UNKNOTTING SINGULAR CHARTS
WITH NO BLACK VERTICES BY
REDUCING NODE-PAIRS

Dedicated to Professor Akio Kawauchi’s 60th birthday

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Abstract

We determine whether singular charts without black vertices can be
deformed to the trivial chart by reducing node-pairs only. It is not true
if the degree of the singular chart is at least four, while it is true if the
degree is at most three.

Key words: singular chart; 2-dimensional braid; crossing change

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1 Introduction

Kamada in [7] showed that any singular chart without black vertices can be
deformed to the trivial chart by introducing and reducing some node-pairs,
which corresponds to the statement that any singular 2-dimensional braid with-
out branch points can be deformed to be trivial by some crossing changes and
inverses of crossing change. Here a crossing change is in the sense of [2].

We will determine whether the above theorem holds by inverses of crossing
change only. It does not hold when the degree is at least four, however it holds
when the degree is at most three.

A *singular 2-dimensional braid* of degree $m$ is a smoothly immersed, compact
and oriented surface $F$ in a bidisk $D^2_1 \times D^2_2$ whose singularities are transverse
double points such that

(i) for an immersion $f : F_0 \to D^2_1 \times D^2_2$ associated with $F$, the composition
\[ \text{pr}_2 \circ f \] is a simple branched covering map,

(ii) $\text{pr}_1(\partial F) = Q_m$.

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(iii) for each \( p \in D^2 \), \( pr_2^{-1}(p) \) contains at most one singular point of \( F \) (that is a double point of \( F \) or the image by \( f \) of a singular point of the simple branched covering map \( pr_2 \circ f \)),

where \( pr_i : D^2_1 \times D^2 \to D^2_i \) (\( i = 1, 2 \)) is the \( i \)-th projection and \( Q_m \) is a fixed set of \( m \) distinct interior points of \( D^2_2 \).

In the case \( F \) has no branch points, \( F_0 \) is \( m \) 2-disks and \( pr_2 \circ f \) is just a covering map.

Two singular 2-dimensional braids are equivalent if there is a fiber-preserving ambient isotopy of \( D^2_1 \times D^2 \) rel \( D^2_1 \times \partial D^2 \) which carries one to the other.

There is a singular chart which corresponds to a singular 2-dimensional braid.

Let \( m \) be a positive integer, and \( \Gamma \) a graph in a 2-disk \( D^2_2 \). Then \( \Gamma \) is called a singular chart of degree \( m \) if it satisfies the following conditions:

(i) \( \Gamma \cap \partial D^2 = \emptyset \).

(ii) Every edge is oriented and labeled, and the label is in \( \{1, \ldots, m-1\} \).

(iii) Every vertex has degree 1, 2, 4, or 6.

(iv) At each vertex of degree 6, there are six edges adhering to which, three consecutive arcs oriented inward and the other three outward, and those six edges are labeled \( i \) and \( i+1 \) alternately for some \( i \).

(v) At each vertex of degree 4, the diagonal edges have the same label and are oriented coherently, and the labels \( i \) and \( j \) of the diagonals satisfy \( |i-j| > 1 \).

(vi) At each vertex of degree 2, the edges attached to it have the same label and their orientations are not coherent at the vertex (Fig. 1.1).

A vertex of degree 1, 2 or 6 is called a black vertex, node or white vertex respectively. Around a white vertex, the middle edge of the three inwardly (or outwardly)-oriented consecutive edges is called a middle edge.

A black vertex (resp. node or white vertex) of a singular chart corresponds to a branch point (resp. singular point or triple point) of the singular 2-dimensional braid associated with the singular chart.
Two singular charts of the same degree are equivalent if we can deform one to the other by a finite sequence of ambient isotopies of $D^2$ and chart moves ([6, 8, 10]). In this paper we use only CI-moves (in particular of type (1), (2) and (3)), and CV-moves (See Figs. 1.2 and 1.3).

More precisely, let $\Gamma$ and $\Gamma'$ be two charts in $D^2$ of the same degree. Then $\Gamma'$ is said to be obtained from $\Gamma$ (or $\Gamma$ is said to be obtained from $\Gamma'$) by a chart move of type I (resp. V), or by a CI-move (resp. CV-move) if there exists a 2-disk $E$ in $D^2$ such that the loop $\partial E$ is in general position with respect to $\Gamma$ and $\Gamma'$ and $\Gamma \cap (D^2 - E) = \Gamma' \cap (D^2 - E)$ and the following condition holds:

(CI) There are neither black vertices nor nodes in $\Gamma \cap E$ nor $\Gamma' \cap E$.

