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"A General Update Rule for Convex Capacities"

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A General Update Rule for Convex Capacities

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A characterization of a general update rule for convex capacities, *the G-updating rule*, is investigated. We introduce a consistency property which bridges between unconditional and conditional preferences, and deduce an update rule for unconditional capacities. The axiomatic basis for the G-updating rule is established through consistent counterfactual acts, which take the form of trinary acts expressed in terms of G, an ordered tripartition of global states.

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1. INTRODUCTION

A decision maker's subjectively uncertain situations are formally described in non-additive probabilities or multiple priors, for which update rules are also extensively investigated in the economic literature on the subject. Among them, the Dempster-Shafer rule(Shafer(1976)) (the DS rule) is one of the most momentous update rules for non-additive probabilities. This rule for convex capacities is examined in Gilboa and Schmeidler(1993) by way of an elegant, systematic method, the *f-Bayesian update rule*. They also showed that the DS rule is equivalent to the maximum likelihood estimation.

Fagin and Halpern(1991) presented an update rule (the DFH rule) for inner/outer measures or belief/plausibility functions, which was originally suggested by Dempster(1967). Tapking(2004) deals with a collection of updated preference relations defined on conditional acts and proposes axioms for the DFH rule. The essence of the axiomatization is the Choquet expected utility (CEU) representation and the fact that a convex capacity updated by the DFH rule conforms to the lower probabilities of the posterior set updated by the full Bayesian update rule (the FB rule) in Jaffray(1992).

Pires (2002) reviews the FB rule or what is called the belief-by-belief updating rule, which is an update rule for a set of priors. The central axiom characterizing the FB rule is Axiom 9, which is called *conditional certainty equivalent consistency* in Eichberger et al. (2007). The FB rule and the DFH rule generates the common

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lower envelope of the updated belief set when the same prior set is given. However, the posterior set updated by the FB rule may be *strictly* included in the core of the updated capacity through the DFH rule as shown in Jaffray(1992). Therefore, the FB rule and the DFH rule should be seen as different rules as pointed out by Horie(2007), which shows that the conditional certainty equivalent consistency for binary gambles is the central property to characterize the DFH rule.

The primary objective of this paper is to characterize a general update rule, called the *G-updating rule* (Horie(2006)). The G-updating rule enables us to deal with apparently different conditioning rules as a single rule with each different parameter G, which takes the form of an ordered tripartition of the global states. To achieve characterization, we introduce a consistency property which bridge between unconditional and conditional preferences, and deduce an update rule for unconditional capacities. The primitives of our decision model are an unconditional and conditional preference relations on the set of Savage acts.

In the conventional Bayesian approach, the conditional preferences are never affected by what have *not* been observed, i.e. any counterfactual event. In the context of the subjective expected utility theory, it is implied by the independency in the sure-thing principle of Savage(1972).

However, under a subjectively ambiguous situation, information in the counterfactual event might affect the formation of posteriors as a result of a decision maker's subjective reasoning to try to reduce ambiguity. In our framework, this is formally embodied in *counterfactual acts*, which are assumed on the unrealized events, counterfactual events.

The axiomatic basis for update rules is established through a set of *consistent* counterfactual acts. This formula is a natural extension of the f-Bayesian update rule, where an act "f" is assumed to be a unique consistent counterfactual act. In case of the G-updating rule, the f-Bayesian update is extended in two directions. (i)Although in the f-Bayesian update rule a consistent counterfactual act has to be unique and common among all acts, consistent counterfactual acts in the G-updating rule are allowed to depend on the conditional certainty equivalence of any (binary) act. Therefore, the set of consistent counterfactual acts is not assumed to be singleton. (ii)The consistency are imposed only on the set of *binary acts*, not *all acts*. The reason is that, the G-updating rule evaluates the observed event quite differently dependent upon which subevent is taken into account. Relaxing the requirement for all acts into binary acts enables us to conform coherently to such inconstancy.

The main result of this paper formally characterizes the G-updating rule through a property that assures the existence of a consistent counterfactual act for binary acts, which formalizes unconditional preferences in terms of the unconditional preferences. The set of consistent counterfactual acts may be chosen from the set of all acts according to the magnitude of its conditional certainty equivalence of any binary acts. As a consequence, it is proved that trinary acts which yields the best, worst, and the conditional certainty equivalence outcomes are the representatives of the set of consistent counterfactual acts.

This paper is organized as follows. The next section provides the basic definitions and introduces the G-updating rule. Section 3 begins with illustrating the main axiom and proves the existence of G that characterizing the conditional preferences. The uniqueness of such a G is investigated in the later subsection.

2. A GENERAL UPDATE RULE

2.1. Basic Set Up

Let Ω be a finite set of *states* with $|\Omega| = n$ and $\Sigma = 2^{\Omega}$. We call a non-empty set in Σ an *event*. Let $X \subset \mathbb{R}$ be a set of *consequences*, or *outcomes* which is assumed to be $X = [\underline{x}, \overline{x}]$ with $\underline{x} < \overline{x}$. A function $f : \Omega \to X$ is called an *act*. Denote the set of all acts by $\mathcal{F}$. For the sake of simplicity, an element x in X also indicates a *constant act* which assigns x for all $\omega \in \Omega$. $f_E g$ denotes the act which yields $f(\omega)$ if $\omega \in E$ and $g(\omega)$ if $\omega \in E^c$.

A finite set function $\mu : \Sigma \to [0,1]$ is called a *capacity* on Ω if it satisfies (i) $\mu(\varnothing) = 0$ and $\mu(\Omega) = 1$, and (ii) for every A and B in Σ with $A \subset B$, we have $\mu(A) \leqq \mu(B)$. A capacity μ is said to be *convex* if for every A and B in Σ, $\mu(A \cup B) + \mu(A \cap B) \geqq \mu(A) + \mu(B)$. Given an event E, a *conditional* or *updated capacity* μ_E is a capacity *on* E, i.e. for all A in Σ with $A \cap E = E$, $\mu_E(A) = 1$. Note that for any event E in Σ, μ_E has domain Σ. When $E = \Omega$, μ_Ω is interpreted as the *unconditional capacity* and we simply write it μ. Let $\int f \, d\mu$ denote the Choquet integral (expected value) of f with respect to μ.

