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An Algebraic Model for the $\mathfrak{su}(2|2)$ Light-Cone String Field Theory

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Abstract

We investigate algebraic structure of light-cone string field theory which respects the spacetime supersymmetry $\mathfrak{su}(2|2)$. Extracting building blocks from the explicit oscillator expressions of light-cone superstring field theory on the pp-wave and the flat background, we propose a model for more generic backgrounds by giving an algebraic procedure to construct supercharges and Hamiltonian without referring to explicit oscillator expressions. Based on this strategy, we systematically construct interaction terms of light-cone string field theory on the pp-wave background and some examples of its generalization. We also study the supergravity limit of the interaction terms on the pp-wave background.

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1 Introduction

There is no overemphasizing the importance of the superalgebra $\mathfrak{su}(2|2) = \mathfrak{psu}(2|2) \ltimes \mathbb{R}$ in the recent progress of AdS/CFT correspondence. The first work which was aware of its importance is probably [1], which shows that the expressions of various superconformal charges of super Yang-Mills theory are strongly constrained in the $\mathfrak{su}(2|3)$ subsector. The appearance of the superalgebra $\mathfrak{su}(2|3)$ was further clarified by the reason that it produces a similar stabilizer subalgebra $[\mathfrak{psu}(2|2)] \ltimes \mathbb{R}$ as the whole symmetry superalgebra $\mathfrak{psu}(2,2|4)$ produces the stabilizer subalgebra $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}$ after fixing the vacuum [2]. Also excitingly, it was discovered that this powerful superalgebra successfully fixes the scattering matrix up to an overall factor [3], which satisfies the integrable Yang-Baxter equation and the Yangian symmetry. Hence, it is tantalizing to generalize the analysis for single objects (trace operators or strings) to their interactions. As on the pp-wave background, the first step to study the string interactions is to construct a light-cone string field theory[1].

Another important related research is to classify the geometrical background preserving half of the supersymmetries [5]. Although the superalgebra was not explicitly mentioned in the original work, it is clear from the recent development that this work amounts to deforming the $AdS_5 \times S^5$ background under the requirement that the background preserves the subisometry $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}$. The deformed background was named bubbling geometries after their construction. Our above direction can be reexpressed as construction of light-cone string field theory on the bubbling geometries.

Compared with AdS/CFT correspondence, construction of light-cone string field theory has a much longer history. The three-string interaction vertex of light-cone superstring field theory on the flat spacetime was constructed in eighties [6], by generalizing the bosonic light-cone string field theory [7] and the extended supergravity [8]. The basic building blocks are *the kinematical overlap* $|V\rangle$ constructed from the local fermionic (and bosonic) momentum conservation on the worldsheet

$$\big(\lambda_{(1)}(\sigma) + \lambda_{(2)}(\sigma) + \lambda_{(3)}(\sigma)\big)|V\rangle = 0, \quad \big(p_{(1)}(\sigma) + p_{(2)}(\sigma) + p_{(3)}(\sigma)\big)|V\rangle = 0, \tag{1.1}$$

with $\lambda_{(i)}(\sigma)$ (and $p_{(i)}(\sigma)$) denoting the fermionic (and bosonic) momentum of the i-th string, and *a real fermionic operator* Y which commutes with the local momentum conservation and is interpreted as the renormalized fermionic momentum at the interaction point $\sigma_{\rm I}$ of the light-cone three-string kinematical overlap,

$$\lambda_{(i)}(\sigma)|V\rangle \sim \frac{1}{\sqrt{\sigma - \sigma_{\rm I}}} Y|V\rangle, \tag{1.2}$$

with *its supersymmetric partners* $X, \widetilde{X}$ defined as

$$\{q, Y\} = X, \quad \{\widetilde{q}, Y\} = \widetilde{X}, \tag{1.3}$$

[1]See [4] for an interesting attempt using the coherent states.

which can also be interpreted as the holomorphic/antiholomorphic part of the bosonic momentum at the interaction point $\sigma_{\rm I}$. The expressions of various charges at the interacting level are then determined from the supersymmetry algebra.

More recently, the light-cone superstring field theory on the pp-wave background was constructed [9] in the development of AdS/CFT correspondence by simply generalizing that on the flat spacetime, which enables the match between Hamiltonian of string theory and the dilatation of super Yang-Mills theory at the interacting level [10]. Subsequently, it was pointed out that another linear combination proportional to the action of the free charges on the kinematical overlap $|V\rangle$ can also solve the algebra as well [11]. In the context of AdS/CFT correspondence it was then proposed [12] that we have to take the original part coming from the flat spacetime and the part proportional to the free action with equal weight in order to match the correlation functions[2]. However, so far there is no concrete reasoning a priori (without referring to AdS/CFT correspondence) why we have to take the two parts with equal rate[3].

In this paper we would like to revisit the construction of light-cone string field theory on the pp-wave background and take a step towards constructing light-cone string field theory on a background with the isometry $\mathfrak{su}(2|2)$. In revisiting the construction on the pp-wave background, we have made several clarifications. Especially, we present a systematic analysis for terms consistent with the superalgebra $\mathfrak{su}(2|2)$ and determine some of the coefficients of these terms from the supergravity limit. After these clarifications, we propose a strategy for a general bubbling geometry and construct some explicit examples of generalization. We hope that our attempts in construction will finally result in the success to match between Hamiltonian of string theory on the bubbling geometries with the isometry $\mathfrak{su}(2|2)$ and anomalous dimension of multi-trace operators as in the $\mathfrak{su}(2|2)$ spin chain.

The construction we adopt in this paper is possible due to several observations. The first observation is that the isometry algebra of the pp-wave background contains $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}$ as its subalgebra and the subalgebra $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}$ acts on the other generators of the pp-wave isometry as an outer automorphism. This fact implies the central role of the subalgebra $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}$. Namely, the stabilizer subalgebra $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}$ is crucial and ubiquitous while the extra enhanced symmetry only appears accidentally in the pp-wave limit. Then, in retrospect to the construction of light-cone superstring field theory on the pp-wave background in the $\mathfrak{so}(4) \times \mathfrak{so}(4)$ formalism [15], we shall notice in the following section that only the subalgebra $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}$ plays an essential role in the construction.

The second observation is the similarity in the situation between the construction of light-cone superstring field theory on the flat spacetime using the super Poincare algebra and that of the spin chain in modeling the computation of the anomalous dimension in super Yang-Mills

[2]For an interesting computation on the match between Hamiltonian and the dilatation using [12], see [13].
[3]See [14] for various stimulating arguments in this direction.

theory using the superalgebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$. In the case of the light-cone superstring field theory, the conventionally written super Poincare algebra on the flat spacetime is

$$\{Q^A, Q^B\} = \{\widetilde{Q}^A, \widetilde{Q}^B\} = 2\delta^{AB} H, \quad \{Q^A, \widetilde{Q}^B\} = 0. \tag{1.4}$$

However, the charges never satisfy these anticommutation relations literally. In general, at the first sight, the Hamiltonian computed from left-moving supercharges $\{Q^A, Q^B\}$ and that computed from right-moving ones $\{\widetilde{Q}^A, \widetilde{Q}^B\}$ do not coincide and the anticommutator $\{Q^A, \widetilde{Q}^B\}$ does not vanish. The algebra (1.4) holds only after we impose the level-matching condition. The situation was more clearly formulated in the case of the AdS/CFT spin chain model [3]. Although the on-shell symmetry is $\mathfrak{psu}(2|2) \ltimes \mathbb{R}$, to take care of the off-shell states we have to consider the centrally extended superalgebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ instead in the spin chain model. We shall exploit this centrally extended off-shell formalism also for the light-cone superstring field theory. Namely, instead of the original algebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}$, we shall choose the centrally extended one $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ as the off-shell symmetry of the light-cone superstring field theory.

A natural question which may arise here is how it can be possible to construct the light-cone string field theory if we have not solved the worldsheet theory. In the typical string field theory, the string worldsheet theory is solved explicitly by string oscillators and all of the building blocks are expressed explicitly in terms of these oscillators. Here, without the explicit solutions to the worldsheet theory[4], we simply assume the building blocks to be some abstract quantities subject to suitable algebraic relations and do not refer to its explicit form. We hope that explicit expressions for the building blocks will be found in the near future.

In the following section we shall revisit light-cone superstring field theory on the pp-wave background. After revisiting the construction on the pp-wave background, in the subsequent section we shall proceed to propose a strategy for a more general background with the isometry $\mathfrak{su}(2|2)$.

2 PP-wave light-cone string field theory revisited

In this section, we shall revisit the three-string interaction vertex of light-cone string field theory on the pp-wave background. Most of the computations here essentially appeared in the previous works [15, 14]. However, we would like to emphasize the following improvements.

- After hiding the explicit oscillator expressions of various building blocks, the algebraic aspect of the construction is much clearer. In particular, from our following analysis, it is clear that most expressions of the three-string interaction vertex are determined purely from the algebraic consistency. The generalization in the subsequent section is based heavily on the analysis here.

[4]The spirit is somehow similar to [16].

- We do not need separate analysis for quantities with various powers, as in [15]. Stimulated by the hyperbolic expression in [17], we shall present a systematic manipulation for various quantities. The formulas in the appendix lead us directly to the final expression.

