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Agreeable Bets with Multiple Priors

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Abstract

This paper considers a two agent model of trade with multiple priors, and characterizes the existence of an agreeable bet on some event in terms of the set of priors. In the model, the existence of an agreeable bet on some event is a strictly stronger condition than the existence of an agreeable trade, whereas the two conditions are equivalent in the standard Bayesian framework. The main result shows that the two conditions are equivalent when the set of priors is the core of a convex capacity.
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Key Words: multiple priors; convex capacity; agreeing and disagreeing; Choquet integral.

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1 Introduction and Summary

Imagine two agents, each of whom has a prior distribution over a finite state space Ω. If the two agents do not share priors, there exists an agreeable *trade* between them: by a trade we mean a function $f : \Omega \to \mathbb{R}$, and it is said to be agreeable if the expected value of f for agent 1 is positive and that of $-f$ for agent 2 is positive as well. The converse is also true: if there exists an agreeable trade, the agents do not share priors. A trade f is called a *bet on event* $E \subseteq \Omega$ if f is constant over E and $\Omega \backslash E$; if f is larger on E than on $\Omega \backslash E$, it can be interpreted that agent 1 wins when E occurs, and agent 2 wins when E does not occur. It is straightforward to show that if there is no agreeable bet on any event, the priors of the two agents must coincide. So disagreement of priors, the existence of an agreeable trade, and the existence of an agreeable bet are all equivalent conditions in this framework.

Now suppose that two agents have multiple priors over the state space, and they use the maximin rule a la Gilboa and Schmeidler (1989) to evaluate a trade. In this context, a trade is agreeable if the *minimum* expected value of f, where the minimum is taken over the set of priors, is positive for agent 1, and that of $-f$ is positive for agent 2 as well. Billot *et al.* (2000) has shown that there exists an agreeable trade if and only if there exists a prior which belongs to the set of priors for each agent, i.e., the agents do not share any priors.

In this paper we show that when the set of priors of each agent is given as the core of a convex (supermodular) capacity, there exists an agreeable bet on some event if and only if there exists a prior which belongs to the set of priors for each agent (Proposition 4 in Section 3). So for the case of convex capacity, disagreement of priors, the existence of an agreeable trade, and the existence of an agreeable bet are equivalent, which is the same for the case of single prior. To show the above result, we give a necessary and sufficient condition for the existence of an agreeable bet for a more general class of multiple priors model, where the set of priors is not necessarily the core of a convex capacity (Lemma 2 in Section 2).

We also show, by an example, that the equivalence of the three conditions fails in general for multiple priors models. Therefore, the existence of an agreeable bet is a strictly stronger condition than the existence of an agreeable trade for general multiple priors models. Unlike in the standard Bayesian framework of single prior, these two concepts must be distinguished in the multiple priors framework, unless a convex capacity is assumed.

2 Characterization of Agreeable Bets on Events

Let $P_1, P_2 \subseteq \Delta(\Omega)$ be non empty closed sets,[1] which will be referred to as the sets of priors for agent 1 and agent 2, respectively. We call a function $f : \Omega \to \mathbb{R}$ a *trade*. A *bet on event* $E \subseteq \Omega$ is a function f which is constant on both E and $\Omega \backslash E$, so a bet is a special type of trade. Agents evaluate a trade f by the minimum of the expected gain, as axiomatized by Gilboa and Schmeidler (1989). We interpret that $f(\omega)$ is a transfer agent 1 receives from agent 2 when the state is ω. Thus, we say that a trade f is an *agreeable trade* if $\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) f(\omega) > 0$ and $\min_{p \in P_2} \sum_{\omega \in \Omega} p(\omega)(-f(\omega)) > 0$. We say that there is an *agreeable bet on event* E if there is an agreeable trade which is a bet on E.

Let us begin with characterizing the existence of an agreeable trade:[2]

Lemma 1 *Suppose that P_1 and P_2 are closed and convex. Then, there exists an agreeable trade if and only if $P_1 \cap P_2 = \emptyset$.*

Proof. Since P_1 and P_2 are compact and convex, by the separation theorem, $P_1 \cap P_2 = \emptyset$ if and only if there exists $f : \Omega \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) f(\omega) > c > \max_{p \in P_2} \sum_{\omega \in \Omega} p(\omega) f(\omega)$. By letting $g = f - c$, it is clear that the latter statement is equivalent to the condition that there exists $g : \Omega \to \mathbb{R}$ such that $\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) g(\omega) > 0$ and $\min_{p \in P_2} \sum_{\omega \in \Omega} p(\omega)(-g(\omega)) > 0$. ∎

Let $\underline{P_i}(E) = \min_{p \in P_i} p(E)$ and $\overline{P_i}(E) = \max_{p \in P_i} p(E)$ be the minimum and maximum probabilities of $E \subseteq \Omega$ for agent $i = 1, 2$. The necessary and sufficient condition for the existence of an agreeable bet on $E \subset \Omega$ is the following.