Let Π be a set of all binary relations on $\mathcal{F}$. We are concerned with preference relations on $\mathcal{F}$ before and after an event E is realized. Given an event E in Σ, a preference relation $\succsim_E$ in Π is called a *conditional preference relation contingent on* E. As usual, $\succ_E$ and $\sim_E$ refer to asymmetric and symmetric parts of $\succsim_E$ respectively. When $E = \Omega$, $\succsim_\Omega$ is interpreted as the *unconditional preference relation* and we simply express it as $\succsim$. Throughout this paper, we focus on preference relations that satisfy the following representation (Schmeidler (1989) and Gilboa(1987)):

(CEU) There exist a continuous, non-constant utility function $u : X \to \mathbb{R}$ unique up to positive linear transformations and a convex capacity μ_E such that for all f, g in $\mathcal{F}$

$$f \succsim_E g \iff \int u \circ f \; d\mu_E \geqq \int u \circ g \; d\mu_E.$$

Let Π_{CE} be the set of binary relations satisfy CEU. A conditional preference relation $\succsim_E \in \Pi_{CE}$ is called *represented by* (u, μ_E), that is, $\succsim_E$ is represented by a Choquet expected utility with respected to u and μ_E. By u's non-constancy of u, it is assumed that for all $E \in \Sigma$, $\overline{x} \succ_E \underline{x}$. Since u is unique up to positive linear transformations, we normalize u so that $u(\underline{x}) = 0$ and $u(\overline{x}) = 1$.

2.2. The G-Updating Rule

A general update rule, called the G-updating rule, is defined as follows (Horie(2006)).

Suppose the set of states Ω is partitioned into three disjoint sets G_i, $i = 1,2,3$, where some G_i is possibly empty. Denote an ordered triplet of G_i, $i = 1,2,3$ by $G = \langle G_1, G_2, G_3 \rangle$ and let $\mathcal{G}$ consist of all such ordered triplets of Ω. Given a $G \in \mathcal{G}$ and an event $E \in \Sigma$, define $T_i^{G,E} \equiv E^c \cap G_i$, $i = 1,2,3$. Although every $T_i^{G,E}$ depends on G and E, we denote it by T_i, $i = 1,2,3$ instead of $T_i^{G,E}$ for brevity's sake when the reference is clear.

Given a $G \in \mathcal{G}$ and an event $E \in \Sigma$, we define the *G-updating rule for a capacity*

μ *given* E through, for every $A \in \Sigma$

$$\mu_E^G(A) = \frac{\mu((A \cap E) \cup T_2) - \mu(T_2)}{[\mu((A \cap E) \cup T_2) - \mu(T_2)] + [\mu(E \cup T_2 \cup T_3) - \mu((A \cap E) \cup T_2 \cup T_3)]}. \tag{1}$$

If $\mu(E) > 0$, then μ_E^G is well-defined.

The G-updating rule has some interesting properties. The first is that, it includes the NB rule, the DS rule, and the DFH rule as special cases. If $G = \langle G_1, \varnothing, \varnothing \rangle$, then (1) is equal to the updated capacity via the naïve Bayesian update rule (NB rule):

$$\mu_E^{NB}(A) = \frac{\mu(A \cap E)}{\mu(E)} \quad \text{for every } A \in \Sigma.$$

When $G = \langle \varnothing, G_2, \varnothing \rangle$, (1) is reduced to the Dempster-Shafer rule (DS rule):

$$\mu_E^{DS}(A) = \frac{\mu((A \cap E) \cup E^c) - \mu(E^c)}{1 - \mu(E^c)} \quad \text{for every } A \in \Sigma.$$

Note that the f-Bayesian update rule includes the NB rule and the DS rule as special cases. Finally, the updated capacity μ_E^{DFH} by the DFH rule is identified as $G = \langle \varnothing, \varnothing, \Omega \rangle$:

$$\mu_E^{DFH}(A) = \frac{\mu(A \cap E)}{\mu(A \cap E) + 1 - \mu((A \cap E) \cup E^c)} \quad \text{for every } A \in \Sigma.$$

The most important property is that the G-updating rule preserves convexity. It is verified by the fact that the G-updating constitutes a 3-step conditioning where one of three rules above, the NB rule, the DS rule and the DFH rule is applied in each step as follows. In the first step, T_1 out of E^c is revised by the NB rule, in the second step T_2 is revised by the DS rule, and in the final step T_3 is revised by the DFH rule. Notice that interchanging step 1 with step 2 generates the same updated capacity (1) since the NB rule and the DS rule are both commutative as in Gilboa and Schmeidler(1993).

3. CHARACTERIZATION OF G-UPDATING RULE

3.1. Axiom

Given an act $f \in \mathcal{F}$, let $R(f) \subset X^n$ be the range of f. Define the set of k-*dimensional* (or, k-*outcome*) *acts* by $\mathcal{F}^k = \{f \in \mathcal{F} \mid \dim R(f) \leqq k\}$. We especially call $\mathcal{F}^1$ the set of constant acts, which stands for X. When $k = n$, $\mathcal{F}^k$ is equal to the set of all acts $\mathcal{F}$. Since $\dim R(f) \leqq n$ for all $f \in \mathcal{F}$, it is effective to set $k \leqq n$.

An act in $\mathcal{F}^2$ is called a *binary act*. Any binary act $f \in \mathcal{F}^2$ is expressed by some $A \in \Sigma$, $f(A)$, and $f(\Omega \backslash A)$ to be $f(A)_A f(\Omega \backslash A)$. We can in turn construct an arbitrary binary act $f \in \mathcal{F}^2$ from some b, $w \in X$, $b \geqq w$ and $A \in \Sigma$ denoted by $b_A w$, which is called a *binary act on* A.

