FILTERED MODULES CORRESPONDING TO POTENTIALLY SEMI-STABLE REPRESENTATIONS

NAOKI IMAI

Abstract. We classify the filtered modules with coefficients corresponding to two-dimensional potentially semi-stable p-adic representations of the absolute Galois groups of p-adic fields under the assumptions that p is odd and the coefficients are large enough.

Introduction

Let p be an odd prime number, and let K be a p-adic field. The absolute Galois group of K is denoted by G_K. By the fundamental theorem of Colmez and Fontaine [CF], there exists a correspondence between potentially semi-stable p-adic representations and admissible filtered (ϕ, N)-modules with Galois action. The aim of this paper is the classification of the admissible filtered (ϕ, N)-modules with Galois action corresponding to two-dimensional potentially semi-stable p-adic representations of G_K with coefficients in a p-adic field E.

If $K = \mathbb{Q}_p$ and $E = \mathbb{Q}_p$, the classification is given in [FM, Appendix A] under the assumption that $p \geq 5$. If $K = \mathbb{Q}_p$ and E is general, these filtered (ϕ, N)-modules are studied in [BM] and [Sav], and the classification is given by Ghate and Mézard in [GM] under the assumptions that p is odd and E is large enough. In this paper, we generalize the results of [GM] to the case where K is a general p-adic field.

In the case where K is a general p-adic field, filtrations are determined by many weights and many elements of $\mathbb{P}^1(E)$. In fact we need $[K : \mathbb{Q}_p]$ elemens of $\mathbb{P}^1(E)$ to parametrize two-dimensional potentially semi-stable p-adic representations. These elements of $\mathbb{P}^1(E)$ play a role similar to Fontaine-Mazur's $\mathfrak{L}$-invariants.

After writing of this paper, the author has known that there is preceding research [Do] on this subject by Dousmanis. The author does not claim priority, but there are some differences. In [Do], a classification is given by Frobenius action, and in this paper, we give a classification by Galois action. Let F be a finite extension of K. A potentially semi-stable representation ρ is said to be F-semi-stable, if the restriction of ρ to the absolute Galois group of F is semi-stable. In [Do], a classification of F-semi-stable representations is given for a general finite Galois extension F of K. In this paper, we give a class of finite Galois extensions of K such that any potentially semi-stable representation is F-semi-stable for a field F in this class, and give a classification of F-semi-stable representations and a more explicit description of Galois action of $\mathrm{Gal}(F/K)$ for F in this class, assuming $p \neq 2$. This difference is conspicuous in the supercuspidal case. Let F_0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in F. In [Do, 5.3], it is proved that $\mathrm{Gal}(F/K)$-action on a filtered $(\phi.N)$-$(F_0 \otimes_{\mathbb{Q}_p} E)$-module comes from a $\mathrm{Gal}(F/K)$-action on the two-dimensional E-vector space in the supercuspidal case. In this paper, we study the $\mathrm{Gal}(F/K)$-action explicitly by using a structure of $\mathrm{Gal}(F/K)$, of coure, assumeing F is in some class. Then, in this paper, we first fix a large enough coefficient field, and do not extend it in the classification.

This paper is clearly influenced by the paper [GM], and we owe a lot of arguments to [GM]. We mention it here, and do not repeat it each times in the sequel.

Acknowledgment. The author is supported by the Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists. He would like to thank Gerasimos Dousmanis for permitting this paper. He is grateful to a referee for a careful reading of this paper and suggestions for improvements.

Notation. Throughout this paper, we use the following notation. Let p be an odd prime number, and $\mathbb{C}_p$ be the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. Let K be a p-adic field. We consider K as a subfield of $\mathbb{C}_p$. The residue field of K is denoted by k, whose cardinality is q. Let K_0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in K. For any p-adic field L, the absolute Galois group of L is denoted by G_L, the inertia subgroup of G_L is denoted by I_L, the Weil group of L is denoted by W_L, the ring of integers of L is denoted by $\mathcal{O}_L$ and the unique maximal ideal of $\mathcal{O}_L$ is denoted by $\mathfrak{p}_L$. For a Galois extension L of K, the inertia subgroup of $\mathrm{Gal}(L/K)$ is denoted by $I(L/K)$. Let v_p be the valuations of p-adic fields normalized by $v_p(p) = 1$.

1. Filtered (ϕ, N)-modules

Let E be a p-adic field. We consider a two-dimensional p-adic representation V of G_K over E, which is denoted by $\rho : G_K \to GL(V)$. As in [Fon], we can construct K_0-algebra B_{st} with a Frobenius endomorphism, a monodromy operator and Galois action. Further, we can define a decreasing filtration on $K \otimes_{K_0} B_{\mathrm{st}}$. Let F be a finite Galois extension of K, and F_0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in F. Then we have $B_{\mathrm{st}}^{G_F} = F_0$. The p-adic representation ρ is called F-semi-stable if and only if the dimension of $D_{\mathrm{st},F}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_F}$ over F_0 is equal to the dimension of V over $\mathbb{Q}_p$. If ρ is F-semi-stable for some finite Galois extension F of K, we say that ρ is potentially semi-stable representation.

Potentially semi-stable representations are Hodge-Tate. To fix a convention, we recall the definition of the Hodge-Tate weights. For $i \in \mathbb{Z}$, we put

$$D^i_{\mathrm{HT}}(V) = \big(\mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V\big)^{G_K}.$$

Here and in the following, (i) means i times twists by the p-adic cyclotomic character of G_K. Then there is a G_K-equivariant isomorphism

$$\bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(-i) \otimes_K D^i_{\mathrm{HT}}(V) \xrightarrow{\sim} \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$$

of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} E)$-modules. The Hodge-Tate weights of the representation V are the integers i such that $D^{-i}_{\mathrm{HT}}(V) \neq 0$, with multiplicities $\dim_E\big(D^{-i}_{\mathrm{HT}}(V)\big)$.

Next, we recall the definition of the filtered $\big(\phi, N, \mathrm{Gal}(F/K), E\big)$-modules. A filtered $\big(\phi, N, \mathrm{Gal}(F/K), E\big)$-module is a finite free $(F_0 \otimes_{\mathbb{Q}_p} E)$-module D endowed with

- the Frobenius endomorphism: an F_0-semi-linear, E-linear, bijective map $\phi : D \to D$,
- the monodromy operator: an $(F_0 \otimes_{\mathbb{Q}_p} E)$-linear, nilpotent endomorphism $N : D \to D$ that satisfies $N\phi = p\phi N$,

- the Galois action: an F_0-semi-linear, E-linear action of $\mathrm{Gal}(F/K)$ that commutes with the action of ϕ and N,
- the filtration: a decreasing filtration $(\mathrm{Fil}^i D_F)_{i\in\mathbb{Z}}$ of $(F\otimes_{\mathbb{Q}_p} E)$-submodules of $D_F = F\otimes_{F_0} D$ that are stable under the action of $\mathrm{Gal}(F/K)$ and satisfy

$$\mathrm{Fil}^i D_F = D_F \text{ for } i \ll 0 \text{ and } \mathrm{Fil}^i D_F = 0 \text{ for } i \gg 0.$$

Let D be a filtered $\big(\phi, N, \mathrm{Gal}(F/K), E\big)$-module. Then, by forgetting the E-module structure, D is also a filtered $\big(\phi, N, \mathrm{Gal}(F/K), \mathbb{Q}_p\big)$-module. We put $d = \dim_{F_0} D$. Then $\bigwedge_{F_0}^d D$ is a filtered $\big(\phi, N, \mathrm{Gal}(F/K), \mathbb{Q}_p\big)$-module of dimension 1 over F_0. We put

$$t_{\mathrm{H}}(D) = \max\{i\in\mathbb{Z} \mid \mathrm{Fil}^i(F\otimes_{F_0} \bigwedge\nolimits_{F_0}^{d} D)\neq 0\},\ t_{\mathrm{N}}(D) = v_p(\lambda)$$

where λ is an element of $F_0^\times$ that satisfies $\phi(x) = \lambda x$ for a non-zero element x of $\bigwedge_{F_0}^d D$. We say that D is admissible if it satisfies the following two conditions:

- $t_{\mathrm{H}}(D) = t_{\mathrm{N}}(D)$.
- For any F_0-submodule D' of D that is stable by ϕ and N, we have $t_{\mathrm{H}}(D') \leq t_{\mathrm{N}}(D')$, where $D'_F \subset D_F$ is equipped with the induced filtration.

