive or negative.

Let

\[
\text{Numerator} = \begin{vmatrix} M & \frac{d}{dv} M \hfill \\
N & \frac{d}{dv} N \end{vmatrix} = \begin{vmatrix} \sum M_i v^i & \sum t M_i v^{i-1} \\
\sum t M_i v^i & \sum t^2 M_i v^{i-1} \end{vmatrix} = \begin{vmatrix} M_0 M_1 M_2 \cdots M_n \hfill \\
0 1 M_1 2 M_2 \cdots \nu M_n \end{vmatrix} \begin{pmatrix} 1 \\
v \\
v^2 \\
\vdots \\
v^n \end{pmatrix} = \sum \begin{vmatrix} M_i \\
j M_j \end{vmatrix} v^i v^{i-1} = \begin{vmatrix} 1 \\
v \\
v^2 \\
\vdots \\
v^n \end{vmatrix} = \sum \begin{vmatrix} 1 \\
j - (j - i)^2 M_i M_j v^{i+j-1} > 0, \text{ since } M_i, M_j, v > 0. 
\end{vmatrix}
\]

Therefore it leads to

\[
(9) \quad \frac{d\pi}{dv} > 0. \quad \text{\textsuperscript{3}}
\]

Thus, the proposition has been proved.

\textsuperscript{3} Unlike the previous method, it can be calculated directly:

\[
\text{Numerator} = \begin{vmatrix} \sum M_i v^i & \sum t M_i v^{i-1} \\
\sum t M_i v^i & \sum t^2 M_i v^{i-1} \end{vmatrix} = 1^1 M_0 M_1 + 2^2 M_0 M_2 v + (3^3 M_0 M_3 + 1^1 M_1 M_2 v^2 + 4^4 M_0 M_4 v^3 + \cdots + (n-1)^{n-1} M_0 M_{n-1} + (n-2)^{n-2} M_0 M_{n-2} v^{n-1} + \cdots + M_{n-1} M_0 v^{2n-1},
\end{vmatrix}
\]

So that Numerator \( >0 \).
These figures also show the proposition in question in accordance with (4). However, only the area within the oblique lines in Figure 2 is actually available owing to the limited area of the rate of interest, that is, $0 \leq r \leq \infty$.

Also, from definition (4) we have

$$
\pi = \frac{\frac{dM}{dv}}{\frac{M}{v}} = \frac{\text{marginal value}}{\text{average value}}
$$

According to this, the average period which is larger than 1 means $\tan \theta_1$ (average value) $< \tan \theta_2$ (marginal value) in regard to a certain value of $v$. 
The sum of the stream of capital values \( M = f(v) \) is convex to the horizontal axis in Figure 3.

Proof: 
\[
\begin{align*}
M(v) &= \sum M_t v^t \\
M'(v) &= \sum tM_t v^{t-1} > 0 \\
M''(v) &= \sum (t-1)tM_t v^{t-2} > 0
\end{align*}
\]

IV Lemma

Let us take Sequence \( \{a_i\} \) for a monotone increasing sequence which is
\[
(11) \quad a_1 < a_2 < \cdots < a_n
\]
and Sequence \( \{w_i\} \) for a positive monotone increasing sequence which is
\[
(12) \quad w_1 < w_2 < \cdots < w_n.
\]
Now, the simple arithmetic mean of (11) denoted by \( a \) can be expressed as
\[
(13) \quad a = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\sum_{i=1}^{n} a_i}{n}.
\]
And let us denote the weighted arithmetic mean by \( \bar{a}_w \) which is obtained by the multiplication of the units having the same subscript in (11) and (12), and we have
\[
(14) \quad \bar{a}_w = \frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n} = \frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i}.
\]
Meanwhile, consider the monotone decreasing sequence placed in the reversed order of (11) put in the form
\[
(15) \quad a_n > a_{n-1} > \cdots > a_1
\]
and denote its weighted arithmetic mean by \( \bar{a}_w' \). The weights corresponding with \( a_n, a_{n-1}, \ldots, a_1 \) are \( w_1, w_2, \ldots, w_n \) in (12) respectively. Then we have
\[
(16) \quad \bar{a}_w' = \frac{w_1 a_n + w_2 a_{n-1} + \cdots + w_n a_1}{w_1 + w_2 + \cdots + w_n} = \frac{\sum_{i=1}^{n} w_i a_{n-i+1}}{\sum_{i=1}^{n} w_i}.
\]
The simple arithmetic mean of (15) is equal to (13) and can be denoted by \( a \) too, especially expressed in the form

4) According to Figure 23 in Hicks's *Value and Capital*, Oxford, 1939, capital value is taken at the horizontal axis and discount ratio at the vertical axis; so that the curve is concave to the horizontal axis.
ON THE "AVERAGE PERIOD" OF J. R. HICKS

(17) \[ a = \frac{a_n + a_{n-1} + \ldots + a_1}{n} = \frac{\sum_{i=1}^{n} a_{n-i+1}}{n}. \]

In the first place, let us show that \( a < a_w \) can be admitted in general\(^5\).

