The B.E. Journal of Theoretical Economics

Contributions

Volume 10, *Issue* 1 2010 *Article* 15

Collusive Behavior of Bidders in English Auctions: A Cooperative Game Theoretic Analysis

Takayuki Oishi*

*Kyoto University, oishi-t@gs.econ.keio.ac.jp

Recommended Citation
Takayuki Oishi (2010) "Collusive Behavior of Bidders in English Auctions: A Cooperative Game Theoretic Analysis," *The B.E. Journal of Theoretical Economics*: Vol. 10: Iss. 1 (Contributions), Article 15.
Available at: http://www.bepress.com/bejte/vol10/iss1/art15

Collusive Behavior of Bidders in English Auctions: A Cooperative Game Theoretic Analysis*

Takayuki Oishi

Abstract

In practice, collusive bidders' rings in English auctions with a single object frequently distribute collusive gains among ring members via sequences of re-auctions called *knockouts*. The present paper introduces a model of sequences of knockouts under the situation in which each bidder has information on his evaluation and the order of the evaluations of all bidders for the object. The present paper examines the distributive function of sequences of knockouts from the viewpoint of cooperative game theory. Each sequence of knockouts yields an element of the *core*, two particular sequences yielding the *Shapley value* and the *nucleolus* respectively. The present paper highlights the sequence of knockouts yielding the nucleolus.

KEYWORDS: bidding rings, knockout, core, Shapley value, nucleolus

---

*The author would like to thank the co-editor, Federico Echenique, and an anonymous referee for very constructive comments. The author would also like to thank Youngsub Chun, Takako Fujiwara-Greve, Toshiyuki Hirai, Shigeo Muto, Shinsuke Nakamura, Mikio Nakayama, Shin Sakaue, and Makoto Yano for their helpful comments and suggestions. The author thanks Toru Hokari for providing him with the paper of Sönmez (1994). The author also thanks the seminar participants at Keio University and the Tokyo Institute of Technology, as well as the session participants of the Fall 2007 National Conference of Operations Research Society of Japan at National Graduate Institute for Policy Studies. This research was supported financially by the Keio University/Kyoto University Joint Global COE Program: "Raising Market Quality-Integrated Design of Market Infrastructure."

1 Introduction

A bidding ring is a group of collusive members who secretly agree not to compete against one another at an auction. Bidding rings have been operated throughout the world. For example, Cassady (1967) reported bidding rings in many commodity fields, such as antique trading, fish trading, and timber rights. Bidding rings reduce or eliminate buyer competition at auctions, thereby securing an advantage over the sellers.

Cassady (1967) explained that a bidding ring allocates an object won at auction and divides collusive gains via a re-auction. This re-auction is called a *knockout.*[1] A knockout is an oral ascending bid auction called an English auction among the ring members. When bidders have different evaluations for the object being auctioned, *a sequence of knockouts* is produced. In practice, some sequences of knockouts have been observed. For example, Cassady (1967) reported the sequence in an antique trade, and Deltas (2002) described the sequence in an auction of used commercial equipment (case of U.S. v. Seville Industrial Machinery Corp.).

The purpose of the present study is to develop a cooperative game theoretic framework in order to analyze the distributive function of sequences of knockouts. In the present framework, we can deal with the following problems. Why would a ring decide to distribute the gains by means of sequences of knockouts? What, if any, are the advantages to sequences of knockouts? Economists have paid very little attention to these problems.

The motivation for the present study stems from the fact that most of the US Department of Justice's bid-rigging convictions are a result of whistleblowers who are dissatisfied with their distributive outcomes, e.g., McAfee and McMillan (1992) and Milgrom (2004). This fact demonstrates that the distribution of gains from collusion among ring members is a serious problem that must be overcome by bidding rings. Therefore, it is important to reveal whether sequential knockouts yield satisfactory outcomes for ring members.

Toward the purpose of the present study, a generic model of sequences of knockouts with a single object is introduced. This model is applied to a bidding ring game, which is a coalitional game with transferable utility. Bidding ring games deal with sharing problems of gains obtained through collusion.

[1] According to Webster's Third New International Dictionary, a knockout is "*an auction or sale or similar transaction at which a combination* (*as of bidders*) *illegitimately forces out other potential competitors and arranges by prior agreement to have one member of the combination secure at a set price the thing being offered so as later to profitably dispose of the thing* (*as by reauctioning*)."

