Riskless versus Risky Bargaining Procedures: the Aumann-Roth Controversy Revisited

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Abstract

In a series of papers on the Shapley NTU value, Aumann and Roth discussed a simple example in which players can cooperate in pairs and a pair of players prefers to form a coalition with each other rather than with the third player. Roth argued that the only rational outcome of the game is that the players who prefer each other form a coalition; Aumann argued that all three coalitions are possible because the players have a problem of expectation coordination. We make a noncooperative analysis of the example and show that the difference between Aumann's and Roth's arguments can be traced back to a difference in the bargaining procedure. In the unified framework of the extensions of Rubinstein's alternating-offers procedure, there is a safe procedure that supports Roth's arguments and a risky procedure that supports Aumann's arguments. Neither bargaining procedure supports the NTU value in another example proposed by Shafer.

1 The Aumann-Roth Controversy

In a series of papers that appeared in Econometrica during the 80s, Aumann and Roth discussed the practical applicability of the Shapley (1969) NTU value or λ-transfer value. The controversy was centered around the following simple example:

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There are three players, $N = \{1, 2, 3\}$. If 1 and 2 form a coalition each of them gets $\frac{1}{2}$, whereas if any of them forms a coalition with player 3 the division is $(p, 1-p)$ with $0 \leq p < \frac{1}{2}$. By choosing a pair of players at random, the grand coalition can achieve any convex combination of the payoff vectors $(\frac{1}{2}, \frac{1}{2}, 0)$, $(p, 0, 1-p)$ and $(0, p, 1-p)$.

Roth's (1980) point was that the only outcome consistent with rationality in this situation is coalition $\{1, 2\}$, associated with the payoff vector $(\frac{1}{2}, \frac{1}{2}, 0)$:

> This is because, when $p < \frac{1}{2}$, the outcome $(\frac{1}{2}, \frac{1}{2}, 0)$ is *strictly* preferred by *both* players 1 and 2 to every other feasible outcome, and because the rules of the game permit players 1 and 2 to achieve this outcome without the cooperation of player 3. So (...) there is really no conflict between players 1 and 2.

Solution concepts like the core and the von Neumann and Morgenstern solution, based on the concept of domination, make the same prediction as Roth.[1] In contrast, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the unique NTU value for $p > 0$ (for $p = 0$, $(\frac{1}{2}, \frac{1}{2}, 0)$ is also an NTU value).

Roth's example can be interpreted in the context of government formation (see Aumann (1986)). Suppose there are three parties in a Parliament and the distribution of seats is proportional to $(\frac{p}{1+p}, \frac{p}{1+p}, \frac{1-p}{1+p})$. Parties 1 and 2 are small parties ($p < \frac{1}{2}$) but any two parties have a majority in Parliament, that is, $2p > \frac{1+p}{2}$, or $p > \frac{1}{3}$. Suppose that, if any two parties form a government, they have to split the payoff in a way proportional to the number of seats. Then this game is equivalent to Roth's example, with the additional restriction that $p > \frac{1}{3}$. The prediction of Roth would then be that the two smaller parties form the government, whereas the largest party (i.e., the party that has won the election, though only with simple majority) is excluded. This phenomenon is known as "strength is weakness" and was early recognized by Caplow (1956) and Gamson (1961a) and observed in experiments.[2]

[1] Notice that Roth's conclusion is based on the fact that both players agree on their most preferred outcome. This condition is much more demanding than the definition of either the core or a vNM solution, and in fact it is never satisfied in TU games.

[2] See e.g. Vinacke and Arkoff (1957), Gamson (1961b) and Murnighan (1978). Payoff division in those experiments was endogenous, so that there were really two issues at stake: the rule of payoff division (conjectured to be proportional to some endowments of the players) and coalition formation given this rule.

Aumann (1985) argues that $(\frac{1}{2}, \frac{1}{2}, 0)$ will not necessarily be the outcome, because players 1 and 2 may accept an offer from player 3 out of security considerations.

> Suppose the players and the rules have just been announced on television. The amount 1 to be shared may be fairly large, so the players are rather excited. Suddenly the phone rings in 1's home; 3 is on the line with an offer. At first 1 is tempted to dismiss it. But then he realizes that if he does so, and if 3 manages to get in touch with 2 before he (1) does, then he won't get anything at all out of the game, unless 2 also rejects 3's offer. "But wait a minute", 1 says now to himself; "2 will only reject 3's offer if he thinks that I will reject it. When he gets 3's phone call, he will go through the agonizing that I am going through now, and will realize that in this situation I would also agonize. (...) I'm beginning not to like this one bit".

