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ON THE NASH SOLUTION

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I Introduction

The aim of this brief paper is to discuss the subject of Nash's solution of the duopoly problem. It will be convenient to give a general outline of the problem before going into any further discussion.

Whenever any description is to be given of a case where one person is participating in a competition or a case where two persons or more are participating, it is very important to discern the difference between the former and the latter situation taking into consideration that the decisions made and the resulting behaviour of the participants in the latter case, unlike the former, are inter-dependent. In addition to this seemingly obvious point, it must also be noted that there is a possibility that any competition in which two or more competitors are participating may give rise to the following two situations—one where the interests of each participant stand completely opposed to each other and another where some of the participants, with their opposing interests on one hand, may be enabled on the other hand to obtain greater advantage by forming a kind of coalition among themselves than other participants who are not so joined. Obviously these situations cannot occur in the case of one participant, namely a case of monopoly. Of the two peculiarities of a two-person competition, as compared with a case of monopoly, what makes the problem fundamentally complicated is the very possibility that the participants may obtain greater advantage through their concerted efforts.

If the acts of each participant cannot be other than mutually restricted, then each participant must determine his own acts by anticipating all possible influences which his act may exert upon all of his rivals. In other words, the optimal strategy for one's own benefit is to be determined in anticipation of all determinate or probable responses that his rivals may take in response to the particular act of his own. Such relationships between *moves* and *responses* observable in a competition is not of much importance when the scale of each participant is small in relation to the size of the market. This is, needless to say, one condition of a pure competition, which constitutes one important prerequisite supporting the hypothesis "ceteris paribus". Consequently it is in the case of a small number of participants that the relationships between moves and responses have an important bearing upon the results of a competition. In particular, an example of two-person participation

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may well be taken as a standard model case, on which basis an analysis of competitive interactions should be undertaken. According to the duopolists in Cournot [4], the optimal output to maximize one's own profit is determined on the assumption that the output of the rival will not be affected by his own decision making under a certain given market demand function. The point where two output reaction curves (each indicating the optimal outputs in response to the outputs of one's rival) intersect is the equilibrium point for the market. What characterises Cournot's equilibrium solution is one specific assumption being formed by each participant about the behavior of his opposing competitor. As is easily imaginable, if there is any change in this behavior, Cournot's solution can no longer be the equilibrium solution. Thus we are led to conclude that many subsequent discussions of Cournot's treatment were based without exception on different assumptions from that of Cournot relating to the competitors' strategies. For examples may be considered a case of market behaviour on the assumption of mutual understanding that the opponent's price is given, a case of a market structure where price fluctuations become possible, or to make inter-dependence more complicated, a case where an output (or price) is determined by anticipating that each opposing competitor will determine his own optimal output (or price) in anticipation of the optimal output (or price) determined by an opposing competior[1]. As to these various kinds of assumption, we must still question further to see whether each of the assumptions can be supported in terms of actual market relationships having a close resemblance to a duopoly. Now, disregarding this aspect, when the consistency of each of the said assumptions is put to question, we find that each of them includes a common contradiction.

Cournot's assumption appears to be inconsistent with the requirement of maximizing profit, because such requirement should make it necessary to take all possible influences to be exercised by a decision as well as its direct effect to which the cournot case was exclusively directed on the assumption that the decision making with regard to one's own output will not have any influence upon decisions regarding his reval's output. With the sole exception of Chamberlin's model in which emphasis is placed on such comprehensive inter- dependency, all other models are not quite free from this kind of contradiction. Therefore, in order to correct this it becomes necessary to consider accurately the effects of inter-dependency upon the choice of actions. Nevertheless, that *each* of them will determine the optimal strategy of his own upon assuming the same patterns of responses from the opposing competiotr would imply that the said patterns of response are not taking place in reality and hence the decision making in that case is based on an erroneous anticipation. In this way Chamberlin's assumption and equilibrium solution are not entirely satisfactory, if viewed from the requirement of maximizing profit. Putting

1) Cf. Aoyama [1], Chapter 3; Baumol [2], Chapter 3; Chamberlin [4], Chapter 3 and Appendix A.

it in another way, the assumptions so far used in any duopoly case, including the one discussed by Cournot, have been based on an idea that the interests of each competitor are perfectly incompatible with each other and efforts have been made to try to find the optimal strategy by providing various patterns of responses common to every rival. However, in order to fulfill the requirement of maximizing profit correctly, it is an absolute necessity to deal not merely with cases wherein the interests of each competitor stand in confrontation but also with other cases wherein it becomes possible for each competitor to gain more profit if one of the competitors acts cooperatively with the other for this purpose.

In Nash [8] we can see his attempt to give a general solution of the duopoly problem, regarding it as a two-person cooperative game in which the interests of each person are not completely incompatible. Because the number of available strategies would increase if the possibility of coalition on the part of some other competitors is realised, the attainable gains for such competitors could not be smaller than in a case of complete confrontation. As will be explained later, it becomes possible to attain a point, Pareto-superior to the profit point attained by the optimum strategy in terms of Cournot's equilibrium. Any point on the boundary of the attainable profit sphere located to the northeastward from Cournot's point becomes Pareto-optimal. Hence it is impossible to apply the Pareto-ranking to the comparison of any point on the boundary. Nash worked out a specific point on the boundary as the solution by making use of the concept of threat.

II Nash's Solution

1. Premises :

(1) Each player 1, 2 has a compact and convex strategy set S_1, S_2 respectively, each of which is composed of mixed strategies s_1, s_2.

(2) The pay-off set (u_1, u_2) due to the acts carried out either independently or in cooperation by the player 1 and 2 is represented by B (u_1, u_2 represents the pay-off of the player 1 or 2 respectively). And B is convex and compact.

(3) If the pay-off for the player 1 or 2, when the strategy $(s_1, s_2) \in S_1 \times S_2$ is being used, are expressed by $p_1\ (s_1, s_2)$, $p_2\ (s_1, s_2)$, p_i $(i=1, 2)$ compose a bi-linear form of s_1, s_2[2].

(4) The players have perfect information available with respect to the game structure and the pay-off relationships of his own and of his rival, and he is supposed to act rationally.

(5) Whenever the player finds that his demand contradicts the rival's demand,

2) This is determined by the nature of the utility function specified for the player. von Neumann=Morgenstern [16], Chapter 1, Part 3: Nash [12].

the player manages himself to develop the game in favour of himself.