The present study provides a rationale for sequences of knockouts by showing that each sequence yields an element of the core; and one sequence yields the Shapley value while another sequence yields the nucleolus. Each solution represents a satisfactory outcome for ring members, and hence a ring would decide to distribute the gains by means of sequences of knockouts. Moreover, in general, each of the Shapley value and the nucleolus is not yielded by a single knockout. Ring members would produce sequential knockouts if they intend toward egalitarianism or minimization of the greatest dissatisfaction of any bidders' coalition. Thus, the results indicate that sequential knockouts are advantageous.

In the model of the present study, *weak asymmetry of information* among buyers is introduced. Namely, each buyer has information on his evaluation and the order of the evaluations of all buyers for the object before the main auction. This situation is more informative than that in the independent private values model, but this situation requires sequences of knockouts in order to elicit the evaluations of buyers for the object. The evaluations of all buyers can be ordered by giving points to each buyer according to (I) the number of times in the past the buyer has participated in bidding, and (II) the size of the buyer's bid on the occasion of his winning at an auction in the past. This method for ordering the evaluations of buyers was adopted by bidding rings in Japan, e.g., McMillan (1991). Using weak asymmetry of information, a ring formation at each knockout is characterized.

In the literature on knockouts, collusive behavior of bidders in knockouts has both non-cooperative and cooperative aspects. Regarding the non-cooperative aspect of collusive behavior, the bidder's strategic behavior has been analyzed under the independent private values model, e.g., Graham and Marshall (1987), Graham *et al.* (1990)[2] and Mailath and Zemsky (1991). On the other hand, there has been little research on the cooperative aspects of collusive behavior.[3] Graham *et al.* (1990) considered only a single sequential knockout yielding the Shapley value. The present study provides a generic model of sequential knockouts including the sequence in Graham *et al.* (1990). Therefore, the sequence of knockouts yielding the nucleolus will be emphasized.

The rest of the present paper is organized as follows. Section 2 explains the bidding ring game and cooperative solutions. Section 3 introduces the model of sequences of knockouts. In addition, this section reports the basic properties of sequential knockouts concerning the core and the Shapley value. Section 4 establishes a particular sequence of knockouts yielding the nucleolus.

[2] Although Graham *et al.* (1990) used the bidding ring game partially, they considered primarily the bidder's strategic behavior under the independent private values model.

[3] The game structure of bidding ring games has been investigated, e.g., Oishi (2006) and van den Brink *et al* (2007). However, they did not deal with sequences of knockouts.

2 Definitions and preliminaries

2.1 Bidding ring games

English auctions are oral auctions in which an auctioneer initially sets a bid at a low price and then gradually increases the price until only one bidder remains active.

Assume that there are n buyers in a single object English auction. Let $v_1 > v_2 > \cdots > v_n > v_0 \geq 0$, where v_i is the evaluation of each buyer $i = 1, \cdots, n$ for the object and v_0 is the reservation price of the only seller for his object. In addition, we assume that there is no asymmetry of information among buyers; that is, each buyer has information on the evaluations of all buyers for the object before the main auction.

A bidding ring game in an English auction is a coalitional game with transferable utility, namely a TU game defined by Graham *et al.* (1990).

Let $N = \{1, \cdots, n\}$ be the finite set of buyers, and let $S \subseteq N$ be a coalition. Consider an English auction with the possibility of a bidding ring among buyers of the auction under non-asymmetric information. This situation can be described as the TU game (N, v) satisfying

$$v(S) = \begin{cases} v_1 - \max_{j \notin S} v_j & \text{if} \quad 1 \in S \\ 0 & \text{if} \quad 1 \notin S, \end{cases} \tag{1}$$

where $\max_{j \notin N} v_j \equiv v_0$. Note that v is a characteristic function as a real valued function on 2^N satisfying $v(\emptyset) = 0$. We refer to this game as the bidding ring game.

The above characteristic function $v(S)$ denotes the net gains that S can obtain by itself. This function is based on the following. First, under the English auction rule, it is a dominant strategy for each bidder to remain active until bidding reaches his evaluation. Second, any coalition including buyer 1 can win the auction with the net gains $v_1 - \max_{j \notin S} v_j$ by making buyer 1 the sole bidder in the coalition. Finally, in any coalition that does not include buyer 1, the coalition does not win the auction, and hence the net gain is 0.

2.2 The core, the Shapley value, and the nucleolus

An n-dimensional vector x of the bidding ring game is a payoff vector if it satisfies $\sum_{i \in N} x_i = v(N)$. Then, the *core* of this game is a set of payoff vectors x satisfying the stability conditions $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$. The core embodies a pattern of coalitional stability of the bidding ring.