Similarly, the set of *trinary acts* is written as $\mathcal{F}^3$. It will be useful to introduce particular trinary acts $f \in \mathcal{F}^3$, whose range $R(f)$ contains the best and the worst outcomes, $\overline{x}$ and $\underline{x}$. Formally, for every $x \in X$ define

$$\mathcal{A}(x) = \left\{ f \in \mathcal{F}^3 \mid f(\omega) \in \{\underline{x}, \overline{x}, x\} \text{ for all } \omega \in \Omega \right\},$$

and let $\mathcal{A} = \bigcup_{x \in X} \mathcal{A}(x)$. Recall that the aforementioned $G \in \mathcal{G}$ is an ordered tripartition of Ω, and so an act in $\mathcal{A}$ is also expressed in terms of G. Define $\alpha : X \times \mathcal{G} \to \mathcal{F}$ by $\alpha(x; G) \equiv \underline{x}_{G_1} \overline{x}_{G_2} x$. Note that α is onto. That is, when $x = \overline{x}$ or $\underline{x}$, $\alpha(x, G) = \alpha(x, G')$ can occur for $G \neq G'$, still every act in $\mathcal{A}$ is uniquely identified by x and G. Although there are many overlaps in G, $\mathcal{A}(x) = \bigcup_{G \in \mathcal{G}} \{\alpha(x; G)\}$.

To characterize the G-updating rule, we are ready to introduce a consistency property defined as follows.

Definition 1. Given an unconditional preferences $\succsim$ in Π and an event E in Σ, a conditional preference $\succsim_E$ in Π has *a conditional certainty-equivalently consistent counterfactual act for binary acts (CCBA)* if for every x in X there exists an act a in $\mathcal{F}$

$$x \sim_E f \Leftrightarrow x_E a \sim f_E a \text{ for all } f \in \mathcal{F}^2.$$

It will be useful to define the set of CCBAs in the following way: Given $\succsim \in \Pi$, $E \in \Sigma$, and $\succsim_E \in \Pi$, define $\psi_E^2(\cdot\,; \succsim_E, \succsim) \subset \mathcal{F}$ as

$$\psi_E^2(x; \succsim_E, \succsim) \equiv \left\{a \in \mathcal{F} \,|\, x \sim_E f \Leftrightarrow x_E a \sim f_E a \text{ for all } f \in \mathcal{F}^2\right\}. \tag{2}$$

$\succsim_E$ has a CCBA if and only if $\psi_E^2(x; \succsim_E, \succsim) \neq \varnothing$ for every $x \in X$.

Although a conditional certainty-equivalently consistent counterfactual act for k-dimensional acts are also defined as $\psi_E^k(\cdot\,; \succsim_E, \succsim)$, there is a real significance in $\psi_E^2(\cdot\,; \succsim_E, \succsim)$ and $\psi_E^n(\cdot\,; \succsim_E, \succsim)$.

In the Bayesian approach, the conditional preferences focus only on the realized event and are never affected by the unrealized events. Formally, in terms of unconditional and conditional preferences, the Bayesian update rule is expressed as: for all f, g, $h \in \mathcal{F}$, $g \succsim_E h \Leftrightarrow g_E f \succsim h_E f$. Based on our formulation above, $\succsim_E \in \Pi$ is such that $\psi_E^n(x; \succsim_E, \succsim) = \mathcal{F}$ for all $x \in X$.

Given that $\succsim \in \Pi_{CE}$, the f-Bayesian update rule is defined through a unique conditional consistent counterfactual act $f \in \mathcal{F}$ satisfying that for all g, $h \in \mathcal{F}$, $g \succsim_E h \Leftrightarrow g_E f \succsim h_E f$, which is equivalent to $f \in \psi_E^n(x; \succsim_E, \succsim)$ for all $x \in X$. Gilboa and Schmeidler(1993) prove that $f = \overline{x}_S \underline{x}$ for some $S \in \Sigma$.

3.2. Main Results

3.2.1. Existence of G

The main result of this paper is the following theorem that characterizes the G-updating rule:

Theorem 1. *Suppose that the unconditional preference relation $\succsim$ in Π_{CE} is represented by (u, μ). Given an event E in Σ with $|E| \geqq 2$ and $\mu(E) > 0$, the following statements are equivalent:*

(i) A conditional preference relation contingent on E, $\succsim_E$ in Π_{CE} has a CCBA.

(ii) There exists a G in $\mathcal{G}$ such that $\succsim_E$ is represented by $\left(u, \mu_E^G\right)$: for any f and g in $\mathcal{F}$

$$f \succsim_E g \iff \int u \circ f \, d\mu_E^G \geqq \int u \circ g \, d\mu_E^G,$$

where μ_E^G is defined as in (1).

Proof. Suppose that $\succsim \in \Pi_{CE}$ represented by (u, μ) and event $E \in \Sigma$ with $|E| \geqq 2$ and $\mu(E) > 0$ are given.

(i)⇒(ii) This part will be proven through the following three lemmas. It is assumed that a conditional preference relation $\succsim_E \in \Pi_{CE}$ has a CCBA. Given $\succsim_E \in \Pi_{CE}$ and an $S \subset E$, let x^S satisfy $x^S \sim_E \overline{x}_S \underline{x}$.

LEMMA 1. *Suppose that $\succsim_E \in \Pi_{CE}$ is has a CCBA and for some $S \subsetneqq E$, x^S is in $(\underline{x}, \overline{x})$. Then, there exists a $G \in \mathcal{G}$ such that $\alpha(x; G) \in \psi_E^2(x; \succsim_E, \succsim)$ for every $x \in X$.*

Proof. Given an act $a \in \mathcal{F}$, define $\overline{P}(a) \equiv \{\omega \in \Omega \mid a(\omega) > x^S\}$ and $\underline{P}(a) \equiv \{\omega \in \Omega \mid a(\omega) < x^S\}$. Denote $\langle \underline{P}(a), \overline{P}(a), \Omega \backslash (\underline{P}(a) \cup \overline{P}(a)) \rangle$ by $P(a)$ which is an element of $\mathcal{G}$. Note that $a(\omega) = x^S$ for every $\omega \in \underline{P}(a) \cup \overline{P}(a)$. Let $T_1 = \underline{P}(a) \cap E^c$, $T_2 = \overline{P}(a) \cap E^c$, and $T_3 = E^c \backslash (T_1 \cup T_2)$.