By [BM, Proposition 3.1.1.5], we may replace the above second condition by the following condition:

- For any $(F_0\otimes_{\mathbb{Q}_p} E)$-submodule D' of D that is stable by ϕ and N, we have $t_{\mathrm{H}}(D') \leq t_{\mathrm{N}}(D')$, where $D'_F \subset D_F$ is equipped with the induced filtration.

Let k_0 be a non-negative integer. By the results of [CF], there is an equivalence of categories between the category of two-dimensional F-semi-stable representations of G_K over E with Hodge-Tate weights in $\{0,\dots,k_0\}$ and the category of admissible filtered $\big(\phi, N, \mathrm{Gal}(F/K), E\big)$-modules of rank 2 over $F_0\otimes_{\mathbb{Q}_p} E$ such that $\mathrm{Fil}^{-k_0}(D_F) = D_F$ and $\mathrm{Fil}^1(D_F) = 0$. This equivalence of categories is given by the functor $D_{\mathrm{st},F}$ defined above. The aim of this paper is the classification of the objects of later categories under the assumption that E is large enough.

2. Preliminaries

Let $\rho : G_K \to GL(V)$ be a two-dimensional potentially-semi-stable representation over E. We assume that ρ is F-semi-stable, and put $D = D_{\mathrm{st},F}(V)$. We recall the definition of Weil-Deligne representation associated to ρ. Now we have $W_K/W_F = \mathrm{Gal}(F/K)$. Let m_0 be the degree of the field extension K_0 over $\mathbb{Q}_p$. We define an F_0-linear action of $g\in W_K$ on D by $(g \mod W_F)\circ\phi^{-m_0\alpha(g)}$, where the image of g in $\mathrm{Gal}(\overline{k}/k)$ is the $\alpha(g)$-th power of the q-th power Frobenius map.

We assume that $F_0\subset E$. According to an isomorphism

$$F_0\otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma_i : F_0\hookrightarrow E} E;\, a\otimes b\mapsto \sigma_i(a)b,$$

we have a decomposition

$$D \xrightarrow{\sim} \prod_{\sigma_i : F_0\hookrightarrow E} D_i.$$

Here and in the sequel, σ_i is an embedding determined by the $(-i)$-th power of the p-th power Frobenius map for $1\leq i\leq [F_0:\mathbb{Q}_p]$. Then D_i, with an induced action of W_K and an induced monodromy operator, defines a Weil-Deligne representation.

The isomorphism class of this Weil-Deligne representation is independent of choice of F and σ_i (cf. [BM, Lemme 2.2.1.2]), and is, by definition, the Weil-Deligne representation $\mathrm{WD}(\rho)$ attached to ρ.

We note that, in the above decomposition of D, the Frobenius endomorphism ϕ induce E-linear isomorphism $\phi : D_i \xrightarrow{\sim} D_{i+1}$. Naturally, we consider a suffix i modulo $[F_0 : \mathbb{Q}_p]$, and we often use such conventions in the sequel.

A Galois type τ of degree 2 is an equivalence class of representations $\tau : I_K \to GL_2(\overline{\mathbb{Q}}_p)$ with open kernel that extend to representations of W_K. We say that an two-dimensional potentially semi-stable representation ρ has Galois type τ if $\mathrm{WD}(\rho)|_{I_K} \simeq \tau$. The potentially semi-stable representation ρ is F-semi-stable if and only if $\tau|_{I_F}$ is trivial.

For a group G, an element $g \in G$, a normal subgroup H of G and a character $\chi : H \to \overline{\mathbb{Q}}_p^\times$, we define a character $\chi^g : H \to \overline{\mathbb{Q}}_p^\times$ by $\chi^g(h) = \chi(ghg^{-1})$ for $h \in H$.

Lemma 2.1. *Let τ be a Galois type of degree* 2*. Then τ has one of the following forms:*

1. $\tau \simeq \chi_1|_{I_K} \oplus \chi_2|_{I_K}$*, where χ_1, χ_2 are characters of W_K finite on I_K,*
2. $\tau \simeq \mathrm{Ind}_{W_{K'}}^{W_K}(\chi)|_{I_K} = \chi|_{I_K} \oplus \chi^\sigma|_{I_K}$*, where K' is the unramified quadratic extension of K, χ is a character of $W_{K'}$ that is finite on $I_{K'}$ and does not extend to W_K, and $\sigma \in W_K$ is a lift of the generator of* $\mathrm{Gal}(K'/K)$*,*
3. $\tau \simeq \mathrm{Ind}_{W_{K'}}^{W_K}(\chi)|_{I_K}$*, where K' is a ramified quadratic extension of K, and χ is a character of $W_{K'}$ such that χ is finite on $I_{K'}$ and $\chi|_{I_{K'}}$ does not extend to I_K.*

Proof. This is a classical lemma, but we briefly recall a proof.

We extend τ to a representation of W_K, which is denoted by $\tilde{\tau}$. If $\tilde{\tau}$ is reducible, we are in the case (1), so we may assume that $\tilde{\tau}$ is irreducible.

First, we treat the case where τ is reducible. In this case, $\tau \simeq \chi \oplus \chi'$ for some characters χ, χ' of I_K. By irreducibility of $\tilde{\tau}$, we have $\chi' = \chi^\sigma$. Then $\tilde{\tau}|_{W_{K'}}$ is already reducible for the unramified quadratic extension K' of K. So we are in the case (2).

Next, we treat the case where τ is irreducible. Let I_K^{w} be the wild inertia subgroup of I_K. Then $\tau|_{I_K^{\mathrm{w}}}$ is reducible, because a dimension of an irreducible representation of a p-group is a power of p and $p \neq 2$. Then $\tilde{\tau}|_{W_{K'}}$ is already reducible for a ramified quadratic extension K' of K. So we are in the case (3). □

To avoid the problem of the rationality, we assume that E is a Galois extension over $\mathbb{Q}_p$, $F \subset E$ and the following:

> For all p-adic fields K' such that $K \subset K' \subset F$ and $[K' : K] \leq 2$, and for all characters χ of $W_{K'}$ that are trivial on I_F, the restrictions $\chi|_{I_{K'}}$ factor through $E^\times$.

For example, if E contains the $|I(F/K)|$-th roots of unity, then this condition is satisfied.

In the sequel, let $\rho : G_K \to GL(V)$ be a two-dimensional potentially semi-stable representation over E with Hodge-Tate weight in $\{0, \ldots, k_0\}$, and τ be its Galois type.

Lemma 2.2. *(cf.* [GM, Lemma 2.3]*) If ρ is not potentially crystalline, then τ is a scalar.*

Therefore, there are following three possibilities:

- Special or Steinberg case: $N \neq 0$ and τ is a scalar.
- Principal series case: $N = 0$ and τ is as in (1) of Lemma 2.1.
- Supercuspidal case: $N = 0$ and τ is as in (2) or (3) of Lemma 2.1.

Next, we study the structure of the filtrations. We assume ρ is F-semi-stable, and take the corresponding filtered $\big(\phi, N, \mathrm{Gal}(F/K), E\big)$-module D. We have a decomposition

$$F \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{j_F : F \hookrightarrow E} E = \prod_{j : K \hookrightarrow E} \Bigg(\prod_{j_F : F \hookrightarrow E,\, j_F|_K = j} E \Bigg) = \prod_{j : K \hookrightarrow E} E_j,$$

where j_F and j are $\mathbb{Q}_p$-embeddings and we put

$$E_j = \prod_{j_F : F \hookrightarrow E,\, j_F|_K = j} E.$$

According to the above decomposition, we have decompositions

$$D_F \cong \prod_{j : K \hookrightarrow E} D_{F,j} \text{ and } \mathrm{Fil}^i D_F \cong \prod_{j : K \hookrightarrow E} \mathrm{Fil}^i_j D_F.$$

Because $\mathrm{Fil}^i D_F$ is $\mathrm{Gal}(F/K)$-stable, $\mathrm{Fil}^i_j D_F$ is free over E_j. We take integers $0 \leq k_{j,1} \leq k_{j,2} \leq k_0$ such that

$$D_{F,j} = \mathrm{Fil}_j^{-k_{j,2}} D_F \supsetneq \mathrm{Fil}_j^{1-k_{j,2}} D_F = \mathrm{Fil}_j^{-k_{j,1}} D_F \supsetneq \mathrm{Fil}_j^{1-k_{j,1}} D_F = 0.$$

Then the Hodge-Tate weights of ρ are $\bigcup_{j : K \hookrightarrow E} \{k_{j,1}, k_{j,2}\}$.

