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Theories of strain analysis from shape fabrics:

A perspective using hyperbolic geometry

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Abstract

A parameter space is proposed for unifying the theories of two-dimensional strain analysis, where strain markers are approximated by ellipses with a prescribed area. It is shown that the theories are unified by hyperbolic geometry, the oldest and simple non-Euclidean geometry. The hyperboloid model of the geometry is used for this purpose. Ellipses normalized by their areas are represented by points on the unit hyperboloid, the curved surface in a non-Euclidean space. Dissimilarity between ellipses is defined by the distance between the points that represent the ellipses. The merit of introducing the geometry comes from the fact that this distance equals the doubled natural strain needed to transform one ellipse to another. Thus, the introduction is natural and convenient for strain and error analyses. Equal-area and gnomonic projections of the hyperboloid are introduced for the R_f/ϕ and kinematic vorticity analyses, respectively. In our formulation, the strain ellipse optimal for a set of R_f/ϕ data is obtained as the centroid of the points corresponding to the data on the hyperboloid, and the dispersion of the points shows the uncertainty of the optimal strain. By means of a bootstrap method, the confidence region of the strain is drawn upon the surface, and equal-area projection from the surface to a Euclidean plane shows the dispersion of the points and the size of the confidence region. In addition, our formulation provides a new graphical technique for kinematic vorticity analysis using the gnomonic projection.

The technique yields the optimal kinematic vorticity number with its uncertainty.

Key words: hyperboloid model, kinematic vorticity, error analysis, general shear, finite deformation, tectonics

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1 Introduction

Strain analysis is an essential technique for complete understanding of the finite deformations of the lithosphere. Various methods have been developed to evaluate strain or deformation of rocks (e.g., Houghton, 1856; Cloos, 1947; Flinn, 1956; Wellman, 1962; Ramsay, 1967; Hossack, 1968; Ramsay and Huber, 1983; Passchier and Trouw, 2005). Their theories are based on continuum mechanics, but are formed ad hoc to deal with specific issues such as the determination of strain from R_f/ϕ data (Ramsay, 1967; Dunnet, 1969). The present paper aims at presenting a common basis for formulating the theories of two-dimensional strain analysis that use ellipses for approximating the shapes of strain markers in deformed rocks.

This work was triggered when I became aware of the equivalence of an equation of two-dimensional strain analysis by Dunnet (1969) with a formula of hyperbolic geometry, the non-Euclidean geometry found by Lobachevskiĭ, Bolyai and Gauss in the early 19th Century (Faber, 1983). This paper demonstrates that the geometry allows us not only to reformulate theories of two-dimensional strain analysis but also to establish a basis for estimating the optimal strain and its error by means of formal statistical methodology.

This paper is organized as follows. Important quantities and symbols used in this

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paper are introduced in §2. Section 3 introduces the connection between two-dimensional deformation and hyperbolic geometry, which has alternative formulations. Among them, we adopt the formulation called the hyperboloid model (§4). The utility of the model for two-dimensional analysis is explained in §5. That is, the model provides the common basis for the theories of the analysis, and leads to new graphical methods. We take R_f/ϕ strain analysis, center-to-center strain analysis and kinematic vorticity analysis as examples. Two different plots, i.e., equal-area and gnomonic nets, for shape fabric are proposed in §4 and used in §5.

In this paper, it is assumed that all the grains in a rock suffered a homogeneous deformation or rotations by spatially steady flow. It is beyond the scope of this paper to deal with the cases to which the assumption does not apply.

2 Notation

Let us, first, define some important symbols for dealing with ellipses. As it is not easy to estimate volume changes accompanied by tectonic deformations from strain markers, the changes are ignored throughout of this paper. And, every ellipse is assumed to have an area of π , which is equal to that of a unit circle. As a result, we deal only with the aspect ratios and major-axis orientations of ellipses. The orientations are measured from a reference line that is taken arbitrarily on the plane on which strain markers are observed. Let ϕ be the major-axis orientation of an ellipse (Fig. 1). Since a strain ellipse with the aspect ratio R results from the principal elongations $R^{1/2}$ and $R^{-1/2}$, $\log R^{1/2}$ equals the logarithmic strain and $\rho = \log R$ is the logarithmic quadratic strain or doubled natural strain. The double angle $\psi = 2\phi$ is theoretically more useful than ϕ . All ellipses are normalized by their area to have the value of π . Accordingly, unstrained state is represented by the unit circle.

Attributes of pre- and post-strain ellipses and strain ellipse are distinguished by the subscripts, ‘i’, ‘f’ and ‘s,’ respectively. For example, ψ_s and ψ_f denote the doubled major-axis orientations of strain ellipse and final ellipse, respectively. ϕ_s indicates the maximum stretching orientation. Accordingly, the logarithmic quadratic strain of pre-, post-strain, and strain ellipses are referred to as ρ_i , ρ_f and ρ_s , respectively. Table 1 shows the symbols for important quantities.

3 Rationale for introducing hyperbolic geometry

Equations introduced by Dunnet (1969, Eq. 28) and Lisle (1985, Eq. A1.1) clearly show the connection between strain analysis and hyperbolic geometry. Dunnet derived the equation

$$\cosh \rho_i = \cosh \rho_f \cosh \rho_s - \sinh \rho_f \sinh \rho_s \cos \psi_f \quad (1)$$

to relate the pre- and post-strain shapes of an elliptical strain marker to the strain ellipse, where the reference orientation is assumed to be parallel to the maximum elongation, i.e., $2\phi_s = \psi_s = 0$. In fact, Eqs. (1) is termed the law of cosines in hyperbolic trigonometry, which relates the sides and interior angles of a triangle (Fig. 2) in a hyperbolic space (Ratcliffe, 2006). On the other hand, Lisle (1985) shows an equation for the change in principal axes of an ellipse,

$$\sinh \rho_i \sin(\pi - \psi_i) = \sinh \rho_f \sin \psi_f. \quad (2)$$

This equation relates the sides to their opposite interior angles of the triangle (Fig. 2), and is called the law of sines in hyperbolic trigonometry.

Hyperbolic geometry gives a useful point of view for statistical processing of two-dimensional strains, because the geometry allows us to evaluate dissimilarity be-

tween strain ellipses. Once dissimilarity or distance between data is properly defined, data analysis can take advantage of various statistical techniques (e.g., Aitchison, 1986; Small, 1996; Duda et al., 2001; Egozcue et al., 2003; Sato and Yamaji, 2006; Yamaji and Sato, 2006). To illustrate this, suppose that we obtained many estimates of the strain ellipse from the assemblage of strain makers through bootstrap or some other resampling techniques. Then, how can we quantify the variation of the ellipses to estimate the uncertainty of the mean strain ellipse? Definition of the spread requires a measure, i.e., distance or dissimilarity between ellipses. If ellipses have large variations in their R and ϕ values, the spread is judged to be large and the mean has a large uncertainty. So, our question becomes what is the appropriate definition of a parameter space, in which a point has a one-to-one correspondence to the paired data R and ψ . In addition, the space requires an essential attribute: distance in the space should be equal to a strain measure needed to transform one ellipse to another. The conventional R_f/ϕ plot is not appropriate for this purpose. Open circles in Fig. 3 indicate two ellipses with the same aspect ratio and different orientations. Diamonds in this figure denote ellipses with a low aspect ratio. Distance between the circles is the same with that between the diamonds on this plot. However, the fat ellipses are much more alike to each other than the thin ellipses. Difference in ϕ is more significant for thin ellipses than for fat ones, but those distances are the same on this plot. It is unclear whether the dissimilarity between the thin ellipses is larger than that between the elongate and circular ones with the same orientations.

The hyperbolic triangle in Fig. 2 not only suggests the connection between strain analysis and hyperbolic geometry, but also indicates that the hyperbolic space fills the statistical necessity. That is, the distances ρ_i , ρ_f and ρ_s equal the logarithmic quadratic strains needed to transform one ellipse to another denoted by the vertices

91 of the triangle.

92 4 Hyperboloid model

93 Hyperbolic space is an abstract non-Euclidean space, and there are a few methods
94 of visualizing the space. However, all the methods have distortion like cartographic
95 projections of the globe. Among them we use the curved surface, called the unit
96 hyperboloid, for the ease of statistical processing. The hyperboloid gives a useful
97 visual expression of hyperbolic space at the cost of losing the important attribute
98 of this space, i.e., the constant curvature of -1 . The content of the next two sub-
99 sections are found in Reynolds (1993), Nakaoka (1993) and Ratcliffe (2006). The
100 notation of this article basically follows that of Reynolds (1993).

101 4.1 The unit hyperboloid

102 The Cartesian coordinates $O-x_0x_1x_2$ is used throughout of this paper, where O is
103 the coordinate origin. Then, the unit hyperboloid is defined by the equation,

$$104 \quad -x_0^2 + x_1^2 + x_2^2 = -1. \quad (3)$$

105 In this article, only the upper half space $x_0 \geq 0$ is considered. The symbol H^2
106 refers to this surface (Fig. 4). We use the cylindrical coordinates $O-r\psi x_0$ as well,
107 where r is the radial coordinate, ψ is the angular coordinate, and x_0 is the height
108 along the cylinder axis. The Cartesian and cylindrical coordinates have the origin in
109 common. H^2 has axial symmetry about the x_0 -axis, and has a round base at the point
110 C with the Cartesian coordinates $(1, 0, 0)$. It follows that H^2 has the expression $x_0 =$
111 $\sqrt{r^2 + 1}$, and has hyperbolic profiles. H^1 is the hyperbola defined by the intersection

112 of H^2 and the x_0x_1 -plane.