2. Negotiation Model

Here a negotiation model composed of four stages as shown below is considered.

(1) Selection of threat t_i $(i=1, 2)$:

Each player uses the mixed strategies t_i whenever the demand $d_i(i=1, 2)$ stands in contradiction.

(2) Each player mutually lets his threats be known.

(3) The demand d_i is held independently.

(4) Determination of pay-off:

Whenever the demand is compatible, the pay-off is settled at the amount demanded and whenever incompatible, the pay-off is what the threat gives. This can be formulated as follows:

$$\begin{array}{lll} d_i & (i=1, 2)\ , & \text{if } (d_1, d_2)\in B. \\ p_i(t_1, t_2) & (i=1, 2)\ , & \text{if } (d_1, d_2)\notin B. \end{array}$$

Since among all four stages, (2) and (4) do not include the decision making by the players, what is called the negotiation game is a game composed of two moves as indicated in (1) and (3). Furthermore, since the decision making for the demand is made in stage (3), which is the second move, upon knowing the decision made as a result of the first move, i. e. the threat in stage (1), a game which includes only the second move, having the pay-off functions which are determined by the threat selected in the manner described in stage (1), can be composed. This is called the demand game.

3. The Demand Game

Supposing that the players 1 and 2 are respectively making selection of the threat t_1, t_2 in stage (1), and that the pay-off for the palyers 1 and 2 under the aforementioned conditions are respectively expressed as $p_1(t_1, t_2)=u_{1N}$, $p_2(t_1, t_2)=u_{2N}$. Then under the following conditions

$$(3\cdot1) \qquad g(d_1, d_2) = \begin{cases} 1 & \text{when } (d_1, d_2)\in B \\ 0 & \text{when } (d_1, d_2)\notin B \end{cases},$$

the pay-off functions for each player can be expressed as follows:

$$(3\cdot2) \qquad \begin{array}{ll} \text{Player 1} & d_1 g+u_{1N}(1-g), \\ \text{Player 2} & d_2 g+u_{2N}(1-g). \end{array}$$

Generally speaking, there is a possibility that countless numbers of equilibrium solutions can be found in the demand game to be player under such pay-off functions.

Now, in order to obtain the game solution, $g\ (d_1,\ d_2)$ is converted into the following continuous function $h\ (d_1,\ d_2)$, i. e.,

$$h \begin{cases} = 1 & \text{if } (d_1,\ d_2) \in B. \\ \to 0 & \text{as } (d_1,\ d_2) \text{ go farther and farther away} \\ & \text{toward the outside direction from } B. \end{cases}$$

Again, if $u_{1N}=0,\ u_{2N}=0$ by making proper transformation of the utility functions, the pay-off functions of the smoothed game can be expressed as follows, taking the place of (3·2):

$$(3\cdot 3) \qquad \begin{aligned} p_1 &= d_1 h, \\ p_2 &= d_2 h. \end{aligned}$$

Since it is reasonable for each player to determine respective demand so that the pay-off can be maximized, the equilibrium point under (3·3) is (d_1, d_2) which satisfies

$$(3\cdot 4) \qquad \begin{aligned} \left[\frac{\partial (d_1 h)}{\partial d_1}\right] &= 0, \\ \left[\frac{\partial (d_2 h)}{\partial d_2}\right] &= 0. \end{aligned}$$

In the meanwhile, $(d_1,\ d_2)$ which satisfies the following conditions also satisfies (3·4):

$$\frac{\partial}{\partial d_1}[d_1 d_2 h] = 0,$$

$$\frac{\partial}{\partial d_2}[d_1 d_2 h] = 0.$$

Now, supposing that the point to maximize $d_1\, d_2\, h$ is P and the maximum value of $u_1\ u_2$ is ρ (because $d_1,\ d_2$ are replaceable with $u_1,\ u_2$), the following relationships are obtainable:

$$[d_1 d_2 h]_{(d_1, d_2)=P} = [u_1 u_2 h]_{(u_1, u_2)=P} \geqslant [u_1 u_2]_{(u_1, u_2)\in B}.$$

Since the maximum value of $[u_1\ u_2]_{(u_1, u_2)\in B}$ is ρ,

$$[u_1\ u_2]_{(u_1, u_2)=P} \geqslant \rho.$$

Supposing that the point which maximises $[u_1\ u_2]$ is on the sphere B, the aforementioned relationships can be shown in Figure I. When $(d_1,\ d_2)$ are incompatible, the position of P is situated in an upward direction from $u_1\, u_2 = \rho$. Further, at this time P must be sufficiently close to B, if the value of h should be as close to 1 as possible. Consequently, P which is the maximum point of $d_1\, d_2\, h$ must move closer and

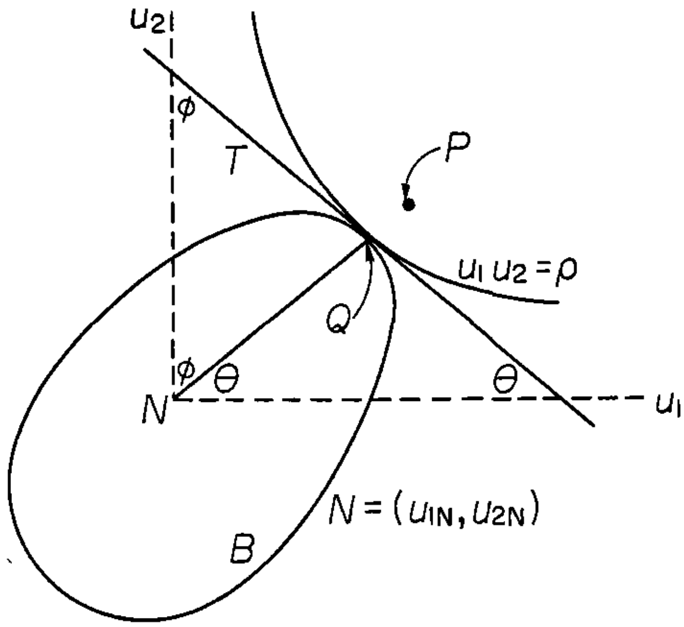

Fig. I Solution of Demand Game

closer to B, as less and less smoothing is applied to $g\ (d_1, d_2)$, that is, the faster and faster h tends to decline as P goes farther and farther away from B. Since Q is the only point of contact of B with the field located in the upward position of $u_1 u_2 = \rho$, P makes a closer approach toward Q.