We are going to prepare some lemmas.

Lemma 2.3. *There is a* $\mathrm{Gal}(F/K)$*-equivariant isomorphism*

$$F \otimes_K E \xrightarrow{\sim} E_j$$

of E*-algebra.*

Proof. Let j_0 be a natural inclusion $K \subset E$. Take an extension $j_E : E \xrightarrow{\sim} E$ of $j : K \hookrightarrow E$. Then a $\mathrm{Gal}(F/K)$-equivariant isomorphism

$$\prod_{j_F : F \hookrightarrow E,\, j_F|_K = j_0} E \xrightarrow{\sim} \prod_{j_F : F \hookrightarrow E,\, j_F|_K = j} E$$

of E-algebra is given by sending j_F-components to $(j_E \circ j_F)$-components. □

Lemma 2.4. *If* $k_{j,1} < k_{j,2}$*, then* $\mathrm{Fil}_j^{-k_{j,1}} D_F \subset D_{F,j}$ *is spanned by a Galois invariant element over* E_j*.*

Proof. A generator of $\mathrm{Fil}_j^{-k_{j,1}} D_F$ over E_j generates an $E_j^\times$-torsor with $\mathrm{Gal}(F/K)$-action. An $E_j^\times$-torsor with $\mathrm{Gal}(F/K)$-action is tirivial, if $H^1\big(\mathrm{Gal}(F/K), E_j^\times\big) = 0$. So it suffices to show that $H^1\big(\mathrm{Gal}(F/K), E_j^\times\big) = 0$. By Lemma 2.3, $E_j^\times$ is isomorphic to $(F \otimes_K E)^\times$, and it is further isomorphic to $\mathrm{Ind}_{\{\mathrm{id}_F\}}^{\mathrm{Gal}(F/K)} E^\times$. By Shapiro's lemma, $H^1\big(\mathrm{Gal}(F/K), \mathrm{Ind}_{\{\mathrm{id}_F\}}^{\mathrm{Gal}(F/K)} E^\times\big) = H^1(\{\mathrm{id}_F\}, E^\times) = 0$. □

Lemma 2.5. *Let* K'*,* M *be* p*-adic fields such that* $K \subset K' \subset M \subset F$ *and* M *is a Galois extension of* K'*. Let* $\chi : \mathrm{Gal}(M/K') \to E^\times$ *be a character. We put* $m = [K' : K]$*. Then there exist* $x_1, \ldots, x_m \in M \otimes_K E$ *that satisfy the followings:*

- *For* $x \in M \otimes_K E$*, we have* $gx = \big(1 \otimes \chi(g)^{-1}\big)x$ *for all* $g \in \mathrm{Gal}(M/K')$ *if and only if* $x = \sum_{i=1}^{m}(1 \otimes a_i)x_i$ *for* $a_i \in E$*.*
- *For* $a_i \in E$*, we have* $\sum_{i=1}^{m}(1 \otimes a_i)x_i \in (M \otimes_K E)^{\times}$ *if and only if* $a_i \neq 0$ *for all* i*.*

Proof. We have a decomposition

$$M \otimes_K E \xrightarrow{\sim} \prod_{j_M : M \hookrightarrow E} E = \prod_{j' : K' \hookrightarrow E} \Bigg(\prod_{j_M : M \hookrightarrow E,\, j_M|_{K'} = j'} E \Bigg) = \prod_{j' : K' \hookrightarrow E} E_{j'},$$

where j_M and j' are K-embeddings and we put

$$E_{j'} = \prod_{j_M : M \hookrightarrow E,\, j_M|_{K'} = j'} E.$$

Let $(x_{j'})_{j'} \in \prod_{j' : K' \hookrightarrow E} E_{j'}$ be the image of x under the above isomorphism. Then, $gx = \big(1 \otimes \chi(g)^{-1}\big)x$ for all $g \in \mathrm{Gal}(M/K')$ if and only if $gx_{j'} = \chi(g)^{-1}x_{j'}$ for all $g \in \mathrm{Gal}(M/K')$ and all $j' : K' \hookrightarrow E$. Further, $x \in (M \otimes_K E)^{\times}$ if and only if $x_{j'} \in E_{j'}^{\times}$ for all j'. As in the proof of Lemma 2.3, we can show there is a $\mathrm{Gal}(M/K')$-equivariant isomorphism $M \otimes_{K'} E \xrightarrow{\sim} E_{j'}$ of E-algebra. So, to prove this Lemma, it suffices to treat the case where $m = 1$.

We assume that $m = 1$. Take $\alpha \in M$ such that $g(\alpha)$ for $g \in \mathrm{Gal}(M/K)$ form a basis of M over K. Then $x \in M \otimes_K E$ can be written uniquely as

$$\sum_{g \in \mathrm{Gal}(M/K)} g(\alpha) \otimes a_g$$

for $a_g \in E$. If $hx = \big(1 \otimes \chi(h)^{-1}\big)x$ for all $h \in \mathrm{Gal}(M/K)$, we have $a_{i,h^{-1}g} = \chi^{-1}(h)a_g$ for all $g, h \in \mathrm{Gal}(M/K)$. By putting $a_1 = a_{\mathrm{id}_M}$, we have

$$x = (1 \otimes a_1) \sum_{g \in \mathrm{Gal}(M/K)} g(\alpha) \otimes \chi(g).$$

It suffices to put $x_1 = \sum_{g \in \mathrm{Gal}(M/K)} g(\alpha) \otimes \chi(g)$. □

3. Classification

3.1. **Special or Steinberg case.** In this case, $\tau \simeq \chi|_{I_K} \oplus \chi|_{I_K}$ for some character χ of W_K that is finite on I_K, and there exists a totally ramified cyclic extension F of K such that $\chi|_{I_F}$ is trivial. So we may assume that ρ is F-semi-stable, and χ determine the action of $\mathrm{Gal}(F/K)$ on D, which is again denoted by χ.

Since $N\phi = p\phi N$, we have that $\mathrm{Ker}\, N$ is ϕ-stable and free of rank 1 over $F_0 \otimes_{\mathbb{Q}_p} E$. So we can take a basis e_1, e_2 of D over $F_0 \otimes_{\mathbb{Q}_p} E$ such that $N(e_1) = e_2$ and $N(e_2) = 0$. Again by $N\phi = p\phi N$, we must have $\phi(e_1) = \frac{p}{\alpha}e_1 + \gamma e_2$ and $\phi(e_2) = \frac{1}{\alpha}e_2$ with $\alpha \in (F_0 \otimes_{\mathbb{Q}_p} E)^{\times}$ and $\gamma \in F_0 \otimes_{\mathbb{Q}_p} E$. Modifying e_1 by a scalar multiple of e_2, we may assume $\gamma = 0$. Let $(\alpha_i)_i \in \prod_{\sigma_i : F_0 \hookrightarrow E} E$ be the image of α under the isomorphism

$$F_0 \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma_i : F_0 \hookrightarrow E} E.$$

Then, by calculations, we have

$$t_{\mathrm{H}}(D) = -[E:K] \sum_{j:K\hookrightarrow E} (k_{j,1}+k_{j,2}),$$
$$t_{\mathrm{N}}(D) = [E:F_0]\bigg(m_0 - 2\sum_i v_p(\alpha_i)\bigg).$$

So the condition $t_{\mathrm{H}}(D) = t_{\mathrm{N}}(D)$ is equivalent to that

$$2[K:K_0]\sum_i v_p(\alpha_i) = \sum_j (k_{j,1}+k_{j,2}+1).$$

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is $\mathrm{Gal}(F/K)$-invariant. We note that $a_j = 0$ or $a_j \in E_j^\times$ and that $b_j = 0$ or $b_j \in E_j^\times$.