A CI-move as in Fig. 1.2 is called a CI-move of type (1), (2) and (3) respectively. (CV) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 1.3, where $|i - j| = 1$.

A node-pair is a pair of nodes connected by an edge. Introducing (resp. reducing) a node-pair to (resp. from) a singular chart corresponds to a crossing change (resp. its inverse) for the associated singular 2-dimensional braid.

In this paper, we show:

**Theorem 3.1.** There is a singular chart without black vertices of degree 4 which cannot be deformed to be trivial by reducing node-pairs only.

and

**Theorem 3.2.** Any singular chart without black vertices whose degree is at
most three can be deformed to be trivial by reducing node-pairs only.

This is similar to Kamada’s theorem that a surface-link chart is ribbon (which is, it can be written without white vertices) if its degree is at most three, while it is not always ribbon if its degree is at least four [5].

2 Preliminaries

As preliminaries, we review a braid system of a singular chart and slide equivalence, and (classical) pure braids and a presentation of the pure braid group.

Let \( \Gamma \) be a singular chart of degree \( m \) in a 2-disk \( D_2^2 \). Let \( q_0 \) be a fixed point on the boundary of \( D_2 \), and \( \Sigma(\Gamma) \) the set of black vertices and nodes in \( \Gamma \). Let \( q_0 \) be a fixed point on the boundary of \( D_2 \), and \( \Sigma(\Gamma) \) the set of black vertices and nodes in \( \Gamma \). Let \( \eta_1, \ldots, \eta_n \) be the Hurwitz generators of \( \pi_1(D_2^2 \setminus \Sigma(\Gamma), q_0) = B_m \) such that each \( \eta_i \) encircles each starting point anti-clockwise. A braid system \( \vec{b} = (b_1, \ldots, b_n) \) of a singular chart is an ordered \( n \)-tuple of elements of \( \pi_1(D_2^2 \setminus Q_m, q_0) = B_m \) such that \( b_i = \rho_{\Gamma}(\eta_i) \), where \( \rho_{\Gamma}(\eta_i) \) means an \( m \)-braid corresponding to \( \eta_i \).

Two braid systems are slide equivalent if we can transform one to the other by applying a finite sequence of the following equivalence relations:

\[
(b_1, \ldots, b_i, b_{i+1}, \ldots, b_n) \sim (b_1, \ldots, b_{i-1}, b_{i+1}, b_{i+1}^{-1} b_i b_{i+1}, b_{i+2}, \ldots, b_n).
\]

Two singular charts of the same degree are equivalent if and only if their braid systems are slide equivalent.

A classical braid whose start point and end point of each string are the same is called a pure braid. We can consider the pure braid group, which is a subgroup of the braid group. There is a presentation of the pure braid group as follows.

Lemma 2.1. [3, Lemma 4.2]. The pure braid group of degree \( m \) \( P_m \) is generated by generators \( A_{ij} \) and relations as follows:

\[
A_{ij} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1} \sigma_{j-1}^{-1}
\]

for \( 1 \leq i < j \leq m \), and the relations are

\[
A_{rs}^{-1} A_{ij} A_{rs} = \begin{cases} 
A_{ij} & \text{if } i < r < s < j \text{ or } r < s < i < j \\
A_{ij} A_{rs} A_{ij} A_{rs}^{-1} & \text{if } r < i = s < j \\
A_{rs} A_{ij} A_{rs} A_{ij} A_{rs}^{-1} A_{rs}^{-1} A_{ij} A_{rs} A_{ij} A_{rs}^{-1} A_{rs}^{-1} & \text{if } r < i < s < j.
\end{cases}
\]

3 Singular charts with no black vertices

Theorem 3.1. The singular chart of degree 4 with no black vertices as in Fig. 3.1 cannot be deformed to the trivial chart by reducing node-pairs only.