- The dependence on the worldsheet coordinates had to be introduced in [6, 15] to show the uncorrelation between the left-moving and right-moving supercharges $\{Q, \widetilde{Q}\} = 0$. Here in subsection 2.2, we shall prove the same equation algebraically without referring to the dependence on the worldsheet coordinates.

- Although it was noticed [11, 14] that action of free charges on the (dressed) kinematical overlaps $|V\rangle$ satisfies the algebra as well, so far there is no explicit analysis that this is the only alternative we have to take into account. Our systematic analysis in subsection 2.3 clarifies all the alternatives.

- The requirement of the abelian duality symmetry in the supergravity limit determines some of the coefficients which is consistent with the previous work [12]. (See also [14].)

2.1 The building blocks

Let us start with the superalgebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ [3],

$$\begin{aligned}
&\{\mathfrak{Q}^\alpha{}_a, \mathfrak{Q}^\beta{}_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathfrak{P}, \\
&\{\mathfrak{S}^a{}_\alpha, \mathfrak{S}^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\mathfrak{K}, \\
&\{\mathfrak{Q}^\alpha{}_a, \mathfrak{S}^b{}_\beta\} = \delta^b_a\mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta\mathfrak{R}^b{}_a + \delta^b_a\delta^\alpha_\beta\mathfrak{C}.
\end{aligned} \tag{2.1}$$

If we redefine various generators by ($\eta = e^{\pi i/4}$)

$$\begin{aligned}
&\mathcal{R}^a{}_b = -i\mathfrak{R}^a{}_b, \quad \mathcal{L}^\alpha{}_\beta = i\mathfrak{L}^\alpha{}_\beta, \quad \mathcal{H} = \mathfrak{C} + i\frac{\mathfrak{P}-\mathfrak{K}}{2}, \quad \widetilde{\mathcal{H}} = \mathfrak{C} - i\frac{\mathfrak{P}-\mathfrak{K}}{2}, \quad \mathcal{N} = \frac{\mathfrak{P}+\mathfrak{K}}{2}, \\
&\mathcal{Q}^\alpha{}_a = \frac{1}{\sqrt{2}}(\eta\mathfrak{Q}^\alpha{}_a + \eta^*\epsilon^{\alpha\beta}\epsilon_{ab}\mathfrak{S}^b{}_\beta), \quad \widetilde{\mathcal{Q}}^\alpha{}_a = \frac{1}{\sqrt{2}}(\eta^*\mathfrak{Q}^\alpha{}_a + \eta\epsilon^{\alpha\beta}\epsilon_{ab}\mathfrak{S}^b{}_\beta),
\end{aligned} \tag{2.2}$$

we can rewrite the algebra into the following expression,

$$\begin{aligned}
&\{\mathcal{Q}^\alpha{}_a, \mathcal{Q}^\beta{}_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathcal{H}, \\
&\{\widetilde{\mathcal{Q}}^\alpha{}_a, \widetilde{\mathcal{Q}}^\beta{}_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\widetilde{\mathcal{H}}, \\
&\{\mathcal{Q}^\alpha{}_a, \widetilde{\mathcal{Q}}^\beta{}_b\} = \epsilon^{\alpha\beta}\epsilon_{ac}\mathcal{R}^c{}_b + \epsilon_{ab}\mathcal{L}^\alpha{}_\gamma\epsilon^{\gamma\beta} + \epsilon^{\alpha\beta}\epsilon_{ab}\mathcal{N}.
\end{aligned} \tag{2.3}$$

Let us expand each generator $\mathcal{J}$ with respect to the string coupling constant g_s, $\mathcal{J} = j + g_s J + \cdots$, where the first term is the free generator j acting on each of three strings and the

second one is their interaction J, which is conventionally expressed in the ket form $|J\rangle$. In terms of these generators, the superalgebra becomes

$$
\begin{aligned}
q^{\alpha}{}_{a}|Q^{\beta}{}_{b}\rangle + q^{\beta}{}_{b}|Q^{\alpha}{}_{a}\rangle &= \epsilon^{\alpha\beta}\epsilon_{ab}|H\rangle, \\
\widetilde{q}^{\alpha}{}_{a}|\widetilde{Q}^{\beta}{}_{b}\rangle + \widetilde{q}^{\beta}{}_{b}|\widetilde{Q}^{\alpha}{}_{a}\rangle &= \epsilon^{\alpha\beta}\epsilon_{ab}|\widetilde{H}\rangle, \\
q^{\alpha}{}_{a}|\widetilde{Q}^{\beta}{}_{b}\rangle + \widetilde{q}^{\beta}{}_{b}|Q^{\alpha}{}_{a}\rangle &= \epsilon^{\alpha\beta}\epsilon_{ab}|N\rangle,
\end{aligned}
\tag{2.4}
$$

where we have assumed the bosonic generators $\mathcal{L}^{\alpha}{}_{\beta}$ and $\mathcal{R}^{a}{}_{b}$ to be kinematical ones without receiving any interacting corrections just as the canonical role these generators played in the $\mathfrak{su}(2|2)$ spin chain model [1, 3].

Now let us turn to the construction of the light-cone string field theory on the pp-wave background [15]. In the following of this subsection, we shall encounter several assumptions which cannot be explained only from the symmetry algebra $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$. The justification for these assumptions is, of course, due to the explicit computations on the pp-wave background, where the worldsheet theory is exactly solvable. After the discussions in this section, it will be clear which part has to be deformed when we extend the background from the pp-wave one to a more general one with the isometry $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$.

In describing the three-string interaction of light-cone string field theory, the main building blocks are the kinematical overlap among three strings $|V\rangle$ and the renormalized fermionic momentum Y as defined in (1.1) and (1.2). The definition of $|V\rangle$ (1.1) refers only to the conservation law of the worldsheet momentum, which should remain valid in a general background. The singular behavior of (1.2) also originates from the worldsheet instead of the spacetime and is interpreted as a two-dimensional operator product expansion [18, 17]. Hence, it is natural to expect that the definition of Y also remains valid in a general background. The indices of Y are determined naturally as being the (fermionic) spacetime momentum.

A strong assumption is that the renormalized momenta transform under the supersymmetry linearly as[5]

$$
\begin{aligned}
\{q^{\alpha}{}_{a}, Y^{\beta}{}_{\dot{b}}\} &= -x\epsilon^{\beta\alpha}(\epsilon X)_{a\dot{b}}, & \{q^{\alpha}{}_{a}, Y'^{b}{}_{\dot{\beta}}\} &= x'\delta^{b}_{a}X'^{\alpha}{}_{\dot{\beta}}, \\
[q^{\alpha}{}_{a}, X^{b}{}_{\dot{b}}] &= w\delta^{b}_{a}W^{\alpha}{}_{\dot{b}}, & [q^{\alpha}{}_{a}, X'^{\beta}{}_{\dot{\beta}}] &= -w'\epsilon^{\beta\alpha}(\epsilon W')_{a\dot{\beta}}, \\
[q^{\alpha}{}_{a}, \widetilde{X}^{b}{}_{\dot{b}}] &= \frac{i}{2}y\delta^{b}_{a}Y^{\alpha}{}_{\dot{b}}, & [q^{\alpha}{}_{a}, \widetilde{X}'^{\beta}{}_{\dot{\beta}}] &= \frac{i}{2}y'\epsilon^{\beta\alpha}(\epsilon Y')_{a\dot{\beta}},
\end{aligned}
\tag{2.5}
$$

with $(\epsilon X)_{a\dot{b}} = \epsilon_{ab}X^{b}{}_{\dot{b}}$, $(\epsilon W')_{a\dot{\beta}} = \epsilon_{ab}W'^{b}{}_{\dot{\beta}}$, $(\epsilon Y')_{a\dot{\beta}} = \epsilon_{ab}Y'^{b}{}_{\dot{\beta}}$. In fact, we can regard the first two equations as the definition of the quantities X, X', W, W' on the right-hand side. Therefore, the only assumption we have actually made is the last one. If we switch the indices by contracting with ϵ such as $Y^{\dot{b}}{}_{\beta} = \epsilon_{\beta\alpha}Y^{\alpha}{}_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{b}}$, $Y'^{\dot{\beta}}{}_{b} = \epsilon_{ba}Y'^{a}{}_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}$, the transformation rule is

[5]The extra minus sign in the first two lines can be regarded as the minus sign appearing in $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^{\alpha}_{\gamma}$, $\epsilon^{ab}\epsilon_{bc} = -\delta^{a}_{c}$, while the meaning of the overall coefficients in the last line will be clear later in this subsection.

rewritten as

$$\{q^\alpha{}_a, Y^{\dot b}{}_\beta\} = xX^{\dot b}{}_a\delta^\alpha_\beta, \qquad \{q^\alpha{}_a, Y'^{\dot\beta}{}_b\} = -x'(X'\epsilon)^{\dot\beta\alpha}\epsilon_{ab},$$
$$[q^\alpha{}_a, X^{\dot b}{}_b] = -w(W\epsilon)^{\dot b\alpha}\epsilon_{ab}, \qquad [q^\alpha{}_a, X'^{\dot\beta}{}_\beta] = w'W'^{\dot\beta}{}_a\delta^\alpha_\beta,$$
$$[q^\alpha{}_a, \widetilde{X}^{\dot b}{}_b] = -\frac{i}{2}y(Y\epsilon)^{\dot b\alpha}\epsilon_{ab}, \qquad [q^\alpha{}_a, \widetilde{X}'^{\dot\beta}{}_\beta] = -\frac{i}{2}y'Y'^{\dot\beta}{}_a\delta^\alpha_\beta. \tag{2.6}$$

Besides, we shall require another strong assumption. We assume that the action of the supersymmetry charge on the kinematical overlap $|V\rangle$ is given as

$$q^\alpha{}_a|V\rangle = \frac{i}{2}[v(YX)^\alpha{}_a + v'(X'Y')^\alpha{}_a]|V\rangle. \tag{2.7}$$

We also assume the supersymmetry generator $\widetilde{q}^\alpha{}_a$ acts similarly as $q^\alpha{}_a$, with all of the quantities replaced by those with tildes (except Y and Y') and complex conjugation taken for complex numbers.