Because Q is the only limiting point of the smoothed game equilibrium, (u_1, u_2) on Q is the optimal demand and at the same time Q is the solution of the original demand game.

The following relationships are established between the optimal solution of the demand game, Q, and the pay-off guaranteed by a given threat, N.

Since Q is the point which makes $u_1 u_2$ maximized on the cordinate with the basic point N, if the boundary of B is put as $f\,(u_1, u_2) = O$, the following relationships are established at Q:

$$\frac{u_2 - u_{2N}}{u_1 - u_{1N}} = -\frac{du_2}{du_1}. \tag{3·5}$$

The geometrical relationships of (3·5) show that Q is the optimal solution under N when NQ makes a positive slope and the support line T of B passing through Q, with the negative-valved slope of NQ (See Figure I).

Whenever Q is on the horizontal line (or vertical line) which passes through N, Q becomes the solution under N. Now, at this time if the slope of the support line at Q is properly interpreted, it may be assumed that the relationships of (3·5) are also being established at this point. Therefore, (3·5) provides necessary and sufficient conditions for Q to be the optimal solution.

In the meanwhile, since the sllope $\frac{du_2}{du_1}$ of the tangent at the boundary of B varies continuously, a straight line with a slope $-\frac{du_2}{du_1}$ (hereafter this line is referred as a complementary line) which goes through its contact point varies continuously, too[3]. Consequently, now, if Q is assumed to be the function of N, $Q(N)$, then there exists an adequate δ for any given small positive number ε, and it becomes possible to choose N, in such manner as to establish the following relationships. In other words, Q is continuous with respect to N.

(3·6) when $\|N-N'\|<\delta, \quad \|Q(N)-Q(N')\|<\varepsilon.$

4. The Threat Game

Presupposing that the demand game solution is the pay-off function, I shall now discuss a particular game which is only concerned with the moves of the first stage of the negotiation model. In short, the determination of the optimal threat is going to be made upon presupposing a certain Q, but because this presupposed Q

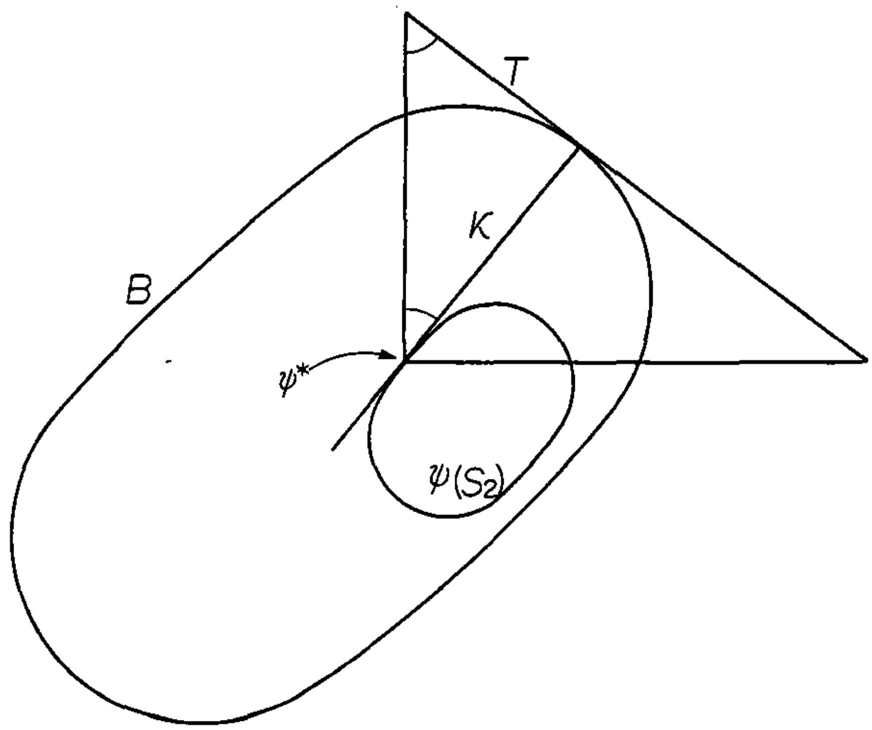

Fig. II Image of Optimal Threat

3) The portion of the boundary which becomes possible solutions to the demand game is the downward sloping part on the northeast boundary to N. Since B is convex, th esupplementary line for a point belonging to this portion begins to increase its slope more and more from left to right. Consequently, if they are to intersect, such intersection takes place only on the boundary.

depends upon the selection of threat as seen in 3, both the demand game and the threat game are mutually correlated with each other, and in this way they compose a negotiation model.

Now, supposing that the threat of the player 1 is fixed at $\bar{t}_1$ we will see from the premise 1 (3) that $p_1(\bar{t}_1, t_2)$, $p_2(\bar{t}_1, t_2)$ are continuous linear functions of t_2. Therefore, it means the existence of the following linear transformation:

$$\psi(t_2) = N[p_1(t_1, t_2),\ p_2(t_1, t_2)].$$

If among all images $\psi(S_2)$ that portion which falls on the most favourable complementary line K for the player 2 is expressed by ψ^*, then ψ^* is the image of the optimal threat of the player 2 in relation to $\bar{t}_1$[4].

There is no guarantee that the optimal strategy for player 1 in relation to $S_2{}^*$ should be $\bar{t}_1$ when the optimal threat of player 2 in relation to $\bar{t}_1$ is $S_2{}^*$. If there exists an equilibrium solution in the threat game at all, then each threat adopted respectively by the two players at that time should be simultaneously an optimal threat in relation to each other. Now, the fact that a pair of such optimal threats does exist can be drawn from the following relationships[5]:

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4) Cf. 3).

5) Proof for each case runs as follows:

(4·1) ...... Trivial since S_2 is compact and convex and ψ is a continuous linear transformation.

