UNKNOTTING SINGULAR CHARTS WITH NO BLACK VERTICES BY REDUCING NODE-PAIRS

Dedicated to Professor Akio Kawauchi's 60th birthday

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Abstract

We determine whether singular charts without black vertices can be deformed to the trivial chart by reducing node-pairs only. It is not true if the degree of the singular chart is at least four, while it is true if the degree is at most three.

Key words: singular chart; 2-dimensional braid; crossing change

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1 Introduction

Kamada in [7] showed that any singular chart without black vertices can be deformed to the trivial chart by introducing and reducing some node-pairs, which corresponds to the statement that any singular 2-dimensional braid without branch points can be deformed to be trivial by some crossing changes and inverses of crossing change. Here a crossing change is in the sense of [2].

We will determine whether the above theorem holds by inverses of crossing change only. It does not hold when the degree is at least four, however it holds when the degree is at most three.

A *singular 2-dimentional braid* of degree m is a smoothly immersed, compact and oriented surface F in a bidisk $D_1^2 \times D_2^2$ whose singularities are transverse double points such that

(i) for an immersion $f : F_0 \to D_1^2 \times D_2^2$ associated with F, the composition $\mathrm{pr}_2 \circ f$ is a simple branched covering map,

(ii) $\mathrm{pr}_1(\partial F) = Q_m$,

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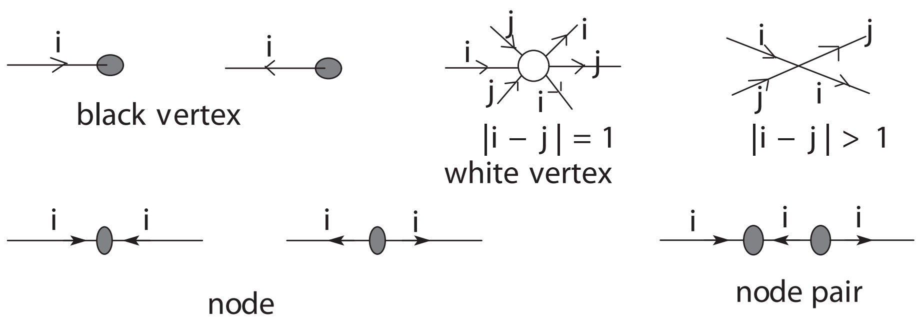

Figure 1.1: Vertices in a singular chart.

(iii) for each $p \in D_2^2$, $\mathrm{pr}_2^{-1}(p)$ contains at most one singular point of F (that is a double point of F or the image by f of a singular point of the simple branched covering map $\mathrm{pr}_2 \circ f$),

where $\mathrm{pr}_i : D_1^2 \times D_2^2 \to D_i^2$ $(i = 1, 2)$ is the i-th projection and Q_m is a fixed set of m distinct interior points of D_1^2.

In the case F has no branch points, F_0 is m 2-disks and $\mathrm{pr}_2 \circ f$ is just a covering map.

Two singular 2-dimentional braids are *equivalent* if there is a fiber-preserving ambient isotopy of $D_1^2 \times D_2^2$ rel $D_1^2 \times \partial D_2^2$ which carries one to the other.

There is a *singular chart* which corresponds to a singular 2-dimentional braid.

Let m be a positive integer, and Γ a graph in a 2-disk D_2^2. Then Γ is called a *singular chart* of degree m if it satisfies the following conditions:

(i) $\Gamma \cap \partial D_2^2 = \emptyset$.

(ii) Every edge is oriented and labeled , and the label is in $\{1, \ldots, m-1\}$.

(iii) Every vertex has degree 1, 2, 4, or 6.

(iv) At each vertex of degree 6, there are six edges adhering to which, three consecutive arcs oriented inward and the other three outward, and those six edges are labeled i and $i+1$ alternately for some i.

(v) At each vertex of degree 4, the diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i - j| > 1$.