Remark. The singular chart of Theorem 3.1 represents an immersion of 4 spheres.
Figure 3.1: Singular chart of Theorem 3.1.
Proof. Take a braid system $\vec{b} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$ of the singular chart, where

$$
\begin{align*}
    b_1 &= \sigma_2\sigma_1^2\sigma_2^{-1} \\
    b_2 &= \sigma_2\sigma_3\sigma_2^{-1} \\
    b_3 &= \sigma_3\sigma_2\sigma_3^{-1} \\
    b_4 &= \sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1} \\
    b_5 &= \sigma_3\sigma_1\sigma_2^{-1}\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_3 \\
    b_6 &= \sigma_3\sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1}\sigma_3^{-1} \\
    b_7 &= \sigma_1\sigma_2^{-2}\sigma_1^{-1} \\
    b_8 &= \sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}.
\end{align*}
$$

It suffices to show that $\vec{b}$ cannot be transformed by slide equivalence to

$$
(b_1', b_2'^{-1}, b_3', b_4'^{-1}, b_5', b_6'^{-1}).
\tag{3.1}
$$

First, we present $b_k$ by $A_{ij}$ ($i, j \in \{1, 2, 3, 4\}$), the generators of the pure braid group, which are

$$
\begin{align*}
    A_{12} &= \sigma_1^2 \\
    A_{13} &= \sigma_2\sigma_1^2\sigma_2^{-1} \\
    A_{14} &= \sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1} \\
    A_{23} &= \sigma_2^2 \\
    A_{24} &= \sigma_3\sigma_2^2\sigma_3^{-1} \\
    A_{34} &= \sigma_3^2.
\end{align*}
$$

Let us compute:

$$
\begin{align*}
    b_1 &= A_{13} \\
    b_2 &= (\sigma_3^{-2})(\sigma_3\sigma_2\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^2) = A_{34}^{-1}A_{24}^{-1}A_{34} \\
    b_3 &= A_{24} \\
    b_4 &= (\sigma_3^2)(\sigma_2\sigma_1^{-1}\sigma_2^{-1})(\sigma_3^{-2}) = A_{34}A_{13}^{-1}A_{34}^{-1} \\
    b_5 &= (\sigma_2^2)(\sigma_3\sigma_2\sigma_1^{-2})(\sigma_1^{-2}) = A_{12}A_{24}^{-1}A_{12} \\
    b_6 &= (\sigma_2^2)(\sigma_3\sigma_2\sigma_1^{-2})(\sigma_2^2)(\sigma_3^{-2}) = A_{34}A_{23}^{-1}A_{13}A_{23}A_{34} \\
    b_7 &= (\sigma_2^2)(\sigma_3\sigma_1^{-2}\sigma_2^{-1})(\sigma_2^2) = A_{23}^{-1}A_{13}^{-1}A_{23} \\
    b_8 &= (\sigma_1^2)(\sigma_3\sigma_2\sigma_1^{-1})(\sigma_3^{-2})(\sigma_1^{-2}) = A_{12}A_{34}^{-1}A_{24}A_{34}A_{12}^{-1}.
\end{align*}
$$

Hence, if $\vec{b}$ can be transformed to the form as (3.1), by slide equivalence and the relations of the pure braid group, $b_1$ must be transformed to $b_4^{-1}$ or $b_7^{-1}$ by some conjugation. In the case of $b_4$,

$$
A_{13} = wA_{34}A_{13}A_{34}^{-1}w^{-1},
$$

where $w$ is a word consisting of some $b_k$’s for $k \in \{1, \ldots, 8\}$ by slide equivalence. Let $\phi : P_4 \rightarrow S_2$ be a homomorphism from the pure braid group of degree 4 to the symmetric group of degree 2 such that $\phi(A_{13}) = \phi(A_{24}) = e$ and $\phi(A_{ij}) = e$ for $i < j$.
(12) for otherwise. This homomorphism is well-defined. By the relations of the pure braid group, a conjugate of \( A_{13} \) has essentially only one relation:

\[
A_{12}^{-1}A_{13}A_{12} = A_{13}A_{23}A_{13}A_{23}^{-1}A_{13}^{-1}, \\
A_{13} = A_{12}A_{13}A_{23}A_{13}A_{23}^{-1}A_{13}^{-1}A_{12}^{-1}.
\]

Since \( \phi(A_{12}A_{13}A_{23}) = e \), \( \phi(wA_{34}) \) must be \( e \), the unit element. However, since \( \phi(b_k) = e \) for \( k \in \{1, 2, \ldots, 8\} \), \( \phi(w) = e \), hence \( \phi(wA_{34}) = (12) \neq e \).

It is similar in the case of \( b_7 \).

**Theorem 3.2.** Any singular chart without black vertices whose degree is at most three can be deformed to the trivial chart by reducing node-pairs only.