Lemma 2 *There exists an agreeable bet on $E \subset \Omega$ if and only if*

$$\underline{P_1}(E) > \overline{P_2}(E) \text{ or } \underline{P_2}(E) > \overline{P_1}(E).$$

Proof. Suppose that f is an agreeable bet on E such that $f(\omega) = a$ if $\omega \in E$ and $f(\omega) = b$ otherwise where $a, b \in \mathbb{R}$. Note if $a = b$ then f cannot be an agreeable bet. Thus, either $a - b > 0$ or $a - b < 0$ is true. If $a - b > 0$, we have

$$\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) f(\omega) = \underline{P_1}(E)(a - b) + b > 0,$$

$$\min_{p \in P_2} \sum_{\omega \in \Omega} p(\omega)(-f(\omega)) = \overline{P_2}(E)(b - a) - b > 0,$$

[1] The set $\Delta(\Omega)$ denotes the collection of all probability distributions over Ω.

[2] The result is the special case of those of Billot *et al.* (2000), who considered multiple risk averse agents, and Kajii and Ui (2004), who considered multiple risk neutral agents with asymmetric information.

and thus

$$\underline{P_1}(E) > -b/(a-b) > \overline{P_2}(E).$$

Similarly, if $a-b<0$, we have $\underline{P_2}(E) > \overline{P_1}(E)$.

Conversely, suppose that $\underline{P_1}(E) > \overline{P_2}(E)$ or $\underline{P_2}(E) > \overline{P_1}(E)$. If $\underline{P_2}(E) > \overline{P_1}(E)$, let $f : \Omega \to \mathbb{R}$ be such that $f(\omega) = c-1$ if $\omega \in E$ and $f(\omega) = c$ otherwise where $\underline{P_2}(E) > c > \overline{P_1}(E)$. Then,

$$\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) f(\omega) = c - \overline{P_1}(E) > 0,$$
$$\min_{p \in P_2} \sum_{\omega \in \Omega} p(\omega)(-f(\omega)) = \underline{P_2}(E) - c > 0,$$

implying that f is an agreeable bet on E. If $\underline{P_1}(E) > \overline{P_2}(E)$, let $f : \Omega \to \mathbb{R}$ be such that $f(\omega) = 1-c$ if $\omega \in E$ and $f(\omega) = -c$ otherwise where $\underline{P_1}(E) > c > \overline{P_2}(E)$. A similar calculation shows that f is an agreeable bet. ∎

The conditions in the results above are not equivalent: we provide an example where there is no agreeable bet on any event whereas the agents do not share any priors, i.e., $P_1 \cap P_2 = \emptyset$. Let $\Omega = \{1,2,3\}$, $P_1 = \{p \in \Delta(\Omega) : (p(1), p(2), p(3)) = t(0.4, 0.4, 0.2) + (1-t)(0.2, 0.2, 0.6),\ 0 \le t \le 1\}$, and $P_2 = \{p \in \Delta(\Omega) : (p(1), p(2), p(3)) = (0.3, 0.25, 0.45)\}$. Since P_1 and P_2 are closed convex sets and $P_1 \cap P_2 = \emptyset$, there exists an agreeable trade by Lemma 1. However, for any event $E \subset \Omega$, $\underline{P_1}(E) < \underline{P_2}(E) = \overline{P_2}(E) < \overline{P_1}(E)$. Thus, by Lemma 2, there is no agreeable bet on any event.

3 Case of Convex Capacity

A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *capacity* if it is monotone and normalized: that is, $v(E) \ge v(F)$ if $E \supseteq F$ and $v(\Omega) = 1$. A capacity v is said to be *supermodular* (or *convex*) if $v(E) + v(F) \le v(E \cap F) + v(E \cup F)$ for $E, F \subseteq \Omega$, and it is said to be *submodular* if $-v$ is supermodular. For supermodular and submodular capacities, the following separation theorem due to Frank (1982) is known.[3]

[3]This is a natural extension of the separation theorem for convex functions, taking into account that the Choquet integral of $x \in \mathbb{R}^\Omega$ with respect to a submodular capacity is a convex function of $x \in \mathbb{R}^\Omega$. For a proof, see for instance Murota (2003).

Theorem 3 *Let* $\mu : 2^\Omega \to \mathbb{R}$ *be supermodular and* $\rho : 2^\Omega \to \mathbb{R}$ *be submodular. If* $\rho(E) \geq \mu(E)$ *for all* $E \subseteq \Omega$*, then there exists* $q : \Omega \to \mathbb{R}$ *such that*

$$\rho(E) \geq q(E) \geq \mu(E) \text{ for all } E \subseteq \Omega$$

where $q(E) = \sum_{\omega \in E} q(\omega)$.