From the consistency with the first line of (2.3) with the full generators $\mathcal{J}$ replaced by the free one j, we obtain the following relations

$$[h, Y^\alpha{}_{\dot a}] = xwW^\alpha{}_{\dot a}, \qquad [h, Y'^a{}_{\dot\alpha}] = x'w'W'^a{}_{\dot\alpha}, \tag{2.8}$$

as well as

$$h|V\rangle = \frac{i}{2}(vx\mathrm{Tr}\,X^2 + v'x'\mathrm{Tr}\,X'^2 - vw\mathrm{Tr}\,YW + v'w'\mathrm{Tr}\,W'Y')|V\rangle, \tag{2.9}$$

while the consistency with the third line of (2.3) requires

$$\frac{\widetilde{x}}{x} = \frac{\widetilde{y}}{y} = \frac{\widetilde{v}}{v}, \quad \frac{\widetilde{x}'}{x'} = \frac{\widetilde{y}'}{y'} = \frac{\widetilde{v}'}{v'}, \quad x\widetilde{y} + \widetilde{x}y = 2, \quad x'\widetilde{y}' + \widetilde{x}'y' = 2. \tag{2.10}$$

Looking back to the definition of (2.5), it is easily seen that x (and w) fix the relative ratio between Y and X (and X and W, respectively). Hence, we can set them to 1 by the redefinition of X and W. Similarly, we can also set $\widetilde{x}$ and $\widetilde{w}$ to 1 by the redefinition of $\widetilde{X}$ and $\widetilde{W}$. In addition, as seen from (2.7), v fixes the overall factor of Y and we also set it to 1. Subsequently, from (2.10) we find $\widetilde{v}$, y and $\widetilde{y}$ are determined to be 1. All these arguments hold also for the variables with primes. Therefore, hereafter without loss of generality, we shall take

$$x = x' = w = w' = y = y' = v = v' = \widetilde{x} = \widetilde{x}' = \widetilde{w} = \widetilde{w}' = \widetilde{y} = \widetilde{y}' = \widetilde{v} = \widetilde{v}' = 1. \tag{2.11}$$

2.2 Anti-chiral terms

Now let us take the interaction terms of the supersymmetry charges to be the most general ones with one bosonic momentum inserted (where the opposite chirality terms with $\widetilde{X}$ and $\widetilde{X}'$

replaced by X and X' are postponed to the next subsection),

$$|Q^\alpha{}_a\rangle = \Big\{\sum_{n,m} q_{nm}(Y^n\widetilde{X}Y'^m)^\alpha{}_a + \sum_{n,m} q'_{mn}(Y^m\widetilde{X}'Y'^n)^\alpha{}_a\Big\}|V\rangle, \tag{2.12}$$

with n denoting odd numbers $1,3$ and m denoting even numbers $0,2,4$. Then, as we study in the following, we can impose a strong constraint on the coefficients q_{nm} and q'_{mn} from the supersymmetry algebra. Although most of the computations essentially appeared in [15], let us repeat the computations here in a much more systematic way.

Before starting the analysis, let us note several properties which will be necessary hereafter. The first important property is the symmetry of the products

$$\begin{aligned}&(Y^4)^\alpha{}_\beta = \delta^\alpha_\beta\frac{1}{2}\mathrm{Tr}\,Y^4, \quad (Y^4)^{\dot a}{}_{\dot b} = -\delta^{\dot a}_{\dot b}\frac{1}{2}\mathrm{Tr}\,Y^4, \quad (X^2)^b{}_a = \delta^b_a\frac{1}{2}\mathrm{Tr}\,X^2,\\ &\mathrm{Tr}\,Y^2 = 0, \quad (Y^2)^\alpha{}_\beta(Y^2)^{\dot a}{}_{\dot b} = 0,\end{aligned} \tag{2.13}$$

where we choose the convention that Tr is taken over the undotted indices: $\mathrm{Tr}\,Y^4 = (Y^4)^\alpha{}_\alpha$, $\mathrm{Tr}\,X^2 = (X^2)^a{}_a$. Secondly, the following formulas also play important roles in the calculation.

$$\begin{aligned}&(V\epsilon)^{\alpha\beta} - (V\epsilon)^{\beta\alpha} = \epsilon^{\alpha\beta}\mathrm{Tr}\,V, \qquad (\epsilon W)_{ab} - (\epsilon W)_{ba} = \epsilon_{ab}\mathrm{Tr}\,W,\\ &V^\alpha{}_aW^\beta{}_b - V^\beta{}_aW^\alpha{}_b = -\epsilon^{\alpha\beta}(\epsilon\bar V W)_{ab}, \quad V^\alpha{}_aW^\beta{}_b - V^\alpha{}_bW^\beta{}_a = -\epsilon_{ab}(V\bar W\epsilon)^{\alpha\beta},\\ &V^\alpha{}_aW^\beta{}_b + V^\beta{}_bW^\alpha{}_a - V^\beta{}_aW^\alpha{}_b - V^\alpha{}_bW^\beta{}_a = -\epsilon^{\alpha\beta}\epsilon_{ab}\mathrm{Tr}\,V\bar W.\end{aligned} \tag{2.14}$$

Note that V, W are some general products, with $\bar V$, $\bar W$ denoting $\bar V = \epsilon V^{\mathrm{T}}\epsilon$, $\bar W = \epsilon W^{\mathrm{T}}\epsilon$, where in the transposes V^{T}, W^{T} the signs coming from exchanges among fermions have to be taken into account.

Now we shall turn to the main formulas. The following simple formulas hold for the action of the free supercharges $q^\alpha{}_a$ on the prefactors $(Y^n\widetilde{X}Y'^m)^\beta{}_b$, $(Y^m\widetilde{X}'Y'^n)^\beta{}_b$ ($n=1,3$, $m=0,2,4$) and the kinematical overlap $|V\rangle$. (For the derivation of these formulas, see appendix A.)

$$\begin{aligned}\big\{q^\alpha{}_a, (Y^n\widetilde{X}Y'^m)^\beta{}_b\big\} &= -\frac{n}{2}\big[\epsilon^{\beta\alpha}(\epsilon XY^{n-1}\widetilde{X}Y'^m)_{ab} + (Y^{n-1}\epsilon)^{\beta\alpha}(\epsilon X\widetilde{X}Y'^m)_{ab}\big]\\ &+ \frac{i}{2}(Y^{n+1}\epsilon)^{\beta\alpha}(\epsilon Y'^m)_{ab} - \frac{m}{2}\big[(Y^n\widetilde{X})^\beta{}_a(X'Y'^{m-1})^\alpha{}_b + (Y^n\widetilde{X}Y'^{m-1}X'\epsilon)^{\beta\alpha}\epsilon_{ab}\big],\end{aligned} \tag{2.15}$$

$$(YX)^\alpha{}_a(Y^n\widetilde{X}Y'^m)^\beta{}_b = -\frac{1}{n+1}\big[\epsilon^{\beta\alpha}(\epsilon XY^{n+1}\widetilde{X}Y'^m)_{ab} - (Y^{n+1}\epsilon)^{\beta\alpha}(\epsilon X\widetilde{X}Y'^m)_{ab}\big], \tag{2.16}$$

$$(X'Y')^\alpha{}_a(Y^n\widetilde{X}Y'^m)^\beta{}_b = -\frac{1}{m+1}\big[(Y^n\widetilde{X})^\beta{}_a(X'Y'^{m+1})^\alpha{}_b - (Y^n\widetilde{X}Y'^{m+1}X'\epsilon)^{\beta\alpha}\epsilon_{ab}\big], \tag{2.17}$$

$$\begin{aligned}\big\{q^\alpha{}_a, (Y^m\widetilde{X}'Y'^n)^\beta{}_b\big\} &= -\frac{m}{2}\big[\epsilon^{\beta\alpha}(\epsilon XY^{m-1}\widetilde{X}'Y'^n)_{ab} + (Y^{m-1}X)^\beta{}_a(\widetilde{X}'Y'^n)^\alpha{}_b\big]\\ &+ \frac{i}{2}(Y^m\epsilon)^{\beta\alpha}(\epsilon Y'^{n+1})_{ab} - \frac{n}{2}\big[(Y^m\widetilde{X}'X'\epsilon)^{\beta\alpha}(\epsilon Y'^{n-1})_{ab} + (Y^m\widetilde{X}'Y'^{n-1}X'\epsilon)^{\beta\alpha}\epsilon_{ab}\big],\end{aligned} \tag{2.18}$$

$$(YX)^\alpha{}_a(Y^m\widetilde{X}'Y'^n)^\beta{}_b = -\frac{1}{m+1}\big[\epsilon^{\beta\alpha}(\epsilon XY^{m+1}\widetilde{X}'Y'^n)_{ab} - (Y^{m+1}X)^\beta{}_a(\widetilde{X}'Y'^n)^\alpha{}_b\big], \tag{2.19}$$

$$(X'Y')^\alpha{}_a(Y^m\widetilde{X}'Y'^n)^\beta{}_b = -\frac{1}{n+1}\big[(Y^m\widetilde{X}'X'\epsilon)^{\beta\alpha}(\epsilon Y'^{n+1})_{ab} - (Y^m\widetilde{X}'Y'^{n+1}X'\epsilon)^{\beta\alpha}\epsilon_{ab}\big]. \tag{2.20}$$