(vi) At each vertex of degree 2, the edges attached to it have the same label and their orientations are not coherent at the vertex (Fig. 1.1).

A vertex of degree 1, 2 or 6 is called a *black vertex*, *node* or *white vertex* respectively. Around a white vertex, the middle edge of the three inwardly (or outwardly)-oriented consecutive edges is called a *middle edge*.

A black vertex (resp. node or white vertex) of a singular chart corresponds to a branch point (resp. singular point or triple point) of the singular 2-dimentional braid associated with the singular chart.

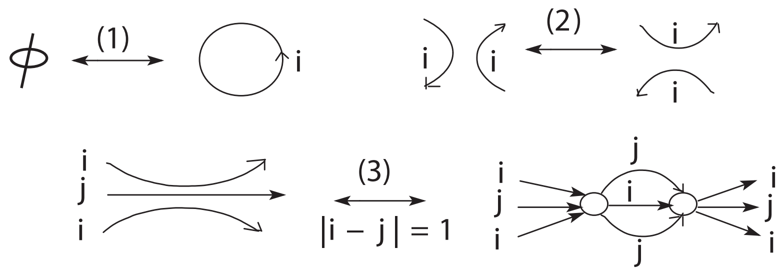

Figure 1.2: CI-moves of type (1), (2) and (3).

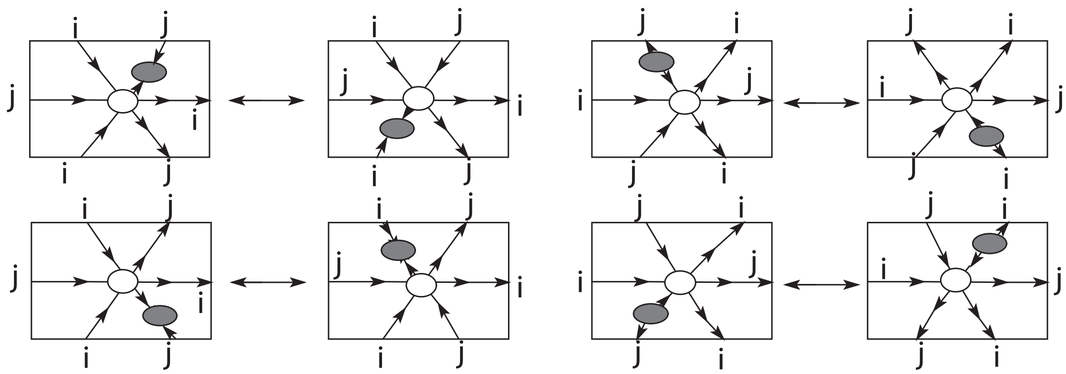

Figure 1.3: CV-moves.

The *trivial singular chart with no black vertices* is a chart represented by an empty graph (cf. [9]).

Two singular charts of the same degree are *equivalent* if we can deform one to the other by a finite sequence of ambient isotopies of D_2^2 and *chart moves* ([6, 8, 10]). In this paper we use only *CI-moves* (in particular of type (1), (2) and (3)), and *CV-moves* (See Figs. 1.2 and 1.3).

More precisely, let Γ and Γ' be two charts in D_2^2 of the same degree. Then Γ' is said to be obtained from Γ (or Γ is said to be obtained from Γ') by a *chart move* of type I (resp. V), or by a *CI-move* (resp. *CV-move*) if there exists a 2-disk E in D_2^2 such that the loop ∂E is in general position with respect to Γ and Γ' and $\Gamma \cap (D_2^2 - E) = \Gamma' \cap (D_2^2 - E)$ and the following condition holds:
(CI) There are neither black vertices nor nodes in $\Gamma \cap E$ nor $\Gamma' \cap E$.
A CI-move as in Fig. 1.2 is called a CI-move of type (1), (2) and (3) respectively.
(CV) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 1.3, where $|i - j| = 1$.