Players 1 and 2 have to solve an expectations coordination problem: if 2 would refuse the offer from 3, it is optimal for 1 to refuse it too; if 2 would accept the offer from 3, it may be optimal for player 1 to accept, since there is a risk that he will get nothing otherwise.

Commenting on Roth's paper, Harsanyi (1980) proposes "to define the solutions for cooperative games by means of suitable bargaining models, having the nature of noncooperative games in extensive (or sometimes in normal) form"; this approach is in line with the Nash (1953) program. Aumann (1985) studied some noncooperative models and showed that coalition $\{1, 2\}$ is not the only outcome consistent with rationality. In the first of the noncooperative models a player i is picked at random to be the proposer. This player chooses another player j and makes him an offer. If j rejects, i makes an offer to k; but k does not know of the previous offer to j. If k also rejects, coalition $\{j, k\}$ forms. In the second noncooperative model, the three pairs of players are ordered at random and given the opportunity to agree; the first pair that does so forms a coalition, and if no pair agrees they all get zero. Players are only informed of proposals involving them.

There are two equilibria of the first model, differing on what happens when player 3 is selected to be the proposer: in one of them both 1 and 2 accept 3's offer and in the other both reject. The second model also admits two (perfect) equilibria if p is large enough: one in which $\{1, 2\}$ is the only pair that forms and other in which the first selected pair forms. The reason why p needs to be large enough is that, for a small p, player 1 or 2 would prefer not to agree with 3 and gamble on the possibility that the pair $\{1, 2\}$ will be selected next.

Notice that with perfect information only coalition $\{1, 2\}$ would form. More generally, in games with a finite horizon the difference between Aumann's and Roth's predictions can be traced to a difference between perfect information and imperfect information (see Aumann (1985), footnote 18). One may argue that cooperative game theory presumes that all players negotiate in public. Furthermore, infinite-horizon games seem more in line with cooperative game theory. We will consider an infinite-horizon noncooperative game with perfect information and show that the different predictions can also be traced to a difference between risky and safe bargaining procedures.

2 Riskless versus Risky Bargaining

We are going to approach Roth's example in the context of a noncooperative model a la Rubinstein (1982). We will consider two variants of the model: the riskless variant and the risky variant.

The riskless variant uses an ordering of the players ρ; we will refer to ρ as the *rule of order*. The risky variant uses a probability vector $\theta \in \mathbb{R}^n$ with $\theta_i > 0$ for all i and $\sum_{i \in N} \theta_i = 1$; we will refer to θ as the *protocol*.

Let (N, V) be an NTU game. Given a rule of order or a protocol, bargaining proceeds as follows: a player is selected according to either the rule of order or the protocol. This player either forms a singleton or proposes a coalition S and a feasible payoff vector $x \in V(S)$. The remaining players in S accept or reject sequentially (in the riskless variant, the order of response follows the rule of order; in the risky variant, the order of response is irrelevant). If the proposal is accepted by all players in S, it is implemented and bargaining continues with the players in $N \backslash S$ (with the rule of order or the protocol adjusted in an appropriate way). If rejected, a new proposer is selected. This is the point where the essential difference between riskless and risky bargaining procedures lies: in the riskless variant, the first player to reject the proposal automatically becomes the next proposer; in the risky variant, the next proposer is randomly selected according to θ.[3] Players who continue bargaining for ever get zero payoffs.

Given a strategy combination, we will denote player i's *expected payoffs* at the beginning of the game by y_i, and his *continuation value* - that is, his expected payoff at a node where he receives a proposal and rejects it - by z_i. Which strategy combination and which node we are referring to will be clear from the context.

We now show how the two variants of the model differ in their predictions for Roth's example. In order to have as little friction as possible we do not introduce discounting in the model, but for simplicity we will assume that players break ties in favor of an early agreement.

Since players 1 and 2 are symmetric, we will assume $\theta_1 = \theta_2$. There are two obvious choices for the vector θ: the *egalitarian protocol*, with $\theta_i = \frac{1}{3}$ for $i = 1, 2, 3$, and the *proportional protocol*, with $\theta = (\frac{p}{1+p}, \frac{p}{1+p}, \frac{1-p}{1+p})$ (consistent with the interpretation of the players as three parties in parliament).

Claim 1 *The game with a rule of order has a unique subgame perfect equilibrium in which coalition* $\{1, 2\}$ *always forms.*

Proof. If the rule of order selects either player 1 or 2 to be the proposer, he will propose coalition $\{1, 2\}$ and this proposal will be accepted, since none of the two players can possibly get a higher payoff from the game and players solve ties in favor of immediate agreement. This implies that each player's continuation value equals $\frac{1}{2}$.