(4·2) ...... (i) ψ^* is an intersection of $\psi(S_2)$ with K. From (4·1), $\psi(S_2)$ is a closed set. If K is regarded as a segmental line divided by the boundary of B, then K is a closed set. Therefore, $\psi^*(=\psi(S_2)\cap K)$ is a closed subset of $\psi(S_2)$. Because ψ is a continuous linear transformation, the inverse-image $\psi^{-1}(\psi^*)=S_2{}^*$ is a closed subset of S_2. As S_2 is compact, so $S_2{}^*$ is compact.
(ii) Since ψ^* is located on a segment K, evidently it is convex. Consequently, if any two points $(\psi_1{}^*, \psi_2{}^*)$ on ψ^* are picked up (where $\psi_1{}^*=\psi(S_2{}^1)$, $\psi_2{}^*=\psi(S_2{}^2)$, and $S_2{}^1$, $S_2{}^2\in S_2{}^*$), for α such that $0\leqslant\alpha\leqslant 1$, $\alpha\psi_1{}^*+(1-\alpha)\psi_2{}^* = \alpha\psi(S_2{}^1)+(1-\alpha)\psi(S_2{}^2)=\psi[\alpha S_2{}^1+(1-\alpha)S_2{}^2]\in\psi^*$. Therefore $\alpha S^1+_2(1-\alpha)S_2{}^2\in S_2{}^*$. From (i) and (ii), $S_2{}^*$ is compact and convex.

(4·3) ...... $N[p_1(t_1, t_2),\ p_2(t_1, t_2)]$ is continuous with respect to t_1, and t_2, and Q is continuous with respect to t_1 and t_2 (Cf. 3). When player 1 takes a threat $t_1{}^\nu(\nu=1, 2, \ldots)$, the optimal threat set for player 2 corresponding to $t_1{}^\nu$ is expressed as $S_2{}^*(t_1{}^\nu)$. Now, since the payoffs of the game to be determined by $t_1{}^\nu$ and the optimal threat $t_2{}^\nu\in S_2{}^*$ for player 2 are $Q(t_1{}^\nu, t_2{}^\nu)$, from the continuity of Q, when $\lim t_1{}^\nu=t_1$, $\lim t_2{}^\nu=t_2$, it leads to $\lim Q(t_1{}^\nu, t_2{}^\nu)=Q(t_1, t_2)$.
Now, supposing that t_2 is not the optimal threat for player 2 in relation to t_1, when the optimal threat $t_2{}^1$ belonging to $S_2{}^*(t_1)$ is taken, $Q(t_1, t_2{}^1)$ gives a greater payoff for player 2 than $Q(t_1, t_2)$. Consequently, if a pair $(t_1{}^N, t_2{}^N)$ which sufficiently closed to (t_1, t_2) is taken, it may be assumed from the continuity of Q that there exists such threat $t_2{}^{N'}$ that gives more desirable $Q(t_1{}^N, t_2{}^{N'})$ for the player 2 than $Q(t_1{}^N, t_2{}^N)$. However, because $t_2{}^N\in S_2{}^*(t_1{}^N)$, this is a contradiction. Therefore $t_2\in S_2{}^*(t_1)$ and $S_2{}^*(t_1)$ is upper semi-continuous with respect to t_1.

(4·4) ...... From (4·3), $S_1{}^*(t_2)$, and $S_2{}^*(t_1)$ are respectively upper semi-continuous with respect to t_2 and t_1. Hence $S_1{}^*(t_2)\times S_2{}^*(t_1)$ are upper semi-continuous with respect to (t_1, t_2). Also, because $S_1{}^*(t_2)$ and $S_2{}^*(t_1)$ are respectively a convex subset of S_1, S_2, $S_1{}^*(t_2)\times S_2{}^*(t_1)$ are a convex subset of $S_1\times S_2$.

(4.1) ϕ (S_2) is compact and convex,

(4.2) $S_2{}^*$ is a compact and convex subset of S_2,

(4·3) If the optimal threat for the player 2 in relation to a threat t_1 for the player 1 is expressed as $S_2{}^*(t_1)$, then $S_2{}^*(t_1)$ is upper semi-continuous with respect to t_1. If t_2 is fixed and the linear transformation ϕ $(t_1)=N[p_1(t_1, \bar{t}_2), p_1(t_1, \bar{t}_2)]$ is applied, the propositions just given above are established respectively in relation with ϕ (S_1), ϕ^*, $S_1{}^*$ (t_2).

For any threat pair (t_1, t_2) there exist the following optimal threasts for two players:

$S_1{}^*(t_2)$ in relation to t_2 for player 1,

$S_2{}^*(t_1)$ in relation to t_1 for player 2.

Now, if R (t_1, t_2) is supposed to be a pair of optimal threasts obtained from $S_1{}^*$ (t_2), $S_2{}^*$ (t_1) then from (4·1)–(4·3) the following propositions are generated:

(4·4) R (t_1, t_2) is upper semi-continuous with respect to (t_1, t_2) and a convex subset of $S_1 \times S_2$.

From (4·4) we see that the conditions of Kakutani's theorem are satisfied, and there exist $(\hat{t}_1, \hat{t}_2)$ such that $(\hat{t}_1, \hat{t}_2) \in R$ $(\hat{t}_1, \hat{t}_2)$. In other words the relationship in which each threat of two players becomes the optimal threat to each other is brought into existence. This is the equilibrium solution of the threat game.

III Economic Significance of the Solution

The optimal solution of the demand game $Q\,(t_1{}^1, t_2{}^1)$, is determined under a given threat $(t_1{}^1, t_2{}^1)$, but because each of two players tries to achieve a greater demand of his own the player 1 uses $t_1{}^2$ such that $t_1{}^2 \in S_1{}^*$ $(t_2{}^1)$ and the player 2 uses $t_2{}^2$ such that $t_2{}^2 \in S_2{}^*$ $(t_1{}^1)$ resulting ultimately in $Q\,(t_1{}^2, t_2{}^2)$. If such reaction is mutually repeated, it leads to $(\hat{t}_1, \hat{t}_2)$ such that $(\hat{t}_1, \hat{t}_2) \in S_1{}^*$ $(\hat{t}_2) \times S_2{}^*$ $(\hat{t}_1)$. Hence, however many times such reaction may be repeated later on, the players always get to the same $Q\,(\hat{t}_1, \hat{t}_2)$. In other words $Q\,(\hat{t}_1, \hat{t}_2)$ is the equilibrium demand solution of this negotiation model.

This model is of an extremely abstract nature as is evident from the foregoing section II. For this reason it is applicable to a variety of cases such as negotiations between employer and employees, commercial problems between two countries and a case of bilateral monopoly, etc. I shall here seek to clarify the meanings of the solution by comparing the two players to the two mineral spring owners in Cournot's case[6].

6) In Mayberry [11], each solution according to Cournot and von Neumann is compared with that of Nash by making use of numerical examples of duopolists having gradually increasing different cost functions. In Bishop [3], a duopoly model composed of a constant average high cost producer and a constant low cost producer is explained with illustrations and a comprehensive discussion of each solution of Raiffa, Shapley, Nash, Zeuthen, etc. is given.