**Proof.** Let \( \Gamma \) be a singular chart in a 2-disk \( D_2^2 \) without black vertices whose degree is at most three. We assume that the two edges connected by a node is an edge with a node on it, and use the phrase that some nodes are on an edge. Because the label of any edge of \( \Gamma \) is either one or two, every vertex of \( \Gamma \) is a white vertex of degree six. Let us call an area with no edges inside it and surrounded by connected edges a **polygon**, and call a polygon with two edges (resp. four edges) a **bigan** (resp. square). Moreover, let us call an edge without vertices a **loop**, and the area with no edges inside it and surrounded by connected edges and \( \partial D_2^2 \) will be called the **boundary polygon**.

**Step 1.** Every polygon of \( \Gamma \) consists of even edges.

**Step 2.** Every edge of \( \Gamma \) has two white vertices at its ends.

Let \( W \) be a white vertex and \( e \) an edge which goes out from and enters \( W \). Then there is another edge \( e_1 \) connected with \( W \), such that around \( W \) are \( e, e_1, e \) consecutively. This edge \( e_1 \) connects \( W \) and some imperfect singular chart which has one vertex of degree five and \( n \) white vertices of degree six for some \( n \). However, if such an imperfect singular chart exists, its number of edges is \((5 + 6n)/2\), which is not an integer value. This is a contradiction.

**Step 3.** We can always assume that each edge of \( \Gamma \) has at most one node on it.

Reduce node-pairs.

**Step 4.** We can assume that \( \Gamma \) does not have two adjacent bigons.

If there are two adjacent bigons, each bigon has two white vertices \( W_1 \) and \( W_2 \) at its end points by Step 2. Let \( e_1, e_2, \) and \( e_3 \) be the three consecutive edges which construct the two adjacent bigons between \( W_1 \) and \( W_2 \).

(Case 4.1) If \( e_1, e_2, \) and \( e_3 \) have no nodes on it, by a CI-move of type (3), we can eliminate their common vertices \( W_1 \) and \( W_2 \).

(Case 4.2) If an edge has nodes on it, from Step 3 we can assume that it has only one node or none. If two of \( e_1, e_2, \) and \( e_3 \) have a node on each, one of the two edges, say \( e_i \), is not a middle edge of \( W_1 \). Hence we can deform the edge \( e_i \) to have no nodes on it by applying a CV-move around \( W_1 \). Now we can assume that only one of \( e_1, e_2, \) and \( e_3 \) has a node on it, and moreover it is a middle edge of \( W_1 \). Let the **orientation** of an edge with respect to \( W_j \) (\( j = 1, 2 \)) be **positive** (resp. **negative**) if the edge goes out of (resp. enters) \( W_j \). It suffices to consider when \( e_1 \) has a node on it and when \( e_2 \) has a node on it. In both cases we can assume that the orientation of \( e_1 \) with respect to \( W_1 \) is positive. Then when \( e_1 \) has a node on it, the orientations of \( e_1, e_2, \) and \( e_3 \) with respect to \( W_1 \)
Let $\Gamma$ be a graph with three consecutive edges entering it and the other three consecutive edges going out of it. When $e_2$ has a node on it, the orientations of $e_1$, $e_2$, and $e_3$ with respect to $W_1$ (resp. $W_2$) are $(+, +, -)$ (resp. $(+, -, +)$). The orientations with respect to $W_2$ $(+, -, +)$ contradict the definition of a white vertex that a white vertex has three consecutive edges entering it and the other three consecutive edges going out of it. When $e_2$ has a node on it, the orientations of $e_1$, $e_2$, and $e_3$ with respect to $W_1$ (resp. $W_2$) are $(+, +, +)$ (resp. $(-, +, -)$), which is also a contradiction (Fig. 3.2).

**Step 5.** Let $\Gamma_0$ be the innermost connected component of $\Gamma$. Then $\Gamma_0$ is a loop with some nodes on it.

If $\Gamma_0$ has white vertices, we obtain a new graph $\Gamma'_0$ from $\Gamma_0$ by ignoring orientations of the edges and regarding a bigon as an edge, or identifying adjacent edges which have two common white vertices. Since $\Gamma_0$ is connected, the new graph $\Gamma'_0$ is connected. Moreover, the new graph $\Gamma'_0$ has no edge which has only one vertex at its ends by Step 2.