Suppose that the rule of order selects player 3 to be proposer. If he proposes a two-player coalition, (say, $\{1, 3\}$) this proposal is rejected because $p < \frac{1}{2}$. Player 1 can confidently reject the proposal and then propose to player 2. The only acceptable proposal player 3 can make is the grand coalition with $(\frac{1}{2}, \frac{1}{2}, 0)$, but this alternative implies in practice that coalition $\{1, 2\}$ forms. ■

Things are not so simple in the game with random proposers. Because the proposer is randomly selected after a proposal has been rejected, the continuation values of players 1 and 2 depend on the strategies of all three players. In equilibrium, the continuation values must be consistent with the

[3] The variant with a rule of order has been studied by Selten (1981), Chatterjee et al. (1993), and Moldovanu and Winter (1995). The variant with random proposers has been studied by Okada (1996).

strategies of the three players and the strategies must be optimal given the continuation values.

Claim 2 *For any protocol θ, the game with random proposers has a subgame perfect equilibrium in which coalition $\{1,2\}$ always forms.*

Proof. If all proposals from player 3 are rejected (except the proposal of the grand coalition with $(\frac{1}{2}, \frac{1}{2}, 0)$), players 1 and 2 have a continuation payoff of $\frac{1}{2}$, and each of them is justified in rejecting any proposals of player 3 other than $(\frac{1}{2}, \frac{1}{2}, 0)$. ■

Depending on the parameters, there may be more equilibria.

Claim 3 *If p is large enough, the game with random proposers has subgame perfect equilibria with expected payoffs other than $(\frac{1}{2}, \frac{1}{2}, 0)$.*

Proof. As in the game with a rule of order, if player $i \in \{1,2\}$ is selected to be the proposer he always proposes coalition $\{1,2\}$ and this proposal is accepted. If there are equilibria with expected payoffs different from $(\frac{1}{2}, \frac{1}{2}, 0)$, this means that *at some node* on the equilibrium path proposals by player 3 other than forming the grand coalition with payoff division $(\frac{1}{2}, \frac{1}{2}, 0)$ are accepted. We concentrate on equilibria in which player 3 can make acceptable proposals already at round 1. We will only consider proposals for a two-player coalition (it can be checked that player 3 cannot make an acceptable proposal to form the grand coalition with expected payoffs other than $(\frac{1}{2}, \frac{1}{2}, 0)$).

Player 3 can only make acceptable proposals if $y_2 \leq p$. If $y_1 > p \geq y_2$, then player 3 always proposes coalition $\{2,3\}$ and this proposal is accepted. But then $y_1 = (1-\theta_3)\frac{1}{2} < y_2 = (1-\theta_3)\frac{1}{2} + \theta_3 p$, a contradiction. Thus, it must be the case that $y_1 \leq p$.

Let π_i be the probability that player 3 proposes to player i ($\pi_1 + \pi_2 = 1$). Then

$$y_i = (1-\theta_3)\frac{1}{2} + \theta_3 \pi_i p; \;\; i = 1,2$$
$$y_3 = \theta_3(1-p).$$

In order for this strategy combination to be an equilibrium we need $y_i \leq p$, $i = 1,2$. For the most favourable case ($\pi_1 = \pi_2 = \frac{1}{2}$) we obtain the following condition

$$p \geq \frac{1-\theta_3}{2-\theta_3}.$$

The larger θ_3, the less demanding this condition. If we interpret the game as a government formation game (thus $p > \frac{1}{3}$) this condition is always satisfied by the proportional protocol. For the egalitarian protocol it is satisfied provided that $p \geq \frac{2}{5}$. ∎

The results of the game with random proposers vindicate Aumann's arguments. Do they also vindicate the NTU value? [4]

Claim 4 *There is an equilibrium with expected payoffs* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *if* $p \geq \frac{1}{3}$ *and* $\theta_3 = \frac{1}{3(1-p)}$.

There is a protocol with expected payoffs $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ but it does not have a natural interpretation (for $p \geq \frac{1}{3}$, $\frac{1}{3(1-p)} \geq \frac{1-p}{1+p}$; only for $p = \frac{1}{3}$ it would be the proportional protocol but then the political interpretation is not valid because it needs $p > \frac{1}{3}$).

3 Shafer's example

Shafer (1980) proposes another example where the NTU value makes an unintuitive prediction. Neither of the two noncooperative models we have considered supports the NTU value for this example.

Consider an exchange economy with two agents and three goods. Initial endowments are $w^1 = (1,0)$, $w^2 = (0,1)$, and $w^3 = (0,0)$. Utility functions are $u(y,z) = \min(y,z)$ for players 1 and 2, and $u(y,z) = \frac{y+z}{2}$ for player 3. The associated cooperative game has $V(i) = 0$ for all i, $V(1,2) = \{(x_1,x_2) : (x_1,x_2) \leq (1,1), x_1+x_2 \leq 1\}$, $V(i,3) = \{(x_i,x_3) : (x_i,x_3) \leq (0,\frac{1}{2})\}$, $i = 1,2$, $V(N) = \{(x_1,x_2,x_3) : (x_1,x_2,x_3) \leq (1,1,1),\ x_1+x_2+x_3 \leq 1\}$.