A *node-pair* is a pair of nodes connected by an edge. Introducing (resp. reducing) a node-pair to (resp. from) a singular chart corresponds to a crossing change (resp. its inverse) for the associated singular 2-dimensional braid.

In this paper, we show:

Theorem 3.1. *There is a singular chart without black vertices of degree 4 which cannot be deformed to be trivial by reducing node-pairs only.*
and
Theorem 3.2. *Any singular chart without black vertices whose degree is at*

most three can be deformed to be trivial by reducing node-pairs only.

This is similar to Kamada's theorem that a surface-link chart is ribbon (which is, it can be written without white vertices) if its degree is at most three, while it is not always ribbon if its degree is at least four [5].

2 Preliminaries

As preliminaries, we review a braid system of a singular chart and slide equivalence, and (classical) pure braids and a presentation of the pure braid group.

Let Γ be a singular chart of degree m in a 2-disk D^2_2. Let q_0 be a fixed point on the boundary of D^2_2, and $\Sigma(\Gamma)$ the set of black vertices and nodes in Γ. Let $\mathfrak{A} = (a_1, \dots, a_n)$ be a Hurwitz arc system with the starting point set $\Sigma(\Gamma)$ and the terminal point q_0, which is, for any i and j, $a_i \cap a_j = \{q_0\}$ and the normal vector of a_i points to a_{i+1}. Let $\eta_1, \dots, \eta_n$ be the Hurwitz generators of $\pi_1(D^2_2 \backslash \Sigma(\Gamma), q_0)$ associated with $\mathfrak{A}$ such that each η_i encircles each starting point anti-clockwise. A *braid system* $\vec{b} = (b_1, \dots, b_n)$ of a singular chart is an ordered n-tuple of elements of $\pi_1(D^2_2 \backslash Q_m, q_0) = B_m$ such that $b_i = \rho_\Gamma(\eta_i)$, where $\rho_\Gamma(\eta_i)$ means an m-braid corresponding to η_i.

Two braid systems are *slide equivalent* if we can transform one to the other by applying a finite sequence of the following equivalence relations:

$$(b_1, \dots, b_i, b_{i+1}, \dots, b_n) \sim (b_1, \dots, b_{i-1}, b_{i+1}, b_{i+1}^{-1} b_i b_{i+1}, b_{i+2}, \dots, b_n).$$

Two singular charts of the same degree are equivalent if and only if their braid systems are slide equivalent.

A classical braid whose start point and end point of each string are the same is called a *pure braid*. We can consider the *pure braid group*, which is a subgroup of the braid group. There is a presentation of the pure braid group as follows.

Lemma 2.1. [3, Lemma 4.2]. *The pure braid group of degree m P_m is generated by generators A_{ij} and relations as follows:*

$$A_{ij} = \sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

for $1 \leq i < j \leq m$, and the relations are

$$A_{rs}^{-1} A_{ij} A_{rs} = \begin{cases} A_{ij} & \text{if } i < r < s < j \text{ or } r < s < i < j \\ A_{rj} A_{ij} A_{rj}^{-1} & \text{if } r < i = s < j \\ A_{rj} A_{sj} A_{ij} A_{sj}^{-1} A_{rj}^{-1} & \text{if } i = r < s < j \\ A_{rj} A_{sj} A_{rj}^{-1} A_{sj}^{-1} A_{ij} A_{sj} A_{rj} A_{sj}^{-1} A_{rj}^{-1} & \text{if } r < i < s < j. \end{cases}$$

3 Singular charts with no black vertices

Theorem 3.1. *The singular chart of degree 4 with no black vertices as in Fig. 3.1 cannot be deformed to the trivial chart by reducing node-pairs only.*

Remark. The singular chart of Theorem 3.1 represents an immersion of 4 spheres.