When the simple arithmetic mean \( a \) takes part in (11), we can write

(18) \[ a_1 < a_2 < \ldots < a_\lambda < a < a_{\lambda+1} < \ldots < a_n \]

and the weights in correspondence with (18) are

(19) \[ w_1 < w_2 < \ldots < w_\lambda < w_{\lambda+1} < \ldots < w_n. \]

Hence, the following two relations are easily found.

(20) \[ \sum_{i=1}^{\lambda} w_i (a_i - a) > \sum_{i=1}^{\lambda} w_i (a_i - a) \]

and

(21) \[ \sum_{i=\lambda+1}^{n} w_i (a_i - a) > \sum_{i=\lambda+1}^{n} w_i (a_i - a). \]

Let these two inequalities be added on each side, and then we have

\[ \sum_{i=1}^{\lambda} w_i a_i - a \sum_{i=1}^{\lambda} w_i > w_\lambda \left( \sum_{i=1}^{\lambda} a_i - \sum_{i=1}^{\lambda} a \right), \]

and

\[ \sum_{i=\lambda+1}^{n} w_i a_i - a \sum_{i=\lambda+1}^{n} w_i > w_\lambda \left( n a - na \right); \quad {\text{See (13)}} \]

therefore

\[ \sum_{i=1}^{\lambda} w_i a_i - a \sum_{i=1}^{\lambda} w_i > 0. \]

Let this inequality be divided by \( \sum_{i=1}^{\lambda} w_i (>0) \) on both sides, and we have

\[ \frac{\sum_{i=1}^{\lambda} w_i a_i}{\sum_{i=1}^{\lambda} w_i} > a. \]

Hence

(22) \[ a_w > a. \quad {\text{See (14)}} \]

In the second place, let us show that \( a_w < a \) can be admitted in general.

---

Using the same method as in (18), we write (15) as follows:

(23) \[ a_n > a_{n-1} > \cdots > a_{m+1} > a > a_{m-1} > \cdots > a_1. \]

When the weights in correspondence with this are placed in the order of (19), the following two relations are easily found:

(24) \[ \sum_{i=1}^j w_i (a_{n-i+1} - a) < \sum_{i=1}^j w_i (a_{m-i+1} - a), \]

(25) \[ \sum_{i=1}^n w_i (a_{n-i+1} - a) < \sum_{i=1}^n w_i (a_{m-i+1} - a). \]

Let these two inequalities be added on each side, and then we have

\[ \sum_{i=1}^n w_i (a_{n-i+1} - a) - a \sum_{i=1}^n w_i < \sum_{i=1}^n w_i (a_{m-i+1} - a); \] See (17)

therefore

\[ \sum_{i=1}^n w_i a_{n-i+1} - a \sum_{i=1}^n w_i < 0. \]

Let this inequality be divided by \( \sum_{i=1}^n w_i (>0) \) on both sides, and we have

\[ \frac{\sum_{i=1}^n w_i a_{n-i+1}}{\sum_{i=1}^n w_i} < \frac{a}{\sum_{i=1}^n w_i}. \]

Hence

(26) \[ \hat{a}_w < a. \] See (16)

Therefore, we obtain from (22) and (26) the following inequality:

(27) \[ \hat{a}_w > a > \hat{a}_w. \]

This is the lemma which we have searched for and which is very useful for explaining the theory of the "Average Period". This will be discussed in the next chapter.

V Theory of the "Average Period"

Let us suppose that receipts and payments compose such streams of capital values over \( n \) periods as
(28) Receipts: \( R_1, R_2, \ldots, R_n \)

and

(29) Payments: \( S_1, S_2, \ldots, S_n \).