The NTU value assigns $\frac{1}{6}$ to player 3, even though he does not have any goods. None of the noncooperative models we have considered supports this outcome.

[4] The vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ remains an NTU value for $p < 0$. In this case, neither the random proposers game nor Aumann's bargaining procedures lead to positive expected payoffs for player 3. Aumann (1986) claims that the case of $p < 0$ is pathological, and should be interpreted as $V(1,3) = V(2,3) = (0,0)$.

Claim 5 *For the procedure with a rule of order, any division of the value of the grand coalition can be supported by a subgame perfect equilibrium.*

Proof. There is a continuum of stationary equilibria with $z_1 + z_2 = 1$, $z_i \geq 0$ for all i (see Selten (1981)). This fact can be used to support any division of the value of the grand coalition by trigger strategies. Given a division of the value of the grand coalition, $(z_i)_{i \in N}$ with $z_i \geq 0$ and $\sum_{i \in N} z_i = 1$, consider the following strategies: all players propose the grand coalition with payoff division $(z_i)_{i \in N}$ and accept any proposal to form the grand coalition that gives them at least z_i. Player 3 accepts any proposal that gives him at least z_3. As for players 1 and 2, they do not accept any offer to form coalitions other than the grand coalition unless they get a payoff of 1; the reason is that after a rejection players play the stationary equilibrium in which the player who rejected the offer gets 1. ■

Thus, the NTU value can be supported by a subgame perfect equilibrium; however, the point $(\frac{5}{12}, \frac{5}{12}, \frac{1}{6})$ plays no special role and claim 5 holds even if $V(1,3) = V(2,3) = 0$, in which case the NTU value is $(\frac{1}{2}, \frac{1}{2}, 0)$.

Claim 6 *In any stationary subgame perfect equilibrium* $z_1 + z_2 \geq 1$*, and thus* z_3 *is either* 0 *or* $\frac{1}{2}$.

Proof. Since 1 can propose to 2, $z_1 \geq 1 - z_2$, thus, $z_1 + z_2 \geq 1$. If both z_1 and z_2 are strictly positive, $z_3 = 0$; only if one of them is 0 $z_3 = \frac{1}{2}$. ■

Claim 7 *If players discount future payoffs and* δ *is close to* 1*, expected payoffs in any stationary subgame perfect equilibrium are close to* $(\frac{1}{2}, \frac{1}{2}, 0)$.

Proof. If players discount future payoffs player 3 will have a positive continuation value and will never receive a proposal. Thus, $z_1 = z_2 = \frac{\delta}{1+\delta}$; when δ is close to 1, z_3 (and y_3) is close to 0. ■

As for the procedure with random proposers, there are similar results provided that θ_1, $\theta_2 > 0$. Any division of the payoff of the grand coalition is supported by a subgame perfect equilibrium; restriction to stationary equilibria implies that expected payoffs satisfy $y_i > 0$ for $i = 1, 2$, $y_2 + y_3 = 1$, $y_3 = 0$. Thus, in the game with random proposers, stationarity alone is enough to exclude player 3 having a positive payoff. If moreover $\theta_1 = \theta_2$ and players discount factors are close to 1, expected payoffs are close to $(\frac{1}{2}, \frac{1}{2}, 0)$.

Of course, it is not to be expected from this sort of bargaining model to support the Shapley value in general, since all coalitions appear in the Shapley value and only some coalitions (the ones that form with positive probability) play a role in the noncooperative game. However, qualitative support would have been found if the player with no endowment had received a positive payoff.

4 Conclusion

We have shown that the different predictions made by Aumann and Roth in Roth's example can be interpreted as coming from different bargaining procedures. The safe bargaining procedure supports Roth's arguments, whereas the risky bargaining procedure supports Aumann's arguments. The safe bargaining procedure assumes away the problem of expectation coordination. We believe that this problem is important in practice. The random proposer model captures an important aspect of bargaining in real situations, namely that if players reject proposals from others they cannot be sure to be the first to make a counterproposal, and thus face a risk of being excluded from a coalition. The noncooperative approach is often criticized because the equilibrium is too sensitive to the details of the bargaining procedures. We believe that this difference between the two bargaining procedures is fundamental and should not be regarded as an unimportant procedural detail. As for the Shafer example, neither of the two procedures supports the NTU value.

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