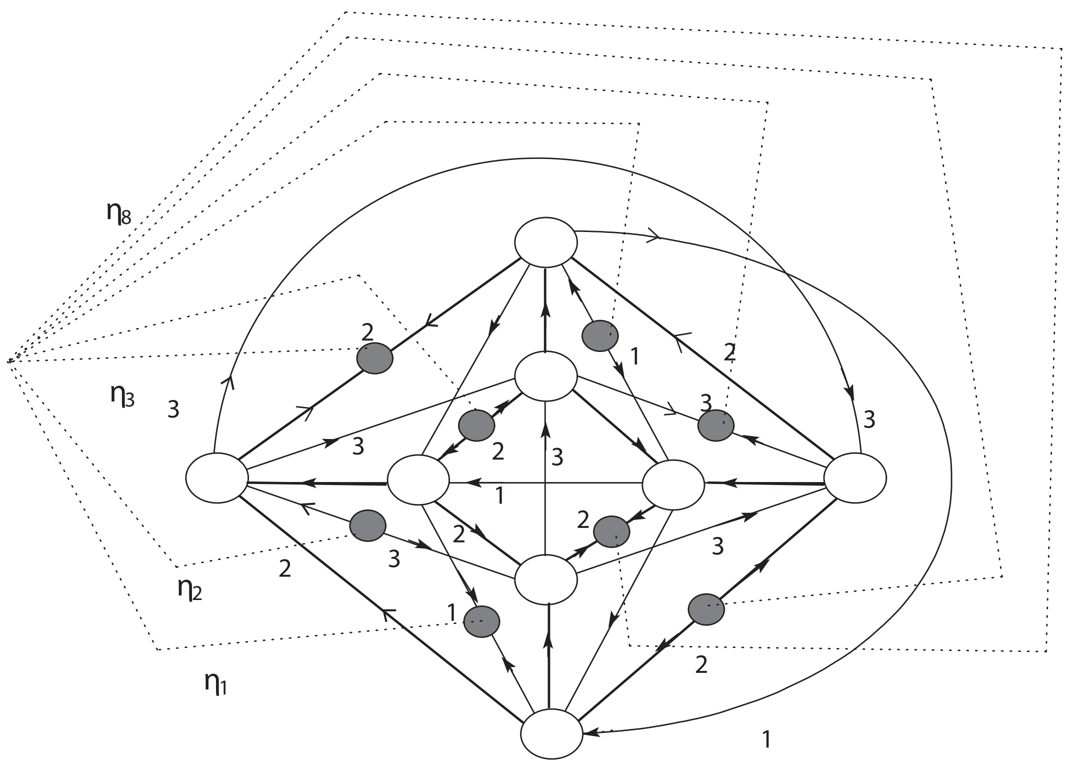

Figure 3.1: Singular chart of Theorem 3.1.

Proof. Take a braid system $\vec{b} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$ of the singular chart, where

$$\begin{aligned}
b_1 &= \sigma_2\sigma_1^2\sigma_2^{-1}\\
b_2 &= \sigma_2\sigma_3^{-2}\sigma_2^{-1}\\
b_3 &= \sigma_3\sigma_2^2\sigma_3^{-1}\\
b_4 &= \sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2^{-2}\sigma_3^{-1}\sigma_1\sigma_2\sigma_3^{-1}\\
b_5 &= \sigma_3\sigma_1\sigma_2^{-1}\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_3^{-1}\\
b_6 &= \sigma_3\sigma_1\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1^{-1}\sigma_3^{-1}\\
b_7 &= \sigma_1\sigma_2^{-2}\sigma_1^{-1}\\
b_8 &= \sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^2\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}.
\end{aligned}$$

It suffices to show that $\vec{b}$ cannot be transformed by slide equivalence to

$$(b_1', {b_1'}^{-1}, b_2', {b_2'}^{-1}, b_3', {b_3'}^{-1}, b_4', {b_4'}^{-1}). \tag{3.1}$$