We obtain the present values of these streams under an assumption\(^6\) that \( R_i \) and \( S_i \) take part in the streams of capital values at the end of each period. Then we have

(30) \( R_1 v^i, R_2 v^i, \ldots, R_n v^n \)

and

(31) \( S_1 v^i, S_2 v^i, \ldots, S_n v^n \).

Let the sums of the present values of (30) and (31) be denoted by \( R \) and \( S \) respectively, then

(32) \( R = R_1 v^i + R_2 v^2 + \cdots + R_n v^n = \sum_{t=1}^{n} R_t v^t \)

and

(33) \( S = S_1 v^i + S_2 v^2 + \cdots + S_n v^n = \sum_{t=1}^{n} S_t v^t \).

Now supposing that \( R = S \) be required for the planning for equating receipts and payments all over \( n \) periods, that is,

(34) \( \sum_{t=1}^{n} R_t v^t = \sum_{t=1}^{n} S_t v^t \).

In other words, what we have to do in this research is to find what the change of discount ratio (or rate of interest) should mean under the condition of (34). We will represent (34) by \( k \)^7,

(35) \( \sum_{t=1}^{n} R_t v^t = \sum_{t=1}^{n} S_t v^t = k \cdot \quad k > 0 \)

The average period of (32), (33) is denoted by \( \pi_r \), \( \pi_s \) respectively, in accordance with the symbol of (4). (From now on, \( \sum_{t=1}^{n} \) will be simply represented by \( \Sigma \).) Thus we have

(36) \( \pi_r = \frac{\sum_t t R_t v^t}{\sum R_t v^t} \)

and

(37) \( \pi_s = \frac{\sum t S_t v^t}{\sum S_t v^t} \).

\(6\) Pay special attention to this assumption; see Chapter VI.

\(7\) By assuming \( R_t \) and \( S_t \) to be given and \( v \) (therefore \( r \)) to be unknown and solving (34), we can search for \( v \) or \( r \) that satisfy the planning for equating receipts and payments in question. But this is not our present subject.
We subtract (37) from (36) on each side. Then we have
\[
\pi_r - \pi_s = \frac{\sum t R_t v_t}{\sum R_t v_t} - \frac{\sum t S_t v_t}{\sum S_t v_t},
\]
therefore
\[
\frac{1}{k} \sum t (R_t v_t - S_t v_t);
\]
See (35)

(38)

transforming (38) into
\[
\pi_r - \pi_s = \frac{1}{k} \left\{ \sum t \cdot \frac{\sum t (R_t v_t - S_t v_t)}{\sum t} \right\}
\]

Now taking the stream of residuals which are the subtractions of the present values of payments (31) from those of receipts (30) in each period:

(40) \( (R_1 v^1 - S_1 v^1), (R_2 v^2 - S_2 v^2), \ldots, (R_n v^n - S_n v^n) \)

And as a method of consideration, let us make a distinction between (40) which results in a monotone increasing stream and that which causes a monotone decreasing stream. Although the practical, especially ex post, stream of capital values does not always reveal either a monotone increasing or a monotone decreasing process, it is often possible that either one of them is produced in the system of the ex ante program of capital.

Case I in which (40) brings about a monotone increasing stream:

(41) \( (R_1 v^1 - S_1 v^1) < (R_2 v^2 - S_2 v^2) < \ldots < (R_n v^n - S_n v^n) \)

The arithmetic mean of (41) weighted by a monotone increasing sequence which is

(42) \( t: 1 < 2 < \ldots < n \)

is quite of the same characteristic as (14). The only difference is that the weight \( w_t \) in (14) is replaced by \( t \) in this case. Let us, therefore, denote it by \( \tilde{a}_w \), and we have

(43) \[
\tilde{a}_w = \frac{1}{1 + 2 + \cdots + n} \left( \sum t (R_t v_t - S_t v_t) \right)
\]

Meanwhile taking the simple arithmetic mean of (41),

(44) \[
\bar{a} = \frac{\sum t (R_t v_t - S_t v_t)}{n} = 0.
\]

According to Lemma (27), we have

(45) \[
\frac{\sum t (R_t v_t - S_t v_t)}{\sum t} > 0.
\]
Applying (45) to (39), the inequality below is led to because of \( k > 0 \) and \( \sum t > 0 \):

(46) \( \pi_r - \pi_s > 0 \).

Case II in which (40) brings about a monotone decreasing stream:

(47) \( (R_1 v^t - S_1 v^t) > (R_2 v^t - S_2 v^t) > \cdots > (R_n v^t - S_n v^t) \).