The core of a capacity v is defined as:

$$\text{Core}(v) = \{q \in \Delta(\Omega) : q(E) \geq v(E) \text{ for all } E \subseteq \Omega\}$$

where $q(E) = \sum_{\omega \in E} q(\omega)$. The core is a closed convex set. It is known that if v is supermodular (convex), then $\text{Core}(v) \neq \emptyset$. The following is the main result of this paper.

Proposition 4 *Suppose that there exists a supermodular (convex) capacity* v_i *such that* $P_i = \text{Core}(v_i)$ *for* $i = 1, 2$*. Then, there exists an agreeable bet on some event if and only if* $P_1 \cap P_2 = \emptyset$.

The "only if" part is an immediate consequence of Lemma 1. The "if" part is an immediate consequence of Lemma 2 and the following lemma.

Lemma 5 *Suppose that there exists a supermodular (convex) capacity* v_i *such that* $P_i = \text{Core}(v_i)$ *for* $i = 1, 2$*. If* $P_1 \cap P_2 = \emptyset$*, then there exists* $E \subset \Omega$ *such that* $\underline{P_1}(E) > \overline{P_2}(E)$.

Proof. Define $v_2' : 2^\Omega \to \mathbb{R}$ by the rule $v_2'(E) = 1 - v_2(\Omega \backslash E)$ for $E \subseteq \Omega$. By construction, v_2' is submodular. We shall show that there exists $E \subset \Omega$ such that $v_2'(E) < v_1(E)$. Seeking a contradiction, suppose that $v_2'(E) \geq v_1(E)$ for all $E \subseteq \Omega$. By Theorem 3, there exists $q : \Omega \to \mathbb{R}$ such that

$$v_2'(E) \geq q(E) \geq v_1(E) \text{ for all } E \subseteq \Omega.$$

Since $v_2'(\Omega) = v_1(\Omega) = 1$ and $v_2'(\emptyset) = v_1(\emptyset) = 0$, we have $q \in \Delta(\Omega)$. Thus, $q \in \text{Core}(v_1)$. In addition,

$$q(E) = 1 - q(\Omega \backslash E) \geq 1 - v_2'(\Omega \backslash E) = v_2(E) \text{ for all } E \subseteq \Omega.$$

Thus, $q \in \text{Core}(v_2)$, which contradicts to the assumption $\text{Core}(v_1) \cap \text{Core}(v_2) = \emptyset$. Therefore, there exists $E \subset \Omega$ such that $v_2'(E) < v_1(E)$. Note that $v_1(E) = \min_{p \in \text{Core}(v_1)} p(E)$ and $v_2'(E) = 1 - v_2(\Omega \backslash E) = 1 - \min_{p \in \text{Core}(v_2)} p(\Omega \backslash E) = \max_{p \in \text{Core}(v_2)} p(E)$. Thus,

$$\underline{P_1}(E) = \min_{p \in \text{Core}(v_1)} p(E) = v_1(E) > v_2'(E) = \max_{p \in \text{Core}(v_2)} p(E) = \overline{P_2}(E),$$

which completes the proof. ∎

Remark 1 If a supermodular capacity v is additive in the sense that $v(E) + v(F) = v(E \cap F)$ for $E \cap F = \emptyset$, then v naturally defines a probability distribution over Ω, and the core of v is a singleton $\{v\}$. In this case, Proposition 4 covers the known result for the standard Bayesian framework.

Remark 2 If v is a supermodular capacity, then

$$\int_\Omega f dv = \min_{p \in \mathrm{Core}(v)} \sum_{\omega \in \Omega} p(\omega) f(\omega)$$

where the left hand side is the Choquet integral of f with respect to v. The decision rule based upon the Choquet integral is axiomatized by Schmeidler (1989). Thus, the characterization in Proposition 4 also applies to the Choquet integral model of trade with convex capacity.

Remark 3 Lemma 5 can be restated in the following way: if $v_i : 2^\Omega \to \mathbb{R}$ is supermodular for $i = 1, 2$ and $\mathrm{Core}(v_1) \cap \mathrm{Core}(v_2) = \emptyset$, then there exists $f : \Omega \to \{0, 1\}$ such that $\min_{p \in \mathrm{Core}(v_1)} p \cdot f > \max_{p \in \mathrm{Core}(v_2)} p \cdot f$. Thus, Lemma 5 is the separation theorem for the core of supermodular capacities. Murota (1998, Theorem 3.6) obtained the separation theorem for the core of discrete supermodular set functions: if $v_i : 2^\Omega \to \mathbb{Z}$ is supermodular for $i = 1, 2$ and $\mathrm{Core}(v_1) \cap \mathrm{Core}(v_2) \cap \mathbb{Z} = \emptyset$, then there exists $f : \Omega \to \{0, 1\}$ such that $\min_{p \in \mathrm{Core}(v_1) \cap \mathbb{Z}} p \cdot f > \max_{p \in \mathrm{Core}(v_2) \cap \mathbb{Z}} p \cdot f + 1$. The proof of Lemma 5 adopts the key elements of Murota's proof.

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