113 Instead of the familiar definition of vector inner product in the Euclidean space, the
114 inner product is defined as

$$115 \quad \mathbf{a} \circ \mathbf{b} = -a_0b_0 + a_1b_1 + a_2b_2, \quad (4)$$

116 which is called hyperbolic inner product. Position vectors are defined as usual: their
117 initial point is fixed at the origin O . Accordingly, they are identified with their end
118 points, and points are denoted by position vectors. In terms of a position vector $\mathbf{x}$,
119 the cone $x_0 = r$ illustrated in Fig. 4 is written as $\mathbf{x} \circ \mathbf{x} = 0$. It is known that the
120 inner product of the position vectors $\mathbf{a}$ and $\mathbf{b}$ with the endpoints above the cone is
121 non-positive in sign, $\mathbf{a} \circ \mathbf{b} \leq 0$ (Ratcliffe, 2006, p. 56).

122 4.2 Distances, geodesics and translations on H^2

123 In terms of a position vector $\mathbf{x}$, H^2 is expressed as $\mathbf{x} \circ \mathbf{x} = -1$. Compared to
124 the equation of the unit sphere, $\mathbf{x} \cdot \mathbf{x} = 1$ and considering the curvature -1 over
125 the entire surface, H^2 is sometimes called a pseudosphere with the radius of $\sqrt{-1}$
126 (Vilenkin, 1968). Accordingly, spherical geometry and spherical statistics are use-
127 ful guides for our study.

128 Given two points on H^2 , the geodesic between the points is the shortest path on H^2 .
129 This is analogous to a great circle, which is the intersection of a plane through the
130 origin and the sphere. Likewise, a geodesic on H^2 is defined by the intersection of a
131 plane through the origin and H^2 . H^1 is a geodesic through the point C , and is defined
132 by the intersection of the x_0x_1 -plane and H^2 (Fig. 4). Accordingly, a geodesic can be
133 identified with the vector $\mathbf{v}$ perpendicular to the intersecting plane and the distance
134 of the plane from O .

Length of the vector $\boldsymbol{x}$ is given by

$$\|\boldsymbol{x}\| = \sqrt{-\boldsymbol{x} \circ \boldsymbol{x}}. \quad (5)$$

Due to the minus sign, the right-hand side of this equation is real for position vectors if their end points exist in the region above the cone. Distances on H^2 are measured along geodesics. That is, the distance between the points $\boldsymbol{a}$ and $\boldsymbol{b}$ upon H^2 is calculated by

$$d_H = \cosh^{-1}(-\boldsymbol{a} \circ \boldsymbol{b}), \quad (6)$$

analogous to the great-circle distance on the unit sphere expressed as $\cos^{-1}(\boldsymbol{a} \cdot \boldsymbol{b})$. It follows from Eqs. (4) and (6) that $\cosh^{-1} x_0$ is the distance between C and $\boldsymbol{x} = (x_0, x_1, x_2)^\top$ on H^2 . Let us use the symbol ϱ to refer to this distance. Then, we have

$$x_0 = \cosh \varrho = \frac{1}{2} \left(R + \frac{1}{R} \right). \quad (7)$$

H^2 can be written in the cylindrical coordinates as $x_0 = \sqrt{r^2 + 1}$. Combining this and Eq. (7) we obtain

$$r = \sinh \varrho = \frac{1}{2} \left(R - \frac{1}{R} \right). \quad (8)$$

Therefore, $C\text{-}\varrho\psi$ can be used as a coordinate system on H^2 with the origin at C (Fig. 4). This is sometimes called hyperbolic spherical coordinates (Barndorff-Nielsen, 1987). A point on H^2 can be expressed as

$$\boldsymbol{x}(\varrho, \psi) = (\cosh \varrho, \sinh \varrho \cos \psi, \sinh \varrho \sin \psi)^\top. \quad (9)$$

Just as translation or movement of a rigid body on the unit sphere is described by an Euler rotation which is further denoted by an orthogonal matrix, translation on H^2 is expressed by a hyperbolic orthogonal matrix with the determinant 1. If the translation is along the geodesic that is defined by a plane containing the x_0 -axis,

the matrix has the form

$$Q(\varrho, \psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} \cosh \varrho & \sinh \varrho & 0 \\ \sinh \varrho & \cosh \varrho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & \sin \psi \\ 0 & -\sin \psi & \cos \psi \end{pmatrix} \quad (10)$$

where ψ denotes the direction of translation.

Translation does not change distances between points so that any figure defined by a set of points does not change its shape and size on H^2 . Consider the congruent triangles ABC and $A'B'C'$ in Fig. 5; A' coincides with C and sides AC and $A'C'$ are on H^1 . They have the sizes a, b and c , and the opposite interior angles α, β and γ . The translation along H^1 from $A'B'C'$ to ABC is denoted by $Q(-b, 0)$. The apices B and B' are denoted by the vectors $x(B) = (\cosh a, \sinh a \cos \gamma, \sinh a \sin \gamma)^\top$ and $x(B') = (\cosh c, \sinh c \cos(\pi - \alpha), \sinh c \sin(\pi - \alpha))^\top$. These vectors are related through an orthogonal matrix as

$$\begin{pmatrix} \cosh c \\ \sinh c \cos(\pi - \alpha) \\ \sinh c \sin(\pi - \alpha) \end{pmatrix} = \begin{pmatrix} \cosh b & -\sinh b & 0 \\ -\sinh b & \cosh b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh a \\ \sinh a \cos \gamma \\ \sinh a \sin \gamma \end{pmatrix}.$$

Extracting the first and third rows of the both sides of this equation, we have

$$\cosh c = \cosh b \cosh a - \sinh b \sinh a \cos \gamma \quad (11)$$

and

$$\sinh c \sin(\pi - \alpha) = \sinh a \sin \gamma. \quad (12)$$

These are the cosine and sine laws of hyperbolic trigonometry. Replacing a, b, c, α and γ by $\rho_f, \rho_s, \rho_i, \psi_i$ and ψ_f , respectively, Eqs. (11) and (12) becomes Eqs. (1) and

(2). Consequently, formulas of strain analysis correspond to those of hyperbolic trigonometry.

The above correspondence demonstrates that the deformation of elliptical objects in the physical space is denoted by a translation of this kind on H^2 . In addition, ϱ equals ρ , meaning that the strain needed to make one ellipse to another equals the geodesic distance between the points representing the ellipses. Consequently, H^2 has ideal properties as the parameter space for dealing with R_f/ϕ data.

As a result, deformation in the physical space is simply denoted by the linear transformation of hyperbolic vectors such that

$$\boldsymbol{x}(\rho_f, \psi_f) = \boldsymbol{Q}(\rho_s, \psi_s) \boldsymbol{x}(\rho_i, \psi_i), \quad (13)$$

where $\boldsymbol{x}$ and $\boldsymbol{Q}$ has been defined in Eqs. (9) and (10). As the strain ellipse is represented by a point $\boldsymbol{x}_s$ on H^2 , Eq. (13) is rewritten as

$$\boldsymbol{x}_f = \boldsymbol{Q}(\boldsymbol{x}_s) \boldsymbol{x}_i. \quad (14)$$

Eq. (13) is simplified to

$$\boldsymbol{x}(\rho_f, \psi_f) = \begin{pmatrix} \cosh \rho_i \cosh \rho_s \\ \cosh \rho_i \sinh \rho_s \\ \sinh \rho_s \end{pmatrix} \quad (15)$$

for the translation along H^1 of the points with the initial hyperbolic spherical coordinates $(\rho_i, \pi/2)$. Eq. (15) denotes the curve with the variable ρ_s that keeps the constant distance ρ_i from H^1 , analogous to the relationship of a small and great circle on a sphere.

194 4.3 *General shear*

195 Various deformation types in the physical space are distinguished by the kinematic
196 vorticity number W_k (e.g., Ramberg, 1975; De Paor, 1993; Passchier and Trouw,
197 2005), which has values 0, 1 and > 1 for pure, simple and super shear, respectively.
198 W_k is not defined in terms of deformations, but of the velocity field resulting in those
199 deformations. That is, W_k is the ratio of asymmetric and symmetric parts of velocity
200 gradient tensor $\mathbf{L}$ (Truesdell, 1954), which is related to the deformation gradient
201 tensor $\mathbf{F}$ and its material derivative $\dot{\mathbf{F}}$ via the equation $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ (Chadwick,
202 1999, p. 65). $\mathbf{L}$ and $\mathbf{F}$ are different quantities even for a steady flow. However, W_k
203 has a qualitative relationship with the hyperboloid model as follows.

204 In analogy with small circles on a sphere, there are three kinds of curves on H^2
205 (Reynolds, 1993), each of which corresponds to a type of deformation in the phys-
206 ical space. The curves are defined by the intersection of planes and H^2 so that a
207 point $\mathbf{x}$ on such a curve satisfies the linear equation

$$208 \quad \mathbf{x} \circ \mathbf{v} = -\kappa. \quad (16)$$

209 Those curves are termed ‘cycles’ in hyperbolic geometry. The vector $\mathbf{v}$ is called
210 the pole of the plane, indicating the attitude of the plane in M^3 , and κ indicates the
211 position of the plane. Planes with $\kappa = 0$ include the origin O so that the curves
212 defined by those planes are geodesics.