Next, the *Shapley value* $\phi(N,v)$ of this game is a payoff vector given by the following formula

$$\phi_i(N,v) = \sum_{S\subseteq N,\ i\notin S} \frac{|S|!(n-|S|-1)!}{n!}\left(v(S\cup i) - v(S)\right) \quad \text{for all } i\in N. \quad (2)$$

As is well known, the Shapley value describes a pattern of egalitarianism. The goal is to distribute the gains from trade equally.[4] For more accounts of the Shapley value, see Mas-Colell *et al.* (1995).

Finally, the nucleolus is defined as the imputation that minimizes the greatest dissatisfaction of any bidders' coalition in the following sense. A payoff vector x is called an imputation if x satisfies the individual rationality conditions $x_i \geq v(\{i\})$ for all $i \in N$. Let X be the set of all imputations of the bidding ring game. Obviously, $X \neq \emptyset$. Given an imputation $x \in X$, the *excess* of a coalition S with respect to x is defined as the number $v(S) - \sum_{i\in S} x_i$. The excess of a coalition S with respect to x is a measure of the dissatisfaction that the coalition S feels when x is proposed. Let $e(x)$ be the (2^n-2)-dimensional vector, the components of which are the excesses of every nonempty coalition $S \neq N$ with respect to x, arranged in decreasing order. The *nucleolus* $\mu(N,v)$ is defined as a set of imputations such that the vector $e(x)$ is *lexicographically minimal* over X. The nucleolus $\mu(N,v)$ is never empty and is a singleton (Schmeidler, 1969).

Remark 1 *Since the bidding ring game is convex,*[5] *(i) the core of this game is nonempty and (ii) the Shapley value and the nucleolus belong to the core.*

3 Sequences of knockouts

3.1 The model

Next, the model of sequences of knockouts is described using the notations in Section 2. At an English auction, bidding rings distribute collusive gains among ring members via sequences of re-auctions called *knockouts*.

Knockouts elicit buyers' evaluations for the object in order to distribute collusive gains among the buyers. Assume *weak asymmetry of information*

[4] The Shapley value $\phi(N,v)$ is defined as the unique value that satisfies the balanced contributions property: $\phi_i(N,v) - \phi_i(N\backslash\{j\},v) = \phi_j(N,v) - \phi_j(N\backslash\{i\},v)$ whenever $i \in N$ and $j \in N$. The balanced contributions property states that what player i gets out of the presence of player j is the same as what player j gets out of the presence of player i. See Osborne and Rubinstein (1994).

[5] See Graham *et al.* (1990)

among n buyers. Then, each buyer has information on his evaluation and the order of the evaluations of all buyers for the object, namely $1 \succ 2 \succ \cdots \succ n$, before the main auction. A relation $i \succ j$ for $i, j \in N$ and $i \neq j$ means that the evaluation of buyer i is higher than that of buyer j. The auctioneers of the main auction and knockouts have no information concerning buyers' evaluations. The reservation price v_0 is publicly announced to n buyers.

There exists the possibility of a bidding ring, which consists of n buyers, in the main auction. This is because, under the English auction rule, the gains of collusive behavior strictly dominate those of non-cooperative behavior.[6] Assume that n buyers form an initial ring $R_0 = N$. In the main auction, (i) buyer 1 remains active in the bidding up to his valuation v_1, and (ii) each of the buyers except for buyer 1 remains active in the bidding up to v_0 or does not participate. The goal of the operation of R_0 is to eliminate competition among the members of R_0. This competitive elimination implies that R_0 can guarantee the maximal gains of R_0. The bidders other than buyer 1 are phantom bidders in this auction. Buyer 1 wins the main auction and pays v_0 to the seller. Since buyer 1 is the representative of R_0, R_0 gains ownership of the object won at the auction.

After the main auction, a sequence of knockouts is produced. The sequence of knockouts has two key points. First, the size of a ring at each knockout shrinks consecutively. Second, at each knockout a *recursive* process on distribution is used.

We assume a ring center.[7] The ring center has two roles, namely, as an auctioneer of each knockout and as a neutral distributor to the members of R_0. After the final knockout, the ring center has neither gain nor loss.

The sequence of knockouts is described in detail as follows.