The only non-trivial (ϕ, N)-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodule of D is $D_2' = (F_0 \otimes_{\mathbb{Q}_p} E)e_2$. By calculations, we have

$$t_{\mathrm{H}}(D_2') = -[E:K]\Bigg\{\sum_{a_j=0} k_{j,1} + \sum_{a_j\neq 0} k_{j,2} + \sum_{k_{j,1}=k_{j,2}} k_{j,2}\Bigg\},$$
$$t_{\mathrm{N}}(D_2') = -[E:F_0]\sum_i v_p(\alpha_i).$$

So the condition $t_{\mathrm{H}}(D_2') \leq t_{\mathrm{N}}(D_2')$ is equivalent to that

$$[K:K_0]\sum_i v_p(\alpha_i) \leq \sum_{a_j=0} k_{j,1} + \sum_{a_j\neq 0} k_{j,2} + \sum_{k_{j,1}=k_{j,2}} k_{j,2}.$$

Since $(a_j e_1 + b_j e_2)$ is $\mathrm{Gal}(F/K)$-invariant, $g \in \mathrm{Gal}(F/K)$ acts on a_j and b_j by $\chi(g)^{-1}$. By Lemma 2.3 and Lemma 2.5, there is $x_1 \in E_j$ such that $a_j = a_j' x_1$ and $b_j = b_j' x_1$ for $a_j', b_j' \in E$. Then, for j such that $a_j \neq 0$,

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a_j' x_1 e_1 + b_j' x_1 e_2) = E_j(e_1 - \mathfrak{L}_j e_2)$$

for $\mathfrak{L}_j \in E$.

Proposition 3.1. *We assume that $N \neq 0$. Then $\tau \simeq \chi|_{I_K} \oplus \chi|_{I_K}$ for some character χ of W_K that is finite on I_K. If we take a totally ramified cyclic extension F of K such that χ is trivial on I_F, then $D = (F_0 \otimes_{\mathbb{Q}_p} E)e_1 \oplus (F_0 \otimes_{\mathbb{Q}_p} E)e_2$ with*

$$N(e_1) = e_2,\ N(e_2) = 0,\ \phi(e_1) = \frac{p}{\alpha} e_1,\ \phi(e_2) = \frac{1}{\alpha} e_2$$

for $\alpha \in (F_0 \otimes_{\mathbb{Q}_p} E)^\times$,

$$g e_1 = \chi(g) e_1,\ g e_2 = \chi(g) e_2$$

for $g \in \mathrm{Gal}(F/K)$ and

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = \begin{cases} E_j e_2 & \text{if } j \in I_1, \\ E_j(e_1 - \mathfrak{L}_j e_2) \text{ for } \mathfrak{L}_j \in E & \text{if } j \in I_2 \end{cases}$$

for j such that $k_{j,1} < k_{j,2}$, where

$$2[K:K_0]\sum_i v_p(\alpha_i) = \sum_j (k_{j,1}+k_{j,2}+1),$$

and I_1, I_2 are any disjoint sets such that $I_1 \cup I_2 = \{j \mid k_{j,1} < k_{j,2}\}$ and

$$[K : K_0] \sum_i v_p(\alpha_i) \leq \sum_{j \in I_1} k_{j,1} + \sum_{j \in I_2} k_{j,2} + \sum_{k_{j,1}=k_{j,2}} k_{j,2}.$$

3.2. **Principal series case.** In this case, $\tau \simeq \chi_1|_{I_K} \oplus \chi_2|_{I_K}$ and $N = 0$. We can take a totally ramified abelian extension F of K such that $\chi_1|_{I_F}$ and $\chi_2|_{I_F}$ are trivial. Then χ_1 and χ_2 determine the action of $\mathrm{Gal}(F/K)$ on D, which is again denoted by the same symbols.

3.2.1. *Irreducible case.* First, we assume that $\chi_1|_{I_K} = \chi_2|_{I_K}$ and D has no non-trivial ϕ-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodule. In this case, we say that ϕ is irreducible. If not, we say that ϕ is reducible. We put $\chi = \chi_1$.

Take bases $e_{i,1}, e_{i,2}$ of D_i over E for $1 \leq i \leq m_0$ so that

$$\phi(e_{1,1}) = ae_{2,1} + ce_{2,2},\ \phi(e_{1,2}) = be_{2,1} + de_{2,2}$$

for $a, b, c, d \in E$, and

$$\phi(e_{i,1}) = e_{i+1,1},\ \phi(e_{i,2}) = e_{i+1,2}$$

for $2 \leq i \leq m_0$. Let e_1, e_2 be a basis of D over $F_0 \otimes_{\mathbb{Q}_p} E$ determined by $(e_{i,1})_i$, $(e_{i,2})_i$ under the isomorphism $D \xrightarrow{\sim} \prod_i D_i$. We will use the same notation in the classification of other cases.

Since ϕ is irreducible, $b \neq 0$ and $c \neq 0$. Modifying $e_{i,1}$ by a scalar multiple of $e_{i,2}$, we may assume $d = 0$. If $X^2 - aX - bc$ is reducible in $E[X]$, by replacing the bases, we can see that ϕ is reducible. This is a contradiction. So $X^2 - aX - bc$ is irreducible in $E[X]$.

Conversely, we suppose that $a, b, c \in E$ are given, $d = 0$, and $X^2 - aX - bc$ is irreducible in $E[X]$. Then the above description determines an endomorphism ϕ. We prove that this endomorphism ϕ is irreducible. If ϕ is reducible, there are $A_i \in GL_2(E)$ such that

$$A_2^{-1} \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} A_1,\ A_3^{-1} A_2,\ A_4^{-1} A_3, \ldots,\ A_1^{-1} A_{m_0}$$

are all upper triangular matrices. Then, multiplying these matrices together, we have that $A_1^{-1} \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} A_1$ is an upper triangular matrix. This contradicts that $X^2 - aX - bc$ is irreducible in $E[X]$.

As above, the endomorphism ϕ is given by $a, b, c \in E$ such that $X^2 - aX - bc$ is reducible in $E[X]$. Now, by calculation, we have

$$t_{\mathrm{H}}(D) = -[E : K] \sum_{j : K \hookrightarrow E} (k_{j,1} + k_{j,2}),$$
$$t_{\mathrm{N}}(D) = [E : F_0]\, v_p(bc).$$

So the condition $t_{\mathrm{H}}(D) = t_{\mathrm{N}}(D)$ is equivalent to that

$$-[K : K_0]\, v_p(bc) = \sum_j (k_{j,1} + k_{j,2}).$$

Since ϕ is irreducible, D has no non-trivial (ϕ, N)-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodule. So there is no condition on the filtrations. For j such that $k_{j,1} < k_{j,2}$, by Lemma 2.3, Lemma 2.4 and Lemma 2.5, we have

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$$

for $(a_j, b_j) \in \mathbb{P}^1(E)$.

By studies of the other cases, ϕ is irreducible only if $N = 0$ and $\tau \simeq \chi|_{I_K} \oplus \chi|_{I_K}$ for some character χ of W_K that is finite on I_K.

Proposition 3.2. *We assume that ϕ is irreducible. Then $N = 0$ and $\tau \simeq \chi|_{I_K} \oplus \chi|_{I_K}$ for some character χ of W_K that is finite on I_K. If we take a totally ramified cyclic extension F of K such that χ is trivial on I_F, then $D = (F_0 \otimes_{\mathbb{Q}_p} E)e_1 \oplus (F_0 \otimes_{\mathbb{Q}_p} E)e_2$ with*

$$\phi(e_{1,1}) = ae_{2,1} + ce_{2,2},\ \phi(e_{1,2}) = be_{2,1}$$

for $a, b \in E^\times$ such that $X^2 - aX - bc$ is irreducible in $E[X]$,

$$\phi(e_{i,1}) = e_{i+1,1},\ \phi(e_{i,2}) = e_{i+1,2}$$

for $2 \leq i \leq m_0$,

$$ge_1 = \chi(g)e_1,\ ge_2 = \chi(g)e_2$$

for $g \in \mathrm{Gal}(F/K)$ and, for j such that $k_{j,1} < k_{j,2}$,

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$$

for $(a_j, b_j) \in \mathbb{P}^1(E)$, where

$$-[K : K_0]\, v_p(bc) = \sum_j (k_{j,1} + k_{j,2}).$$

3.2.2. *Non-split reducible case.* If D has two or more non-trivial ϕ-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodules, we say that ϕ is split. If not, we say that ϕ is non-split. We assume that $\chi_1|_{I_K} = \chi_2|_{I_K}$ and that ϕ is non-split and reducible. We put $\chi = \chi_1$.