Let $|E^c| = M > 0$. An act $a \in \mathcal{F}$ has at most M different outcomes on E^c. Let $|T_2| = l$ and number a superscript so that $a(\omega^1) \geqq a(\omega^2) \geqq \cdots \geqq a(\omega^l)$ for every $\omega \in T_2$ and $a(\omega^{l+1}) \geqq \cdots \geqq a(\omega^m)$ for every $\omega \in T_1$ where $m \leqq M$.

Take an arbitrary act from $\psi_E^2(x^S; \succsim_E, \succsim)$ and set it a^S. We can always find such an a^S since $\psi_E^2(x^S; \succsim_E, \succsim) \neq \varnothing$. For this a^S consider $P(a^S)$.

Fix an arbitrary $A \subsetneqq E$. Since $x^S \in (\underline{x}, \overline{x})$ and $\succsim_E$ is continuous, for an event $A \subsetneqq E$ such that $x^A \in (\underline{x}, \overline{x})$, we can always find sufficiently small $\varepsilon > 0$ and $\delta > 0$ such that $x^S \sim_E (x^S + \varepsilon)_A (x^S - \delta)$ with $x^S + \varepsilon < a^l$ and $x^S - \delta > a^{l+1}$. When $x^A = \overline{x}$ (resp. $\underline{x}$), ε (resp. δ) is equal to zero, but the argument below is still valid. Then

$$\begin{aligned} & x^S \sim_E (x^S + \varepsilon)_A (x^S - \delta) \\ \Leftrightarrow \quad & \int u \circ x^S \, d\mu_E - \int u \circ (x^S + \varepsilon)_A (x^S - \delta) \, d\mu_E = 0 \\ \Leftrightarrow \quad & u(x^S) - \{\mu_E(A) u(x^S + \varepsilon) + [1 - \mu_E(A)] u(x^S - \delta)\} = 0. \end{aligned} \tag{3}$$

Since $(x^S + \varepsilon)_A (x^S - \delta)$ is a binary act and $a^S \in \psi_E^2(x^S; \succsim_E, \succsim)$, a^S satisfies

$$\begin{aligned} & x_E^S a^S \sim (x^S + \varepsilon)_A (x^S - \delta)_{E \backslash S} a^S \\ \Leftrightarrow \quad & \int u \circ x_E^S a^S \, d\mu - \int u \circ (x^S + \varepsilon)_A (x^S - \delta)_{E \backslash S} a^S \, d\mu = 0 \\ \Leftrightarrow \quad & \{[\mu(A \cup T_2) - \mu(T_2)] + [\mu(E \cup T_2 \cup T_3) - \mu(A \cup T_2 \cup T_3)]\} u(x^S) \\ & - [\mu(A \cup T_2) - \mu(T_2)] u(x^S + \varepsilon) \\ & - [\mu(E \cup T_2 \cup T_3) - \mu(A \cup T_2 \cup T_3)] u(x^S - \delta) = 0. \end{aligned} \tag{4}$$

Rearranging terms in (3) and (4), we obtain

$$\mu_E(A) = \frac{\mu(A \cup T_2) - \mu(T_2)}{[\mu(A \cup T_2) - \mu(T_2)] + [\mu(E \cup T_2 \cup T_3) - \mu(A \cup T_2 \cup T_3)]}. \tag{5}$$

Since the choice of A is arbitrary and the denominator is always strictly positive by assumption, (5) holds for all $A \subset E$.

Now consider $\alpha\left(x;P\left(a^S\right)\right)$. Any binary act is written in the form of $b_A w$ for some b, $w \in X$ with $b \geqq w$ and $A \subset E$. Then

$$
\begin{aligned}
& x_E \alpha\left(x, P\left(a^S\right)\right) \sim b_A w_{E\setminus A} \alpha\left(x, P\left(a^S\right)\right) \\
\Leftrightarrow \quad & \int u \circ x_E \alpha\left(x, P\left(a^S\right)\right)\, d\mu - \int u \circ b_A w_{E\setminus A} \alpha\left(x, P\left(a^S\right)\right)\, d\mu = 0 \\
\Leftrightarrow \quad & \{[\mu(A\cup T_2) - \mu(T_2)] + [\mu(E\cup T_2\cup T_3) - \mu(A\cup T_2\cup T_3)]\} \times \\
& \Big\{u(x) - \Big[\tfrac{\mu(A\cup T_2)-\mu(T_2)}{[\mu(A\cup T_2)-\mu(T_2)]+[\mu(E\cup T_2\cup T_3)-\mu(A\cup T_2\cup T_3)]} u(b) \\
& \qquad + \tfrac{\mu(E\cup T_2\cup T_3)-\mu(A\cup T_2\cup T_3)}{[\mu(A\cup T_2)-\mu(T_2)]+[\mu(E\cup T_2\cup T_3)-\mu(A\cup T_2\cup T_3)]} u(w)\Big]\Big\} = 0 \\
\Leftrightarrow \quad & u(x) - \{\mu_E(A)\, u(b) + [1-\mu_E(A)]\, u(w)\} = 0 \qquad \text{(by (5))} \\
\Leftrightarrow \quad & \int u \circ x\, d\mu_E - \int u \circ b_A w\, d\mu_E = 0 \\
\Leftrightarrow \quad & x \sim_E b_A w. \qquad (6)
\end{aligned}
$$

Therefore, we have $x \sim_E b_A w \Leftrightarrow x_E \alpha\left(x;P\left(a^S\right)\right) \sim b_A w_{E\setminus A}\alpha\left(x;P\left(a^S\right)\right)$ for any $b_A w \in \mathcal{F}^2$. It follows that $\alpha\left(x;P\left(a^S\right)\right) \in \psi_E^2\left(x; \succsim_E, \succsim\right)$ for any $x \in X$. ∎