213 4.3.1 *Pure shear*

214 Pure shear has been explained in the previous subsection: such a deformation car-
215 ries points on H^2 along a geodesic on the planes parallel to the x_0 -axis or along a
216 line that keeps a constant hyperbolic distance from such a geodesic (Fig. 6). The

217 curve shown by Eq. (15) is the one along H^1 , and is regarded as the ‘strain path’ of
218 an initial ellipse represented by a point on the curve for the growing strain increas-
219 ing with ρ_s . Those planes have a common pole vector $\boldsymbol{v}$ lying on the x_1x_2 -plane. In
220 this case, the pole vector satisfies $\boldsymbol{v} \circ \boldsymbol{v} > 0$ (Reynolds, 1993). And, the W_k value is
221 0, equivalent to the ratio v_0/v_r of the cylindrical components of the vector $\boldsymbol{v}$.

222 4.3.2 General shear between pure and simple

223 General shear between pure and simple has strain paths on the intersection of H^2
224 and the plane defined by the pole vectors between the x_1x_2 -plane and the cone in
225 Fig. 4. This case is considered in detail for kinematic vorticity analysis in §5.2. The
226 geodetic corresponding to this vector is asymptote to the cone in the two differ-
227 ent directions ψ_1 and ψ_2 in Fig. 7, corresponding to the fact that the straining and
228 destraining of an ellipse leads to ellipses with different stretching orientations in
229 physical space. In this case, we have $0 \leq W_k \leq 1$, concordant to the range of the
230 ratio v_0/v_r .

231 Regarding kinematic vorticity analysis (§5.2), the fabric attractor (flow plane) is
232 represented by a geodetic on H^2 , and the ratio v_0/v_r of the geodetic is exactly equal
233 to the mean kinematic vorticity number (Appendix B).

234 4.3.3 Simple shear

235 The strain path of simple shear is represented by a ‘horocycle’ on H^2 , i.e., a curve
236 lying on a plane with $\boldsymbol{v}$ tangent to the cone in Fig. 4. This vector satisfies $\boldsymbol{v} \circ \boldsymbol{v} = 0$.
237 As a result, the curves are parabolic on H^2 (Fig. 8). A horocycle is asymptote to the
238 same point at infinity, corresponding to the fact that infinite simple shear results in
239 an infinitely long and thin ellipse irrespective of the sense of shear. Horocycles in

Fig. 8 are asymptote to H^1 in the $+x_1$ direction. In this case, the value of W_k equals 1. This is equivalent to the ratio of the cylindrical coordinates of the vector v_0/v_r .

4.3.4 *Super shear*

Super shear or super simple shear is characterized by W_k values greater than 1, and is the behavior of fluid inclusions immersed in a fluid with different viscosity subject to shearing (e.g., De Paor, 1983; Mulchrone and Walsh, 2006). Those inclusions change their aspect ratios and orientations synchronously. Such a behavior is illustrated by a circular orbit on H^2 (Fig. 9). The orbit lies on a plane with v included by the cone. In this case, the pole vector satisfies $v \circ v < 0$ (Reynolds, 1993). The range of W_k is concordant with that of the ratio v_0/v_r .

4.3.5 *Rigid body rotation*

Rigid-body rotations in the physical space are represented by the circular orbits on the planes perpendicular to the x_0 -axis, i.e., $v = 1$. Rigid-body rotations has the value $W_k = \infty$, concordant with $v_0/v_r = \infty$.

4.4 *Cartography*

Projections from H^2 onto a Euclidean plane are convenient tools for investigating spatial data on H^2 , although the projections inevitably have distortion like those from the globe onto a plane. Consider the mapping from a point with polar coordinates (ρ, ψ) on H^2 to the polar coordinates (ζ, θ) on the Euclidean plane (Fig. 10). We consider only azimuthal projections, meaning that $\theta = \psi$. All the projections in this subsection preserve azimuths from C to any point on H^2 , but the conventional

261 R_f/ϕ plot (Fig. 3) does not.

262 4.4.1 Equal-area projection

263 The projection denoted by the equation

$$264 \quad \zeta = 2 \sinh(\rho/2) = R^{1/2} - R^{-1/2} \quad (17)$$

265 preserves areas (Reynolds, 1993), meaning that the area of any closed region on H^2
266 equals that of the corresponding region on the plane (Fig. 11). $(R^{1/2} - R^{-1/2})$ is the
267 diameter of the Mohr circle of strain (Brace, 1961). This projection is useful for
268 displaying the variation of ellipses, because the variation is visualized by the dis-
269 persion of data points on H^2 . The area preservation guarantees that density of points
270 on H^2 is preserved on this map. Straining of ellipses in the physical space corre-
271 sponds to the translation of points on H^2 , which does not affect their density. So,
272 the points indicating pre- and post-strain ellipses have the same dispersion on this
273 map. For this reason, this projection gives the best visualization for the dispersion
274 of R_f/ϕ data.

275 4.4.2 Gnomonic projection

276 Gnomonic projection transforms a point on H^2 along a line through the origin O
277 onto the disk that is denoted by $x_0 = 1$ and $x_1^2 + x_2^2 < 1$. Points on the fringe of
278 this disk (∂D in Fig. 4) represent points at infinity, and the entire surface of H^2 is
279 mapped onto the disk (Fig. 12). This projection has the expression,

$$280 \quad \zeta = \tanh \rho = (R^2 - 1)/(R^2 + 1), \quad (18)$$

281 where $0 \leq \zeta < 1$ (Fig. 10). Geodesics on H^2 are mapped to straight lines on the
282 disk. A horocycle (Fig. 8) is mapped to an ellipse that inscribes ∂D . Specifically, a

283 horocycle through C is shown on this projection by a circle.

284 This projection is convenient for determining the mean kinematic vorticity number
285 (§5.2). The plot by Jessup et al. (2007) was devised for this purpose, and takes
286 $\tanh \rho$ and ϕ as rectangular Cartesian coordinates instead of the polar coordinates
287 of the gnomonic net proposed in this article.

288 4.4.3 *Equidistant projection*

289 The plot of Elliott (1970) is the equidistant projection of H^2 , and has the relation-
290 ship $\zeta = \rho = \log R$. In spite of the adjective ‘equidistant,’ distances on the map is
291 correct only from the map center C. Distances between other points are not. The
292 name of this projection is used by Reynolds (1993) to emphasize the correspondence
293 to the equidistant projection of the globe.

294 This projection is useful, if one needs the directions and distances of points from
295 a specific location on H^2 . In this case, the location along with the points to the
296 point C on H^2 using Eq. (14). Then, the equidistant projection of the points shows
297 the directions and distances from the specified location to the original points. This
298 method is used in §5.2 for error estimation of R_f/ϕ strain analysis. The hyperbolic
299 net of De Paor (1988) is a variant of this projection in that the hyperbolic net uses
300 the angle ϕ (Fig. 1), instead of the double angle ψ , as the tangential components.

301 4.4.4 *Orthographic projection*

302 The plot by Wheeler (1984) is the orthographic projection of H^2 , i.e., the orthogonal
303 projection onto the x_1x_2 -plane. This projection is denoted by the equation, $\zeta =$
304 $\sinh \rho$ (Eq. 8). It is the strong point of this projection that strain paths for coaxial

305 strain are expressed by parallel lines on the plot. The trend of the lines corresponds
306 to the maximum stretching axis in the physical space. Therefore, it is easy to see if
307 an ellipse can be transformed from another by strains with a prescribed stretching
308 axis.

309 5 Strain analysis

310 Aspect ratios and orientations of pre- and post-strain ellipses are related to each
311 other by a couple of complicated equations (Ramsay, 1967, pp. 205–209). By
312 means of hyperbolic geometry, this relationship is simplified to the linear trans-
313 formation in Eq. (13). And, the theories of two-dimensional strain analysis are sim-
314 plified as well. We consider a few of them in this section.

315 5.1 Coaxial strain determined from R_f/ϕ data

316 First, we consider the determination of coaxial strain from R_f/ϕ data obtained from
317 elliptical strain markers such as pebbles (Flinn, 1956; Hossack, 1968) and ooids
318 (Cloos, 1947) or from non-elliptical fossils (e.g., Lisle, 1985) observed on a section
319 of a deformed rock. A homogeneous deformation of the markers and their matrix is
320 assumed as usual. It is shown that the confidence region of inferred strain is easily
321 obtained in our formulation.

322 Suppose that we observed n elliptical strain markers, and that the shape and orien-
323 tation of the markers are transformed into the data points $\mathbf{x}_f^{(1)}, \dots, \mathbf{x}_f^{(n)}$ on H^2 via
324 Eq. (9). Here, our task is to determine the point $\mathbf{x}_s$ indicating the optimal strain for
325 the data and its error.