Note that although the formulas are originally defined for $n = 1, 3$, (2.15) and (2.18) also holds for $n = 5$. Consistency with the first equation of (2.4) requires the coefficients in the supersymmetry charge to satisfy

$$\begin{aligned} &-3q_{3m} + \frac{i}{2}q_{1m} = 0, \qquad &&-4q_{n4} - \frac{i}{3}q_{n2} = 0, \qquad &&-2q_{n2} - \frac{i}{1}q_{n0} = 0, \\ &-3q'_{m3} - \frac{i}{2}q'_{m1} = 0, \qquad &&-4q'_{4n} + \frac{i}{3}q'_{2n} = 0, \qquad &&-2q'_{2n} + \frac{i}{1}q'_{0n} = 0, \end{aligned} \tag{2.21}$$

which can be solved by

$$q_{nm} = q\frac{\eta^n \eta^{*m}}{n!m!}, \quad q'_{mn} = q'\frac{\eta^m \eta^{*n}}{m!n!}. \tag{2.22}$$

Consequently, the supersymmetry charge and the Hamiltonian are given by

$$\begin{aligned} |Q^\alpha{}_a\rangle &= \Big\{ q\big[(\sinh \overline{Y})\widetilde{X}(\cosh \overline{Y}')\big]^\alpha{}_a + q'\big[(\cosh \overline{Y})\widetilde{X}'(\sinh \overline{Y}')\big]^\alpha{}_a \Big\}|V\rangle, \\ |H\rangle &= \Big\{ b\big[\mathrm{Tr}\, Y^4 - \mathrm{Tr}\, Y'^4\big] \\ &\quad + q\big[\eta \mathrm{Tr}\, X \cosh \overline{Y}\widetilde{X} \cosh \overline{Y}' + \eta^* \mathrm{Tr}\, \sinh \overline{Y}\widetilde{X} \sinh \overline{Y}' X'\big] \\ &\quad + q'\big[\eta^* \mathrm{Tr}\, \cosh \overline{Y}\widetilde{X}' \cosh \overline{Y}' X' + \eta \mathrm{Tr}\, X \sinh \overline{Y}\widetilde{X}' \sinh \overline{Y}'\big] \Big\}|V\rangle, \end{aligned} \tag{2.23}$$

with $\overline{Y} = Y\eta$, $\overline{Y}' = Y'\eta^*$ and b defined as

$$b = \frac{q\eta}{12} = \frac{q'\eta^*}{12}. \tag{2.24}$$

Similarly, the second equation of (2.4) determines $|\widetilde{Q}^\alpha{}_a\rangle$ and $|\widetilde{H}\rangle$.

The expression of supersymmetry charge and Hamiltonian in (2.23) is one of the most famous results from the pp-wave light-cone string field theory [15], though it has never been written down in this succinct form using the hyperbolic function. The hyperbolic expression of the light-cone string field theory basically originates from our experience on the flat spacetime with $\mathfrak{so}(8)$ symmetry [17] (with slightly change of notation):

$$\begin{aligned} |Q^{\dot{a}}\rangle &= \sqrt{-\alpha}[\sinh \not{Y}]^{\dot{a}i}\widetilde{X}^i|V\rangle, \\ |\widetilde{Q}^{\dot{a}}\rangle &= i\sqrt{-\alpha}X^i[\sinh \not{Y}]^{i\dot{a}}|V\rangle, \\ |H\rangle &= X^i[\cosh \not{Y}]^{ij}\widetilde{X}^j|V\rangle. \end{aligned} \tag{2.25}$$

Decomposition from the $\mathfrak{so}(8)$-invariant expression [17] into the current $\mathfrak{so}(4)\times\mathfrak{so}(4)$-invariant expression is reminiscent of the summation formulas of the hyperbolic functions: $\cosh(\chi_1 + \chi_2) = \cosh\chi_1 \cosh\chi_2 + \sinh\chi_1 \sinh\chi_2$, $\sinh(\chi_1 + \chi_2) = \sinh\chi_1 \cosh\chi_2 + \cosh\chi_1 \sinh\chi_2$.

In studying the third equation of (2.4), let us note that

$$\begin{aligned}\widetilde{q}^{\beta}{}_{b}|Q^{\alpha}{}_{a}\rangle = \Big\{ & -iq_{10}\big[\eta(\sinh\overline{Y}\widetilde{W}\epsilon)^{\alpha\beta} - \frac{i}{2}\mathrm{Tr}\,\widetilde{X}^2(\cosh\overline{Y}\epsilon)^{\alpha\beta}\big](\epsilon\cosh\overline{Y}')_{ba} \\ & - iq'_{01}\big[\eta^*(\epsilon\widetilde{W}'\sinh\overline{Y}')_{ba} - \frac{i}{2}\mathrm{Tr}\,\widetilde{X}'^2(\epsilon\cosh\overline{Y}')_{ba}\big](\cosh\overline{Y}\epsilon)^{\alpha\beta} \\ & - i(q'_{01} - q_{10})(\sinh\overline{Y}\widetilde{X})^{\alpha}{}_{b}(\widetilde{X}'\sinh\overline{Y}')^{\beta}{}_{a}\Big\}|V\rangle. \end{aligned} \tag{2.26}$$

To satisfy the third equation, we have to first require that

$$q_{10} = q'_{01}(=: q_1/2), \tag{2.27}$$

for the cancellation of terms of odd powers in Y and Y'. Furthermore, using (A.18) proved in appendix, we have the formulas

$$\begin{aligned}\big[\widetilde{h}, (\cosh\overline{Y}\epsilon)^{\alpha\beta}\big] &= \eta(\sinh\overline{Y}\widetilde{W}\epsilon)^{\alpha\beta} - \frac{i}{2}(\mathrm{Tr}\,Y\widetilde{W})(\cosh\overline{Y}\epsilon)^{\alpha\beta}, \\ \big[\widetilde{h}, (\epsilon\cosh\overline{Y}')_{ba}\big] &= \eta^*(\epsilon\widetilde{W}'\sinh\overline{Y}')_{ba} + \frac{i}{2}(\mathrm{Tr}\,\widetilde{W}'Y')(\epsilon\cosh\overline{Y}')_{ba}. \end{aligned} \tag{2.28}$$

Hence, the third equation becomes

$$q^{\alpha}{}_{a}|\widetilde{Q}^{\beta}{}_{b}\rangle + \widetilde{q}^{\beta}{}_{b}|Q^{\alpha}{}_{a}\rangle = \frac{i}{2}\big[\widetilde{q}_1 h - q_1\widetilde{h}\big](\cosh\overline{Y}\epsilon)^{\alpha\beta}(\epsilon\cosh\overline{Y}')_{ba}|V\rangle. \tag{2.29}$$

This final result requires

$$\widetilde{q}_1 = q_1, \tag{2.30}$$

with the level matching condition $h = \widetilde{h}$.

Note that to study the third equation of (2.4), the dependence of the kinematical overlap on the worldsheet coordinate has to be introduced in [6, 15]. Here we have studied the same equation purely algebraically without referring to the worldsheet coordinates.