The arithmetic mean of (47) weighted by (42) is quite of the same characteristic as (16), which is denoted by \( \overline{a_w} \).

Then in the same way as Case I, we will have

(48) \[ \overline{a_w} = \frac{\sum t (R_t v^t - S_t v^t)}{\sum t} \]

Applying (48) to Lemma (27),

(49) \[ \frac{\sum t (R_t v^t - S_t v^t)}{\sum t} < 0 \] \( \text{See (44)} \)

Applying (49) to (39), we have the inequality below for the same reason as in Case I:

(50) \( \pi_r - \pi_s < 0 \)

The above-stated can briefly be put into a table below:

<table>
<thead>
<tr>
<th>(Case)</th>
<th>(40)</th>
<th>(46) (50)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>Monotone Increasing Stream</td>
<td>( \pi_r - \pi_s &gt; 0 )</td>
</tr>
<tr>
<td>(II)</td>
<td>Monotone Decreasing Stream</td>
<td>( \pi_r - \pi_s &lt; 0 )</td>
</tr>
</tbody>
</table>

Next let us obtain from (30) and (31) the stream of present values of residuals denoted by \( G_t \), that is,

(52) \[ G_1 = (R_1 v^t - S_1 v^t) , \quad G_2 = (R_2 v^t - S_2 v^t) , \cdots , \quad G_n = (R_n v^t - S_n v^t) \]

and represent the sum of these by \( G \), then

(53) \[ G = \sum G_t = \sum (R_t v^t - S_t v^t) . \]

Differentiating \( G \) in regard to \( v \), we have

(54) \[ dG = \left\{ \sum t R_t v^{t-1} - \sum t S_t v^{t-1} \right\} dv . \]

We assume \( \frac{dv}{v} = \theta v \),

(55) \[ dv = \theta v . \]

Let (55) be substituted in (54), and we obtain

\[ dG = \left\{ \sum t R_t v^{t-1} - \sum t S_t v^{t-1} \right\} \theta v \]

therefore
Transforming (56) slightly into
\[
dG = \theta \left\{ \sum_t R_t v^t - \sum_t S_t v^t \right\}.
\]

Then using (35), (36), and (37), we will come to the following conclusion:
\[
dG = \theta k \{ \pi_r - \pi_s \}.
\]

In this case, since \( k > 0 \), we can find the following relations:

<table>
<thead>
<tr>
<th>( \pi_r - \pi_s &gt; 0 ) (Planning to be a Borrower)</th>
<th>Rate of Interest ( r )</th>
<th>Discount Ratio ( v )</th>
<th>Rate of Change of Discount Ratio ( \theta )</th>
<th>Marginal Surplus ( dG )</th>
<th>Balance of Payments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Down</td>
<td>Up</td>
<td>Positive</td>
<td>Positive</td>
<td>Better</td>
<td></td>
</tr>
<tr>
<td>Up</td>
<td>Down</td>
<td>Negative</td>
<td>Negative</td>
<td>Worse</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \pi_r - \pi_s &lt; 0 ) (Planning to be a Lender)</th>
<th>Rate of Interest ( r )</th>
<th>Discount Ratio ( v )</th>
<th>Rate of Change of Discount Ratio ( \theta )</th>
<th>Marginal Surplus ( dG )</th>
<th>Balance of Payments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Down</td>
<td>Up</td>
<td>Positive</td>
<td>Negative</td>
<td>Positive</td>
<td>Better</td>
</tr>
<tr>
<td>Up</td>
<td>Down</td>
<td>Negative</td>
<td>Positive</td>
<td>Better</td>
<td></td>
</tr>
</tbody>
</table>

Some explanation about (59) seems to be useful.

That the average period of the stream of receipts is larger than that of the stream of payments, that is, \( \pi_r - \pi_s > 0 \), indicates a planning that the receipts in each period are relatively little at present or in the near future and relatively large in the far future; whereas the payments in each period are relatively large at present or in the near future and relatively little in the far future. The deficit caused by the greater payment over the receipts in each period at present or in the near future are to be made amends for by the surplus which will be brought about by the greater receipts over the payments in each period in the far future. This is the "planning to be a borrower" as Hicks names it.

That the average period of the stream of payments is larger than that of the stream of receipts, that is, \( \pi_r - \pi_s < 0 \), indicates a planning that the payments in each period are relatively little at present or in the near future and relatively large in the far future; whereas the receipts in each period are relatively large at present or in the near future and relatively little in the far future. The surplus caused by the greater receipts over the payments in each period at present or in the near future is to be provided so as to make amends for the deficit which will be brought about by the greater payments over the receipts in each period in the far future. This is the "planning to be a lender" as Hicks names it.