Now, supposing that the market demand function is given in the following formula:

$$p = 10-2(q_1+q_2) \begin{cases} p \text{ ... price} \\ q_1 \text{ ... supply of duopolist 1,} \\ q_2 \text{ ... supply of duopolist 2} \end{cases}$$

profits π_1, π_2 for duopolist 1 and 2 are expressed as

$$\begin{aligned} \pi_1 &= 10q_1-2q_1^2-2q_1q_2, \\ \pi_2 &= 10q_2-2q_1^2-2q_1q_2. \end{aligned} \tag{3·1}$$

If the duopolist is supposed to act as Cournot thinks, then

$$\frac{\partial\pi_1}{\partial q_1} = 0, \quad \frac{\partial\pi_2}{\partial q_2} = 0. \tag{3·2}$$

From (3·1), (3·2), Cournot's equilibrium output and profit can be expressed as

$$q_1 = q_2 = \frac{3}{5},$$

$$\pi_1 = \pi_2 = \frac{50}{9}.$$

On the other hand, if the duopolists 1 and 2 are supposed to determine the outputs cooperatively, an attainable sphere for (π_1, π_2) widens and the following relation is established at the boundary of the sphere:

$$\begin{vmatrix} \dfrac{\partial\pi_1}{\partial q_1} & \dfrac{\partial\pi_2}{\partial q_1} \\ \dfrac{\partial\pi_1}{\partial q_2} & \dfrac{\partial\pi_2}{\partial q_2} \end{vmatrix} = 0.$$

Therefore, when $q_1=\frac{5}{2}-q_2$, (π_1, π_2) are located on the boundary, satisfying

$$\pi_2 = \frac{25}{2}-\pi_1. \tag{3·3}$$

Now, when the output of the duopolist 1 is $\hat{q}_1$, a pair of (π_1, π_2) which satisfies the following formula is obtainable:

$$\pi_2 = \frac{1}{2\hat{q}_1^2}\pi_1\{\pi_1-2\hat{q}_1(5-\hat{q}_1)\}. \tag{3·4}$$

If the output of duopolist 2 is fiexd at $\hat{q}_2$, then

(3·5) $$\pi_1 = -\frac{1}{2\hat{q}_2^2}\pi_2\{\pi_2-2\hat{q}_2(5-\hat{q}_2)\}.$$

In order to achieve the greatest profit duopolist 1 makes selection of the threat so that (π_1, π_2), under the restriction of (3·5), will come on the most favourable complementary line in his favour. In a similar manner, duopolist 2 also determines his threat so that (π_1, π_2) of (3·4) will come on the most favourable complementary line. In this way duopolist 1 adopts a certain optimal threat in relation to $\hat{q}_2$ and duopolist 2 determines his optimal threat in relation to $\hat{q}_1$, Now, if $\hat{q}_1$ and $\hat{q}_2$ are the optimal threats for the two players respectively, then the necessary and sufficient conditions are:

(i) The same π_1, π_2 are established in (3·4) and (3·5),

(ii) The slope $\frac{d\pi_2}{d\pi_1}$ of the tangent of (3·4) and (3·5) at the point of $(\hat{\pi}_1, \hat{\pi}_2)$ are identical, their values being 1.

From the conditions of the former half of (i) and (ii),

(3·6) $$(\hat{\pi}_1, \hat{\pi}_2) = (0, 0).$$

Again, from (3·4) and (3·5), using

$$\left[\frac{d\pi_2}{d\pi_1}\right]_{\substack{\pi_1=0\\ \pi_2=0}} = \frac{5-q_1}{q_1} \quad \text{and} \quad \left[\frac{d\pi_2}{d\pi_1}\right]_{\substack{\pi_1=0\\ \pi_2=0}} = \frac{q_2}{5-q_2},$$

(3·7) $$q_2 = 5-q_1.$$

Since $\frac{d\pi_2}{d\pi_1}=-1$ at the boundary (3·3) of the sphere of attainable profit, $\hat{q}_1=\hat{q}_2=\frac{5}{2}$ can be drawn from $\frac{5-\hat{q}_1}{\hat{q}_1}=\frac{\hat{q}_2}{5-\hat{q}_2}=1$. When each player is adopting the threat which satisfies the conditions of (3·7) and particularly when the optimal threat is being used, the attainable profit is given in (3·6) (Cf. $N(\hat{t}_1, \hat{t}_2)$ in II 4).

If $\hat{q}_1$, $\hat{q}_2$ is supposed to represent the optimal output threat $\frac{5}{2}$, then (3·4) and (3·5) correspond respectively to $S_2^*(t_1)$, $S_1^*(t_2)$ of II 4. These relationships are illustrated in Figure 3.

It is evident that the equilibrium output of the duopoly case according to Nash's method is more restricted than it is in the case of Cournot's solution. The duopolists gain greater profits by imposing cooperative restriction on the quantity of supply or by raising the price than the case of Cournot's equilibrium.

Next let us examine the stability of Nash's solution. Now, supposing some threats q_1, q_2 which satisfy $p=10-2(q_1+q_2)\geqq 0$, duopolist 1 determines a new threat

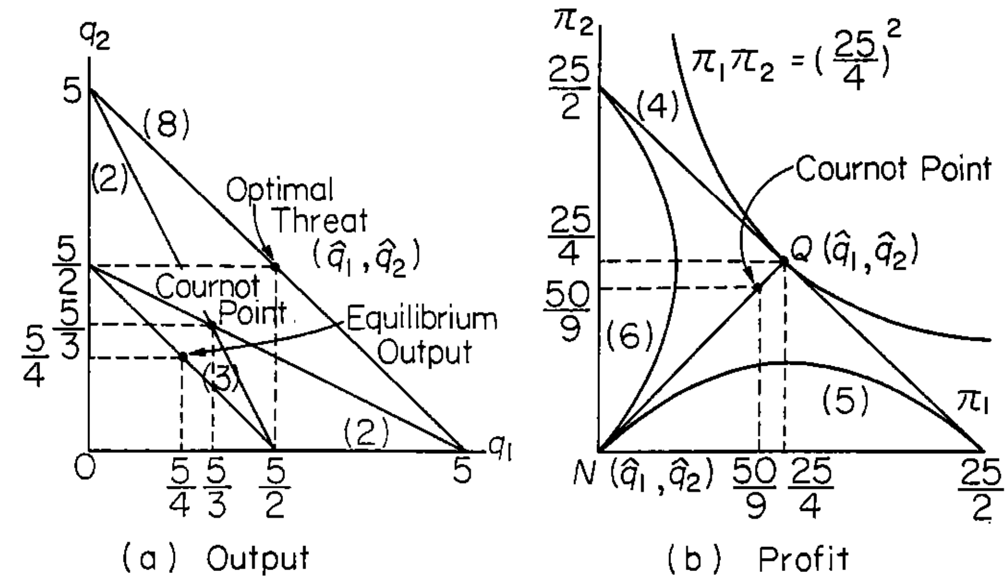

Fig. III Equilibrium Solution of Duopoly Model

so that (π_1, π_2) of (3·5) will make contact with the most favourable complementary line. Since the slope of the complementary line is 1 in the given example, it leads to