First, we present b_k by A_{ij} $(i, j \in \{1, 2, 3, 4\})$, the generators of the pure braid group, which are

$$\begin{aligned}
A_{12} &= \sigma_1^2\\
A_{13} &= \sigma_2\sigma_1^2\sigma_2^{-1}\\
A_{14} &= \sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}\\
A_{23} &= \sigma_2^2\\
A_{24} &= \sigma_3\sigma_2^2\sigma_3^{-1}\\
A_{34} &= \sigma_3^2.
\end{aligned}$$

Let us compute:

$$\begin{aligned}
b_1 &= A_{13}\\
b_2 &= (\sigma_3^{-2})(\sigma_3\sigma_2^{-2}\sigma_3^{-1})(\sigma_3^2) = A_{34}^{-1}A_{24}^{-1}A_{34}\\
b_3 &= A_{24}\\
b_4 &= (\sigma_3^2)(\sigma_2\sigma_1^{-2}\sigma_2^{-1})(\sigma_3^{-2}) = A_{34}A_{13}^{-1}A_{34}^{-1}\\
b_5 &= (\sigma_1^2)(\sigma_3\sigma_2^{-2}\sigma_3^{-1})(\sigma_1^{-2}) = A_{12}A_{24}^{-1}A_{12}^{-1}\\
b_6 &= (\sigma_3^2)(\sigma_2^{-2})(\sigma_2\sigma_1^2\sigma_2^{-1})(\sigma_2^2)(\sigma_3^{-2}) = A_{34}A_{23}^{-1}A_{13}A_{23}A_{34}^{-1}\\
b_7 &= (\sigma_2^{-2})(\sigma_2\sigma_1^{-2}\sigma_2^{-1})(\sigma_2^2) = A_{23}^{-1}A_{13}^{-1}A_{23}\\
b_8 &= (\sigma_1^2)(\sigma_3^{-2})(\sigma_3\sigma_2^2\sigma_3^{-1})(\sigma_3^2)(\sigma_1^{-2}) = A_{12}A_{34}^{-1}A_{24}A_{34}A_{12}^{-1}.
\end{aligned}$$

Hence, if $\vec{b}$ can be transformed to the form as (3.1), by slide equivalence and the relations of the pure braid group, b_1 must be transformed to b_4^{-1} or b_7^{-1} by some conjugation. In the case of b_4,

$$A_{13} = wA_{34}A_{13}A_{34}^{-1}w^{-1},$$

where w is a word consisting of some b_k's for $k \in \{1, \ldots, 8\}$ by slide equivalence. Let $\phi : P_4 \to S_2$ be a homomorphism from the pure braid group of degree 4 to the symmetric group of degree 2 such that $\phi(A_{13}) = \phi(A_{24}) = e$ and $\phi(A_{ij}) =$

(12) for otherwise. This homomorphism is well-defined. By the relations of the pure braid group, a conjugate of A_{13} has essentially only one relation:

$$\begin{aligned} A_{12}^{-1}A_{13}A_{12} &= A_{13}A_{23}A_{13}A_{23}^{-1}A_{13}^{-1}, \\ A_{13} &= A_{12}A_{13}A_{23}A_{13}A_{23}^{-1}A_{13}^{-1}A_{12}^{-1}. \end{aligned}$$

Since $\phi(A_{12}A_{13}A_{23}) = e$, $\phi(wA_{34})$ must be e, the unit element. However, since $\phi(b_k) = e$ for $k \in \{1, 2, \ldots, 8\}$, $\phi(w) = e$, hence $\phi(wA_{34}) = (12) \neq e$.

It is similar in the case of b_7. □

Theorem 3.2. *Any singular chart without black vertices whose degree is at most three can be deformed to the trivial chart by reducing node-pairs only.*

Proof. Let Γ be a singular chart in a 2-disk D_2^2 without black vertices whose degree is at most three. We assume that the two edges connected by a node is an edge with a node on it, and use the phrase that some nodes are on an edge. Because the label of any edge of Γ is either one or two, every vertex of Γ is a white vertex of degree six. Let us call an area with no edges inside it and surrounded by connected edges a *polygon*, and call a polygon with two edges (resp. four edges) a *bigon* (resp. *square*). Moreover, let us call an edge without vertices a *loop*, and the area with no edges inside it and surrounded by connected edges and ∂D_2^2 will be called the *boundary polygon*.