5.1.1 Hyperbolic vector mean

To determine the optimal strain, we make a simple assumption for the pre-strain distribution of the points $\{\mathbf{x}_i^{(1)}, \dots, \mathbf{x}_i^{(n)}\}$. That is, the points have the centroid at the point C on H^2 . This is denoted by the equation,

$$[\mathbf{x}_i^{(1)} + \dots + \mathbf{x}_i^{(n)}] / \|\mathbf{x}_i^{(1)} + \dots + \mathbf{x}_i^{(n)}\| = (1, 0, 0)^\top. \quad (19)$$

Note that strain is denoted by translation upon H^2 , and the translation does not affect distances between points on H^2 . It follows that, on the one hand, the strain that carries $\{\mathbf{x}_i^{(1)}, \dots, \mathbf{x}_i^{(n)}\}$ to $\{\mathbf{x}_f^{(1)}, \dots, \mathbf{x}_f^{(n)}\}$ transforms the centroid of the former set to that of the latter. On the other hand, the point C represents the unit circle in the physical space, and the strain ellipse is the result of strain from the unit circle. Therefore, the latter centroid coincides with the point on H^2 that represents the strain ellipse. Determination of the optimal strain is surprisingly simple in the present formulation: the strain for the data is represented by the mean,

$$\hat{\mathbf{x}}_s = \boldsymbol{\lambda} / \|\boldsymbol{\lambda}\|, \quad (20)$$

where

$$\boldsymbol{\lambda} = \mathbf{x}_f^{(1)} + \dots + \mathbf{x}_f^{(n)} \quad (21)$$

and $\|\boldsymbol{\lambda}\|$ are termed the resultant vector and the resultant length of data vectors, respectively (Jensen, 1981). The same terms are used in spherical statistics (Mardia, 1972; Mardia and Jupp, 1999). It is straightforward to take into account measurement errors of data. The weighted mean

$$\hat{\mathbf{x}}_s = \frac{w^{(1)}\mathbf{x}_f^{(1)} + \dots + w^{(n)}\mathbf{x}_f^{(n)}}{\|w^{(1)}\mathbf{x}_f^{(1)} + \dots + w^{(n)}\mathbf{x}_f^{(n)}\|}$$

deals with those data, where $w^{(i)}$ is the weight of the i th datum.

Once the optimal point $\hat{\mathbf{x}}_s$ is obtained, the corresponding ρ - and ψ -values are cal-

culated by the inverse of Eq. (9). That is,

$$\rho = \cosh^{-1} x_0 = \log \left(x_0 + \sqrt{x_0^2 - 1} \right) \quad (22)$$

$$\psi = \text{atan2}(x_2, x_1) \quad (23)$$

where $\text{atan2}(x_2, x_1)$ denotes the arctangent of x_2/x_1 and has the range $(-\pi, \pi]$ corresponding to the quadrant of the $x_1 x_2$ -plane. In this way, the optimal values $\hat{\rho}_s, \hat{\psi}_s$, $\hat{R}_s = \exp(\hat{\rho}_s)$ and $\hat{\phi}_s = \hat{\psi}_s/2$ are derived from $\hat{x}_s$.

5.1.1.1 Example 1 The method of hyperbolic vector mean was tested, first, with the natural R_f/ϕ data from the photomicrograph of 282 deformed ooids in Ramsay and Huber (1983). The data were transformed into position vectors $\mathbf{x}_f^{(1)}, \dots, \mathbf{x}_f^{(282)}$ via Eq. (15), and the hyperbolic vector mean was calculated using Eqs. (20) and (21). As a result, I obtained the optimal values, $\hat{R}_s = 1.50$ and $\hat{\phi}_s = -46.9^\circ$ (Fig. 13), which are included by the 95% confidence intervals $\hat{R}_s = 1.51 \pm 0.06$ and $\hat{\phi}_s = -46.9^\circ \pm 2.3^\circ$ determined by the method of Yamaji (2005).

5.1.1.2 Example 2 The second data set is artificial for simulating R_f/ϕ data deformed from fossils, which are not necessarily elliptical. The graphical method of Wellman (1962) is usually employed to estimate strain from such fossils, but R_f/ϕ data can be obtained from the ellipses inscribing rectangles that are fitted on certain parts of the fossils (Fig. 14a) (Lisle, 1985). A specific part of fossils has a certain shape with individual variation to some extent if the fossils belong to one paleontological taxon. Accordingly, the ellipses abstracted from that part have a narrow range of R_i values. The artificial data in Fig. 14b were generated with the parameters $R_s = 2$ and $\phi_s = 0$ to simulate such a data set. The present method yielded the solution, $\hat{R}_s = 2.000$ and $\hat{\phi}_s = 0.000^\circ$. That is, the optimal strain ellipse

374 exactly coincided with the assumed one.

375 Theoretically, the present method is perfectly accurate, provided that pre-strain
376 points had the centroid at C. Deviation of the centroid from C directly gives rise to
377 the inaccuracy of the optimal solution. The artificial data in Fig. 14 were generated
378 with the pre-strain points exactly at C. Therefore, the strain estimation was perfect.
379 Uncertainty of the estimated strain comes from the finite number of data.

380 5.1.2 Error analysis

381 The bootstrap method (e.g., Davison and Hinkley, 1997) is employed to draw a
382 confidence region of the optimal strain on H^2 . Suppose that we observed n strain
383 markers, from which n data are resampled with replacement to make $m = 10n$ boot-
384 strap data sets. Then, by means of Eq. (20), the optimal solutions $\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(m)}$ are
385 determined for the bootstrap data sets. The spread of those points around their cen-
386 troid $\hat{\mathbf{b}}$ indicates the uncertainty of the optimal solution $\hat{\mathbf{x}}_s$ (Fig. 15). This centroid
387 is obtained by Eqs. (20) and (21), where $\mathbf{x}_f^{(i)}$ in the latter equation is replaced by
388 $\mathbf{b}^{(i)}$.

389 Following Michael (1987) who estimated the uncertainty of stresses obtained from
390 focal mechanism data by means of bootstrap resampling, a confidence region of
391 the optimal strain is defined as a closed region on H^2 . Namely, if a 95% confi-
392 dence region is required, the solutions $\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(m)}$ are sorted in ascending order of
393 their distances from the centroid. Then, the confidence region is the polygon that
394 contains only the first $0.95m$ solutions.

395 We consider a method for error estimation for the optimal strain determined from
396 strain markers that may have significant pre-strain anisotropy expressed by the clus-

ter of the points $x_i^{(1)}, \dots, x_i^{(n)}$ with non-rotational symmetry around the point C on H^2 . Sedimentary grain fabric has such anisotropy to some extent (Griffiths, 1967). That is, Potter and Pettijohn (1963, p. 44) pointed out that sedimentary particles often exhibit bimodal orientations separated by $\sim 90^\circ$. This tendency is expressed by an eccentric cluster of data points centered by the point C on H^2 (Yamaji and Masuda, 2005). Other types of initial grain fabric, including strong imbrication, lead to inaccuracy of the present technique. Given the stretching orientation of the sediment, the method of De Paor (1988) can be employed to evaluate the strain from imbricated fabrics.

The artificial R_f/ϕ data in Fig. 14b have such asymmetry as described by Potter and Pettijohn (1963) and Yamaji and Masuda (2005). In those cases, cluster of the points $b^{(1)}, \dots, b^{(m)}$ exhibits significant deviation from rotational symmetry about the point $\hat{b}$, and shows elongation similar to the cluster of the post-strain points $x_f^{(1)}, \dots, x_f^{(n)}$. Dashed line in Fig. 15b shows an example. This line is elliptical on this figure, the eccentricity of which is due to the distortion of the equal-area projection. The region encircled by this line is a circle on H^2 . However, the points $b^{(1)}, \dots, b^{(m)}$ make an elliptical cluster on H^2 . This is evidenced by the dots that are not enclosed by the dashed line in Fig. 15b. They are scattered out of the region depicted by the dashed line. To cope with the eccentric shape of the cluster on H^2 , the confidence region of $\hat{x}_s$ drawn on H^2 should be an oval.

To meet the requirement above, the spread of $b^{(1)}, \dots, b^{(m)}$ is evaluated by the Mahalanobis distances (Duda et al., 2001) from the cluster center $\hat{b}$ on H^2 . For this purpose, this center along with the points is translated to the area around the point C on H^2 so that the new centroid coincides with this point and that the azimuthal equidistant projection can be used to estimate the azimuth and distance of points

from their centroid. Then, the i th point moves to $\tilde{\mathbf{b}}^{(i)} = [\mathbf{Q}(\hat{\mathbf{b}})]^{-1} \mathbf{b}^{(i)}$ (Eq. 14). Since the translation does not affect the relative positions between the points, the centroid of $\tilde{\mathbf{b}}^{(1)}, \dots, \tilde{\mathbf{b}}^{(m)}$ exists at C. Then, the azimuthal equidistant projection of the points preserves the azimuths and distances of the points from C (§4.4.3). Let $\mathbf{y}^{(i)}$ be the projection of $\tilde{\mathbf{b}}^{(i)}$. Then, the points $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}$ make a cluster at the center of equidistant chart. The covariance matrix (Johnson and Wichern, 2003),

$$\mathbf{C} = \frac{1}{m} \left\{ \mathbf{y}^{(1)} [\mathbf{y}^{(1)}]^\top + \dots + \mathbf{y}^{(m)} [\mathbf{y}^{(m)}]^\top \right\},$$

characterizes the eccentric cluster shape, thereby the Mahalanobis distance of $\mathbf{y}^{(i)}$ from the center is written as $\sqrt{[\mathbf{y}^{(i)}]^\top \mathbf{C}^{-1} \mathbf{y}^{(i)}}$. This distance is used as the key to sort the bootstrap solutions $\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(m)}$. Namely, the solutions are renumbered to have the ascending order of the Mahalanobis distances of $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}$. If we want to determine the $(100\alpha)\%$ confidence region, the polygon enveloping only the points $\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(100\alpha m)}$ on H^2 approximates the confidence region of the optimal solution $\hat{\mathbf{x}}_s$, where $0 < \alpha < 1$ (Fig. 15). For example, the value $\alpha = 0.95$ is used when 95% confidence region is required.