$$\frac{d\pi_1}{d\pi_2} = 1. \qquad \therefore \quad \pi_2 = q_2(5-2p_2).$$

Seeing that π_2 can be attained under the unknown threat $\hat{q}_1$, and a given threat q_2, the following expression is drawn from (3·1):

$$q_2(5-2q_2) = 10q_2 - 2q_2^2 - 2q_2\hat{q}_1. \qquad \therefore \quad \hat{q}_1 = \frac{5}{2}.$$

Determining the optimal threat $\hat{q}_2$ for duopolist 2 in the same manner, we get $\hat{q}_2 = \frac{5}{2}$. Since we can confirm that the optimal threat to a threat of $\left(\frac{5}{2}, \frac{5}{2}\right)$ are also $\left(\frac{5}{2}, \frac{5}{2}\right)$, $\left(\frac{5}{2}, \frac{5}{2}\right)$ are the optimal threats of the duopoly model.

Putting it in other way, because an unquestionable sequence, originating from any given threat (q_1, q_2), comes into existence as demonstrated below, Nash's solution is stable[7].

7) The equilibrium solution of the threat game is stable. Supposing that the optimal threat is $(\hat{t}_1, \hat{t}_2)$ and the payoff of the threat game $[Q_1(t_1, t_2), Q_2(t_1, t_2)]$, because the boundary of B slopes downward,

$$Q_1(t_1, \hat{t}_2) \leqslant Q_1(\hat{t}_1, \hat{t}_2) \leqslant Q_1(\hat{t}_1, t_2),$$
$$Q_2(\hat{t}_1, t_2) \leqslant Q_2(\hat{t}_1, \hat{t}_2) \leqslant Q_2(t_1, \hat{t}_2),$$

and $(\hat{t}_1, \hat{t}_2)$ is a saddle-point solution.

$$(q_1, q_2) \Rightarrow \underbrace{N(q_1, q_2) \rightarrow Q(\hat{q}_1, \hat{q}_2)}_{\text{Demand Game}} \Rightarrow \overbrace{\left(\frac{5}{2}, \frac{5}{2}\right)}^{\text{Threat Game}}$$

$$\Rightarrow N\left(\frac{5}{2}, \frac{5}{2}\right) \rightarrow Q\left(\frac{5}{2}, \frac{5}{2}\right) \Rightarrow \left(\frac{5}{2}, \frac{5}{2}\right) \Rightarrow \cdots\cdots$$

IV Interpretation of the Threat

1. Axiomatic Approach

It is known that Nash attempted to solve the problem in question on the presumption of a few axioms which were laid down on the basis of several specific features with which the solution should be characterised.

Supposing that S_1, S_2 represent a set of mixed strategies for respective player 1 and 2 and that B represents a set of pay-offs, the solution v_1, v_2 of the game for players 1 and 2 should satisfy the following axioms:

Axiom 1—Pareto-Optimality:

There exists a unique solution, (v_1, v_2) and this solution shall not be governed by any other one in B. If

$$(u_1, u_2) \in B \text{ for } u_1 \geqslant v_1 \text{ and } u_2 \geqslant v_2,$$

then (u_1, u_2) is the solution.

Axiom 2—Invariance with respect to utility scales:

Even if the linear transformation which maintains order is applied to pay-off, both the solution before the linear transformation and the solution after the transformation can be correlated by the same transformation. In other words, the solution is substantially constant, though transformed[8].

Axiom 3—Symmetry of the Solution:

If B is symmetrical, each solution of two players is identical. In other words, if (v_1, v_2) belongs to B, and (v_2, v_1) also belong to B, the solution is $v_1 = v_2$, not affected by the names of players.

Axiom 4—Independence of irrelevant Alternatives:

In a new game such that the strategy-sets are S_1, S_2 and the pay-off-set is B', $(B' \subset B)$, if the solution (v_1, v_2) of B is included in B', then (v_1, v_2) is the solution of B'.

Since it can be proved that the only solution which would satisfy these axioms

8) Under linear utility functions like von Neumann=Morgenstern's, it is possible to treat both the payoff frontier and the utility frontier as if they were of the same nature.

is no less than the solution of the non-cooperative game in the case of negotiation[9], of which explanation was given above, Nash's method has a wide range of application as was asserted by Nash himself. Yet a little further investigation will, show us some shortcomings in those axioms which appear to be seemingly reasonable[10]. We shall take them up with an emphasis on the problem of the interpersonal comparison of utility which has much to do with our further argument.

It can well be assumed that Nash himself, needless to say, holds the basic standpoint that such interpersonal comparison of utility is impossible. It is unquestionable that the structure of his theory is not explicitly based on the presumption of the the interpersonal comparison of utility. However, when the solution obtained therefrom is closely examined in its relation to his axiom of symmetry it becomes quite clear that some interpersonal comparison of utility is presupposed.