Lemma 2. *Suppose that $\succsim_E \in \Pi_{CE}$ has a CCBA and for all $A \subset E$, $x^A = \overline{x}$ or $\underline{x}$. Then, there exists a $G \in \mathcal{G}$ such that $\alpha(x;G) \in \psi_E^2(x;\succsim_E,\succsim)$ for every $x \in X$.*

Proof. If there is no proper subset $B \subsetneqq E$ such that $x^B = \overline{x}$, then there is no binary act indifferent to $\overline{x}$ other than itself. In this case, it is clear that for any $a \in \mathcal{F}$, $x^B \sim_E \overline{x} \Leftrightarrow x_E^B a \sim \overline{x}_E a$, hence for all $G \in \mathcal{G}$, $\alpha(\overline{x}, G) \in \psi_E^2(\overline{x};\succsim_E,\succsim)$.

Suppose that, for some $B \subsetneqq E$, $x^B = \overline{x}$. Pick an element of $\psi_E^2(\overline{x};\succsim_E,\succsim)$ and call it $\overline{a}$. Consider $P(\overline{a})$. $x^B = \overline{x}$ implies that $\mu_E(B) = 1$. Then we always find sufficiently small $\delta > 0$ such that $\overline{x} \sim_E \overline{x}_B(\overline{x}-\delta)$ with $\overline{x}-\delta > a^{l+1}$. Since $\overline{a} \in \psi_E^2(\overline{x};\succsim_E,\succsim)$, we have

$$
\begin{aligned}
& \overline{x}_E \overline{a} \sim \overline{x}_B (\overline{x}-\delta)_{E\setminus B}\, \overline{a} \\
\Leftrightarrow \quad & \int u \circ \overline{x}_E \overline{a}\, d\mu - \int u \circ \overline{x}_B \underline{x}_{E\setminus B} \overline{a}\, d\mu = 0 \\
\Leftrightarrow \quad & \mu(E\cup T_2\cup T_3) - \mu(B\cup T_2\cup T_3) \\
& \quad + [\mu(E\cup T_2\cup T_3) - \mu(B\cup T_2\cup T_3)]\, u(\overline{x}-\delta) = 0.
\end{aligned}
$$

However, $u(\overline{x}-\delta) > 0$ implies that $\mu(E\cup T_2\cup T_3) = \mu(B\cup T_2\cup T_3)$. Then we put

$$
\mu_E(B) = \tfrac{\mu(B\cup T_2)-\mu(T_2)}{[\mu(B\cup T_2)-\mu(T_2)]+[\mu(E\cup T_2\cup T_3)-\mu(B\cup T_2\cup T_3)]} = 1.
$$

On the other hand, consider $E\setminus B$. Since $\mu_E(B) = 1$, we have $\mu_E(E\setminus B) = 0$. Then for every $\varepsilon \in [0, \overline{x}-\underline{x}]$, we have $\underline{x} \sim_E (\underline{x}+\varepsilon)_{E\setminus B}\, \underline{x}$. Now consider $\alpha(\underline{x}, P(\overline{a}))$. Then

$$
\begin{aligned}
& \underline{x}_E \alpha(\underline{x}, P(\overline{a})) \sim (\underline{x}+\varepsilon)_{E\setminus B}\, \underline{x}_B \alpha(\underline{x}, P(\overline{a})) \\
\Leftrightarrow \quad & \int u \circ \underline{x}_E \alpha(\underline{x}, P(\overline{a}))\, d\mu - \int u \circ (\underline{x}+\varepsilon)_{E\setminus B}\, \underline{x}_B \alpha(\underline{x}, P(\overline{a}))\, d\mu = 0 \\
\Leftrightarrow \quad & [\mu((E\setminus B)\cup T_2\cup T_3) - \mu(T_2\cup T_3)]\, u(\underline{x}+\varepsilon) = 0.
\end{aligned}
$$

However, by convexity of μ, we have

$$\mu\left((E\backslash B)\cup T_2\cup T_3\right)-\mu\left(T_2\cup T_3\right)\leqq\mu\left(E\cup T_2\cup T_3\right)-\mu\left(B\cup T_2\cup T_3\right)=0,$$

hence $\mu\left((E\backslash B)\cup T_2\cup T_3\right)=\mu\left(T_2\cup T_3\right)$. For event $E\backslash B$, we have $\mu_E\left(E\backslash B\right)=0$, which is also written as

$$\begin{aligned}\mu_E\left(E\backslash B\right) &= \tfrac{\mu(E\cup T_2\cup T_3)-\mu(B\cup T_2\cup T_3)}{[\mu(B\cup T_2)-\mu(T_2)]+[\mu(E\cup T_2\cup T_3)-\mu(B\cup T_2\cup T_3)]}\\ &= 1-\mu_E\left(B\right)\\ &= 0.\end{aligned}$$

Therefore, we have $\underline{x}\sim_E\left(\underline{x}+\varepsilon\right)_{E\backslash B}\,\underline{x}\Leftrightarrow\underline{x}_E\alpha\left(\underline{x},P\left(\overline{a}\right)\right)\sim\left(\underline{x}+\varepsilon\right)_{E\backslash B}\,\underline{x}_B\alpha\left(\underline{x},P\left(\overline{a}\right)\right)$ for every $\varepsilon>0$, hence $\alpha\left(\underline{x},P\left(\overline{a}\right)\right)\in\psi_E^2\left(\overline{x};\succsim_E,\succsim\right)$.