(A) *A knockout:* Given the $j-1_{th}$ and j_{th} rings, $R_{j-1} = \{1, 2, \cdots, m_{j-1}\} \supsetneq R_j = \{1, 2, \cdots, m_j\}$, where $m_j < m_{j-1}$ and $m_0 \equiv n$, the j_{th} knockout is a tuple

$$\left\langle K_j, (v_\alpha)_{\alpha \in K_j}, E \right\rangle. \tag{3}$$

Note that $K_j = \{1\} \cup \tilde{K}_j \cup (R_{j-1} \backslash R_j)$, where $\tilde{K}_j \subseteq R_j \backslash \{1\}$. K_j is the set of participants of the j_{th} knockout, $(v_\alpha)_{\alpha \in K_j}$ are the private values of the participants, and E is the English auction rule in the knockout. Here, let k_j be the number of the participants of the j_{th} knockout, namely $k_j = |K_j|$.

[6] For example, see the introduction in Graham *et al* (1990).

[7] A ring center is considered in the literature concerning non-cooperative analysis of knockouts as well, e.g., Graham and Marshall (1987).

In the j_{th} knockout, (a) the members of R_j appoint buyer 1 to remain active in the bidding up to his valuation v_1, and (b) they appoint each of the buyers other than buyer 1 to remain active in the bidding up to v_{m_j+1} or not to participate. The valuation v_{m_j+1} is observable to the members of R_j at the the j_{th} knockout. The goal of the operation of R_j is the same as that of the operation of R_0. Each member of $R_{j-1} \backslash R_j$ bids competitively at the j_{th} knockout. Thus, $K_j = \{1\} \cup \tilde{K}_j \cup (R_{j-1} \backslash R_j)$, where $\tilde{K}_j \subseteq R_j \backslash \{1\}$. Note that $m_{j-1} - m_j + 1 \leq k_j \leq m_{j-1}$. The lower bound of k_j means that the members of R_{j-1} other than $R_j \backslash \{1\}$ participate in the j_{th} knockout. The upper bound of k_j means that the members of R_{j-1} participate in the j_{th} knockout.

It is essential to form a ring by consecutive buyers at each knockout. For example, $N = \{1, 2, 3, 4\}$ and $R_1 = \{1, 2\}$. Buyer 1 wins the first knockout and pays v_3 to the ring center. Even if buyer 4 joins R_1, buyer 1 wins the knockout and pays v_3 to the ring center. In this case, buyer 4 does not have a role of reducing the payments of buyer 1. If R_1 intends to reduce the payments of buyer 1, R_1 will be reformed as $R_1 = \{1, 2, 3\}$. Thus, R_1 consists of consecutive buyers. Similarly, each ring consists of consecutive buyers at the other knockouts. Using weak asymmetry of information, ring members at the j_{th} knockout can form $R_j = \{1, 2, \cdots, m_j\}$. Note that buyer m_j is the ring member whose private value is the lowest in R_j.

(B) *A sequence of knockouts:* A sequence of knockouts is given by

$$\left(\left\langle K_j, (v_\alpha)_{\alpha \in K_j}, E \right\rangle\right)_{j=1}^{t}, \text{ given } (R_j)_{j=0}^{t} \text{ such that } R_t = \{1\}. \tag{4}$$

It is possible for the knockouts to continue in the manner stated in (A) for $j = 1, 2, \cdots, t$ such that $R_t = \{1\}$. In the final knockout, buyer 1 is determined as the final owner of the object. In addition, the distributive outcome of the members of R_0 is determined.

(C) *Distributive process:* If an arbitrary sequence of knockouts is fixed, the distributive outcome of the sequence of knockouts is defined as follows:

Definition 1 *Let $(z_i)_{i \in N}$ be a distributive outcome of the sequence of knockouts. Let $R_j = \{1, 2, \cdots, m_j\}$, where $m_j < m_{j-1}$, and let the number of participants of the j_{th} knockout be given by k_j, where $m_{j-1} - m_j + 1 \leq k_j \leq m_{j-1}$. Given $N = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_t = \{1\}$ and $(k_j)_{j=1}^{t}$, where $k_t = m_{t-1}$, z_i is defined inductively by $z_1 = v_1 - v_2 + z_t$ and*

$$z_i = \frac{v_{m_j+1} - v_0 - \sum_{l=m_{j-1}+1}^{n} z_l}{k_j} \quad \textit{for} \quad m_j + 1 \leq i \leq m_{j-1}, \tag{5}$$

beginning with $m_0 \equiv n$, $\sum_{l=m_0+1}^{n} z_l \equiv 0$ *and continuing for* $j = 1, \cdots, t$.