According to (59), which is our conclusion, the fall (rise) of rate of interest improves (deteriorates) the balance of payments in the planning to be a borrower; the fall (rise) of rate of interest deteriorates (improves) it
in the planning to be a lender.

A remark on \( dG \geq 0 \): in short, this indicates a positive or negative sign of the quantity of change (which is the approximate value) of the sum of residuals \( G \) which is caused by a small change of \( v \) (or \( r \)) out of a certain balance of payments over \( n \) periods. In the above, we have substituted (35) in (57) to lead (58). If (35) be substituted in (53), \( G \) should be equal to 0, that is, \( G = 0 \); in other words, the balance of payments over \( n \) periods is to be attained. This means that \( v \), obtained from solving (35) is the root of \( G = 0 \). \( dG \) is the approximate value of the quantity of change of \( G \) in response to the change of \( v \) out of \( G = 0 \) in this root.

From (53), we have

\[
G = \frac{1}{v_i} k \left\{ \pi_r - \pi_s \right\} dv
\]

Let one of the roots of \( G = 0 \) be denoted by \( v_1 \) and substituted in \( \frac{dv}{v} = \theta \), and it becomes \( \theta_i = \frac{dv}{v_i} \). So (58) can be written in the form

\[
\frac{dG}{\theta_i} = \frac{1}{v_i} k \left\{ \pi_r - \pi_s \right\} dv
\]

(62) indicates the approximate value of the quantity of change of \( G \) in response to the change of \( v \) out of \( v = v_i \). Both \( \pi_r \) and \( \pi_s \) on the right side of (62), however, are the values with respect to \( v = v_i \).

VI Advanced Inquiry into the Theory of the "Average Period"

In the former chapters we have proved by mathematical analysis and revealed the utility of the theory of the "Average Period" which Hicks developed — though only verbally and rather simply — in his excellent work, *Value and Capital*. We, furthermore, would like to examine this theory a little more, because it seems possible that it might not necessarily be applicable to all cases depending upon the kind and nature of an assumption presupposed.

We have assumed that, in (28) and (29), receipts and payments take part in the streams of capital values respectively at the end of each period. Now we assume, in the first place\(^3\), that the receipts take part in the stream of capital values at the beginning and the payments at the end of each period.

\(^3\) In the latter part of this chapter, it will be assumed that "receipts take part in the stream of capital values at the end, and payments at the beginning of each period".
period. (In the mathematics of life insurance such an assumption is rather common.) In this case the streams of present values of receipts and payments over \( n \) periods would become

\[
R_1 v^0, \quad R_2 v^1, \quad \ldots, \quad R_n v^{n-1}
\]

and

\[
S_1 v^1, \quad S_2 v^2, \quad \ldots, \quad S_n v^n.
\]

Denoting the sums of these streams of present values by \( R \) and \( S \) respectively, thus

\[
R = R_1 v^0 + R_2 v^1 + \cdots + R_n v^{n-1} = \sum R_i v^{i-1}
\]

and

\[
S = S_1 v^1 + S_2 v^2 + \cdots + S_n v^n = \sum S_i v^i.
\]

For the planning of equating receipts and payments over all periods, \( R = S \) is required. Let us assume \( R = S = k \), which can be expressed as

\[
\sum R_i v^{i-1} = \sum S_i v^i = k.
\]

Then, let us search for the elasticities of the capital values of \( R \) and \( S \) with regard to the discount ratio, that is, the average periods, \( \pi_r \) and \( \pi_s \):

\[
\pi_r = \frac{v}{R} \frac{dR}{dv}
\]

\[
= \frac{v}{R} \left[ R_2 + 2R_3 v^1 + \cdots + (n-1) R_n v^{n-2} \right]
\]

\[
= \frac{1R_2 v^1 + 2R_3 v^2 + \cdots + (n-1) R_n v^{n-1}}{R_1 v^0 + R_2 v^1 + \cdots + R_n v^{n-1}}
\]

\[
= \frac{(1) R_2 v^1 + (2) R_3 v^2 + \cdots + (n) R_n v^{n-1}}{R_1 v^0 + R_2 v^1 + \cdots + R_n v^{n-1}}
\]

therefore

\[
\pi_r = \frac{\sum t R_i v^{t-1}}{\sum R_i v^{t-1}} - 1.
\]