The attainable sphere of the solution in the numerical example of III is obviously of a symmetrical nature and Nash' solution is determined at the point $\left(\frac{25}{4}, \frac{25}{4}\right)$ which gives increased pay-off in an equal amount to each of those duopolists, compared with the pay-off basic point (0, 0) which is to be determined by the optimal threat $\left(\frac{5}{2}, \frac{5}{2}\right)$. Because the solution satisfies axiom 3, it must be considered that it carries a certain kind of desirability or fairness entrused to the content of the axiom. If one of the duopolists is supposed to have a marginal utility to decline rapidly, while the other a marginal utility to decline slowly, it will lead to a conclusion that the increased portion of the share of pay-off in an equivalent amount does not satisfy social justice[11]. Such difference in the appraisal of that increased portion of pay-off for the duopolists is not a special case of rare occurrence, but might just as well be considered as a very ordinary matter which is caused by the difference in the extent of the pressing need for funds in view of the status of accumulation achieved up to that time and the business policy to be pursued in the future. And it is also natural that each one of the parties concerned has his own way of appraisal of the pay-off which is different from that of others in a case of ordinary negotiation. If the interpersonal comparison of utility is to be completely excluded under such circumstance, then

9) The axioms listed here are taken from Nash [12]. In [13] two more axioms are added and it is proved there that the solution which satisfies the axioms is identical with that of the negotiation model. About the proof of identity under axioms 1–4 see Luce = Raiffa [10], pp. 127–128.

10) More detailed discussion of this point is developed in Luce=Raiffa [10], pp. 128–137 and Bishop [3], pp. 578–582. To further better understanding on this point it is necessary to make references to many experiments such as working out the design of other kinds of games which satisfy Nash' axioms and examining the relationships between the results obtained therefrom and these axioms. The book compiled by Suzuki [15] contains some of the results of such experiments conducted by the author and others in Chapter 9. In addition, in Chapters, 7 and 8 a survey of the theory of a negotiation game is also given, which is worthy of notice.

11) Bishop [3], pp. 576–577.

Nash solution would diverge from social justice, which would only serve to satisfy a certain kind of desirability. It is only when both parties concerned happen to have the same marginal utility with respect to the pay-off that his solution conforms to social justice. On the other hand, if the interpersonal comparison of utility is to be explicity presupposed, then what must be made clear is the relationships between the solution to maximize the product of increased shares of pay-off to be received by both parties concerned, and social welfare[12]. Be that as it may, it is seen that this dilemma pointed out in the symmetrical nature of Nash's solution does exert no small influence on the positive or normative nature with which the solution should be characterised, as will be made clear later in relation to threat.

2. Appraisal of Threat

There is no need of argument that the significance of Nash's solution, unlike all other conventional theories, lies in the fact that one point on the frontier of negotiation pay-off set, or one point on the contract curve has been designated as an equilibrium point. We pointed out above that the use of threat constituted a very important element in settling this one point. In short, the solution of the demand game is to come to a settlement, based on a fixed basic point to be given by a certain optimal threat, but each player arrives at a new solution by using his own threat to counteract his rival's threat so that he can obtain more favourable demand in the subsequent demand game than before. The equilibrium solution of negotiation is the point where the repetion of these moves is ultimately arrived at. Since the use of threat is one competitive means to determine the equilibrium solution by continuously playing demand games in succession one after another, if the shortcomings disclosed in the demand game which were pointed out in the foregoign section are to be disregarded, the rationality of the negotiation solution, primarily depends on whether use of threat is in conformity with actual facts or not. The same holds with the solution by the axiomatic approach. The significance and effectiveness of an arbitration proposal which satisfies Nash's axioms lies in eliminating the actual dispute or fight to be continued at the sacrifice of no small expense by guessing what the result will be when the non-cooperative competition between the two parties concerned is left to follow its own course without entrusting its settlement to arbitration. But when the arbitration proposal is based on erroneous judgements with respect to elements such as the pay-off or utility function of the two parties concerned and the strategy-sets, particularly threat, such arbitration will be turned down

12) It is clear enough according to Harsanyi [8] that the solution by way of maximizing the product of the pay-offs is identical with the solution of Zeuthen. In this way Nash's solution has been backed up by the behavior of the players who make every effort respectively to maximize the predictable amount of their pay-off, but the relation of the solution to the social welfare still remains to be explained.

by at least one of the two parties concerned. Therefore, a further discussion on the subject of the actualities of threat is essential not only to further the better understanding of Nash's solution which is the immediate aim of this paper but also to improve the game-model in such way that the behaviour of each party concerned can be explained with more exactness.

The strategy to produce desirable conditions in negotiation in favour of one's own interests by inflicting damage on one's opposing competitor is what is meant by threat. In this connection what should be noted are realities in which such strategy is inflicting certain damages upon one's enemy while on the other hand there is a possibility that a certain degree of damage is being suffered at the same time by the strategist himself, too. In order to emphasise the specific features of threat, it would be correct, in my opinion, to define threat rather as a strategy to inflict certain damages simultaneously on both sides. For an instance in the numerical example of the duopoly in III, supposing that the duopolists 1 and 2 start the game under conditions of $q_1=1$, $q_2=2$, then the basic point becomes (4,8), which means that the immediate equilibrium solution is the distribution of profit at the rate of $\left(\frac{17}{4}, \frac{33}{4}\right)$. Now, supposing one of the duopolists who objects to this distribution succeeds in making use of the profit (0,0), which is to be determined by a pair of the optimal threat $\left(\frac{5}{2}, \frac{5}{2}\right)$, as a final basic point of negotiation by exercising the threat $\frac{5}{2}$. Because of the use of such threat as his strategy, he can inflict damage to the extent of $\frac{33}{4}$ on his rival, but he himself is also obliged to suffer damage to the extent of $\frac{17}{4}$. Any threat, as is self-explanatory from this example, must be not in the form of mere psychological threatening, but someting which must take the form of practical action or behaviour.

If any threat is to be effective at all, it must satisfy the following conditions. Now, it is defined that, speaking of the pay-off set, the respective distances along the two axes on a coordinate from the basic point $U(T)$ to be given by a certain pair of threats, T, to the maximum point $(\bar{U}_1, \bar{U}_2)$ of the pay-off products are the cost of conflict respectively to the player 1 and 2[13]. This particular threat becomes effect tive, on condition that T is not optimal and the new threat used by player 1 increases the cost to player 2 by bU_2 and the cost to player 1 by aU_1 and that $b>a$. Putting it in another way, the demand solution $(\bar{U}_1', \bar{U}_2')$ under a new basic point $U(T')$ as results of a pair of new threats T', is always $\bar{U}_1'>\bar{U}_1$ and it is more favourable to player 1 than it is to player 2.

The condition $b>a$ can be divided into the following three cases:

13) Harsanyi [8], pp. 152–153.

(1) $b > a \geqslant 0$,
(2) $b \geqslant 0 > a$,
(3) $0 > b > a$.