Step 1. Every polygon of Γ consists of even edges.

Every polygon consists of even edges, for its edges have labels one and two alternately.

Step 2. Every edge of Γ has two white vertices at its ends.

Let W be a white vertex and e an edge which goes out from and enters W. Then there is another edge e_1 connected with W, such that around W are e, e_1, e consecutively. This edge e_1 connects W and some imperfect singular chart which has one vertex of degree five and n white vertices of degree six for some n. However, if such an imperfect singular chart exists, its number of edges is $(5 + 6n)/2$, which is not an integer value. This is a contradiction.

Step 3. We can always assume that each edge of Γ has at most one node on it.

Reduce node-pairs.

Step 4. We can assume that Γ does not have two adjacent bigons.

If there are two adjacent bigons, each bigon has two white vertices W_1 and W_2 at its end points by Step 2. Let e_1, e_2, and e_3 be the three consecutive edges which construct the two adjacent bigons between W_1 and W_2.

(Case 4.1) If e_1, e_2, and e_3 have no nodes on it, by a CI-move of type (3), we can eliminate their commom vertices W_1 and W_2.

(Case 4.2) If an edge has nodes on it, from Step 3 we can assume that it has only one node or none. If two of e_1, e_2, and e_3 have a node on each, one of the two edges, say e_i, is not a middle edge of W_1. Hence we can deform the edge e_i to have no nodes on it by applying a CV-move around W_1. Now we can assume that only one of e_1, e_2, and e_3 has a node on it, and moreover it is a middle edge of W_1. Let the *orientation* of an edge with respect to W_j ($j = 1, 2$) be *positive* (resp. *negative*) if the edge goes out of (resp. enters) W_j. It suffices to consider when e_1 has a node on it and when e_2 has a node on it. In both cases we can assume that the orientation of e_1 with respect to W_1 is positive. Then when e_1 has a node on it, the orientations of e_1, e_2, and e_3 with respect to W_1

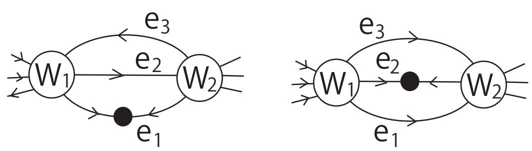

Figure 3.2: White vertices W_1, W_2 and the edges $e_1\,e_2$, and e_3.

(resp. W_2) are $(+,+,-)$ (resp. $(+,-,+)$). The orientations with respect to W_2 $(+,-,+)$ contradict the definition of a white vertex that a white vertex has three consecutive edges entering it and the other three consecutive edges going out of it. When e_2 has a node on it, the orientations of e_1, e_2, and e_3 with respect to W_1(resp. W_2) are $(+,+,+)$ (resp. $(-,+,-)$), which is also a contradiction (Fig. 3.2).

Step 5. Let Γ_0 be an innermost connected component of Γ. Then Γ_0 is a loop with some nodes on it.

If Γ_0 has white vertices, we obtain a new graph Γ_0' from Γ_0 by ignoring orientations of the edges and regarding a bigon as an edge, or identifying adjacent edges which have two common white vertices. Since Γ_0 is connected, the new graph Γ_0' is connected. Moreover, the new graph Γ_0' has no edge which has only one vertex at its ends by Step 2.

We show that the degree of each vertex of the new graph Γ_0' is at least four. Let W' be a vertex of Γ_0' with degree 1, 2 or 3, and W a white vertex of Γ_0 corresponding to W'.

(Case 5.1) If W' has degree one or two, there are two adjacent bigons in the singular chart Γ_0, which contradicts Step 4.