The underlying scenario of Definition 1 is as follows. Consider an arbitrarily fixed j_{th} knockout. Buyer 1 wins the j_{th} knockout and pays v_{m_j+1} to the ring center. Since buyer 1 is the representative of R_j, R_j gains ownership of the object. As a result of the j_{th} knockout, the ring center has the surplus $v_{m_j+1} - v_0 - \sum_{l=m_{j-1}+1}^{n} z_l$. The ring center divides this surplus among the participants of the j_{th} knockout equally and then disburses this equal distribution to the members of $R_{j-1} \backslash R_j$. Each member of $R_{j-1} \backslash R_j$ receives the above equal distribution z_i for $m_j + 1 \leq i \leq m_{j-1}$ and leaves the ring. It is possible for the knockouts to continue the above distributive behavior for $j = 1, 2, \cdots, t$ such that $R_t = \{1\}$.

The following remark illustrates money transfer among the ring members and the ring center.

Remark 2 *Let* $v_1 = \$120$, $v_2 = \$90$, $v_3 = \$70$, *and* $v_0 = \$10$. *Consider a sequence of knockouts:* $\left\langle K_1 = R_0, (v_\alpha)_{\alpha \in K_1}, E \right\rangle, \left\langle K_2 = R_1, (v_\alpha)_{\alpha \in K_2}, E \right\rangle$ *given* $R_0 = \{1, 2, 3\}, R_1 = \{1, 2\}$, *and* $R_2 = \{1\}$.

Step 1: Before the first knockout, R_0 *exchanges a single object for* $\$10$ *via the ring center, and the ring center has a debt of* $\$10$. *Here,* R_0 *gives* $\$10$ *to buyer 1 in compensation for buyer 1's payment in the main auction. In the first knockout, buyer 1 wins and pays* $\$70$ *to the ring center. The ring center receives the surplus* $\$60$ *because* $\$70 - \$10$. *The ring center pays* $\$60/3$ $(= \$20)$ *only to buyer 3. The ring center then has* $\$40$.

Step 2: Before the second knockout, R_1 *exchanges the object for* $\$70$ *via the ring center. The ring center has a debt of* $\$30$ $(= \$70 - \$40)$. R_1 *gives* $\$70$ *to buyer 1 in compensation for buyer 1's payment in the first knockout. In the second knockout, buyer 1 wins and pays* $\$90$ *to the ring center. Thus, buyer 1 gains final ownership of the object and obtains a net gain of* $\$120 - \$90 = \$30$. *The ring center obtains the surplus* $\$60$ $(= \$90 - \$30)$. *The ring center pays* $\$60/2$ $(= \$30)$ *only to buyer 2. Finally, the ring center distributes the remaining* $\$30$ *to buyer 1. Thus, the distributive outcome of this sequence is* $(\$60, \$30, \$20)$.

3.2 Basic properties of sequential knockouts

Next, the basic properties of the sequences of knockouts are stated. Each sequence of knockouts yields an element of the core. In addition, a particular sequence yields the Shapley value.

Proposition 1 *Each distributive outcome of the sequential knockouts belongs to the core.*

Proof. Let $z \in \mathbb{R}^n$ be the distributive outcome of the sequential knockouts. By Definition 1, z is a payoff vector. If $1 \notin S$, then $\sum_{i\in S} z_i > 0 = v(S)$. If $1 \in S$, it is sufficient to show $\sum_{i\in S} z_i \geq v(S)$ such that $S = R_j (= \{1, \cdots, m_j\})$, because $z_i > 0$.

By Definition 1, we have

$$\begin{aligned}\sum_{i\in S} z_i - v(S) &= v_1 - v_0 - \sum_{i\in N\setminus R_j} z_i - \left(v_1 - v_{m_j+1}\right) \\ &= \left(1 - \frac{m_{j-1} - m_j}{k_j}\right)\left(v_{m_j+1} - v_0 - \sum_{i\in N\setminus R_{j-1}} z_i\right) \\ &> 0, \end{aligned} \tag{6}$$

which completes the proof. ■

Proposition 2 *Let the ϕ-sequence of knockouts be given by $\left(\left\langle K_j, (v_\alpha)_{\alpha\in K_j}, E\right\rangle\right)_{j=1}^{n-1}$ satisfying $R_j = \{1, 2, \cdots, n-j\}$ and $K_j = R_{j-1}$ consecutively for $j = 1, 2, \cdots, n-1$. Then, the distributive outcome of the ϕ-sequence of knockouts is the Shapley value.*

Proof. By calculation, we find that the outcome of the ϕ-sequence is the same as Theorem 2 in Graham *et al.* (1990). ■

4 A sequence of knockouts yielding the nucleolus

In the above propositions, the distributive outcomes of the sequential knockouts under weak asymmetric information are characterized by cooperative game solutions under non-asymmetric information. In this section, the same statement can be made for the case of the nucleolus.