Case (3) is the situation where the new basic point $U(T')$ belongs to the sphere surrounded by a line connecting $U(T)$ and $\bar{U}$, the axis U_1 and the frontier. In the case of (2), $U(T')$ belongs to the sphere of $U_1 \geqslant 0$, $U_2 \leqslant 0$. In both cases (2) and (3), the new threat strategy gives player 1 a greater pay-off of its own accord, i. e. without depending upon negotiation and for that reason it is a better strategy in the ordinary sense of the word.

The peculiarity of a threat strategy lies in that it includes case (1). So-called cut-throat competition is, in my opinion, the situation most close to such case among all traditional concepts of competition. This price-war is mainly based on two different motives. One is where a competitior takes the course of cutting down prices in order to increase profits by depriving his rival of the market. Another is a situation where the price-cut competition is engaged in by disregarding the immediate profit in order to drive his rival completely from the market. Because the former is a development of price-cut competition in an ordinary sense, the type of competition closest to case (1) is the latter.

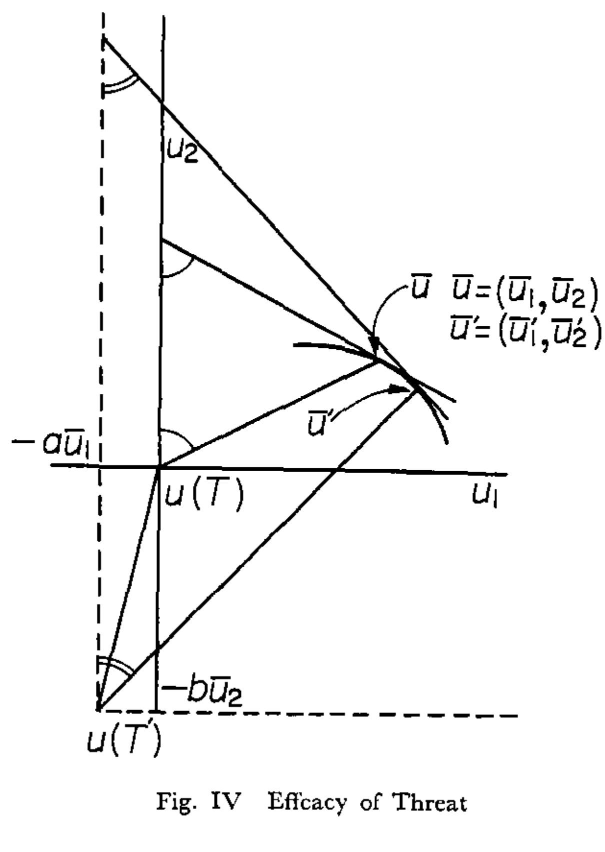

Fig. IV Effcacy of Threat

Nevertheless, in reality there are a few circumstances where such use of threats as those in case (1) appear to be unfavourable. In the first place we have the problem of the time required for negotiation. In the numerical example of duopoly of Bishop [3], the optimal threat gives positive immediate profit to the low cost duopolist and negative immediate profit to the high cost duopolist. If it takes a long time for a settlement of negotiation, the utility function of the high cost duopolist's profit (or money) will not be a linear function as conceived by von Neumann and Morgenstern, but will be transformed into a shape indicating the decrease of the marginal utility[14]. Therefore, it is most likely that the high cost duopolist may restrain himself from using a threat which might give negative immediate profit, even if it is of such a nature as to satisfy Nash's optimal requirements.

Another circumstance to be considered is the difference in economic power. In this connection it is difficult to give a clear and accurate definition of the term 'economic power', but at this moment for convenience it may be defined as the power to resist damage. Let us take an example from a labour dispute, where there is every reasonable expectation that greater damage can be inflicted upon the business operator than that to be suffered by the union members when a labour union uses the threat of declaring a strike upon refusing a compromise. For the reason that the effective conditions for such threat are all satisfied, calling a strike is supposed to lead the next negotiation to be conducted in favour of the union. But, if the strike fund of the union is not sufficient to cover the loss of wages and other expenditure caused by the strike, the union cannot help but give up the strike. In this case, as in the foregoing case, the marginal utility for money on the part of the union is decreased.

Now, let us take up the same situation from the angle of the low cost duopolist. The low cost manufacturer is not obliged to be in an disadvantageous position even when a new threat which presses him to bear additional cost of bU_1 is used. The same holds true with the case of the business administrator. Therefore, if the threat is to be used really effectively, the extent of real suffering to be inflicted upon player 2 through the damage $b\bar{U}_2$ must be greater than the real suffering to be inflicted upon player 1 through the damage $a\bar{U}_1$. However, this involves the problem of interpersonal comparison of utility as was made clear when the axiom of symmetry was discussed above. In the case of (2) and (3) it is possible to avoid this problem, but it is evident that in the case where a real threat is to be used, it is impossible to do so.

14) Needless to say, under linear utility functions marginal utility is constant whether profit is positive or negative. Therefore, there is no possibility for such appraisal of threat.

15) It is pointed out that the Shapley's solution does not reflect the influences of threat arising from the difference in productivity because its basic point is based on the minimum security level. But if the foregoing appraisal of the threat is to be accepted, the appropriateness of that criticism should be duly diminished.

In the meanwhile, if the above comments on the actuality of threat are correct, there is a possibility that the solution of the negotiation problem should become somewhat different from Nash's solution. Even if the method of Nash's solution of the demand game is used, it is possible to take the minimum security level of the players as their basic points in a manner as demonstrated in Shapley's solution and it is possible, also, that a case is likely to occur where a competition may be carried on in a manner as demonstrated by Cournot, whereby Counrot's point is regarded as their basic point. Bishop suggests the latter case as a solution when the negotiation turned out to be unsuccessful. Or, it may so happen, though it is a matter beyond the frame of all models taken up here, that the duopolists may restrain themselves from using any threat and endeavor to make their position more favourable by investing what they have in the form of profit on hand for market development.

3. Conclusion

So far there have been several discussions of the subject of threat which constitutes the core of the Nash solution. These discussions, I believe, must be elucidated along the following two directions:

One is the attempt to make a dynamic interpretation of threat along the lines of the context maintained by Bishop [4], Cross [7], etc. who are attempting a dynamic extension of Nash's solution by introducing an element of time in the process of negotiation.

The other is by speculation about the threat in a n-person cooperative game. "To what extent and effect should the relative decline of the weight of one player caused by the increased number of players and the diversified kind of coalition prescribe the efficacy of threat?" is a question of paramount concern in this line of thought.

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