2.3 Chiral terms

We can also consider the possibility of

$$|Q^{\alpha}{}_{a}\rangle = \Big\{\sum_{n,m} p_{nm}(Y^n X Y'^m)^{\alpha}{}_{a} + \sum_{n,m} p'_{mn}(Y^m X' Y'^n)^{\alpha}{}_{a}\Big\}|V\rangle. \tag{2.31}$$

The supersymmetry algebra puts constraints on the coefficients $p_{12} = p_{32} = p'_{21} = p'_{23} = 0$ as well as

$$\frac{p_{10}}{i/2} = \frac{p'_{01}}{i/2}(=: p_1), \quad \frac{p_{34}}{-2} = \frac{p'_{43}}{2}(=: p_7), \quad \frac{p_{30}}{-2} = \frac{p'_{41}}{i/2}(=: p_>), \quad \frac{p_{14}}{i/2} = \frac{p'_{03}}{2}(=: p_<). \tag{2.32}$$

Therefore, consistency with the first equation of (2.4) requires

$$
\begin{aligned}
|Q^\alpha{}_a\rangle &= \Big\{p_1\big[\frac{i}{2}(YX)^\alpha{}_a + \frac{i}{2}(X'Y')^\alpha{}_a\big] + p_7\big[-2(Y^3XY'^4)^\alpha{}_a + 2(Y^4X'Y'^3)^\alpha{}_a\big] \\
&\quad + p_>\big[-2(Y^3X)^\alpha{}_a + \frac{i}{2}(Y^4X'Y')^\alpha{}_a\big] + p_<\big[\frac{i}{2}(YXY'^4)^\alpha{}_a + 2(X'Y'^3)^\alpha{}_a\big]\Big\}|V\rangle, \\
|H\rangle &= \Big\{p_1\frac{i}{2}\big(\mathrm{Tr}\,X^2 + \mathrm{Tr}\,X'^2 - \mathrm{Tr}\,YW + \mathrm{Tr}\,W'Y'\big) \\
&\quad + \frac{p_7}{4}\big[\frac{i}{2}\mathrm{Tr}\,Y^4\mathrm{Tr}\,Y'^4\big(\mathrm{Tr}\,X^2 + \mathrm{Tr}\,X'^2\big) + 4\mathrm{Tr}\,Y^3W\mathrm{Tr}\,Y'^4 + 4\mathrm{Tr}\,Y^4\mathrm{Tr}\,W'Y'^3\big] \\
&\quad + \frac{p_>}{2}\big[\frac{i}{2}\mathrm{Tr}\,Y^4\big(\mathrm{Tr}\,X^2 + \mathrm{Tr}\,X'^2 + \mathrm{Tr}\,W'Y'\big) + 4\mathrm{Tr}\,Y^3W\big] \\
&\quad + \frac{p_<}{2}\big[\frac{i}{2}\mathrm{Tr}\,Y'^4\big(\mathrm{Tr}\,X^2 + \mathrm{Tr}\,X'^2 - \mathrm{Tr}\,YW\big) + 4\mathrm{Tr}\,W'Y'^3\big]\Big\}|V\rangle.
\end{aligned}
\tag{2.33}
$$

Similarly $|\widetilde{Q}^\alpha{}_a\rangle$ and $|\widetilde{H}\rangle$ is determined from the second equation of (2.4), while the third equation holds identically with $|N\rangle = 0$ if we require that

$$
p_1 = \widetilde{p}_1, \quad p_7 = \widetilde{p}_7, \quad p_> = \widetilde{p}_>, \quad p_< = \widetilde{p}_<. \tag{2.34}
$$

Furthermore, it is not difficult to notice that the result under this ansatz can be summarized as

$$
|Q^\alpha{}_a\rangle = q^\alpha{}_a|W\rangle, \quad |H\rangle = h|W\rangle, \tag{2.35}
$$

if we define the dressed kinematical overlap $|W\rangle$ as

$$
|W\rangle = \Big(p_1 + \frac{p_>}{2}\mathrm{Tr}\,Y^4 + \frac{p_<}{2}\mathrm{Tr}\,Y'^4 + \frac{p_7}{4}\mathrm{Tr}\,Y^4\mathrm{Tr}\,Y'^4\Big)|V\rangle. \tag{2.36}
$$

Previously, it was noticed [11, 14] that the free action on the dressed kinematical overlap satisfies the supersymmetry algebra as well. However, there was no systematic analysis claiming that this is the only alternative. Here by listing all the possible terms with one X or X' inserted (2.31), we find that all of the alternatives can be written as the free action on the dressed kinematical overlap $|W\rangle$.

2.4 Supergravity limit

Now let us study the supergravity limit. In the supergravity limit, there is no difference between left-moving modes and right-moving modes. Therefore, the bosonic momenta $\widetilde{X}$ and $\widetilde{X}'$ should coincide with X and X', while the excited fermionic momenta W, W', $\widetilde{W}$ and $\widetilde{W}'$ without contribution from zero modes should vanish identically.

It was further noticed [8] that in the supergravity Hamiltonian on the flat spacetime, there appears an extra $\mathfrak{u}(1)$ symmetry coming from the duality algebra of the theory which can be expressed as

$$u = 2 - \frac{1}{2}\lambda^a \vartheta^a. \tag{2.37}$$

In the $\mathfrak{so}(8)$-invariant formalism, the $\mathfrak{u}(1)$ action on various quantities is[6]

$$[u, Y] = -\frac{1}{2}Y, \quad [u, X] = 0, \quad u|V\rangle = 2|V\rangle. \tag{2.38}$$

Therefore, the contribution other than

$$|H\rangle \sim Y^4|V\rangle \tag{2.39}$$

should vanish identically in the supergravity limit. In our current $\mathfrak{so}(4) \times \mathfrak{so}(4)$-invariant formalism, Y and Y' (or X and X') decomposed from the same quantity Y (or X) in the $\mathfrak{so}(8)$ formalism should have the same charges.

Hereafter, let us determine some of the coefficients in (2.33) by requiring the $\mathfrak{u}(1)$ invariance in the supergravity limit[7]. Before starting it, let us first note that there are no Y^2 or Y^6 terms in (2.33). The condition of requiring that those terms in (2.23) cancel among themselves reduces to the same condition as (2.27). Next, if we require the Y^0 (and Y^8) terms from (2.23) and from (2.33) cancel with each other, we find two equations for p_1 (and p_7 respectively). The two equations have the same solution only if (2.27) holds and the result is

$$p_1 = iq_1, \quad p_7 = \frac{-iq_1}{(4!)^2}. \tag{2.40}$$

3 The $\mathfrak{su}(2|2)$ light-cone string field theory

After revisiting the pp-wave light-cone string field theory, it is now clear to distinguish the property of general backgrounds with the $\mathfrak{su}(2|2)$ isometry from that special to the pp-wave background. In fact, in obtaining the most important result (2.23) and (2.33), the main assumptions we have made is the last equation of (2.5) and (2.7). By generalizing these equations, we hope that we can explore the light-cone string field theory on general $\mathfrak{su}(2|2)$ backgrounds. Since we start our analysis from the fermionic momenta Y and Y' and act with the fermionic charges $q^\alpha{}_a$ to define other quantities X, X', W, W', we can assign gradings and

[6]The last relation can be understood from the computation: $\left(\sum_{r=1,2,3} u_{(r)}\right) \delta^8(\lambda_{(1)} + \lambda_{(2)} + \lambda_{(3)}) = \left[2 \times 3 - (1/2) \times 8\right] \delta^8(\lambda_{(1)} + \lambda_{(2)} + \lambda_{(3)})$.

[7]Corresponding discussions were given previously in [14]. It seems that their notation is different from ours.

dimensions to all of these building blocks from the outer automorphism of the superalgebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$:

$$\begin{aligned} \mathrm{grd}\, Y &= 0, & \mathrm{grd}\, X &= -\mathrm{grd}\, \widetilde{X} = 1/2, & \mathrm{grd}\, W &= -\mathrm{grd}\, \widetilde{W} = 1, \\ \dim Y &= 0, & \dim X &= \dim \widetilde{X} = 1/2, & \dim W &= \dim \widetilde{W} = 1. \end{aligned} \tag{3.1}$$

We have to generalize the commutation relations (2.5) respecting the gradings[8], while the dimension serves in the role of the expansion parameter.

The generalization of the right-hand side of the last equation in (2.5) can contain various terms including

$$[q^\alpha{}_a, \widetilde{X}^b{}_{\dot{b}}] = \sum_{m,n} y_{mn} (Y'^m)^b{}_a (Y^n)^\alpha{}_{\dot{b}} + \cdots . \tag{3.2}$$

The next corrections with bosonic momenta X, X', $\widetilde{X}$ and $\widetilde{X}'$ inserted will contain about 100 terms. Similarly, generalization of (2.7) can be

$$q^\alpha{}_a |V\rangle = \Big\{ \sum_{n,m} v_{nm} (Y^n X Y'^m)^\alpha{}_a + \sum_{n,m} v'_{mn} (Y^m X' Y'^n)^\alpha{}_a \Big\} |V\rangle, \tag{3.3}$$

as well as terms obtained by replacing X and X' by $\widetilde{X}$ and $\widetilde{X}'$.

Our final direction will be to determine all of the coefficients so that corrections to (2.5) and (2.7) are consistent with the algebra (2.3). For example, as previously in the case of the pp-wave background, consistency with

$$\begin{aligned} \{[q^\alpha{}_a, \widetilde{X}^c{}_{\dot{c}}], q^\beta{}_b\} + \{q^\alpha{}_a, [q^\beta{}_b, \widetilde{X}^c{}_{\dot{c}}]\} &= \epsilon^{\alpha\beta} \epsilon_{ab} [h, \widetilde{X}^c{}_{\dot{c}}], \\ \{q^\alpha{}_a, q^\beta{}_b\} |V\rangle &= \epsilon^{\alpha\beta} \epsilon_{ab} h |V\rangle, \\ \{q^\alpha{}_a, \widetilde{q}^\beta{}_b\} |V\rangle &= \epsilon^{\alpha\beta} \epsilon_{ab} n |V\rangle, \end{aligned} \tag{3.4}$$

gives strong constraints to the corrections. In the third equation, we have imposed the condition $r^a{}_b |V\rangle = l^\alpha{}_\beta |V\rangle = 0$ because we assume that the invariance of the kinematical overlap $|V\rangle$ under the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ transformation remains unmodified.