In the same way

\[
\pi_s = \frac{v}{S} \frac{dS}{dv},
\]

therefore

\[
\pi_s = \frac{\sum t S_i v^t}{\sum S_i v^t}.
\]

Subtracting (69) from (68) on each side:
ON THE "AVERAGE PERIOD" OF J. R. HICKS

\[
\pi_r - \pi_s = \frac{\sum t R_t v^{t-1}}{\sum R_t v^{t-1}} - \frac{\sum t S_t v^t}{\sum S_t v^t} - 1
\]

\[
= \frac{1}{k} \sum t (R_t v^{t-1} - S_t v^t) - 1
\]

transferring into

\[
(70) \quad = \frac{1}{k} \left\{ \sum t \frac{(R_t v^{t-1} - S_t v^t)}{\sum t} \right\} - 1
\]

Next, let us also take from (63) and (64) the stream of residuals in each period, and we have

\[
(71) \quad (R_t v^0 - S_t v^0), \quad (R_t v^1 - S_t v^1), \quad \ldots \ldots \ldots, \quad (R_n v^{n-1} - S_n v^n).
\]

The arithmetic mean of these residuals weighted by

\[
(72) \quad t : \quad 1, 2, \ldots \ldots \ldots \ldots, \quad n
\]

becomes

\[
(73) \quad \tilde{a}_w, \quad \tilde{a}_w = \frac{\sum t (R_t v^{t-1} - S_t v^t)}{\sum t}
\]

When (71) is a monotone increasing stream, (73) becomes \( \tilde{a}_w \); When (71) is a monotone decreasing stream, (73) becomes \( \tilde{a}_w \). And the simple arithmetic mean \( a \) of (71) becomes

\[
a = \frac{1}{n} \left\{ (R_t v^0 - S_t v^0) + (R_t v^1 - S_t v^1) + \ldots \ldots \ldots + (R_n v^{n-1} - S_n v^n) \right\}
\]

\[
= \frac{1}{n} \left\{ \sum R_t v^{t-1} - \sum S_t v^t \right\},
\]

therefore

\[
(74) \quad = 0; \quad \text{See (67)}
\]

Applying Lemma (27) to (70), we can find the following relations:

<table>
<thead>
<tr>
<th>( \pi_r - \pi_s )</th>
<th>( \text{(71)} )</th>
<th>( \text{(73)} )</th>
<th>( \text{(70)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monotone Increasing Stream</td>
<td>Positive</td>
<td>Negative</td>
<td>Impossible to Judge Positive or Negative</td>
</tr>
<tr>
<td>Monotone Decreasing Stream</td>
<td>Negative</td>
<td>Negative</td>
<td></td>
</tr>
</tbody>
</table>

This leads to a different result from that of (51) in the previous chapter which was reached under the assumption that both receipts and payments take part in each stream of capital values at the end of each period.

In the second place, unlike the previous assumption let us assume that
receipts take part in the stream of capital values at the end, and payments at the beginning of each period. In this case each stream of present values of receipts and payments over \( n \) periods will become respectively

\[
R_1 v^1, R_2 v^2, \ldots, R_n v^n
\]

and

\[
S_1 v^1, S_2 v^2, \ldots, S_n v^{n-1}.
\]

Suppose that \( R = S = k \) be required for the planning of equating receipts and payments all over \( n \) periods, that is,

\[
\sum R_t v^t = \sum S_t v^{t-1} = k.
\]

Let us search for \( \pi_r \) and \( \pi_s \), the average periods of \( R \) and \( S \); in the same way as before,

\[
\pi_r = \frac{\sum t R_t v^t}{\sum R_t v^t}
\]

and

\[
\pi_s = \frac{\sum t S_t v^{t-1}}{\sum S_t v^{t-1}} - 1.
\]

Subtracting (80) from (79) on each side, then

\[
\pi_r - \pi_s = \frac{1}{k} \left\{ \sum t \cdot \frac{\sum t (R_t v^t - S_t v^{t-1})}{\sum t} \right\} + 1.
\]

Taking the stream of residuals in each period from (76) and (77) which becomes

\[
(R_1 v^1 - S_1 v^0), (R_2 v^2 - S_2 v^1), \ldots, (R_n v^n - S_n v^{n-1}),
\]

we have an arithmetic mean weighted by (72) as

\[
\bar{w}_w = \frac{\sum t (R_t v^t - S_t v^{t-1})}{\sum t}.
\]