At the j_{th} knockout, the value of the distribution of the ring center is given by formula (5). Consider the situation in which the value of formula (5) is minimal with respect to $|R_j|$, given $K_j = \{1\} \cup (R_{j-1}\setminus R_j)$ consecutively for $j = 1, 2, \cdots, t$, such that $R_t = 1$. Note that at the j_{th} knockout the value of the formula (5) depends only on $|R_j|$, given $(R_l)_{l=0}^{j-1}$ and $K_j = \{1\} \cup (R_{j-1}\setminus R_j)$.

Definition 2 *The μ-sequence of knockouts is* $\left(\left\langle K_j, (v_\alpha)_{\alpha \in K_j}, E \right\rangle\right)_{j=1}^{t}$ *satisfying the condition that the value of the distribution at the j_{th} knockout is minimal with respect to $|R_j|$, given $(R_l)_{l=0}^{j-1}$ and $K_j = \{1\} \cup (R_{j-1} \backslash R_j)$ consecutively for $j = 1, 2, \cdots, t$, such that $R_t = 1$.*

The following proposition gives the distributive outcome of the μ-sequence. Each step of the algorithm in this proposition describes the distributive process at each knockout of the μ-sequence.

Proposition 3 *Let $(x_i^*)_{i \in N}$ be a distributive outcome of the μ-sequence of knockouts. Let $M_i = v_i - v_0$ for each $i \in N$. Then, x_i^* is given by $x_i^* = \Lambda_t^*$ and $x_1^* = v_1 - v_2 + \Lambda_{t'}^*$ for $m_{t-1}^* - k_t^* + 2 \leq i \leq m_{t-1}^*$ and $1 \leq t \leq t'$ such that $m_{t'}^* = 1$, where Λ_t^*, k_t^* and m_t^* are defined inductively by*

$$\Lambda_t^* = \min_{k=2,\cdots,m_{t-1}^*} \left\{ \frac{M_{m_{t-1}^*-k+2} - \sum_{i=0}^{t-1}(k_i^* - 1)\Lambda_i^*}{k} \right\}, \tag{7}$$

where k_t^* denotes the largest value of k for which the above expression attains its minimum, and $m_t^* = m_{t-1}^* - k_t^* + 1$ (beginning with $m_0^* \equiv n$, $\Lambda_0^* \equiv 0$ and continuing for $t = 1, \cdots, t'$, such that $m_{t'}^* = 1$).

Proof. The proof will be completed by the following two steps.

Step 1 : At the first knockout, the value of the distribution is given by

$$\Lambda_1 = \min_{k=2,\cdots,n} \left\{ \frac{M_{n-k+2}}{k} \right\}. \tag{8}$$

Each member of $R_0 \backslash R_1$ receives Λ_1 as his distributive outcome. Take k_1 as a k giving Λ_1. Let $R_1 = \{1, \cdots, m_1\}$. Then, $m_1 = n - k_1 + 1$.

At the second knockout, the value of the distribution is given by

$$\Lambda_2 = \min_{k=2,\cdots,m_1} \left\{ \frac{M_{m_1-k+2} - (k_1 - 1)\Lambda_1}{k} \right\}. \tag{9}$$

Each member of $R_1 \backslash R_2$ receives Λ_2 as his distributive outcome. Take k_2 as a k giving Λ_2. Let $R_2 = \{1, \cdots, m_2\}$. Then, $m_2 = m_1 - k_2 + 1$. Similarly, the same procedure can be repeated until the distributive outcome of each member of R_0 is determined.

Step 2 : Next, it is shown that, at the i_{th} knockout, we can take $k_i = k_i^*$, such that k_i^* is the largest k giving Λ_i^* consecutively for $i = 1, 2, \cdots, t$ such that $R_t = 1$.[8]

Let k_p and k_q ($k_q < k_p$) be two distinct ks giving Λ_1^*, namely, $\Lambda_1^* = \frac{M_{n-k_p+2}}{k_p} = \frac{M_{n-k_q+2}}{k_q}$. Take k_q as a k giving Λ_1^* in the first knockout. Let $s_1 = \Lambda_1^*(k_q - 1)$, $m_1 = n - k_q + 1$, and $\Lambda_2^* = \min_{k=2,\cdots,m_1} \left\{ \frac{M_{m_1-k+2}-s_1}{k} \right\}$. In the following, it will be proven that (i) $\frac{M_{n-k_p+2}}{k_p} = \frac{M_{m_1-k'+2}-s_1}{k'}$, where $k' = k_p - k_q + 1$, and (ii) for any $k = 2, \cdots, m_1$, $\frac{M_{n-k''+2}}{k''} \leq \frac{M_{m_1-k+2}-s_1}{k}$, where $k'' = k + k_q - 1$. It is sufficient to show (i) and (ii) for the purpose of Step 2 because the same procedure can be repeated until the t'_{th} knockout.