We show that the degree of each vertex of the new graph $\Gamma'_0$ is at least four. Let $W'$ be a vertex of $\Gamma'_0$ with degree 1, 2 or 3, and $W$ a white vertex of $\Gamma_0$ corresponding to $W'$.

(Case 5.1) If $W'$ has degree one or two, there are two adjacent bigons in the singular chart $\Gamma_0$, which contradicts Step 4.

Let $W'$ have degree one. Let $W'_1$ be the other vertex of the edge connected to $W'$ in $\Gamma'_0$. The existence of $W'_1$ is by Step 2. Then the two corresponding vertices $W$ and $W_1$ in the singular chart $\Gamma_0$ are connected by six edges, for $W$ has degree six. Then there are more than two adjacent bigons between $W$ and $W_1$. The case $W'$ has degree two can be shown likewise. Remark that there are two vertices connected with $W'$ by Step 2.

(Case 5.2) Let $W'$ have degree three. Then denote the three white vertices in $\Gamma_0$ connected with $W$ by $W_1$, $W_2$ and $W_3$ anti-clockwise. The existence of these three vertices is also by Step 2. By Step 4, there are exactly two adjacent edges connecting $W$ and $W_j$ ($j = 1, 2, 3$). Let us denote them by $e_j$, $e'_j$ anti-clockwise, and denote the edges by $f_j, f'_j$ ($j = 1, 2, 3$) such that around $W_1$ are consecutive edges $f_2, e'_1, e_1, f'_3$, around $W_2$ are $f_3, e'_2, e_2, f'_1$, and around $W_3$ are $f_1, e'_3, e_3, f'_2$ anti-clockwise (Fig. 3.3). Then let the orientation of each edge be positive (resp. negative) if the edge goes out of (resp. enters) $W_j$. We can assume that each $e_j$ (resp. $e'_j$) has the opposite orientation with $f_j$ (resp. $f'_j$), for if some $e_j$ (resp. $e'_j$) has the same orientation with $f_j$ (resp. $f'_j$), we can apply a CI-move of type (2) between them and have two adjacent bigons between $W$ and $W_{j-1}$ (resp. $W_{j+1}$), where $W_0 = W_3$ and $W_4 = W_1$. We can assume that $(e_1, e'_1, e_2, e'_2, e_3, e'_3) = (+, +, +, +, +, +)$ by the definition of a white vertex that a white vertex has three consecutive edges entering it and the other three consecutive edges going out of it. Then $(f_1, f'_1, f_2, f'_2, f_3, f'_3) = (-, +, +, +, +, +)$, and hence the consecutive edges around $W_j$ have the ori-
entations \((f_2, e_1', e_1, f_3') = (-, +, +, +), (f_3, e_2', e_2, f_1') = (+, -, +, -)\) and \((f_1, e_3', e_3, f_2') = (-, - , - , +)\). The second set of orientations \((f_3, e_2', e_2, f_1') = (+, -, +, -)\) contradicts the definition of a white vertex. This can be applied regardless of whether each edge has a node or not.

It remains to show that if each vertex of \(\Gamma_0'\) has at least degree four, there is a contradiction. The connected graph \(\Gamma_0'\) consists of polygons. By Step 1 and Step 2, the number of the edges of each polygon is even and at least four. Let \(G_0'\) be another new graph obtained from \(\Gamma_0'\) by adding some edges, which divide each polygon of \(\Gamma_0'\) into squares. Then the new connected graph \(G_0'\) consists of squares, and each vertex of \(G_0'\) has degree at least four. Let \(n\) be the number of the squares including the boundary square and \(x\) be the number of the vertices of \(G_0'\). Consider \(G_0'\) to be a polyhedron in the three space. Since its faces are square, we can get the following equation by computing its Euler characteristic:

\[
n - 2n + x = 2.
\]

Hence, \(x = 2 + n\). However, since the degree of each vertex is at least four,

\[
4(2 + n) \leq 4n,
\]

which is a contradiction.

**Step 6.** We can eliminate \(\Gamma_0\), an innermost connected component of the singular chart \(\Gamma\).

By Step 5, \(\Gamma_0\) is a loop with some nodes on it. By Step 3, it has no nodes, for the number of nodes on a loop must be even. Then applying a CI-move of type (1), we can eliminate \(\Gamma_0\).

Therefore we can deform \(\Gamma\) to the trivial chart by repeating these steps. \(\square\)

**References**