Once the confidence region is drawn on H^2 , we use the method used by Yamaji (2005) to evaluate the confidence intervals of the optimal R - and ϕ -values. First, the circles that inscribe and circumscribe the confidence region are drawn on H^2 to determine $\Delta_1\rho$ and $\Delta_2\rho$ (Fig. 16). The sector that has the apex at the origin with the smallest apical angle to include the region is also used to define $\Delta_1\psi$ and $\Delta_2\psi$. The former pair are transformed into the lower and upper confidence limits for the optimal aspect ratio $\hat{R}$ as $\exp(\hat{\rho}_s - \Delta_1\rho)$ and $\exp(\hat{\rho}_s + \Delta_2\rho)$, respectively. Those of the optimal major axis orientation are $(\hat{\psi}_s - \Delta_1\psi)/2$ and are $(\hat{\psi}_s + \Delta_2\psi)/2$. If the confidence region includes the origin, these error bounds become indeterminate. The 95% confidence intervals for the example in Fig. 13 are $\hat{R}_s = 1.50 \pm 0.06$ and

447 $\hat{\phi}_s = -46.9^\circ \pm 2.6^\circ$, largely identical with those of Yamaji (2005). The confidence
448 intervals for the example in Fig. 14 are $\hat{R}_s = 2.000^{+0.114}_{-0.108}$ and $\hat{\phi}_s = 0.000^{+2.167^\circ}_{-2.157^\circ}$. For
449 the case of the data with strong anisotropy in Fig. 15, the optimal strain has the
450 parameters, $\hat{R}_s = 2.717^{+0.236}_{-0.219}$ and $\hat{\phi}_s = 0.556^{+1.806^\circ}_{-2.071^\circ}$.

451 5.2 Kinematic vorticity analysis

452 Rotational wakes of rigid particles are used to infer the vorticity of the paleo flow
453 field in sheared rocks. Specifically, kinematic vorticity analysis estimates the kine-
454 matic vorticity compatible with those flow patterns (e.g., Ghosh and Ramberg,
455 1976; Means et al., 1980; Passchier and Simpson, 1986; Passchier, 1987; Wallis
456 et al., 1993; Wallis, 1995; Simpson and De Paor, 1997). The gnomonic projection
457 gives a simple graphical technique to solve this problem.

458 Assuming that particle shapes are approximated by ellipses with the aspect ratio
459 $R = \exp(\rho)$ and that the flow field is the superposition of simple and pure shears,
460 Bobyarchick (1986) derived a formula for describing the attractor of the major-axis
461 orientations of the particles as a function of the aspect ratio of a particle and W_m , the
462 mean kinematic vorticity number of the flow field. According to Passchier (1987,
463 p. 685), the function is rewritten as

$$464 \quad \psi = \sin^{-1} \left[\frac{W_m}{\tanh \rho} \left(\sqrt{1 - W_m^2} - \sqrt{\tanh^2 \rho - W_m^2} \right) \right] \quad (24)$$

465 where ρ and ψ are hyperbolic spherical coordinates. The angle ψ is measured from
466 the shearing direction. Circular particles rotate in the rotation sense concordant with
467 the vorticity of the flow, but long particles can rotate backward. The line denoted
468 by Eq. (24) is the separatrix, meaning that the line separates H^2 into two regions.
469 That is, particles represented by points on the different sides of the line rotate in

different senses. So, Simpson and De Paor (1993, 1997) proposed the porphyroclast hyperbolic distribution method, which uses the rotation senses in conjunction with ρ and ψ values of particles to determine the position of the separatrix and further to constrain a W_m value. Simpson and De Paor showed that a hyperbola enclosing all points representing back-rotated particles indicates this line on their hyperbolic net (De Paor, 1988). A limb of the hyperbola is asymptote to the shearing direction.

The method of Simpson and De Paor (1993, 1997) is simplified by means of the gnomonic net, where the entire surface of H^2 is mapped onto the disk with the unit radius (Fig. 12), whereby W_m value is obtained by simple graphical procedures. On this net, the separatrix is a chord, an end of which exists at the point of infinity with the parameters $\rho = \infty$ and $\psi = 0$. The other end indicates the W_m value (Fig. 17). The W_m value is determined by finding a straight line on the net to separate points of back-rotated particles from other points (Appendix A).

Fig. 17 shows an example of the graphical method applied to artificial data. This is a difficult example in that the cluster of points corresponding to back-rotated particles partly overlaps that of other particles. Dashed lines in this figure are chords meeting at the point of infinity parallel to the foliation, and show the possible range of the separatrix. That is, if the line through the data point with the label 'a' is chosen, the upper-right side of the line includes not only the data with backward rotations but also four data points with forward rotations. If the other line is chosen, the lower-left side of the line includes all the data points with forward rotations and those with backward rotations as well. There is no line for the clear separation of rotation senses.

This problem is a special case of linear discriminant analysis, which seeks a line to separate data points on a plane according to the labels linked to the data (Duda et

al., 2001). In the case of Fig. 17a, the separator is a ray from the point indicated by the triangle in the figure, and the labels are the rotation senses.

Here, we evaluate the goodness of position of the borderline to determine the optimal value and uncertainty of W_m . Points corresponding to back-rotated particles should be plotted on the upper-right side of the borderline, but the points are plotted on the both sides of a borderline in this range. Let n_B be the number of those points on the lower-left side of a borderline, and n_F be the number of points corresponding to forward-rotated particles on the upper-right side of the line. The total number of data is n . Accordingly,

$$ER = \max(n_B, n_F)/n \quad (25)$$

is the error rate of the position of the borderline. Namely, the optimal value of W_m is determined by positioning the separatrix to minimizing this rate.

In order for counting the numbers n_F and n_B , we need a simple criteria to judge which side of the separating line a data point exists on the gnomonic map. To understand the criteria, suppose the point P in Fig. 18. The W_m value corresponding to the line AP is obtained from the polar coordinates, ζ and ψ , of P as follows. Since the right triangles ABC and CBE in Fig. 18 are similar to each other, the line AP is expressed in terms of the Cartesian coordinates A- pq as $q = p \tan(\pi/2 - \theta_0)$. This tangent equals the ratio of the lengths FP and AF, which further equal $\zeta \sin \psi$ and $(1 - \zeta \cos \psi)$, respectively. It follows that

$$\tan(90^\circ - \theta_0) = \zeta \sin \psi / (1 - \zeta \cos \psi). \quad (26)$$

It follows from Eqs. (26) and (A.4) that

$$W_m = RHS / \sqrt{RHS^2 + 1}, \quad (27)$$

where RHS is the right-hand side of Eq. (26).

519 In the case of Fig. 17, the range $W_m = 0.493\text{--}0.583$ indicates the maximum possible
520 uncertainty. The ER value is minimized in the interval $W_m = 0.535\text{--}0.556$, the
521 midpoint of which gives the optimal W_m value at 0.546. In case where the clusters of
522 different rotation senses are clearly separated on the gnomonic projection, there is
523 the interval of W_m where $n_B = n_F = 0$ (Fig. 19). The interval marks the uncertainty
524 of W_m value for the given data set, and the optimal W_m value is obtained as the
525 midpoint of the interval.

526 6 Comparison with previous methods

527 6.1 Techniques for determining coaxial strain

528 6.1.1 Method of Mulchrone et al. (2003)

529 The method of Mulchrone et al. (2003) for determining optimal strain from R_f/ϕ
530 data is equivalent to the present one, though formulations are different. The methods
531 have different techniques for error estimation.

532 First, Mulchrone et al. (2003) normalize an ellipse by its area being equal to π .
533 Then, they express an ellipse using its moving radius $\ell(\varphi)$ (Fig. 1). Their basic idea
534 is that the mean of moving radii of post-strain ellipses may approximate the strain
535 ellipse, if the ellipses have the centers at the same point (Fig. 20). To this end, they
536 introduce the expression

$$537 \quad 1/\ell^2 = p - (m \cos \psi) \cos 2\varphi - (m \sin \psi) \sin 2\varphi \quad (28)$$

538 for an ellipse with the major-axis orientation ϕ , where the parameters $p = R/2 +$
539 $1/2R$ and $m = R/2 - 1/2R$ characterize the ellipse shape (Mulchrone et al., 2003,

Eq. 10). Using Eqs. (7) and (8), we obtain $p = \cosh \rho = x_0$ and $m = \sinh \rho = r$, where r and x_0 are the cylindrical coordinates of a point $\mathbf{x} = (p, m \cos \psi, m \sin \psi)^\top$ on H^2 (Fig. 4). It follows from Eq. (28) that

$$1/\ell^2 = -\mathbf{x} \circ \boldsymbol{\chi}, \quad (29)$$

where $\boldsymbol{\chi} = (1, \cos 2\varphi, \sin 2\varphi)^\top$ indicates a point on ∂D in Fig. 4. The right-hand side of Eq. (29) is guaranteed to be positive in sign, because both $\mathbf{x}$ and $\boldsymbol{\chi}$ have the end points above and on the cone (§4.1).