From the above argument, for every $A\subset E$, the relationship in (6) of Lemma 1 also holds, hence $\alpha\left(x,P\left(\overline{a}\right)\right)\in\psi_E^2\left(x;\succsim_E,\succsim\right)$ for any $x\in X$. ∎

Lemma 3. *Given a $G\in\mathcal{G}$, if $\alpha\left(x;G\right)\in\psi_E^2\left(x;\succsim_E,\succsim\right)$ for any $x\in X$, then $\mu_E=\mu_E^G$.*

Proof. Given a $G\in\mathcal{G}$, suppose $\alpha\left(x;G\right)\in\psi_E^2\left(x;\succsim_E,\succsim\right)$ for any $x\in X$. By Lemma 1, $\mu_E\left(A\right)\in(0,1)$ is determined as in (5) for every $A\subset E$. In addition, Lemma 2 tells that $\mu_E\left(A\right)=1$ if $\mu\left(E\cup T_2\cup T_3\right)=\mu\left(A\cup T_2\cup T_3\right)$ and $\mu_E\left(A\right)=0$ if $\mu\left(A\cup T_2\right)=\mu\left(T_2\right)$ for any G. It is also expressed by (5).

Furthermore, by convexity of μ, we have $\mu_E\left(E\cup A\right)-\mu_E\left(A\right)\geqq\mu_E\left(E\right)-\mu_E\left(A\cap E\right)$ for any $A\in\Sigma$. However, $E\subset E\cup A$, so $\mu_E\left(E\cup A\right)=\mu_E\left(E\right)=1$ by monotonicity. Hence $\mu_E\left(A\cap E\right)\geqq\mu_E\left(A\right)$. Furthermore we also have $\mu_E\left(A\cap E\right)\leqq\mu_E\left(A\right)$ by monotonicity. Thus we obtain $\mu_E\left(A\cap E\right)=\mu_E\left(A\right)=\mu_E^G\left(A\right)$ for each $A\in\Sigma$. ∎

(ii)⇒(i) To show this part, assume that $\succsim_E$ is represented by $\left(u,\mu_E^G\right)$ for some $G\in\mathcal{G}$.

For any $A\in\Sigma$, we have

$$\mu\left((A\cap E)\cup T_2\right)-\mu\left(T_2\right)+\mu\left(E\cup T_2\cup T_3\right)-\mu\left((A\cap E)\cup T_2\cup T_3\right)>0,$$

by the assumption that $\mu\left(E\right)>0$ and μ's convexity.

Any binary act in $\mathcal{F}^2$ is expressed in the form of b_Aw, where b, $w\in X$ with $b\geqq w$ and $A\in\Sigma$. Then

$$\begin{aligned}
& x_E\alpha\left(x;G\right)\sim\left(b_Aw\right)_E\alpha\left(x;G\right)\\
\Leftrightarrow\ & \int u\circ x_E\alpha\left(x;G\right)\ d\mu-\int u\circ b_{A\cap E}w_{A^c\cap E}\alpha\left(x;G\right)\ d\mu=0\\
\Leftrightarrow\ & \left\{\mu\left((A\cap E)\cup T_2\right)-\mu\left(T_2\right)+\mu\left(E\cup T_2\cup T_3\right)-\mu\left((A\cap E)\cup T_2\cup T_3\right)\right\}\times\\
& \left\{u\left(x\right)-\left[\tfrac{\mu((A\cap E)\cup T_2)-\mu(T_2)}{\mu((A\cap E)\cup T_2)-\mu(T_2)+\mu(E\cup T_2\cup T_3)-\mu((A\cap E)\cup T_2\cup T_3)}u\left(b\right)\right.\right.\\
& \qquad\left.\left.+\tfrac{\mu(E\cup T_2\cup T_3)-\mu((A\cap E)\cup T_2\cup T_3)}{\mu((A\cap E)\cup T_2)-\mu(T_2)+\mu(E\cup T_2\cup T_3)-\mu((A\cap E)\cup T_2\cup T_3)}u\left(w\right)\right]\right\}=0\\
\Leftrightarrow\ & u\left(x\right)-\left\{\mu_E\left(A\right)u\left(b\right)+\left[1-\mu_E\left(A\right)\right]u\left(w\right)\right\}=0\\
\Leftrightarrow\ & \int u\circ x\ d\mu_E^G-\int u\circ b_Aw\ d\mu_E^G=0\\
\Leftrightarrow\ & x\sim_E b_Aw.
\end{aligned}$$

It follows that for every $A \in \Sigma$, we have $x \sim_E b_A w \Leftrightarrow x_E \alpha(x;G) \sim b_A w_{E\setminus A} \alpha(x;G)$ for all $b_A w \in \mathcal{F}^2$. Thus $\alpha(x;G) \in \psi_E^2(x;\succsim_E,\succsim)$ for every $x \in X$, which completes the proof. ∎

Theorem 1 proves that, when an event E occurred, a $G \in \mathcal{G}$ represents a decision maker's conditioning pattern on every state in E^c. It is supposed to illustrate a condition for a decision maker to have a coherent conditioning pattern with any event E observed.

Let $\widehat{\Sigma}$ be the set of events which is removed $\varnothing$ and Ω from Σ. Let $\{\succsim_E\}_{E\in\widehat{\Sigma}}$ be a collection of conditional preferences where every $\succsim_E \in \Pi_{CE}$ is represented by (u,μ_E). With convenience, put $\psi_E^2(x;\succsim_E,\succsim) = \mathcal{F}$ for any $x \in X$ when $|E| = 1$ or $\mu(E) = 0$.

Definition 2. A collection of conditional preferences $\{\succsim_E\}_{E\in\widehat{\Sigma}}$ has *a certainty-equivalently consistent counterfactual act for binary acts (CBA)* if for every x in X there exists an act a in $\mathcal{F}$ such that for every $E \in \widehat{\Sigma}$

$$x \sim_E f \Leftrightarrow x_E a \sim f_E a \text{ for all } f \in \mathcal{F}^2.$$

On a parallel with the CCBA case, $\{\succsim_E\}_{E\in\widehat{\Sigma}}$ has a CBA if and only if $\bigcap_{E\in\widehat{\Sigma}} \psi_E^2(x;\succsim_E,\succsim) \neq \varnothing$ for every $x \in X$.

Corollary 1. *Suppose that the unconditional preference relation $\succsim$ in Π_{CE} is represented by (u,μ). Then, the following statements are equivalent:*

(i) A collection of conditional preferences $\{\succsim_E\}_{E\in\widehat{\Sigma}}$ has a CBA.