When (82) is a monotone increasing stream, (83) becomes \( \bar{w}_w \); when (82) is a monotone decreasing stream, (83) becomes \( \bar{w}_w \). And the simple arithmetic mean \( a \) becomes

\[
a = \frac{1}{n} \left\{ (R_1 v^1 - S_1 v^0) + (R_2 v^2 - S_2 v^1) + \cdots + (R_n v^n - S_n v^{n-1}) \right\}
\]

\[
= \frac{1}{n} \left\{ \sum R_t v^t - \sum S_t v^{t-1} \right\},
\]

therefore

\[
a = 0.
\]

Applying Lemma (27) to (81), we can find the following relations:
ON THE "AVERAGE PERIOD" OF J. R. HICKS

\[ \pi_r - \pi_s \]

<table>
<thead>
<tr>
<th>( (82) )</th>
<th>( (83) )</th>
<th>( (81) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monotone Increasing Stream</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>Monotone Decreasing Stream</td>
<td>Negative</td>
<td>Impossible to Judge Positive or Negative</td>
</tr>
</tbody>
</table>

This leads to a different result from that of (51) in the previous chapter in which both receipts and payments are assumed to take part in each stream of capital values at the end of each period.

The result of (51), (75), and (85) which are cases derived from different assumptions is as follows.

\[ \pi_r - \pi_s \]

<table>
<thead>
<tr>
<th>Assumptions</th>
<th>Streams of residuals of present values</th>
<th>M. I. S.</th>
<th>M. D. S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) Both receipts and payments taking part in the streams at the end of each period</td>
<td></td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>(B) Receipts taking part in the stream at the beginning, and payments at the end of each period</td>
<td></td>
<td>Impossible to Judge Positive or Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>(C) Receipts taking part in the stream at the end, payments at the beginning of each period</td>
<td></td>
<td>Positive</td>
<td>Impossible to Judge Positive or Negative</td>
</tr>
</tbody>
</table>

In short, \( \pi_r - \pi_s \) is always settled as either positive or negative under Assumption (A); whereas there can be some cases in which it is impossible to judge positive or negative under Assumptions (B) and (C). In other words, in the cases in which the stream of residuals of present values indicates monotone increase under Assumption (B) and cases which it has a monotone decrease under Assumption (C), \( \pi_r - \pi_s \) cannot be judged either positive or negative. In such cases, therefore, it is impossible to judge whether the balance of payments in (59) improves or deteriorates.

In the last place, we will take, as a concrete case, the management of life insurance into account, applying the theory which we have treated so far.

In the case of life insurance, insurance premiums which are receipts for the insurance company are received at the beginning of each period;
while insurance amounts which are payments for the company are paid at any time when the insured die. So the time when insurance amounts are paid is, on an average, to be supposed at the middle of each respective period.

We will explain, as an example, the whole life insurance system, the most representative of all the life-insurance contract systems. The stream of present values of receipts and payments is

\[(87) \text{Receipts: } R_1 v^0, R_2 v^1, \ldots, R_n v^{n-1}\]

and

\[(88) \text{Payments: } S_1 v^1, S_2 v^{1+1}, \ldots, S_n v^\left(n-1+\frac{1}{2}\right)\]

(In the mathematics of life insurance, it is often assumed that insurance amounts be paid at the end of each period for the purpose of simple calculation. In this case, (88) is to be quite the same as (64). But we assume here that they be paid by the company at the middle of each period for the purpose of strict calculation.)

The sums of (87) and (88) denoted by \(R\) and \(S\) respectively become

\[(89) R = R_1 v^0 + R_2 v^1 + \ldots + R_n v^{n-1} = \sum R_i v^{t-1}\]

and

\[(90) S = S_1 v^1 + S_2 v^{1+1} + \ldots + S_n v^\left(n-1+\frac{1}{2}\right) = \sum S_i v^{t-\frac{1}{2}}\]

Again \(R = S = k\) is required for the planning of equating receipts and payments all over \(n\) periods, and therefore

\[(91) \sum R_i v^{t-1} = \sum S_i v^{t-\frac{1}{2}} = k\]

Let us search for \(\pi_r\) and \(\pi_s\) with regard to \(R\) in (89) and \(S\) in (90), and we have

\[(92) \pi_r = -\frac{\sum t R_i v^{t-1}}{\sum R_i v^{t-1}} - 1 : \quad \text{this is the same as (68)};\]

and

\[(93) \pi_s = \frac{\sum t S_i v^{t-\frac{1}{2}}}{\sum S_i v^{t-\frac{1}{2}}} - \frac{1}{2}\]