Due to a vast of possibilities, it is difficult to write down all of the possible terms. Instead, here we shall present some interesting possible corrections.

3.1 Toy model I

Let us consider a toy model for generalizations. We assume the deformation of (3.3) with $n = 1, 3$ and $m = 0, 2, 4$ instead of (2.7), keeping the commutation relations (2.5) unmodified.

[8] It is interesting to note that the grading appeared as an extra symmetry in the $\mathfrak{su}(2|2)$ spin chain model [19] also plays an important role in constructing light-cone string field theory.

From the consistency with the algebra on the overlap (3.4), the coefficients of (3.3) should satisfy $v_{12} = v_{32} = v_{34} = v'_{21} = v'_{23} = v'_{43} = 0$ and

$$v_{30} = \widetilde{v}_{30}, \quad v_{34} = \widetilde{v}_{34}, \quad v'_{03} = \widetilde{v}'_{03}, \quad v'_{43} = \widetilde{v}'_{43}, \quad v_{34} + v'_{43} = 0, \tag{3.5}$$

in addition to $v_{10} = v'_{01} = i/2$.

By solving (2.4) with the ansatz (2.12), the supersymmetry charges $|Q^\alpha{}_a\rangle$ and the Hamiltonian $|H\rangle$ (2.23) are deformed as

$$\begin{aligned}
|Q^\alpha{}_a\rangle &= \Big\{ q\big[(\sinh\overline{Y})\widetilde{X}\big(\cosh\overline{Y}' - \widehat{v}'Y'^4\big)\big]^\alpha{}_a + q'\big[\big(\cosh\overline{Y} + \widehat{v}Y^4\big)\widetilde{X}'\sinh\overline{Y}'\big]^\alpha{}_a \Big\}|V\rangle, \\
|H\rangle &= \Big\{ b\big[\mathrm{Tr}\,Y^4(1-(\widehat{v}'/2)\mathrm{Tr}\,Y'^4) - \mathrm{Tr}\,Y'^4(1+(\widehat{v}/2)\mathrm{Tr}\,Y^4)\big] \\
&\quad + q\big[\eta\mathrm{Tr}\,X(\cosh\overline{Y} - \widehat{v}Y^4)\widetilde{X}(\cosh\overline{Y}' - \widehat{v}'Y'^4) - \eta\widehat{b}\mathrm{Tr}XY^4\widetilde{X}Y'^4 \\
&\qquad + \eta^*\mathrm{Tr}\,\sinh\overline{Y}\widetilde{X}\sinh\overline{Y}'X'\big] \\
&\quad + q'\big[\eta^*\mathrm{Tr}\,(\cosh\overline{Y} + \widehat{v}Y^4)\widetilde{X}'(\cosh\overline{Y}' + \widehat{v}'Y'^4)X' - \eta^*\widehat{b}\mathrm{Tr}\,Y^4\widetilde{X}'Y'^4X' \\
&\qquad + \eta\mathrm{Tr}\,X\sinh\overline{Y}\widetilde{X}'\sinh\overline{Y}'\big]\Big\}|V\rangle,
\end{aligned} \tag{3.6}$$

where we have defined $\widehat{v} = v_{30}/2$, $\widehat{v}' = v'_{03}/2$ and $\widehat{b} = v_{34}/4 = -v'_{43}/4$. In this case, (2.29) becomes

$$q^\alpha{}_a|\widetilde{Q}^\beta{}_b\rangle + \widetilde{q}^\beta{}_b|Q^\alpha{}_a\rangle = \frac{i}{2}q_1\big[h - \widetilde{h}\big]\big[(\cosh\overline{Y}\epsilon)^{\alpha\beta}(\epsilon\cosh\overline{Y}')_{ba} + 4\widehat{b}(Y^4\epsilon)^{\alpha\beta}(\epsilon Y'^4)_{ba}\big]|V\rangle. \tag{3.7}$$

On the other hand, if we take the ansatz of the form (2.31), (2.4) is solved by

$$\begin{aligned}
|Q^\alpha{}_a\rangle = \Big\{ &\frac{i}{2}p_1\big[(YX)^\alpha{}_a + (X'Y')^\alpha{}_a\big] \\
&+ (-2p_7 + v_{30}p_< + 4v_{34}p_1)(Y^3XY'^4)^\alpha{}_a + (2p_7 + v'_{03}p_> + 4v'_{43}p_1)(Y^4X'Y'^3)^\alpha{}_a \\
&+ (-2p_> + v_{30}p_1)(Y^3X)^\alpha{}_a + \frac{i}{2}p_>(Y^4X'Y')^\alpha{}_a \\
&+ \frac{i}{2}p_<(YXY'^4)^\alpha{}_a + (2p_< + v'_{03}p_1)(X'Y'^3)^\alpha{}_a\Big\}|V\rangle,
\end{aligned} \tag{3.8}$$

where we have redefined p_1, p_7, $p_>$ and $p_<$ by

$$\begin{aligned}
&\frac{p_{10}}{i/2} = \frac{p'_{01}}{i/2} =: p_1, \quad \frac{p_{34}}{-2} + \frac{v_{30}}{2}p_< + 2v_{34}p_1 = \frac{p'_{43}}{2} - \frac{v'_{03}}{2}p_> - 2v'_{43}p_1 =: p_7, \\
&\frac{p_{30}}{-2} + \frac{v_{30}}{2}p_1 = \frac{p'_{41}}{i/2} =: p_>, \quad \frac{p_{14}}{i/2} = \frac{p'_{03}}{2} - \frac{v'_{03}}{2}p_1 =: p_<,
\end{aligned} \tag{3.9}$$

instead of (2.32). We find that the above expression can also be rewritten as the action of free charges on a dressed kinematical overlap (2.35) with the dressed kinematical overlap formally taking the same expression as (2.36).

3.2 Toy model II

As a next example, let us consider a type of deformations of the last equation of (2.5) with $q^\alpha{}_a|V\rangle$ (2.7) unchanged. We introduce the coefficient of Y^3, namely y_{03} in (3.2), and a counterpart in $[q^\alpha{}_a, \widetilde{X}'^\beta{}_{\dot\beta}]$, which we denote as y'_{30}. In this case, the Jacobi identities such as the first equation of (3.4) are broken. To solve this problem, further deformation terms in the commutation relations are required. Actually, we find a consistent deformation of commutation relations, up to terms of dim 1,

$$
\begin{aligned}
[q^\alpha{}_a, \widetilde{X}^{\dot b}{}_b] &= -\frac{i}{2}(Y\epsilon)^{\dot b\alpha}\epsilon_{ab} + y_{03}\big[(Y^3\epsilon)^{\dot b\alpha}\epsilon_{ab} + 2i(Y\epsilon)^{\dot b\alpha}(\epsilon X\widetilde{X})_{ab} + 4i(YX)^\alpha{}_a\widetilde{X}^{\dot b}{}_b\big],\\
[q^\alpha{}_a, \widetilde{X}'^\beta{}_{\dot\beta}] &= \frac{i}{2}\epsilon^{\beta\alpha}(\epsilon Y')_{a\dot\beta} - y'_{30}\big[\epsilon^{\beta\alpha}(\epsilon Y'^3)_{a\dot\beta} + 2i(\widetilde{X}'X'\epsilon)^{\beta\alpha}(\epsilon Y')_{a\dot\beta} + 4i(X'Y')^\alpha{}_a\widetilde{X}'^\beta{}_{\dot\beta}\big],
\end{aligned}
\tag{3.10}
$$

and their counterparts of $[\widetilde{q}^\alpha{}_a, X^{\dot b}{}_b]$ and $[\widetilde{q}^\alpha{}_a, X'^\beta{}_{\dot\beta}]$, which satisfy the first equation[9] of (3.4) up to terms of dim 1/2. From the consistency with the third equation of (3.4),

$$
y_{03} = \widetilde{y}_{03}, \quad y'_{30} = \widetilde{y}'_{30}, \tag{3.11}
$$

are required.

With the above deformation and the ansatz (2.12), the supersymmetry charges are obtained by the same formula as the first equation of (2.23), which satisfy the first algebraic relation of (2.4) up to terms of dim 1. Although the relation (2.29) is unchanged, the Hamiltonian $|H\rangle$ is deformed from the second equation of (2.23) by

$$
\Big[\eta y_{03} q \mathrm{Tr}\, Y^4\Big(1 - \frac{1}{48}\mathrm{Tr}\, Y'^4\Big) + \eta^* y'_{30} q' \mathrm{Tr}\, Y'^4\Big(1 - \frac{1}{48}\mathrm{Tr}\, Y^4\Big)\Big]|V\rangle. \tag{3.12}
$$

4 Discussions

We have revisited the construction of light-cone string field theory on the pp-wave background and clarified its algebraic structure. Among others, we find the centrally extended subalgebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ of the whole pp-wave isometry plays an essential role in the construction. In the meanwhile, we distinguish the properties satisfied by general $\mathfrak{su}(2|2)$ backgrounds from those special to the pp-wave background. We also make several improvements for the construction on the pp-wave backgrounds:

- Our systematic analysis reproduces the famous previous results [15] in a succinct form (2.23).