Since ϕ is reducible, we can take bases $e_{i,1}, e_{i,2}$ of D_i over E and $a_i, b_i, d_i \in E$ for all i so that

$$\phi(e_{i,1}) = a_i e_{i+1,1},\ \phi(e_{i,2}) = b_i e_{i+1,1} + d_i e_{i+1,2}$$

for all i. Replacing the bases, we may assume that $a_i = d_i = 1$ and $b_i = 0$ for $2 \leq i \leq n$. Since ϕ is non-split, $a_1 = d_1 \neq 0$ and $b_1 \neq 0$. We put $a = a_1$ and $b = b_1$.

Conversely, we suppose that $a, b \in E^\times$ are given. Then the above description determines an endomorphism ϕ. We prove that this endomorphism ϕ is non-split. If ϕ is split, there are $A_i \in GL_2(E)$ such that

$$A_2^{-1} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} A_1,\ A_3^{-1} A_2,\ A_4^{-1} A_3, \ldots,\ A_1^{-1} A_{m_0}$$

are all diagonal matrices. Then, multiplying these matrices together, we have that $A_1^{-1} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} A_1$ is a diagonal matrix. This contradicts that $b \neq 0$.

As above, the endomorphism ϕ is given by $a, b \in E^\times$. The condition $t_{\mathrm{H}}(D) = t_{\mathrm{N}}(D)$ is equivalent to that

$$-2[K : K_0]\, v_p(a) = \sum_j (k_{j,1} + k_{j,2}).$$

Now we have bases $e_{i,1}, e_{i,2}$ of D_i over E such that

$$\phi(e_{1,1}) = ae_{2,1},\ \phi(e_{1,2}) = be_{2,1} + ae_{2,2}$$

for $a, b \in E^\times$, and

$$\phi(e_{i,1}) = e_{i+1,1},\ \phi(e_{i,2}) = e_{i+1,2}$$

for $2 \leq i \leq m_0$.

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is $\mathrm{Gal}(F/K)$-invariant.

The only non-trivial (ϕ, N)-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodule of D is $D'_1 = (F_0 \otimes_{\mathbb{Q}_p} E)e_1$. The condition $t_{\mathrm{H}}(D'_1) \leq t_{\mathrm{N}}(D'_1)$ is equivalent to that

$$-[K : K_0]\, v_p(a) \leq \sum_{b_j = 0} k_{j,1} + \sum_{b_j \neq 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}.$$

As in the special or Steinberg case, for j such that $b_j \neq 0$,

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(-\mathfrak{L}_j e_1 + e_2)$$

for $\mathfrak{L}_j \in E$.

By studies of the other cases, ϕ is non-split reducible only if $N = 0$ and $\tau \simeq \chi|_{I_K} \oplus \chi|_{I_K}$ for some character χ of W_K that is finite on I_K.

Proposition 3.3. *We assume that ϕ is non-split reducible. Then $N = 0$ and $\tau \simeq \chi|_{I_K} \oplus \chi|_{I_K}$ for some character χ of W_K that is finite on I_K. If we take a totally ramified cyclic extension F of K such that χ is trivial on I_F, then $D = (F_0 \otimes_{\mathbb{Q}_p} E)e_1 \oplus (F_0 \otimes_{\mathbb{Q}_p} E)e_2$ with*

$$\phi(e_{1,1}) = a e_{2,1},\ \phi(e_{1,2}) = b e_{2,1} + a e_{2,2}$$

for $a, b \in E^{\times}$,

$$\phi(e_{i,1}) = e_{i+1,1},\ \phi(e_{i,2}) = e_{i+1,2}$$

for $2 \leq i \leq m_0$,

$$g e_1 = \chi(g) e_1,\ g e_2 = \chi(g) e_2$$

for $g \in \mathrm{Gal}(F/K)$ and

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = \begin{cases} E_j e_1 & \text{if } j \in I_1, \\ E_j(-\mathfrak{L}_j e_1 + e_2) \text{ for } \mathfrak{L}_j \in E & \text{if } j \in I_2 \end{cases}$$

for j such that $k_{j,1} < k_{j,2}$, where

$$-2[K : K_0]\, v_p(a) = \sum_j (k_{j,1} + k_{j,2}),$$

and I_1, I_2 are any disjoint sets such that $I_1 \cup I_2 = \{j \mid k_{j,1} < k_{j,2}\}$ and

$$-[K : K_0]\, v_p(a) \leq \sum_{j \in I_1} k_{j,1} + \sum_{j \in I_2} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}.$$

3.2.3. *Split case.* The remaining cases are the following two cases:

- $\chi_1|_{I_K} = \chi_2|_{I_K}$ and ϕ is split.
- $\chi_1|_{I_K} \neq \chi_2|_{I_K}$.

First, we assume that $\chi_1|_{I_K} \neq \chi_2|_{I_K}$. Let e_1, e_2 be a basis of D over $F_0 \otimes_{\mathbb{Q}_p} E$ such that $\mathrm{Gal}(F/K)$ acts on e_1 by χ_1 and e_2 by χ_2. We put

$$\phi(e_1) = \alpha e_1 + \gamma e_2,\ \phi(e_2) = \beta e_1 + \delta e_2,$$

where $\alpha, \beta, \gamma, \delta \in F_0 \otimes_{\mathbb{Q}_p} E$. Since ϕ commutes with the action of $\mathrm{Gal}(F/K)$ and $\chi_1|_{I_K} \neq \chi_2|_{I_K}$, we have $\beta = \gamma = 0$. So, in the both cases, we may assume that ϕ is split.

We take bases $e_{i,1}, e_{i,2}$ of D_i over E so that

$$\phi(e_{1,1}) = a e_{2,1},\ \phi(e_{1,2}) = b e_{2,2}$$

for some $a, b \in E^\times$ and

$$\phi(e_{i,1}) = e_{i+1,1},\ \phi(e_{i,2}) = e_{i+1,2}$$

for $2 \leq i \leq m_0$. Let e_1, e_2 be a basis of D over $F_0 \otimes_{\mathbb{Q}_p} E$ determined by $(e_{i,1})_i$, $(e_{i,2})_i$ under the isomorphism $D \xrightarrow{\sim} \prod_i D_i$.

Then the condition $t_{\mathrm{H}}(D) = t_{\mathrm{N}}(D)$ is equivalent to that

$$(S) \qquad [K : K_0]\, v_p(ab) = \sum_j (k_{j,1} + k_{j,2}).$$

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is $\mathrm{Gal}(F/K)$-invariant.

Since $(a_j e_1 + b_j e_2)$ is $\mathrm{Gal}(F/K)$-invariant, $g \in \mathrm{Gal}(F/K)$ acts on a_j and b_j by $\chi_1(g)^{-1}$ and $\chi_2(g)^{-1}$ respectively. By Lemma 2.3 and Lemma 2.5, there are $x_1, x_2 \in E_j$ such that $a_j = a'_j x_1$ and $b_j = b'_j x_2$ for $a'_j, b'_j \in E$. Then, for j such that $a_j \neq 0$ and $b_j \neq 0$, we have

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a'_j x_1 e_1 + b'_j x_2 e_2) = E_j(e_1 - \mathfrak{L}_j x_0 e_2)$$

for $\mathfrak{L}_j \in E^\times$, where we put $x_0 = x_1^{-1} x_2$.