Subtract (93) from (92) on each side in consideration of (91), and transform into

\[(94) \pi_r - \pi_s = \frac{1}{k} \left\{\sum t, \frac{\sum t (R_i v^{t-1} - S_i v^{t-\frac{1}{2}})}{\sum t} - \frac{1}{2}\right\}.\]

Furthermore we obtain the stream of residuals of present values from (87) and (88), which is

\[(95) (R_1 v^0 - S_1 v^{\frac{1}{2}}), (R_2 v^1 - S_2 v^{1+\frac{1}{2}}), \ldots, (R_n v^{n-1} - S_n v^{(n-1)+\frac{1}{2}}).\]
Taking the arithmetic mean weighted by
\[(96) \quad t : 1, 2, \ldots, n\]
which becomes
\[(97) \quad \bar{a}_w = \frac{\sum_{t} t (R_t v^{t-1} - S_t v^{t-\frac{1}{2}})}{\sum_{t} t} = \bar{a}_w.\]

When (95) is a monotone increasing stream, (97) becomes \(\bar{a}_w\); when (95) is a monotone decreasing stream, (97) becomes \(\bar{a}_w\). But, in life insurance, (95) always indicates a monotone decreasing stream. The reason: in life insurance the system of paying insurance premiums, which are receipts for the insurance company, is most commonly the uniform premiums system in which the sum of premiums paid from policy-holder is constant at every period.

Therefore,
\[R_1 = R_2 = \ldots = R_n.\]
Whereas the sequence of discount ratio is decreasing:
\[(98) \quad v^0 > v^1 > v^2 > \ldots > v^{n-1}.\]
So that the present values of premium receipts indicate a monotone decreasing stream which is
\[(99) \quad R_1 v^0 > R_2 v^1 > \ldots > R_n v^{n-1}.\]
Against this tendency, the insurance amounts paid become steadily higher and higher as the probability of dying of the insured grows high, period to period. So they make a monotone increasing stream which is
\[(100) \quad S_1 < S_2 < \ldots < S_n.\]
But the increasing trend of (100) is much greater than the decreasing trend of the discount ratio which is a monotone decreasing sequence, that is,
\[(101) \quad v^{\frac{1}{2}} > v^{1+\frac{1}{2}} > \ldots > v^{\left(\frac{n-1}{2}\right)+\frac{1}{2}}.\]
So that the stream of present values of insurance amount payments becomes a monotone increasing stream such as
\[(102) \quad S_1 v^{\frac{1}{2}} < S_2 v^{1+\frac{1}{2}} < \ldots < S_n v^{\left(\frac{n-1}{2}\right)+\frac{1}{2}}.\]
The result of this shows that the stream of residuals of present values in each period obtained from (87) and (88) which is
\[(103) \quad (R_1 v^d - S_1 v^{\frac{1}{2}}) > (R_2 v^d - S_2 v^{1+\frac{1}{2}}) > \ldots > (R_n v^{n-1} - S_n v^{\left(\frac{n-1}{2}\right)+\frac{1}{2}})\]
very clearly indicates a monotone decreasing stream.
In life insurance the method of keeping the balance of payments equal
is projected in such a way that the surplus of receipts brought about in the former half of all the periods (the area bounded by \( MPN \) in Figure 4) can meet the deficit brought about in the latter half (the area bounded by \( QPT \) in Figure 4). For this reason it is clear that (103) indicates a monotone decreasing sequence. According to Lemma (27), therefore, (97) can be represented by

\[
(104) \quad \sum_t \frac{R_t \nu^{t-1} - S_t \nu^{t-\frac{1}{2}}}{\sum_t} < 0;
\]

substituting (104) in (94), we obtain

\[
(105) \quad \pi_r - \pi_s < 0.
\]

In terms of our conclusion which has been made clear by (86), the management of life insurance belongs to the case in which the stream of residuals of present values takes a monotone decreasing stream under Assumption (B). We, therefore, are able to see by (59) how the change of discount ratio (therefore, that of the rate of interest) influences the balance of payments of an insurance company that manages by planning for equating receipts and payments.

In conclusion, Hicks's theory of the "Average Period" is fully applicable as far as life insurance is concerned.

Figure 4