(ii) There exists a G in $\mathcal{G}$ such that for any event E in $\widehat{\Sigma}$, $\succsim_E$ is represented by $\left(u,\mu_E^G\right)$.

3.2.2. Uniqueness of G

For various applications, it is especially useful to clarify the sufficient condition for the uniqueness of G. As noted in the previous subsection, it is quite natural that one expects *any* $G \in \mathcal{G}$ to represent $\succsim_E$ when μ is additive. The conditional preference $\succsim_E$ given E is represented by $\left(u,\mu_E^G\right)$ through a counterfactual act $\alpha(x;G) = \underline{x}_{G_1}\overline{x}_{G_2}x$, which is identified by G. Therefore, some different Gs to represent $\succsim_E$ implies that they generates the same conditional capacity μ_E^G which represents $\succsim_E$ by $\left(u,\mu_E^G\right)$. To obtain the uniqueness of G, we need to know some restrictions on the unconditional preference, hence the unconditional capacity, μ, in the followings:

Definition 3. A capacity μ is *strictly positive* if for all A in $\widehat{\Sigma}$, $\mu(A) > 0$.

In general, a μ may assign 0 to some events. In those cases, we can transform μ into a convex combination with a uniform distribution υ. Formally, given a μ, let $\widehat{\mu} = (1-\varepsilon)\mu + \varepsilon\upsilon$ for some $\varepsilon > 0$ sufficiently small, where $\upsilon : \Sigma \to [0,1]$, $\upsilon(A) = \frac{|A|}{n}$ for any $A \in \Sigma$. Remind that this transformation does not alter analyses below.

Definition 4. A capacity μ is called *strictly convex* if for all A and B in $\widehat{\Sigma}$ with $A \nsubseteq B$ nor $B \nsubseteq A$, $\mu(A\cup B) > \mu(A) + \mu(B) - \mu(A\cap B)$.

The strict convexity is interpreted as strictly increasing in marginal increments in the lower probability of any event. However, one cannot hope for the uniqueness of G only from μ's strict convexity (and strict positiveness). Let us see the following example:

- $\Omega = \{1, 2, 3\}$
- $\mu(A) = \begin{cases} \alpha & \text{if } |A| = 1 \\ 3\alpha & \text{if } |A| = 2 \\ 7\alpha & \text{if } A = \Omega \end{cases}$, where $\alpha = \frac{1}{7}$.

It is easily verified that this μ is strictly positive and strictly convex. However, the NB and DS update generates the same conditional capacity:

$$\mu_E^{NB}(\{1\}) = \frac{\alpha}{3\alpha} = \frac{1}{3},\ \mu_E^{DS}(\{1\}) = \frac{3\alpha - \alpha}{7\alpha - \alpha} = \frac{1}{3},$$

although

$$\mu_E^{DFH}(\{1\}) = \frac{\alpha}{\alpha + 7\alpha - 3\alpha} = \frac{1}{5}.$$

This example shows that μ's strict convexity is not enough to discriminate between the NB and DS rules. We introduce a family of μs that generate $\mu_E^{NB} = \mu_E^{DS}$ for any $E \in \widehat{\Sigma}$:

DEFINITION 5. A capacity μ is *marginally increasing by a constant ratio* if for some $t > 0$

$$\begin{aligned} \mu(A) &= T \sum_{\iota=1}^{|A|} (t+1)^{\iota-1} \text{ for every } A \in \Sigma, \\ \text{where } T &= \left[\sum_{\iota=1}^{n} (t+1)^{\iota-1} \right]^{-1}. \end{aligned}$$

LEMMA 4. *Suppose that μ is strictly positive and strictly convex. $\mu_E^{NB} = \mu_E^{DS}$ for every $E \in \widehat{\Sigma}$, if and only if μ is marginally increasing by a constant ratio.*

Proof. Suppose that μ is marginally increasing by a constant ratio for some $t > 0$. It is immediately obvious that μ is strictly positive. It is also verified that for any $t > 0$, μ is strictly positive as follows. For A and B in $\widehat{\Sigma}$ with $A \not\subseteq B$ nor $B \not\subseteq A$

$$\begin{aligned} & \mu(A \cup B) - \mu(A) \\ = \; & T \sum_{\iota=1}^{|A|+|B|-|A\cap B|} (t+1)^{\iota-1} - T \sum_{\iota=1}^{|A|} (t+1)^{\iota-1} \\ = \; & (t+1)^{|B|} T \sum_{\iota=1}^{|B|-|A\cap B|} (t+1)^{\iota-1}. \end{aligned}$$

On the other hand

$$\begin{aligned}
& \mu(B) - \mu(A \cap B) \\
= \; & T \sum_{\iota=1}^{|B|} (t+1)^{\iota-1} - T \sum_{\iota=1}^{|A \cap B|} (t+1)^{\iota-1} \\
= \; & (t+1)^{|A \cap B|} T \sum_{\iota=1}^{|B|-|A \cap B|} (t+1)^{\iota-1} .
\end{aligned}$$

Since $|B| > |A \cap B|$ and $t > 0$ by assumption, we conclude that $\mu(A \cup B) - \mu(A) > \mu(B) - \mu(A \cap B)$.