Suppose that we have n ellipses. We take the generalized mean with the power -2 of the moving radii of the n ellipses ℓ_s such that

$$(\ell_s)^{-2} \equiv \frac{1}{n} \left\{ [\ell^{(1)}]^{-2} + \cdots + [\ell^{(n)}]^{-2} \right\}, \quad (30)$$

where $\ell^{(i)}$ is the i th radius. Combining Eqs. (29) and (30), we have

$$\begin{aligned} (\ell_s)^{-2} &= \frac{1}{n} \left[-\mathbf{x}_f^{(1)} \circ \boldsymbol{\chi} - \cdots - \mathbf{x}_f^{(n)} \circ \boldsymbol{\chi} \right] \\ &= -\frac{1}{n} \left[\mathbf{x}_f^{(1)} + \cdots + \mathbf{x}_f^{(n)} \right] \circ \boldsymbol{\chi} \end{aligned}$$

Rearranging this equation, we obtain

$$\left(\sqrt{\|\boldsymbol{\lambda}\|/n} \ell_s \right)^{-2} = -\hat{\mathbf{x}}_s \circ \boldsymbol{\chi}, \quad (31)$$

where $\hat{\mathbf{x}}_s$ is the hyperbolic vector mean in Eq. (20) and $\boldsymbol{\lambda}$ is the resultant vector (Eq. 21). Comparison of Eqs. (29) and Eq. (31) leads us to the interpretation that $\sqrt{\|\boldsymbol{\lambda}\|/n} \ell_s$ is the moving radius of the strain ellipse that is represented by $\hat{\mathbf{x}}_s$. Jensen (1981, p. 200) shows $\|\boldsymbol{\lambda}\| \geq n$, where equality holds only if all the n ellipses are identical. Therefore, ℓ_s denotes the moving radius of an ellipse similar to and smaller than the strain ellipse (Fig. 20). The explanation on ℓ_s by Mulchrone et al. (2003) is misleading, because they suggest as if ℓ_s is the arithmetic mean of the moving radii (especially their Fig. 2).

564 It is obvious in our formulation that the limitations made by Mulchrone et al. (2003,
565 p. 530) are not essential. They limited the applicability of their method to such
566 cases where (1) ψ_i is a uniform random variable on the $[0, 2\pi)$ interval, and (2) the
567 variable R_i is independent from ϕ_i . That is, distribution of the points $\mathbf{x}_i^{(1)}, \dots, \mathbf{x}_i^{(n)}$
568 is assumed to have rotational symmetry with respect to the point C. Their optimal
569 strain is identical with that indicated by our hyperbolic vector mean. The necessary
570 condition of the latter is that the centroid of the points $\mathbf{x}_i^{(1)}, \dots, \mathbf{x}_i^{(n)}$ is located at
571 the point C. This is looser than the two conditions. For example, the data in Fig.
572 14 satisfy neither of their conditions. That is, those points are distributed along
573 an elliptical line on the equal-area projection, but we have obtained the accurate
574 solution via the hyperbolic vector mean.

575 Mulchrone et al. (2003) estimate the confidence intervals of R_s and ϕ_s by means
576 of the bootstrap resampling, and the cluster size of bootstrap solutions on the con-
577 ventional $\log R$ - ϕ plot (Fig. 3) is used to evaluate the intervals. Accordingly, their
578 estimate is largely identical with ours (Fig. 16), when the cluster does not include
579 the point C on H^2 like the region A in Fig. 21. If a confidence region on H^2 include
580 this point like the region B in this figure, the intervals cannot be evaluated.

581 In contrast, our error estimation firstly draw a closed loop on H^2 for denoting a
582 confidence region. So, a confidence region is clearly defined for the both cases in
583 Fig. 21. Then, the confidence intervals of R_s and ϕ_s are evaluated from the loop,
584 but this second step is not done for the case B. Definition of appropriate distance
585 in a parameter space is essential to define such a confidence region whatever the
586 parameter space is. The hyperboloid model meets this demand.

587 6.1.2 Method of Yamaji (2005)

588 The method of Yamaji (2005) calculates an optimal strain ellipse and its error from
589 R_f/ϕ data. It can deal with data with pre-strain anisotropy, but has a weakness. That
590 is, optimal solutions become inaccurate when the spread of data is large (Yamaji,
591 2005, Figs. 9, 10), though the inaccuracy is practically small. The reason for this is
592 now understood. The present method has no such inaccuracy.

593 Yamaji (2005) assumed that the pre-strain points obeys a bivariate normal distri-
594 bution (Johnson and Wichern, 2003) with the mean at the origin of the equidistant
595 projection, which has the polar coordinates ρ and ψ . Let us use the bold capital let-
596 ter $\mathbf{X}$ to represent a point and the corresponding position vector on this projection.
597 Once a point on the projection $\mathbf{X}_s = (\rho_s \cos \psi_s, \rho_s \sin \psi_s)^\top$ is assumed to represent
598 strain, the pre-strain points $\mathbf{X}_i^{(1)}, \dots, \mathbf{X}_i^{(n)}$ are easily obtained from the post-strain
599 points $\mathbf{X}_f^{(1)}, \dots, \mathbf{X}_f^{(n)}$ using Eqs. (1) and (2). So, the method of Yamaji (2005) seeks
600 out the point $\mathbf{X}_s$, thereby the likelihood of the bivariate normality applies best to the
601 pre-strain points. The unlikeness is evaluated by Hotelling's T^2 statistic (Johnson
602 and Wichern, 2003),

$$603 \quad T^2 = [\mathbf{X}_i^{(1)}]^\top \mathbf{S} \mathbf{X}_i^{(1)} + \dots + [\mathbf{X}_i^{(n)}]^\top \mathbf{S} \mathbf{X}_i^{(n)}$$

604 where

$$605 \quad \mathbf{S} = \frac{1}{n-1} \left\{ [\mathbf{X}_i^{(1)}]^\top \mathbf{X}_i^{(1)} + \dots + [\mathbf{X}_i^{(n)}]^\top \mathbf{X}_i^{(n)} \right\}$$

606 is the covariance matrix of the pre-strain points about the origin. $\mathbf{S}$ and T^2 are
607 functions of the data $\mathbf{X}_f^{(1)}, \dots, \mathbf{X}_f^{(n)}$ and of the strain to be determined. Accordingly,
608 the optimal strain for a given data set is obtained by seeking out the strain that
609 minimizes T^2 .

610 This method gradually becomes inaccurate with the increasing spread of those vec-

tors. Two factors affect this accuracy. Distortion of the equidistant projection increases with the distance from the point C on H^2 . Accordingly, if the cluster of the pre-strain points is large, this effect becomes significant. To avoid the distortion, a hyperboloid distribution (Barndorff-Nielsen, 1987; Jensen, 1981) should be utilized instead of the bivariate normal distribution. More important factor is statistical. Maximum likelihood estimation is a popular statistical method used to calculate the best way of fitting a mathematical model for some data (van den Bos, 2007). The estimation has optimal properties for statistical parameter estimation. For example, the estimated parameters asymptotically approaches the true ones with increasing number of data. The minimization of T^2 has no such properties. The maximum likelihood estimate of strain with the assumption above is obtained by maximizing

$$\mathcal{L} = -\frac{n}{2}T^2 - \frac{n}{2}\log |S| - n\log 2\pi. \quad (32)$$

This is called the likelihood function, and is derived in Appendix C. Since S indicates dispersion of the pre-strain points, the effect of the second term in the right-hand side of this equation become significant relative to the first term with the increase of this spread. This leads to the inaccuracy of Yamaji's (2005) method.

The fact is that the hyperbolic vector mean (Eq. 20) is the maximum likelihood estimate of the mean of a hyperboloid distribution (Jensen, 1981). Random work from a point on a Euclidean plane results in a density distribution obeying a bivariate normal distribution, whereas that on a unit sphere obeys a von Mises-Fisher distribution (e.g., Mardia and Jupp, 1999). The distribution on H^2 is known to obey a hyperboloid distribution (Jensen, 1981). Accordingly, the strain ellipse represented by the mean is the most appropriate estimate if we have no *a priori* information on data or strain. Arithmetic mean and standard deviation are useful in many cases when the vectors are in a Euclidean space, even if the statistical distribution of the

vector population is uncertain. The variation of shapes and orientations of deformed fossil part such as those in Fig. 14 does not obey a hyperboloid distribution. However, the hyperbolic vector mean gives accurate strain ellipse in both cases. This is because coaxial strain is represented by a rigid body movement of the cluster of points on H^2 , and the centroid of the initial cluster is mapped to the centroid of the final one. Therefore, the present method determines perfectly accurate strain, provided that the initial centroid was at the point C on H^2 .

6.1.3 *Theta-curve method*

The θ -curve method of Lisle (1977) is the first computerized technique for determining strain from R_f/ϕ data, and utilizes a mathematical inversion with the assumption that the pre-strain orientations $\phi_1^{(1)}, \dots, \phi_1^{(n)}$ obey a uniform distribution. Namely, the optimal strain is determined so as to maximize the χ^2 statistic that is the function of R_s and ϕ_s indicating the uniformity of the orientations.

This inverse method has weakness. First, pre-strain grain fabric is assumed to be isotropic, but sedimentologists have described anisotropic fabrics possibly suggesting sedimentary environments (Potter and Pettijohn, 1963; Griffiths, 1967; Middleton and Southard, 1977). Second, this method is known to be numerically unstable, thereby error estimation based on the method is difficult (Yamaji, 2005). Specifically, χ^2 has multiple peaks for a dataset. As a result, the uncertainty of the optimal strain for the data is denoted not by a simply connected confidence region but by multiple confidence regions on the R_s - ϕ_s plot. In order to stabilize the mathematical inversion to estimate strain, not only the initial orientations but also initial aspect ratios of strain markers should be taken into account (Yamaji, 2005). The methods of Mulchrone et al. (2003) and Yamaji (2005) and of the present article has no such

661 instability.

662 6.2 *Kinematic vorticity analysis*

663 There are a few techniques to estimate mean kinematic vorticity number W_m . Those
664 of Passchier (1987) and Wallis (1992) utilize the critical aspect ratio R_c below
665 which rigid particles continuously rotate and hence their long-axis orientations have
666 a large variation, and above which they display a preferred orientation. R_c has a one-
667 to-one correspondence with W_m . Finding this threshold from shape fabric is feasi-
668 ble, but R_c value is often ambiguous. Many samples show a gradual decrease in the
669 variation of long-axis orientations. As a result, the interval in which the variation
670 exhibits a rapid drop is used as the uncertainty (e.g., Law et al., 2004). Depending
671 on tectonic settings, the gradual change may have several origins, including initial
672 preferred orientations prior to the onset of deformation, heterogeneous flow field
673 owing to the interaction between particles or to the heterogeneous rheology of the
674 matrix.