Let W' have degree one. Let W_1' be the other vertex of the edge connected to W' in Γ_0' . The existence of W_1' is by Step 2. Then the two corresponding vertices W and W_1 in the singular chart Γ_0 are connected by six edges, for W has degree six. Then there are more than two adjacent bigons between W and W_1. The case W' has degree two can be shown likewise. Remark that there are two vertices connected with W' by Step 2.

(Case 5.2) Let W' have degree three. Then denote the three white vertices in Γ_0 connected with W by W_1, W_2 and W_3 anti-clockwise. The existence of these three vertices is also by Step 2. By Step 4, there are exactly two adjacent edges connecting W and W_j $(j=1,2,3)$. Let us denote them by e_j, e_j' anti-clockwise, and denote the edges by f_j, f_j' $(j=1,2,3)$ such that around W_1 are consecutive edges f_2, e_1', e_1, f_3', around W_2 are f_3, e_2', e_2, f_1', and around W_3 are f_1, e_3', e_3, f_2' anti-clockwise (Fig. 3.3). Then let the *orientation* of each edge be *positive* (resp. *negative*) if the edge goes out of (resp. enters) W_j. We can assume that each e_j (resp. e_j') has the opposite orientation with f_j (resp. f_j'), for if some e_j (resp. e_j') has the same orientation with f_j (resp. f_j'), we can apply a CI-move of type (2) between them and have two adjacent bigons between W and W_{j-1} (resp. W_{j+1}), where $W_0=W_3$ and $W_4=W_1$. We can assume that $(e_1, e_1', e_2, e_2', e_3, e_3')=(+,+,+,-,-,-)$ by the definition of a white vertex that a white vertex has three consecutive edges entering it and the other three consecutive edges going out of it. Then $(f_1, f_1', f_2, f_2', f_3, f_3')=(-,-,-,+,+,+)$, and hence the consecutive edges around W_j have the ori-

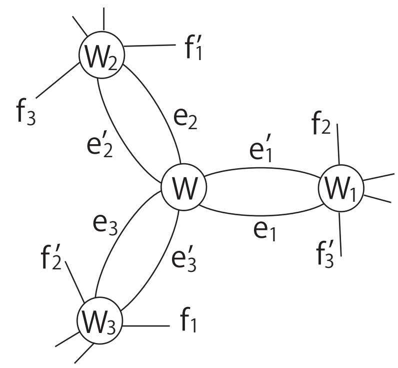

Figure 3.3: White vertices W, W_1, W_2, W_3 and the edges.

entations $(f_2, e_1', e_1, f_3') = (-, +, +, +)$, $(f_3, e_2', e_2, f_1') = (+, -, +, -)$ and $(f_1, e_3', e_3, f_2') = (-, -, -, +)$. The second set of orientations $(f_3, e_2', e_2, f_1') = (+, -, +, -)$ contradicts the definition of a white vertex. This can be applied regardless of whether each edge has a node or not.

It remains to show that if each vertex of Γ_0' has at least degree four, there is a contradiction. The connected graph Γ_0' consists of polygons. By Step 1 and Step 2, the number of the edges of each polygon is even and at least four. Let G_0' be another new graph obtained from Γ_0' by adding some edges, which divide each polygon of Γ_0' into squares. Then the new connected graph G_0' consits of squares, and each vertex of G_0' has degree at least four. Let n be the number of the squares including the boundary square and x be the number of the vertices of G_0'. Consider G_0' to be a polyhedron in the three space. Since its faces are square, we can get the following equation by computing its Euler characteristic:

$$n - 2n + x = 2.$$

Hence, $x = 2 + n$. However, since the degree of each vertex is at least four,

$$4(2 + n) \leq 4n,$$

which is a contradiction.

Step 6. We can eliminate Γ_0, an innermost connected component of the singular chart Γ.

By Step 5, Γ_0 is a loop with some nodes on it. By Step 3, it has no nodes, for the number of nodes on a loop must be even. Then applying a CI-move of type (1), we can eliminate Γ_0.

Therefore we can deform Γ to the trivial chart by repeating these steps. $\square$

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