[9] We note that the Jacobi identity composed of $q^\alpha{}_a, \widetilde{q}^\alpha{}_a$ and $\widetilde{X}^c{}_{\dot c}$ does not give any constraints because $\{q^\alpha{}_a, \widetilde{W}^\beta{}_b\}$ is not defined yet.

- Previously, a dependence of $|V\rangle$ on the worldsheet coordinate had to be introduced in [6, 15] to study the orthogonality between two supercharges $\mathcal{Q}^\alpha{}_a$ and $\widetilde{\mathcal{Q}}^\alpha{}_a$. Here we show that the same relation can be derived purely algebraically.
- We present a thorough study of possible contributions from the chiral terms. We find that all of the interaction terms $|J\rangle$ can be put into the forms $|J\rangle = j|W\rangle$ (2.35) where free charges j are acting on a dressed kinematical overlap $|W\rangle$.
- We determine some of the coefficients of the chiral terms from the abelian duality group in the supergravity limit.

After the clarification, we propose a strategy for extending the background from the pp-wave one to a more general one with the isometry $\mathfrak{su}(2|2)$ and present some examples of generalizations. As further directions, we hope to identify some of these generalizations as $AdS_5 \times S^5$, and investigate AdS/CFT correspondence with the string interactions.

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A Some useful formulas

In this appendix we shall prove some useful formulas including (2.15)–(2.20). For Grassmann odd quantities $Y^\alpha{}_{\dot a}$ ($\alpha = 1, 2, a = 1, 2$), the following identities hold:

$$
\begin{aligned}
Y^\alpha{}_{\dot a} Y^\beta{}_{\dot b} &= -\frac{1}{2}(Y^2\epsilon)^{\alpha\beta}\epsilon_{\dot a\dot b} - \frac{1}{2}(\epsilon Y^2)_{\dot a\dot b}\epsilon^{\alpha\beta}, \\
Y^\alpha{}_{\dot a} Y^\beta{}_{\dot b} Y^\gamma{}_{\dot c} &= \frac{1}{9}(Y^3)^\delta{}_{\dot d}\big(\delta^\alpha_\delta\epsilon^{\beta\gamma}(\delta^{\dot d}_{\dot c}\epsilon_{\dot a\dot b} - \delta^{\dot d}_{\dot b}\epsilon_{\dot c\dot a}) + \delta^\beta_\delta\epsilon^{\gamma\alpha}(\delta^{\dot d}_{\dot a}\epsilon_{\dot b\dot c} - \delta^{\dot d}_{\dot c}\epsilon_{\dot a\dot b}) + \delta^\gamma_\delta\epsilon^{\alpha\beta}(\delta^{\dot d}_{\dot b}\epsilon_{\dot c\dot a} - \delta^{\dot d}_{\dot a}\epsilon_{\dot b\dot c})\big), \\
Y^\alpha{}_{\dot a} Y^\beta{}_{\dot b} Y^\gamma{}_{\dot c} Y^\delta{}_{\dot d} &= -\frac{1}{36}\mathrm{Tr}\, Y^4 \\
&\quad\times\big(\epsilon^{\delta\alpha}\epsilon^{\beta\gamma}(\epsilon_{\dot d\dot c}\epsilon_{\dot a\dot b} - \epsilon_{\dot d\dot b}\epsilon_{\dot c\dot a}) + \epsilon^{\delta\beta}\epsilon^{\gamma\alpha}(\epsilon_{\dot d\dot a}\epsilon_{\dot b\dot c} - \epsilon_{\dot d\dot c}\epsilon_{\dot a\dot b}) + \epsilon^{\delta\gamma}\epsilon^{\alpha\beta}(\epsilon_{\dot d\dot b}\epsilon_{\dot c\dot a} - \epsilon_{\dot d\dot a}\epsilon_{\dot b\dot c})\big),
\end{aligned}
\tag{A.1}
$$

where

$$
\begin{aligned}
(Y^2\epsilon)^{\alpha\beta} &= Y^\alpha{}_{\dot a} Y^{\dot a}{}_\gamma \epsilon^{\gamma\beta} = Y^\alpha{}_{\dot a}(\epsilon_{\gamma\delta} Y^\delta{}_{\dot b}\epsilon^{\dot b\dot a})\epsilon^{\gamma\beta} = (Y^2\epsilon)^{\beta\alpha}, \\
(\epsilon Y^2)_{\dot a\dot b} &= \epsilon_{\dot a\dot c} Y^{\dot c}{}_\alpha Y^\alpha{}_{\dot b} = \epsilon_{\dot a\dot c}(\epsilon_{\alpha\beta} Y^\beta{}_{\dot d}\epsilon^{\dot d\dot c}) Y^\alpha{}_{\dot b} = (\epsilon Y^2)_{\dot b\dot a}.
\end{aligned}
\tag{A.2}
$$

Let us consider commutation relations between $\mathrm{Tr}\,\xi q \equiv \xi^a{}_\alpha q^\alpha{}_a$ and Y, Y', where Grassmann odd parameters $\xi^a{}_\alpha$ are introduced for simplicity of indices. The results are given by

$$
\begin{aligned}
[\mathrm{Tr}\,\xi q, Y^\alpha{}_{\dot a}] &= -(\xi X)^\alpha{}_{\dot a}, & [\mathrm{Tr}\,\xi q, \bar Y^{\dot a}{}_\alpha] &= (\bar X\xi)^{\dot a}{}_\alpha, \\
[\mathrm{Tr}\,\xi q, Y'^a{}_{\dot\alpha}] &= (\xi X')^a{}_{\dot\alpha}, & [\mathrm{Tr}\,\xi q, \bar Y'^{\dot\alpha}{}_a] &= -(\bar X'\bar\xi)^{\dot\alpha}{}_a,
\end{aligned}
\tag{A.3}
$$

with[10] $\bar\xi^\alpha{}_a = \epsilon_{ab}\xi^b{}_\beta\epsilon^{\beta\alpha}$, $\bar Y^{\dot a}{}_\alpha = \epsilon_{\alpha\beta} Y^\beta{}_{\dot b}\epsilon^{\dot b\dot a}$, $\bar Y'^{\dot\alpha}{}_a = \epsilon_{ab} Y^b{}_{\dot\beta}\epsilon^{\dot\beta\dot\alpha}$ as well as $\bar X^{\dot a}{}_a = \epsilon_{ab} X^b{}_{\dot b}\epsilon^{\dot b\dot a}$, $\bar X'^{\dot\alpha}{}_\alpha = \epsilon_{\alpha\beta} X'^\beta{}_{\dot\beta}\epsilon^{\dot\beta\dot\alpha}$. If we define 4×4 matrices $\boldsymbol{Y}, \boldsymbol{Y}'$ as

$$
\boldsymbol{Y} = (\boldsymbol{Y}_{(\alpha,\dot a),(\beta,\dot b)}) = \begin{pmatrix} 0 & Y^\alpha{}_{\dot b} \\ \bar Y^{\dot a}{}_\beta & 0 \end{pmatrix}, \quad \boldsymbol{Y}' = (\boldsymbol{Y}'_{(a,\dot\alpha),(b,\dot\beta)}) = \begin{pmatrix} 0 & Y'^a{}_{\dot\beta} \\ \bar Y'^{\dot\alpha}{}_b & 0 \end{pmatrix},
\tag{A.4}
$$

(A.3) can be rewritten as

$$
[\mathrm{Tr}\,\xi q, \boldsymbol{Y}] = \boldsymbol{X}_\xi, \quad [\mathrm{Tr}\,\xi q, \boldsymbol{Y}'] = \boldsymbol{X}'_\xi,
\tag{A.5}
$$

with $\boldsymbol{X}_\xi$ and $\boldsymbol{X}'_\xi$ defined by

$$
\boldsymbol{X}_\xi = \begin{pmatrix} 0 & -(\bar\xi X)^\alpha{}_{\dot b} \\ (\bar X\xi)^{\dot a}{}_\beta & 0 \end{pmatrix}, \quad \boldsymbol{X}'_\xi = \begin{pmatrix} 0 & (\xi X')^a{}_{\dot\beta} \\ -(\bar X'\bar\xi)^{\dot\alpha}{}_b & 0 \end{pmatrix}.
\tag{A.6}
$$