If $a \neq b$, the non-trivial (ϕ, N)-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodules of D are $D'_1 = (F_0 \otimes_{\mathbb{Q}_p} E)e_1$ and $D'_2 = (F_0 \otimes_{\mathbb{Q}_p} E)e_2$. The condition $t_{\mathrm{H}}(D'_1) \leq t_{\mathrm{N}}(D'_1)$ is equivalent to that

$$[K : K_0]\, v_p(a) \leq \sum_{b_j = 0} k_{j,1} + \sum_{b_j \neq 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}.$$

The condition $t_{\mathrm{H}}(D'_2) \leq t_{\mathrm{N}}(D'_2)$ is equivalent to that

$$[K : K_0]\, v_p(b) \leq \sum_{a_j = 0} k_{j,1} + \sum_{a_j \neq 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}.$$

If $a = b$, the non-trivial (ϕ, N)-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodules of D are D'_1, D'_2 and $D'_{\mathfrak{L}} = (F_0 \otimes_{\mathbb{Q}_p} E)(e_1 - \mathfrak{L} e_2)$ for $\mathfrak{L} \in E^\times$. For $\mathfrak{L} \in E^\times$, the condition $t_{\mathrm{H}}(D'_{\mathfrak{L}}) \leq t_{\mathrm{N}}(D'_{\mathfrak{L}})$ is equivalent to that

$$\begin{aligned}(S_{\mathfrak{L}}) \qquad [K : K_0]\, v_p(a) \leq & \sum_{a_j b_j = 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2} \\ & + \sum_{a_j b_j \neq 0} \bigl\{ t_j(\mathfrak{L}, \mathfrak{L}_j) k_{j,1} + \bigl(1 - t_j(\mathfrak{L}, \mathfrak{L}_j)\bigr) k_{j,2} \bigr\},\end{aligned}$$

where

$$t_j(\mathfrak{L}, \mathfrak{L}_j) = \frac{\bigl|\{ j_F : F \hookrightarrow E \mid j_F\text{-component of } \mathfrak{L}_j x_0 \in E_j \text{ is } \mathfrak{L} \}\bigr|}{[F : K]}.$$

If $t_j(\mathfrak{L}, \mathfrak{L}_j) \leq 1/2$, the condition $(S_{\mathfrak{L}})$ is automatically satisfied by the condition (S).

We assume that $t_j(\mathfrak{L}, \mathfrak{L}_j) > 1/2$. Then we have

$$\frac{\Bigl|\mathrm{Ker}\Bigl(\chi_1 \chi_2^{-1} : \mathrm{Gal}(F/K) \to \overline{\mathbb{Q}}_p^\times\Bigr)\Bigr|}{[F : K]} > \frac{1}{2},$$

because $\mathrm{Gal}(F/K)$ act on x_0 by $\chi_1 \chi_2^{-1}$. This implies that $\chi_1|_{I_K} = \chi_2|_{I_K}$ and

$$x_0 = (x_E)_{j_F} \in \prod_{j_F : F \hookrightarrow E,\, j_F|_K = j} E$$

for some $x_E \in E^\times$. Then $\mathfrak{L}_j x_E = \mathfrak{L}$ and $t_j(\mathfrak{L}, \mathfrak{L}_j) = 1$.

Proposition 3.4. *We assume that $N = 0$ and ϕ is split reducible and $\tau \simeq \chi_1|_{I_K} \oplus \chi_2|_{I_K}$ for some character χ_1, χ_2 of W_K that are finite on I_K. If we take a totally ramified cyclic extension F of K such that χ_1, χ_2 is trivial on I_F, then $D = (F_0 \otimes_{\mathbb{Q}_p} E)e_1 \oplus (F_0 \otimes_{\mathbb{Q}_p} E)e_2$ with*

$$\phi(e_{1,1}) = ae_{2,1},\ \phi(e_{1,2}) = be_{2,2}$$

for $a, b \in E^\times$ and

$$\phi(e_{i,1}) = e_{i+1,1},\ \phi(e_{i,2}) = e_{i+1,2}$$

for $2 \leq i \leq m_0$ and

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = \begin{cases} E_j e_1 & \text{if } j \in I_1, \\ E_j e_2 & \text{if } j \in I_2, \\ E_j(e_1 - \mathfrak{L}_j x_0 e_2) \text{ for } \mathfrak{L}_j \in E^\times & \text{if } j \in I_3 \end{cases}$$

for j such that $k_{j,1} < k_{j,2}$, where

$$[K : K_0]\, v_p(ab) = \sum_j (k_{j,1} + k_{j,2}),$$

and I_1, I_2, I_3 are any disjoint sets such that $I_1 \cup I_2 \cup I_3 = \{j \mid k_{j,1} < k_{j,2}\}$ and

$$[K : K_0]\, v_p(a) \leq \sum_{j \in I_1} k_{j,1} + \sum_{j \in I_2 \cup I_3} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2},$$
$$[K : K_0]\, v_p(b) \leq \sum_{j \in I_2} k_{j,1} + \sum_{j \in I_1 \cup I_3} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2},$$

and, if $a = b$ and $\chi_1|_{I_K} = \chi_2|_{I_K}$, further

$$[K : K_0]\, v_p(a) \leq \sum_{j \in I_3,\, \mathfrak{L}_j x_E = \mathfrak{L}} k_{j,1} + \sum_{j \in I_3,\, \mathfrak{L}_j x_E \neq \mathfrak{L}} k_{j,2} + \sum_{j \in I_1 \cup I_2} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}$$

for all $\mathfrak{L} \in E^\times$.

3.3. **Supercuspidal case.** In this case, $N = 0$ and $\tau \simeq \mathrm{Ind}_{W_{K'}}^{W_K}(\chi)|_{I_K}$ for a quadratic extension K' of K and a character χ of $W_{K'}$ that is finite on $I_{K'}$. Let k' be the residue field of K'. We take a totally ramified abelian extension L of K' such that $\chi|_{I_L}$ is trivial.

For a uniformizer π' of K' and a positive integer n, let $K'_{\pi',n}$ be the Lubin-Tate extension of K' generated by the ${\pi'}^n$-torsion points. For any p-adic field M and a positive integer n, we put $U_M^{(n)} = 1 + \mathfrak{p}_M^n$. Then we have

$$\mathrm{Gal}(K'_{\pi',n}/K') \cong (\mathcal{O}_{K'}/\mathfrak{p}_{K'}^n)^\times \cong {k'}^\times \times \big(U_{K'}^{(1)}/U_{K'}^{(n)}\big).$$

For any p-adic field M and a positive integer m, let M_m be the unramified extension of M of degree m.

3.3.1. *Unramified case.* We first treat the case in (2) of Lemma 2.1, where K' is unramified over K and χ does not extend to W_K. We take a uniformizer π of K. This is also a uniformizer of K'. We take positive integers m_1 and n_1 so that L is contained in $K'_{m_1}K'_{\pi,n_1}$, and put $F = K'_{m_1}K'_{\pi,n_1}$. Then ρ is crystalline over F, and F is a Galois extension of K.

We put $f(X) = \pi X + X^{q^2}$. For a positive integer n, let $f^{(n)}(X)$ be the n-th iterate of $f(X)$. We take a root θ of $f^{(n_1)}(X)$ in K'_{π,n_1} that is not a root of $f^{(n_1-1)}(X)$. Then $K'_{\pi,n_1} = K'(\theta)$. We can see that $K(\theta)$ is a totally ramified extension of K and that F is an unramified extension of $K(\theta)$ of degree $2m_1$. Now the restriction $\mathrm{Gal}\big(F/K(\theta)\big) \to \mathrm{Gal}(K'_{m_1}/K)$ is an isomorphism, and $\mathrm{Gal}(F/K)$ is a semi-direct product of $\mathrm{Gal}\big(F/K(\theta)\big)$ by $\mathrm{Gal}(F/K'_{m_1})$. We take a generator σ of $\mathrm{Gal}(F/K(\theta))$. Then the restriction $\sigma|_{K'}$ is the non-trivial element of $\mathrm{Gal}(K'/K)$.

We consider a decomposition

$$U_{K'}^{(1)}/U_{K'}^{(n_1)} = U_{n_1,+} \times U_{n_1,-}$$

of abelian groups such that $\sigma(\gamma_1) = \gamma_1$ for $\gamma_1 \in U_{n_1,+}$ and $\sigma(\gamma_2) = \gamma_2^{-1}$ for $\gamma_2 \in U_{n_1,-}$. There is an exact sequence

$$1 \to U_K^{(1)}/U_K^{(n_1)} \to U_{K'}^{(1)}/U_{K'}^{(n_1)} \to U_{K'}^{(1)}/U_{K'}^{(n_1)}$$

where the first map is induced from a natural inclusion and the second map is induced from a map

$$U_{K'}^{(1)} \to U_{K'}^{(1)};\ g \mapsto \sigma(g)g^{-1}.$$

Then, by the above exact sequence, we see that

$$U_{n_1,+} \cong U_K^{(1)}/U_K^{(n_1)},\ U_{n_1,-} \cong U_{K'}^{(1)}/\big(U_K^{(1)}U_{K'}^{(n_1)}\big)$$

and $|U_{n_1,+}| = |U_{n_1,-}| = q^{n_1-1}$.