Proof of (i):

$$\begin{aligned}
\frac{M_{m_1-k'+2} - s_1}{k'} &= \frac{M_{n-k_q+1-(k_p-k_q+1)+2} - s_1}{k_p - k_q + 1} \\
&= \frac{M_{n-k_p+2} - \frac{M_{n-k_p+2}}{k_p}(k_q - 1)}{k_p - k_q + 1} \\
&= \frac{M_{n-k_p+2}\left(1 - \frac{k_q-1}{k_p}\right)}{k_p - k_q + 1} = \frac{k_p - k_q + 1}{k_p} \cdot \frac{M_{n-k_p+2}}{k_p - k_q + 1} \\
&= \frac{M_{n-k_p+2}}{k_p},
\end{aligned} \tag{10}$$

which completes the proof of (i).

Proof of (ii): For any $k = 2, \cdots, m_1$, it will be proven that $k''(M_{m_1-k+2} - s_1) - k \cdot M_{n-k''+2} \geq 0$. By calculation, we have

$$\begin{aligned}
& k''(M_{m_1-k+2} - s_1) - k \cdot M_{n-k''+2} \\
= \;& (k + k_q - 1)\left(M_{n-k_q+1-k+2} - \Lambda_1^*(k_q - 1)\right) - k \cdot M_{n-(k+k_q-1)+2} \\
= \;& (k_q - 1)M_{n-k_q-k+3} - (k + k_q - 1)\Lambda_1^*(k_q - 1) \\
= \;& (k_q - 1)(k + k_q - 1)\left(\frac{M_{n-k_q-k+3}}{k + k_q - 1} - \Lambda_1^*\right) \\
= \;& (k_q - 1)(k + k_q - 1)\left(\frac{M_{n-k''+2}}{k''} - \Lambda_1^*\right) \\
\geq \;& 0,
\end{aligned} \tag{11}$$

[8]This implies that the distributive outcome of the μ-sequence is independent of the number of knockouts.

because $\frac{M_{n-k''+2}}{k''} \geq \Lambda_1^*$ from the definition of Λ^*. This completes the proof of (ii). ■

Example 1 ***A seven-buyers case***

$$v_1 = 30, v_2 = 25, v_3 = 21, v_4 = 15, v_5 = 14, v_6 = 11, v_7 = 10, v_0 = 5.$$

The first knockout:

$$\Lambda_1^* = \min\left\{\frac{M_7}{2}, \frac{M_6}{3}, \cdots, \frac{M_2}{7}\right\} = \min\left\{\frac{5}{2}, \frac{6}{3}, \frac{9}{4}, \frac{10}{5}, \frac{16}{6}, \frac{20}{7}\right\} = 2$$
$$k_1^* = 5, \quad x_i^* = \Lambda_1^* = 2 \quad (i = 4, 5, 6, 7)$$
$$m_1^* = n - k_1^* + 1 = 3$$

The second knockout:

$$\Lambda_2^* = \min\left\{\frac{M_3 - (k_1^* - 1)\Lambda_1^*}{2}, \frac{M_2 - (k_1^* - 1)\Lambda_1^*}{3}\right\} = \min\left\{\frac{8}{2}, \frac{12}{3}\right\} = 4$$
$$k_2^* = 3, \quad x_i^* = \Lambda_2^* = 4 \quad (i = 2, 3)$$
$$m_2^* = m_1^* - k_2^* + 1 = 1 = m_{t'}^*$$
$$x_1^* = v_1 - v_2 + \Lambda_{t'}^* = v_1 - v_2 + \Lambda_2^* = 9$$

The distributive outcome of the μ-sequence is given by $x^* = (9, 4, 4, 2, 2, 2, 2)$.

The following proposition is the main result in this section.