Take any $E \in \widehat{\Sigma}$ and consider μ_E^{NB} and μ_E^{DS}. Then for every $A \subsetneq E$

$$\begin{aligned}
\mu_E^{NB}(A) &= \frac{\mu(A)}{\mu(E)} = \frac{T \sum_{\iota=1}^{|A|} (t+1)^{\iota-1}}{T \sum_{\iota=1}^{|E|} (t+1)^{\iota-1}}, \\
\mu_E^{DS}(A) &= \frac{\mu(A \cup E^c) - \mu(E^c)}{1 - \mu(E^c)} = \frac{T \sum_{\iota=1}^{|A \cup E^c|} (t+1)^{\iota-1} - T \sum_{\iota=1}^{|E^c|} (t+1)^{\iota-1}}{1 - T \sum_{\iota=1}^{|E^c|} (t+1)^{\iota-1}}.
\end{aligned}$$

However, $|E^c| = n - |E|$ and $|A \cup E^c| = |A| + |E^c|$,

$$\begin{aligned}
\frac{\mu(E)}{1 - \mu(E^c)} &= \frac{T \sum_{\iota=1}^{|E|} (t+1)^{\iota-1}}{1 - T \sum_{\iota=1}^{|E^c|} (t+1)^{\iota-1}} \\
&= \frac{\sum_{\iota=1}^{|E|} (t+1)^{\iota-1}}{\sum_{\iota=1}^{n} (t+1)^{\iota-1} - \sum_{\iota=1}^{|E^c|} (t+1)^{\iota-1}} \\
&= (t+1)^{-|E^c|}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{\mu(A)}{\mu(A \cup E^c) - \mu(E^c)} &= \frac{\sum_{\iota=1}^{|A|} (t+1)^{\iota-1}}{\sum_{\iota=1}^{|A \cup E^c|} (t+1)^{\iota-1} - \sum_{\iota=1}^{|E^c|} (t+1)^{\iota-1}} \\
&= (t+1)^{-|E^c|} .
\end{aligned}$$

It follows that $\mu_E^{NB}(A) = \mu_E^{DS}(A)$ for every $A \subsetneq E$.

As for the other direction, suppose that for some $S \in \widehat{\Sigma}$, $\mu(S) \neq T \sum_{\iota=1}^{|S|} (t+1)^{\iota-1}$. When $|S| > 1$, take $S = E$. Then, for every $A \subsetneq S$

$$\frac{\mu(S)}{1 - \mu(S^c)} \neq \frac{\mu(A)}{\mu(A \cup S^c) - \mu(S^c)}.$$

When $|S| = 1$, take $S = E^c$. Then, for every $A \subsetneq E$

$$\frac{\mu(E)}{1 - \mu(S)} \neq \frac{\mu(A)}{\mu(A \cup S) - \mu(S)}.$$

∎

THEOREM 2. *Suppose that* $|\Omega| \geqq 3$ *and the unconditional preference relation* $\succsim$ *in* Π_{CE} *is represented by* (u, μ) *where* μ *is strictly positive and strictly convex but not marginally increasing by any constant ratio. If a collection of conditional preferences* $\{\succsim_E\}_{E\in\widehat{\Sigma}}$ *has a CBA, then there exists a unique* G *in* $\mathcal{G}$ *such that for any event* E *in* $\widehat{\Sigma}$, $\succsim_E$ *is represented by* $\left(u, \mu_E^G\right)$.

Proof. To lead contradiction, assume that G and G' with $G \neq G'$ generate $\mu_E^G = \mu_E^{G'}$ for all $E \in \widehat{\Sigma}$.

At first, suppose that $G_3 \cap (G_1' \cup G_2') \neq \varnothing$, that is, there exists a state $\omega \in G_3 \cap (G_1' \cup G_2')$. There are two cases where (1) $\omega \in G_3 \cap G_1'$ and (2) $\omega \in G_3 \cap G_2'$.

Case 1: $\omega \in G_3 \cap G_1'$.

Let $E = \Omega \backslash \{\omega\}$. Then, for every $A \subsetneqq E$

$$\mu_E^{G'}(A) - \mu_E^G(A) = \frac{\mu(A)}{\mu(\Omega \backslash \{\omega\})} - \frac{\mu(A)}{\mu(A) + 1 - \mu(A \cup \{\omega\})} = 0.$$

However, we have

$$\mu(\Omega \backslash \{\omega\}) < \mu(A) + 1 - \mu(A \cup \{\omega\}),$$

since $\widehat{\mu}$ is strictly positive and strictly convex. Hence $\mu_E^G(A) < \mu_E^{G'}(A)$, which leads contradiction.

Case 2: $\omega \in G_3 \cap G_2'$.

Consider $E = \Omega \backslash \{\omega\}$. For every $A \subsetneqq E$ we have

$$\begin{aligned}
& \mu_E^{G'}(A) - \mu_E^G(A) \\
= & \frac{\mu(A \cup \{\omega\}) - \mu(\{\omega\})}{1 - \mu(\{\omega\})} - \frac{\mu(A)}{\mu(A) + 1 - \mu(A \cup \{\omega\})} \\
= & \frac{[1 - \mu(A \cup \{\omega\})][\mu(A \cup \{\omega\}) - \mu(A) - \mu(\{\omega\})]}{[1 - \mu(\{\omega\})][\mu(A) + 1 - \mu(A \cup \{\omega\})]} \\
> & 0.
\end{aligned}$$

The last inequality is due to μ's strict positiveness and strict convexity. It again contradicts the assumption that $\mu_E^G = \mu_E^{G'}$.

From the above arguments, we have $G_3 \cap (G_1' \cup G_2') = \varnothing$, hence $G_3 = G_3'$. Since $G \neq G'$, it only remains to assume $G_1 \cap G_2' \neq \varnothing$ without loss of generality.

Suppose that μ is not constant ratio for any $t > 0$.

Then, by lemma above, there exists an $E \in \widehat{\Sigma}$ such that for every $A \subsetneqq E$

$$\frac{\mu(E)}{1 - \mu(E^c)} \neq \frac{\mu(A)}{\mu(A \cup E^c) - \mu(E^c)}.$$

Therefore, for this event E, $\mu_E^{NB} \neq \mu_E^{DS}$, hence $G_1 \cap G_2' = \varnothing$.

It follows that $G \neq G'$ cannot generate the same conditionals, hence G is unique.

∎

4. CONCLUDING REMARKS

We have investigated a characterization of the G-updating rule. To relax the Bayesian hypothesis, we concentrate on the set of binary acts. However, it might also suggest the limitation in the method of unconditional-conditional preferences approach, since further widening the sets of counterfactual acts may not generate any consistent preferences, and become disconnected with reality. Although it is quite difficult to test and measure out how to revise subjective uncertainty, it is the matter for future investigation anticipated eagerly in a behavioral sense.

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