Now, the restriction $\mathrm{Gal}(F/K'_{m_1}) \to \mathrm{Gal}(K'_{\pi,n_1}/K')$ is an isomorphism. Then we can prove that, under an identification

$$\mathrm{Gal}(F/K'_{m_1}) \cong \mathrm{Gal}(K'_{\pi,n_1}/K') \cong k'^{\times} \times U_{n_1,+} \times U_{n_1,-},$$

we have

$$(*) \qquad \sigma^{-1}\delta\sigma = \delta^q,\ \sigma^{-1}\gamma_1\sigma = \gamma_1 \text{ and } \sigma^{-1}\gamma_2\sigma = \gamma_2^{-1}$$

for $\delta \in k'^{\times}$, $\gamma_1 \in U_{n_1,+}$ and $\gamma_2 \in U_{n_1,-}$.

Considering $\chi|_{I_K}$ as a character of

$$I(F/K) \cong k'^{\times} \times U_{n_1,+} \times U_{n_1,-},$$

we write $\chi = \omega^s \cdot \chi_1 \cdot \chi_2$, where ω is the Teichmüller character, s is an integer, and χ_1 and χ_2 are characters of $U_{n_1,+}$ and $U_{n_1,-}$ respectively. The condition that χ does not extend to W_K is equivalent to that $\chi \neq \chi^{\sigma}$ on $W_{K'}$, and it is further equivalent to that $\chi \neq \chi^{\sigma}$ on $I_{K'}$. This last condition is equivalent to that $s \not\equiv 0 \mod q+1$ or $\chi_2^2 \neq 1$.

Now we have $[F_0 : \mathbb{Q}_p] = 2m_0m_1$. We take bases $e_{i,1}$, $e_{i,2}$ of D_i over E for $1 \leq i \leq 2m_0m_1$ so that

$$\begin{aligned}
&\delta e_{i,1} = \omega^s(\delta)e_{i,1}, && \gamma_1 e_{i,1} = \chi_1(\gamma_1)e_{i,1}, && \gamma_2 e_{i,1} = \chi_2(\gamma_2)e_{i,1},\\
&\delta e_{i,2} = \omega^{qs}(\delta)e_{i,2}, && \gamma_1 e_{i,2} = \chi_1(\gamma_1)e_{i,2}, && \gamma_2 e_{i,2} = \chi_2(\gamma_2)^{-1}e_{i,2}
\end{aligned}$$

for $\delta \in k'^{\times}$, $\gamma_1 \in U_{n_1,+}$ and $\gamma_2 \in U_{n_1,-}$.

Remark 3.5. *A normalization of bases here is different from that in* [GM, 3.3.2]. *We prefer that the action of δ on $e_{i,1}$, $e_{i,2}$ is the same form for all i. In stead of this, the action of σ does not preserve lines generated by e_1 and e_2 as we see in the below.*

Since σ takes D_i to D_{i+m_0}, we have that

$$\sigma e_{i,1} = a_{i+m_0} e_{i+m_0,2},\ \sigma e_{i,2} = b_{i+m_0} e_{i+m_0,1}$$

for some $a_{i+m_0}, b_{i+m_0} \in E^\times$ by $(*)$. Because $\sigma^{2m_1} = 1$, we see that

$$\prod_{l=1}^{m_1} (a_{i+2lm_0-m_0} b_{i+2lm_0}) = 1$$

for all i. Replacing $e_{i,1}$ and $e_{i,2}$ by their scalar multiples, we may assume that

$$\sigma e_{i,1} = e_{i+m_0,2},\ \sigma e_{i,2} = e_{i+m_0,1}.$$

Since ϕ takes D_i to D_{i+1} and commutes with the action of $I(F/K)$, we have that

$$\phi(e_{i,1}) = \frac{1}{\alpha_{i+1}} e_{i+1,1},\ \phi(e_{i,2}) = \frac{1}{\beta_{i+1}} e_{i+1,2}$$

for some $\alpha_{i+1}, \beta_{i+1} \in E^\times$ for all i. Since ϕ commutes with the action of σ, we have $\alpha_i = \beta_{i+m_0}$ and $\beta_i = \alpha_{i+m_0}$ for all i. Replacing $e_{i,1}$ and $e_{i,2}$ by their scalar multiples, we may further assume that $\alpha_i = \beta_i = 1$ for $2 \leq i \leq m_0$.

Let e_1, e_2 be a basis of D over $F_0 \otimes_{\mathbb{Q}_p} E$ determined by $(e_{i,1})_i$, $(e_{i,2})_i$ under the isomorphism $D \xrightarrow{\sim} \prod_i D_i$. Then $\sigma e_1 = e_2$ and $\sigma e_2 = e_1$.

The condition $t_{\mathrm{H}}(D) = t_{\mathrm{N}}(D)$ is equivalent to that

$$(U) \qquad [K : K_0]\, v_p(\alpha_1 \beta_1) = \sum_j (k_{j,1} + k_{j,2}).$$

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is $\mathrm{Gal}(F/K)$-invariant. By $\sigma(a_j e_1 + b_j e_2) = (a_j e_1 + b_j e_2)$, we get $\sigma(a_j) = b_j$ and $\sigma(b_j) = a_j$. So $a_j \in E_j^\times$ if and only if $b_j \in E_j^\times$.

Since $(a_j e_1 + \sigma(a_j) e_2)$ is $\mathrm{Gal}(F/K)$-invariant, $\sigma^2(a_j) = a_j$ and $g \in I(F/K)$ acts on a_j by $\chi(g)^{-1}$. We prove that there are $x_{j,1}, x_{j,2} \in E_j$ such that

- a_j satisfies the above condition if and only if $a_j = a_{j,1} x_{j,1} + a_{j,2} x_{j,2}$ for some $a_{j,1}, a_{j,2} \in E$,
- for $a_{j,1}, a_{j,2} \in E$, we have $a_{j,1} x_{j,1} + a_{j,2} x_{j,2} \in E_j^\times$ if and only if $a_{j,1} \neq 0$ and $a_{j,2} \neq 0$.

By Lemma 2.3, we may replace E_j by $F \otimes_K E$. Then $\sigma^2(a_j) = a_j$ if and only if $a_j \in K'_{\pi,n_1} \otimes_K E$. By Lemma 2.5, we get the claim. We put $x_j(a_{j,1}, a_{j,2}) = a_{j,1} x_{j,1} + a_{j,2} x_{j,2}$ and $x_j^\sigma(a_{j,1}, a_{j,2}) = \sigma\big(x_j(a_{j,1}, a_{j,2})\big)$. Then we have

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j \big(x_j(a_{j,1}, a_{j,2}) e_1 + x_j^\sigma(a_{j,1}, a_{j,2}) e_2 \big)$$

for $(a_{j,1}, a_{j,2}) \in \mathbb{P}^1(E)$.

The non-trivial (ϕ, N)-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodules of D are $D'_1 = (F_0 \otimes_{\mathbb{Q}_p} E) e_1$, $D'_2 = (F_0 \otimes_{\mathbb{Q}_p} E) e_2$ and $D'_{\mathfrak{L}} = (F_0 \otimes_{\mathbb{Q}_p} E)(e_1 - \mathfrak{L} e_2)$ for $\mathfrak{L} \in (F_0 \otimes_{\mathbb{Q}_p} E)^\times$ satisfying the following:

If $\mathfrak{L}$ corresponds to $(\mathfrak{L}_i)_i$ under the isomorphism

$$F_0 \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma_i : F_0 \hookrightarrow E} E,$$

then $\mathfrak{L}_{i+1} = \frac{\alpha_{i+1}}{\beta_{i+1}} \mathfrak{L}_i$ for all i.