Using the relations

$$
\boldsymbol{Y}\boldsymbol{X}_\xi\boldsymbol{Y} = \frac{1}{2}(\boldsymbol{X}_\xi\boldsymbol{Y}^2 + \boldsymbol{Y}^2\boldsymbol{X}_\xi), \quad \boldsymbol{Y}'\boldsymbol{X}'_\xi\boldsymbol{Y}' = \frac{1}{2}(\boldsymbol{X}'_\xi\boldsymbol{Y}'^2 + \boldsymbol{Y}'^2\boldsymbol{X}'_\xi),
\tag{A.7}
$$

which follow from (A.1), the commutation relations (A.5) can be easily generalized into

$$
[\mathrm{Tr}\,\xi q, \boldsymbol{Y}^k] = \frac{k}{2}(\boldsymbol{X}_\xi\boldsymbol{Y}^{k-1} + \boldsymbol{Y}^{k-1}\boldsymbol{X}_\xi), \quad [\mathrm{Tr}\,\xi q, \boldsymbol{Y}'^k] = \frac{k}{2}(\boldsymbol{X}'_\xi\boldsymbol{Y}'^{k-1} + \boldsymbol{Y}'^{k-1}\boldsymbol{X}'_\xi),
\tag{A.8}
$$

for non-negative integers k. More explicitly, they can be rewritten as

$$
\begin{aligned}
[\mathrm{Tr}\,\xi q, \boldsymbol{Y}^n] &= \frac{n}{2}\begin{pmatrix} 0 & -\bar\xi X\bar Y^{n-1} - Y^{n-1}\bar\xi X \\ \bar X\xi Y^{n-1} + \bar Y^{n-1}\bar X\xi & 0 \end{pmatrix}, \\
[\mathrm{Tr}\,\xi q, \boldsymbol{Y}'^n] &= \frac{n}{2}\begin{pmatrix} 0 & \xi X'\bar Y'^{n-1} + Y'^{n-1}\xi X' \\ -\bar X'\bar\xi Y'^{n-1} - \bar Y'^{n-1}\bar X'\bar\xi & 0 \end{pmatrix},
\end{aligned}
\tag{A.9}
$$

[10] We attach bars $(\bar{\ })$ to avoid confusions in the following matrix notation. These bars should not be confused with the overlines of $\overline{Y}$ and $\overline{Y}'$.

for $n=1,3$, and

$$
\begin{aligned}
[\mathrm{Tr}\,\xi q, \boldsymbol{Y}^m] &= \frac{m}{2}\begin{pmatrix} -\bar{\xi}X\bar{Y}^{m-1}+Y^{m-1}\bar{X}\xi & 0 \\ 0 & \bar{X}\xi Y^{m-1}-\bar{Y}^{m-1}\bar{\xi}X \end{pmatrix}, \\
[\mathrm{Tr}\,\xi q, \boldsymbol{Y}'^m] &= \frac{m}{2}\begin{pmatrix} \xi X'\bar{Y}'^{m-1}-Y'^{m-1}\bar{X}'\bar{\xi} & 0 \\ 0 & -\bar{X}'\bar{\xi}Y'^{m-1}+\bar{Y}'^{m-1}\xi X' \end{pmatrix},
\end{aligned}
\tag{A.10}
$$

for $m=0,2,4$, where we denote the matrix products of Y and Y' as

$$
Y^k = Y\bar{Y}Y\cdots, \quad \bar{Y}^k = \bar{Y}Y\bar{Y}\cdots, \quad Y'^k = Y'\bar{Y}'Y'\cdots, \quad \bar{Y}'^k = \bar{Y}'Y'\bar{Y}'\cdots. \tag{A.11}
$$

By removing $\xi^a{}_\alpha$ from the above formulas (A.9) and (A.10), we obtain

$$
\begin{aligned}
\{q^\alpha{}_a, (Y^n)^\beta{}_{\dot{c}}\} &= -\frac{n}{2}\left[\epsilon^{\beta\alpha}(\epsilon X\bar{Y}^{n-1})_{a\dot{c}} + (Y^{n-1}\epsilon)^{\beta\alpha}(\epsilon X)_{a\dot{c}}\right], \\
[q^\alpha{}_a, (Y^m)^\beta{}_\gamma] &= -\frac{m}{2}\left[\epsilon^{\beta\alpha}(\epsilon X\bar{Y}^{m-1})_{a\gamma} + (Y^{m-1}\bar{X})^\beta{}_a\delta^\alpha_\gamma\right], \\
\{q^\alpha{}_a, (\bar{Y}'^n)^{\dot{\gamma}}{}_b\} &= -\frac{n}{2}\left[(\bar{X}'\epsilon)^{\dot{\gamma}\alpha}(\epsilon\bar{Y}'^{n-1})_{ab} + (\bar{Y}'^{n-1}\bar{X}'\epsilon)^{\dot{\gamma}\alpha}\epsilon_{ab}\right], \\
[q^\alpha{}_a, (Y'^m)^c{}_b] &= \frac{m}{2}\left[\delta^c_a(X'\bar{Y}'^{m-1})^\alpha{}_b + (Y'^{m-1}\bar{X}'\epsilon)^{c\alpha}\epsilon_{ab}\right],
\end{aligned}
\tag{A.12}
$$

where $n=1,3$ and $m=0,2,4$. These formulas are essentially (2.15) and (2.18).

Next we shall turn to the derivation of the remaining formulas. Noting $\epsilon_{\alpha\beta}\delta^\delta_\gamma + \epsilon_{\beta\gamma}\delta^\delta_\alpha + \epsilon_{\gamma\alpha}\delta^\delta_\beta = 0$ and similar identities, we find

$$
\mathbf{1}_4\mathrm{Tr}\,(\xi Y\bar{X}) = \boldsymbol{X}_\xi\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{X}_\xi, \quad \mathbf{1}_4\mathrm{Tr}\,(\xi X'\bar{Y}') = \boldsymbol{X}'_\xi\boldsymbol{Y}' - \boldsymbol{Y}'\boldsymbol{X}'_\xi, \tag{A.13}
$$

where $\mathbf{1}_4$ is the identity matrix of 4×4. Using the above formulas repeatedly, we obtain

$$
\boldsymbol{Y}^{k-1}\mathrm{Tr}\,(\xi Y\bar{X}) = \frac{1}{k}(\boldsymbol{X}_\xi\boldsymbol{Y}^k - \boldsymbol{Y}^k\boldsymbol{X}_\xi), \quad \boldsymbol{Y}'^{k-1}\mathrm{Tr}\,(\xi X'\bar{Y}') = \frac{1}{k}(\boldsymbol{X}'_\xi\boldsymbol{Y}'^k - \boldsymbol{Y}'^k\boldsymbol{X}'_\xi), \tag{A.14}
$$

with k being a positive integer. These relations are nothing but (2.16)-(2.17) and (2.19)-(2.20).

Similarly, noting (2.8) or

$$
[h, \boldsymbol{Y}] = \boldsymbol{W}, \quad [h, \boldsymbol{Y}'] = \boldsymbol{W}', \tag{A.15}
$$

where

$$
\boldsymbol{W} = \begin{pmatrix} 0 & W^\alpha{}_{\dot{b}} \\ \bar{W}^{\dot{a}}{}_\beta & 0 \end{pmatrix}, \quad \boldsymbol{W}' = \begin{pmatrix} 0 & W'^a{}_{\dot{\beta}} \\ \bar{W}'^{\dot{\alpha}}{}_b & 0 \end{pmatrix}, \tag{A.16}
$$

we find the multiplication formulas for $\boldsymbol{Y}, \boldsymbol{Y}', \boldsymbol{W}$ and $\boldsymbol{W}'$

$$
\begin{aligned}
\boldsymbol{Y}\boldsymbol{W}\boldsymbol{Y} &= \frac{1}{2}(\boldsymbol{W}\boldsymbol{Y}^2 + \boldsymbol{Y}^2\boldsymbol{W}), & \boldsymbol{Y}'\boldsymbol{W}'\boldsymbol{Y}' &= \frac{1}{2}(\boldsymbol{W}'\boldsymbol{Y}'^2 + \boldsymbol{Y}'^2\boldsymbol{W}'), \\
\boldsymbol{Y}^k\boldsymbol{W} &= \boldsymbol{W}\boldsymbol{Y}^k + k\boldsymbol{Y}^{k-1}\mathrm{Tr}\,(Y\bar{W}), & \boldsymbol{Y}'^k\boldsymbol{W}' &= \boldsymbol{W}'\boldsymbol{Y}'^k - k\boldsymbol{Y}'^{k-1}\mathrm{Tr}\,(W'\bar{Y}'),
\end{aligned}
\tag{A.17}
$$

as well as the h action,

$$[h, \boldsymbol{Y}^k] = k\boldsymbol{W}\boldsymbol{Y}^{k-1} + \frac{k(k-1)}{2}\boldsymbol{Y}^{k-2}\mathrm{Tr}\,(Y\bar{W}) = k\boldsymbol{Y}^{k-1}\boldsymbol{W} - \frac{k(k-1)}{2}\boldsymbol{Y}^{k-2}\mathrm{Tr}\,(Y\bar{W}),$$
$$[h, \boldsymbol{Y}'^k] = k\boldsymbol{W}'\boldsymbol{Y}'^{k-1} - \frac{k(k-1)}{2}\boldsymbol{Y}'^{k-2}\mathrm{Tr}\,(W'\bar{Y}') = k\boldsymbol{Y}'^{k-1}\boldsymbol{W}' + \frac{k(k-1)}{2}\boldsymbol{Y}'^{k-2}\mathrm{Tr}\,(W'\bar{Y}'), \tag{A.18}$$

where k is a positive integer. This final result is helpful in studying (2.28).

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