675 Statistically relevant error estimation is not easy for the technique of finding R_c
676 because of the difficulty of statistical modeling of shape fabric below the thresh-
677 old. It is inappropriate to assume a uniform distribution for long-axis orientations,
678 because rotation velocity depends on the orientation. Although particles fatter than
679 this threshold rotate continuously, their angular velocities depend on long-axis ori-
680 entations (Jeffrey, 1922). As a result, the orientations have tendency to have a pre-
681 ferred orientation around which the rotations are retarded.

682 The hyperbolic distribution method by Simpson and De Paor (1997) and its variants
683 including the techniques using the gnomonic net (§5.2) utilize not only R_c but also

the decreasing pattern to estimate W_m . The advantage of the technique using the gnomonic net is that both an optimal W_m value and its uncertainty are estimated graphically only with a ruler and the net in Fig. 12.

The uncertain statistical distribution of long-axis orientations hinders the quantitative estimation of the specific percentile interval of W_m , e.g., a 95% confidence interval. Owing to this difficulty, we estimate the uncertainty by the interval of misclassification in Fig. 17b or by the gap.

By means of the simplified hyperbolic distribution method, Forte and Bailey (2007) applied bootstrap method only to back-rotated particles for evaluating the confidence interval of W_m . Namely, they dealt only with those particles, and the maximum ψ value of the particles was assumed to be equal with θ_0 in Figs. 18 and A1. Then, the W_m value corresponding to the maximum was calculated through Eq. (A.4). Bootstrap samples were taken only from those particles, and the maximum was determined for each of the samples to calculate W_m . The maximum ψ value of the samples was assumed to be equal to that of the population which is represented by the samples. It follows that the true population maximum was smaller than the observed maximum. The particles with large ψ values are thought of as ‘outliers,’ meaning that increasing number of back-rotated particles was expected to clarify those being exceptional. However, the assumption does not always hold. Instead, it is improbable that there is a generally relevant statistical distribution for the shape fabric of porphyroclasts. It is equally possible that the statistical distribution of ψ of back-rotated particles have a heavy tail, in which samples have large ψ values. Their error analysis is not appropriate in this case.

707 Acknowledgments

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709 their comments and criticisms to improve the manuscript.

710 A Gnomonic projection and vorticity analysis

711 The line defined by Eq. (24) is demonstrated in this appendix to be a geodesic on
712 H^2 using the gnomonic projection. In this case, the geodesic lies on a plane the x_0 -
713 axis. In terms of the polar coordinates (ζ, θ) on the gnomonic projection (§4.4.2),
714 Eq. (24) is simplified to

$$715 \quad \zeta \sin \theta = W_m \left(\sqrt{1 - W_m^2} - \sqrt{\zeta^2 - W_m^2} \right), \quad (A.1)$$

716 where the relationship between ζ and R is given by Eqs. (18), and $\theta = \psi = 2\phi$ is
717 assumed for the projection. Points at infinity from the point C on H^2 have the radial
718 coordinate $\zeta = 1$. The polar coordinates constrained by Eq. (A.1) indicate a line on
719 the gnomonic projection. This line is referred to as L .

720 The points A and B in Fig. A1 are the ends of L . First, we show that the triangle
721 ABC has a right angle at B. The point A is defined by $\zeta = 1$, and represents the
722 infinitely long ellipse, $R = \infty$ and $\phi = 0$. The length AC equals 1, because points
723 at infinity from C on H^2 are mapped onto the unit circle. The end B of L is defined
724 by $\zeta = W_m$, because $\sqrt{\zeta^2 - W_m^2}$ in Eq. (A.1) is imaginary for $\zeta < W_m$. This reflects
725 the fact that circular particles in the physical space cannot rotate backward. At the
726 point B, Eq. (A.1) reduces to

$$727 \quad \sin \theta_0 = \sqrt{1 - W_m^2}. \quad (A.2)$$

The lengths of AC and BC are 1 and W_m , respectively. It follows from Eq. (A.2) that ABC is a right triangle with the side AC being the hypotenuse and $\angle ABC = 90^\circ$. The circle with the diameter AC circumscribes this triangle and inscribes ∂D . Such a circle on the gnomonic projection is a horocycle through the point C on H^2 .

In order to show that the point $P(\zeta, \theta)$ satisfying Eq. (A.1) is constrained on the straight line AB, we assume that P can deviate from this line. The distance of P from AB is referred to as s . It is obvious from Fig. A1 that $\overline{PU} = \zeta \sin \theta$, which is rewritten by Eq. (A.1) as

$$\overline{PU} = W_m \left(\sqrt{1 - W_m^2} - \sqrt{\zeta^2 - W_m^2} \right). \quad (A.3)$$

The right triangle ABC has the side $\overline{BC} = \sqrt{1 - W_m^2}$. So, $\overline{BV} = \cos \theta_0 \sqrt{1 - W_m^2}$. The right triangle CPS has the side $\overline{SP} = \sqrt{\zeta^2 - (W_m + s)^2}$, where s is the distance of P from AB. It follows that $\overline{SE} = \cos \theta_0 \sqrt{\zeta^2 - (W_m + s)^2}$. Since W_m has a value in the range from 0 to 1, the formula (Bobyarchick, 1986; Passchier, 1986),

$$\cos \theta_0 = W_m, \quad (A.4)$$

is obtained from Eq. (A.2). Accordingly, we have $\overline{BV} = W_m \sqrt{1 - W_m^2}$. Substituting this into Eq. (A.3), we obtain $\overline{PU} = \overline{BV} - W_m \sqrt{\zeta^2 - W_m^2}$ and $\overline{PF} = W_m \sqrt{\zeta^2 - W_m^2}$. It follows that $\overline{SE} = \overline{PF} + s \sqrt{1 - W_m^2}$. Rewriting this, we arrive at the equation,

$$W_m \sqrt{\zeta^2 - (W_m + s)^2} = W_m \sqrt{\zeta^2 - W_m^2} + s \sqrt{1 - W_m^2}.$$

The deviation s should vanish in order for this equation to hold for any ζ greater than W_m . That is, the line L must coincide with the straight line AB in Fig. A1. Straight lines on the gnomonic projection are images of geodesics on H^2 . Thus, the line defined by Eq. (24) has been proved to be such a geodesic.

Given a W_m value, the point Q can be uniquely determined. That is, CB is the

perpendicular bisector of the chord AQ. So, $\angle ACQ = 2\theta_0$. From Eq. (A.4), we obtain the equation (Bobyarchick, 1986)

$$2\theta_0 = 2 \cos^{-1} W_m. \quad (\text{A.5})$$

B Relationship between pole vector and mean kinematic vorticity number

The separating line AQ in Fig. A1 is a straight line on the gnomonic projection, meaning that it is a shadow of a geodesic on H^2 . The pole vector $\boldsymbol{v}$ of this geodesic is determined as follows. The point B in this figure indicates the point on the geodesic nearest from the point C on H^2 . From Fig. A1 and Eq. (18), we have $\overline{CB} = W_m = \tanh \varrho$, where $\overline{CB}$ and ϱ are the lengths of CB on the projection and on H^2 , respectively. It follows that the point corresponding to B on H^2 has the Cartesian coordinates $(\cosh \varrho, \sinh \varrho \cos \theta_0, \sinh \varrho \sin \theta_0)$. The position vector with those components is perpendicular to the pole vector $\boldsymbol{v}$, which exists on the plane containing the point and the x_0 -axis. Therefore, we obtain $\boldsymbol{v} = (\sinh \varrho, \cosh \varrho \cos(\theta_0 + \pi), \cosh \varrho \sin(\theta_0 + \pi))^T$. Then, inclination of this vector is $v_0/v_r = \sinh \varrho / \cosh \varrho = \tanh \varrho = W_m$, where v_0 and v_r are the cylindrical components of the pole vector.

C Maximum likelihood estimation of bivariate normal distribution

Eq (32) is derived as follows. The pre-strain vectors were assumed to obey the bivariate normal distribution with the probability density

$$p(\boldsymbol{X}_i) = \frac{1}{2\pi|\boldsymbol{S}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{X}_i^T \boldsymbol{S} \boldsymbol{X}_i\right).$$

If data are statistically independent from each other, the probability of obtaining a specific set of R_f/ϕ data equals the product, $p(\mathbf{X}_i^{(1)}) \cdots p(\mathbf{X}_i^{(n)})$. Consequently, the optimal strain ellipse is determined by maximizing this probability or equivalently the logarithm of the probability. Eq. (32) denotes this logarithm. The optimal strain is the most likely estimate for given R_f/ϕ data.

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907 **Figure and Table Captions**

908 Fig. 1. Ellipse with aspect ratio $R = a/b$ and major-axis orientation ϕ . The polar
909 coordinates ℓ and φ denote a point on the ellipse.

910 Fig. 2. Schematic illustration showing a hyperbolic triangle ABC, the vertices of
911 which represent the ellipses A, B and C in physical space. Sides of the triangle
912 are geodesics, the shortest paths connecting the vertices. The symbols ρ_f , ρ_i and ρ_s
913 denote lengths of the sides.

914 Fig. 3. Conventional R/ϕ plot. Open circles and diamonds denote the points on the
915 plot corresponding to black ellipses. R and ϕ are the aspect ratio and major-axis
916 orientation of an ellipse, respectively.