Proposition 4 *The distributive outcome of the μ-sequence of knockouts is the nucleolus.*

Proof. Let $(x_i^*)_{i\in N}$ be a distributive outcome of the μ-sequence of knockouts. Let $M_i = v_i - v_0$ for each $i \in N$. The outcome of the μ-sequence is given by the following alternative formula:

$$\begin{array}{lclllll}
x_n^* & = & \min\{\frac{M_n}{2}, & \frac{M_{n-1}}{3}, & \frac{M_{n-2}}{4}, & \cdots, & \frac{M_2}{n} \quad \} \\
x_{n-1}^* & = & \min\{ & \frac{M_{n-1}-x_n^*}{2}, & \frac{M_{n-2}-x_n^*}{3}, & \cdots, & \frac{M_2-x_n^*}{n-1} \quad \} \\
x_{n-2}^* & = & \min\{ & & \frac{M_{n-2}-x_n^*-x_{n-1}^*}{2}, & \cdots, & \frac{M_2-x_n^*-x_{n-1}^*}{n-2} \quad \} \\
\vdots & = & \min\{ & & & \ddots & \vdots \quad \} \\
x_2^* & = & \min\{ & & & & \frac{M_2-x_n^*-x_{n-1}^*-\cdots-x_3^*}{2} \quad \} \\
x_1^* & = & M_1- & \sum_{i=2}^n x_i^* & & &
\end{array} \tag{12}$$

This alternative formula follows from Proposition 3.

Let (N, c) be a simple version of a cost sharing game referred to as the airport game, as proposed by Littlechild (1974), with one aircraft of each type, namely $c(S) = \max_{i \in S} C_i$ for each $S \subseteq N$ satisfying $c(\emptyset) = 0$. Note that C_i is the construction cost of the runway for the aircraft of type i. Let $C_i = M_i$ for each $i \in N$, so that $C_1 > C_2 > \cdots > C_n$. Then, the alternative formula of the μ-sequence can be regarded as Sönmez's formula of the nucleolus of the airport game (N, c) (Sönmez, 1994). Owing to the anti-dual relation introduced by Oishi and Nakayama (2009), the nucleolus of the bidding ring game (N, v) coincides with that of the airport game (N, c). Therefore, the alternative formula of the μ-sequence gives the nucleolus of the bidding ring game. Thus, the proof is completed. ■

References

[1] Cassady, R.Jr., 1967. *Auctions and Auctioneering*. University of California Press, Berkeley.

[2] Deltas, G., 2002. Determining damages from the operation of bidding rings: An analysis of the post-auction knockout sale. *Economic Theory* 19, 243-269.

[3] Graham, D.A. and R.C. Marshall, 1987. Collusive bidder behavior at single-object second-price and English auctions. *Journal of Political Economy* 95, 1217-1239.

[4] Graham, D.A., R.C. Marshall and J.-F. Richard, 1990. Differential payments within a bidder coalition and the Shapley value. *The American Economic Review* 80, 493-510.

[5] Littlechild, S.C., 1974. A simple expression for the nucleolus in a special case. *International Journal of Game Theory* 3, 21-29.

[6] Mailath, G. and P. Zemsky, 1991. Collusion in second price auctions with heterogeneous bidders. *Games and Economic Behavior* 4, 467-486.

[7] Mas-Colell, A., M.D. Whinston and J.R. Green, 1995. *Microeconomic Theory*, Oxford University Press, New York and Oxford.

[8] McAfee, R.P. and J. McMillan, 1992. Bidding rings. *The American Economic Review* 82, 579-599.

[9] McMillan, J., 1991. Dango: Japan's price-fixing conspiracies. *Economics and Politics* 3, 201-218.

[10] Milgrom, P., 2004. *Putting Auction Theory to Work*. Cambridge University Press, Cambridge.

[11] Oishi, T., 2006. The nucleolus in a game with collectively contributing coalitional leaders. DP2005-040, *KUMQRP Discussion Paper Series*.

[12] Oishi, T. and M. Nakayama, 2009. The anti-dual of economic coalitional TU games. *The Japanese Economic Review* 60, 560-566.

[13] Osborne, M. and A. Rubinstein, 1994. *A Course in Game Theory*. MIT Press, Cambridge, Massachusetts.

[14] Schmeidler, D., 1969. The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics* 17, 1163-1170.

[15] Sönmez, T., 1994. Population-monotonicity on a class of public good problems. University of Rochester, mimeo.

[16] van den Brink R., G. van der Laan and V. Vasil'ev, 2007. Component efficient solutions in line-graph games with applications. *Economic Theory* 33, 349-364.