917 Fig. 4. The hyperboloid H^2 defined by Eq. (3) for visualizing hyperbolic geome-
918 try. Cartesian and cylindrical coordinates, $O-x_0x_1x_2$ and $O-r\psi x_0$, have the origin O
919 in common. Intersection of H^2 and the x_0 -axis defines the point C, which has the
920 Cartesian coordinates $(1, 0, 0)$. This point represents the unit circle in the physical
921 space, and other points on H^2 represent different ellipses with different combina-
922 tions of R - and ϕ -values. Thick line H^1 is the geodesic defined by the intersection
923 of H^2 and the x_0x_1 -plane. Dotted line ∂D indicates the circle $x_1^2 + x_2^2 = 1$ in the plane
924 $x_0 = 1$. Dashed lines show the cone that has its apex at the origin and includes ∂D .
925 The hyperboloid asymptotically approaches the cone.

926 Fig. 5. Congruent triangles ABC and A'B'C' on H^2 . The triangles have the apices C
927 and A' at the same point. C is the point where the x_0 -axis meets H^2 (Fig. 4). Sides
928 of the triangles have the lengths a , b and c . The side b is on H^1 .

929 Fig. 6. Solid lines showing the curves on H^2 corresponding to the strain paths of

coaxial strain. White lines on the x_1x_2 -plane are the orthogonal projections of the
curves.

Fig. 7. Geodesic (thick line) on H^2 lying on a plane through the origin. The plane
has the pole vector v between the x_1x_2 -plane and the cone in Fig. 4. White line is
the orthogonal projection of the geodesic onto this plane.

Fig. 8. The horocycles (thin lines) on H^2 that are defined by the intersections of H^2
and the planes with the pole vector v tangent to the cone in Fig. 4. Those curves
represent strain paths for simple shear in the reference orientation. White lines on
the x_1x_2 -plane show the orthogonal projections of the horocycles.

Fig. 9. Solid line on H^2 showing the orbit corresponding to super shear. The orbit
lies on the plane with the pole vector v . Dashed line shows the cone in Fig. 4.

Fig. 10. Section of H^2 through the x_0 -axis for illustrating projections from H^2 onto
the Euclidean x_1x_2 -plane. The hyperbolic distance of P from C is ρ , and P' is the
midpoint. Q' is the orthogonal projection of P', and is the midpoint of the segment
between O and Q_A. The points Q_G, Q_D, Q_O and Q_A are the gnomonic, equidistant,
orthographic and equal-area projection of P, respectively. Dashed lines indicate the
cone in Fig. 4.

Fig. 11. Equal-area net of H^2 . Radial lines and concentric circles are iso- R and ϕ
lines, respectively, for indicating the shapes and orientations of ellipses. Radius of
the circles are given by Eq. (17). Center of this net corresponds to the point C in
Fig. 4, and arrow indicates the reference orientation.

Fig. 12. Gnomonic net of H^2 . Center of this net corresponds to the point C on H^2 ,
and triangle indicates the reference orientation. Eq. (18) gives the radii of iso- R cir-
cles. Mean kinematic vorticity number W_m can be determined with this net (§5.2).

954 The outermost circle with tick marks indicate ∂D in Fig. 4, points on which repre-
955 sent ellipses with $R = \infty$. Decimals attached on the marks indicate the kinematic
956 vorticity number by Eq. (A.5).

957 Fig. 13. (a) Equal-area projection (§4.4) showing the optimal strain (cross) and its
958 95% confidence region (polygon) determined by the present method for the R_f/ϕ
959 data from the deformed ooids in Ramsay and Huber (1983, p. 83). Solid triangle
960 indicates the reference orientation. The parameters ζ (Eq. 17) and ψ are the polar
961 coordinates of the equal-area projection. (b) Close-up of the 95% confidence re-
962 gions and the optimal strains strains determined by the present and Yamaji's (2005)
963 methods. Cross and diamond indicate the optimal solutions by the methods, respec-
964 tively.

965 Fig. 14. Example of non elliptical strain markers. (a) Schematic illustration showing
966 deformed fossil leaves. Ellipses are fitted on parallelograms with the sides being
967 parallel to specific veins. The ellipses give R_f/ϕ data. (b) Equal-area projection
968 showing artificial data simulating the cases like (a). Points corresponding to 200
969 pre-strain ellipses have the centroid at the origin of this plot. Post-strain ellipses
970 were generated from the pre-strain ones with the assumed strain, $R_s = 2$ and $\phi_s = 0$.
971 The pre-strain points have the centroid at C. Solid line centered by the cross denotes
972 the 95% confidence region of the optimal strain ellipse.

973 Fig. 15. (a) Equal-area projection (§4.4) of data points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(100)}$ on H^2 . The
974 points make an elliptical cluster, indicating a pre-strain anisotropic fabric. Cross
975 and white polygon indicate the optimal strain and its 95% confidence region. The
976 parameters ζ (Eq. 17) and ψ are the polar coordinates of the equal-area projec-
977 tion. (b) Close-up of the confidence region (solid line). The optimal solution and
978 its confidence region were determined upon H^2 , and visualized on this Euclidean

979 plane. The 95% confidence region determined not by the Mahalanobis but by the
980 hyperbolic distances are shown by dotted line.

981 Fig. 16. Confidence region (thick line) on H^2 determined as the envelope containing
982 the specified fraction of bootstrap solutions (dots) near the optimal solution (cross).

983 Fig. 17. (a) Schematic illustration showing the determination of the mean kinematic
984 vorticity number W_m on the gnomonic net for the data that consist of the shapes,
985 orientations and rotation senses of 40 rigid particles. Solid lines through the points
986 with the labels ‘a’ and ‘b’ indicate the possible range of the line separating rotation
987 senses. The upper-left ends of the lines indicate the lower and upper limits of the
988 number for this artificial data set. Dotted line from solid circle to triangle indicates
989 the horocycle on H^2 denoted by Eq. (A.2). (b) The optimal W_m value for the data
990 determined as the midpoint of the interval of minimum error rate (Eq. 25).

991 Fig. 18. Schematic illustration for deriving Eq. (26).

992 Fig. 19. Optimal value and uncertainty of W_m .

993 Fig. 20. Three ellipses illustrating three R_f/ϕ data. Dashed and thick lines indicate
994 the strain ellipse determined as the hyperbolic vector mean $\hat{x}_s$ and ℓ_s determined
995 by Eq. (31) for the data.

996 Fig. 21. Confidence regions A and B on H^2 , the latte of which includes the point C.
997 A’ and B’ are the orthogonal projections of the regions onto the x_1x_2 -plane.

998 Fig. A1. Gnomonic projection of H^2 for the explanation of vorticity analysis. Thick
999 curve L is the line denoted by Eq. (A.1). A and B are the ends of the curve. ∂D is
1000 the circle shown in Fig. 4, and represent points at infinity from C on H^2 . P indicates
1001 the point with the polar coordinates (ζ, θ) , which satisfy Eq. (A.1). Dashed line is

- 1002 the semicircle that circumscribes $\triangle ABC$ and inscribes ∂D . Line ST through P is
- 1003 parallel to the side AB . Lines PU and BV are perpendicular to AC .
- 1004 Talbe 1. List of important symbols.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

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Fig. 21

Fig. A1

Table 1: List of important symbols.

$\hat{\mathbf{b}}$	centroid of $\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(m)}$	§5.1.2
$\mathbf{b}^{(i)}$	hyperbolic vector mean of the i th bootstrap sample	§5.1.2
$\mathbf{C}$	point on $\mathbf{H}^2$ with the cartesian coordinates (1,0,0)	Fig. 4
∂D	circle with the radius 1 on the plane $x_0 = 1$ and centered by $\mathbf{C}$	Fig. 4
$\mathbf{H}^1$	one-dimensional hyperbolic space, a special geodesic on $\mathbf{H}^2$	Fig. 4
$\mathbf{H}^2$	two-dimensional hyperbolic space, hyperboloid in $\mathbf{M}^3$	Fig. 4
i, f, s	subscripts for quantities of pre- and post-strain and strain ellipses, respectively	§2
ℓ	moving radius of ellipse	Fig. 1
m	number of bootstrap sample sets	§5.1.2
R	aspect ratio of ellipse	Fig. 1
$\mathbf{v}$	pole vector for defining a geodesic on $\mathbf{H}^2$	§4.3
W_k	kinematic vorticity number	§4.3
W_m	mean kinematic vorticity number	§5.2
$\mathbf{x}$	position vector in $\mathbf{M}^3$ or point on $\mathbf{H}^2$	§4.1, Eq. (9)
$\hat{\mathbf{x}}_s$	hyperbolic vector mean	Eq. (20)
x_0, x_1, x_2	Cartesian coordinates of $\mathbf{M}^3$	Fig. 4
(ζ, θ)	radial and tangential components on the azimuthal projection of $\mathbf{H}^2$	§4.4
$\boldsymbol{\lambda}$	resultant vector	Eq. (21)
ϕ	major-axis orientation of ellipse	Fig. 1
φ	angular coordinate used with ℓ	Fig. 1
ρ	logarithm of R , logarithm of quadratic strain	§2
r, ψ, x_0	cylindrical coordinates of $\mathbf{M}^3$	Fig. 4
ϱ	hyperbolic distance from the point $\mathbf{C}$ on $\mathbf{H}^2$	§4.2
$\circ$	hyperbolic inner product	Eq. (4)
$\ \parallel$	hyperbolic vector length	Eq. (5)