The condition $t_{\mathrm{H}}(D'_1) \leq t_{\mathrm{N}}(D'_1)$ is equivalent to that

$$[K : K_0]\, v_p(\alpha_1) \leq \sum_{a_{j,1}a_{j,2}=0} \frac{k_{j,1}+k_{j,2}}{2} + \sum_{a_{j,1}a_{j,2}\neq 0} k_{j,2} + \sum_{k_{j,1}=k_{j,2}} k_{j,2},$$

the condition $t_{\mathrm{H}}(D'_2) \leq t_{\mathrm{N}}(D'_2)$ is equivalent to that

$$[K : K_0]\, v_p(\beta_1) \leq \sum_{a_{j,1}a_{j,2}=0} \frac{k_{j,1}+k_{j,2}}{2} + \sum_{a_{j,1}a_{j,2}\neq 0} k_{j,2} + \sum_{k_{j,1}=k_{j,2}} k_{j,2},$$

and the condition $t_{\mathrm{H}}(D'_{\mathfrak{L}}) \leq t_{\mathrm{N}}(D'_{\mathfrak{L}})$ is equivalent to that

$$(U_{\mathfrak{L}}) \quad [K : K_0]\, \frac{v_p(\alpha_1\beta_1)}{2} \leq \sum_{a_{j,1}a_{j,2}=0} k_{j,2} + \sum_{k_{j,1}=k_{j,2}} k_{j,2} + \sum_{a_{j,1}a_{j,2}\neq 0} \Big\{ t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big) k_{j,1} + \Big(1 - t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big)\Big) k_{j,2} \Big\},$$

where

$$t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big) = \frac{\Big|\big\{ j_F : F \hookrightarrow E \;\big|\; j_F\text{-component of } \frac{x_j^{\sigma}(a_{j,1},a_{j,2})}{x_j(a_{j,1},a_{j,2})} \in E_j \text{ is } -\mathfrak{L}_{j_F} \big\}\Big|}{[F : K]}.$$

Here and in the sequel, $\mathfrak{L}_{j_F}$ is the j_F-component of $\mathfrak{L} \in F_0 \otimes_{\mathbb{Q}_p} E \subset F \otimes_{\mathbb{Q}_p} E$. If $t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big) \leq 1/2$, the condition $(U_{\mathfrak{L}})$ is automatically satisfied by the condition (U).

To prove that $t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big) \leq 1/2$, we assume that $t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big) > 1/2$. We consider a decomposition

$$E_j = \prod_{j_F : F \hookrightarrow E,\, j_F|_K = j} E = \prod_{j_{F_0} : F_0 \hookrightarrow E,\, j_{F_0}|_K = j} \Bigg(\prod_{j_F : F \hookrightarrow E,\, j_F|_{F_0} = j_{F_0}} E \Bigg).$$

Then there is $j_{F_0} : F_0 \hookrightarrow E$ such that $j_{F_0}|_K = j$ and

$$\frac{\Big|\Big\{ j_F : F \hookrightarrow E \;\Big|\; j_F|_{F_0} = j_{F_0} \text{ and } j_F\text{-component of } \frac{x_j^{\sigma}(a_{j,1},a_{j,2})}{x_j(a_{j,1},a_{j,2})} \in E_j \text{ is } -\mathfrak{L}_{j_F} \Big\}\Big|}{[F : F_0]}$$

is greater than $1/2$. Here $\mathfrak{L}_{j_F}$ is independent of j_F such that $j_F|_{F_0} = j_{F_0}$, because $\mathfrak{L} \in F_0 \otimes_{\mathbb{Q}_p} E$. Then we have

$$\frac{\Big|\mathrm{Ker}\Big(\chi(\chi^{\sigma})^{-1} : I(F/K) \to \overline{\mathbb{Q}}_p^{\times} \Big)\Big|}{[F : F_0]} > \frac{1}{2},$$

because $I(F/K')$ act on $x_j^{\sigma}(a_{j,1}, a_{j,2}) / \big(x_j(a_{j,1}, a_{j,2})\big)$ by $\chi(\chi^{\sigma})^{-1}$. This implies that $\chi|_{I_{K'}} = \chi^{\sigma}|_{I_{K'}}$, and contradicts the condition that χ does not extend to W_K. Thus we have proved that $t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big) \leq 1/2$.

Proposition 3.6. *We assume $\tau \simeq \mathrm{Ind}_{W_{K'}}^{W_K}(\chi)|_{I_K}$ for the unramified quadratic extension K' of K and a character χ of $W_{K'}$ that is finite on $I_{K'}$ and does not extend to W_K. We take a uniformizer π of K and a totally ramified abelian extension L of K' such that χ is trivial on I_L, and take positive integers m_1 and n_1 so that L is contained in $K'_{m_1}K'_{\pi,n_1}$. We put $F = K'_{m_1}K'_{\pi,n_1}$. Then $N = 0$ and $D = (F_0 \otimes_{\mathbb{Q}_p} E)e_1 \oplus (F_0 \otimes_{\mathbb{Q}_p} E)e_2$ with*

$$\begin{aligned}
&\phi(e_{i,1}) = \frac{1}{\alpha_1}e_{i+1,1}, && \phi(e_{i,2}) = \frac{1}{\beta_1}e_{i+1,2}, && \text{if } i \equiv 0 \pmod{2m_0},\\
&\phi(e_{i,1}) = \frac{1}{\beta_1}e_{i+1,1}, && \phi(e_{i,2}) = \frac{1}{\alpha_1}e_{i+1,2}, && \text{if } i \equiv m_0 \pmod{2m_0},\\
&\phi(e_{i,1}) = e_{i+1,1}, && \phi(e_{i,2}) = e_{i+1,2}, && \text{if } i \not\equiv 0 \pmod{m_0}
\end{aligned}$$

for $\alpha_1, \beta_1 \in E^\times$,

$$\sigma e_1 = e_2,\ \sigma e_2 = e_1,\ g e_1 = \big(1 \otimes \chi(g)\big)e_1,\ g e_2 = \big(1 \otimes \chi^\sigma(g)\big)e_2$$

for $g \in I(F/K)$ and, for j such that $k_{j,1} < k_{j,2}$,

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j\big(x_j(a_{j,1},a_{j,2})e_1 + x_j^\sigma(a_{j,1},a_{j,2})e_2\big)$$

for $(a_{j,1},a_{j,2}) \in \mathbb{P}^1(E)$ where

$$[K:K_0]\, v_p(\alpha_1\beta_1) = \sum_j (k_{j,1}+k_{j,2})$$

and

$$\sum_j k_{j,1} + \sum_{a_{j,1}a_{j,2}=0} \frac{k_{j,2}-k_{j,1}}{2} \leq [K:K_0]\, v_p(\alpha_1) \leq \sum_j k_{j,2} - \sum_{a_{j,1}a_{j,2}=0} \frac{k_{j,2}-k_{j,1}}{2}.$$

The definition of σ is in the above discussion.

3.3.2. *Ramified case.* Next, we treat the case in (3) of Lemma 2.1, where K' is ramified over K and $\chi|_{I_{K'}}$ does not extend to I_K.

Let ι_0 be the non-trivial element of $\mathrm{Gal}(K'/K)$. We take a uniformizer π' of K' such that $\iota_0(\pi') = -\pi'$. Then we have $(K'_{\pi',n})^\iota = K'_{-\pi',n}$ for a positive integer n and any lift $\iota \in G_K$ of ι_0. So $K'_{\pi',n}K'_{-\pi',n}$ is a Galois extension of K. By the class field theory, the abelian extensions $K'_{\pi',n}$ and $K'_{-\pi',n}$ of K' correspond to $\langle \pi' \rangle \times (1+\mathfrak{p}_{K'}^n)$ and $\langle -\pi' \rangle \times (1+\mathfrak{p}_{K'}^n)$ respectively. Then the abelian extension $K'_{\pi',n}K'_{-\pi',n}$ of K' corresponds to $\langle {\pi'}^2 \rangle \times (1+\mathfrak{p}_{K'}^n)$. So we see that $K'_{\pi',n}K'_{-\pi',n} = K'_2K'_{\pi',n}$.

We take positive integers m_1 and n_1 so that L is contained in $K'_{2m_1}K'_{\pi',2n_1+1}$, and put $F = K'_{2m_1}K'_{\pi',2n_1+1}$. Then F is a Galois extension of K, and ρ is crystalline over F because $\tau|_{I_F}$ is trivial.

We consider an exact sequence

$$(\diamondsuit) \qquad 1 \to \mathrm{Gal}(F/K') \to \mathrm{Gal}(F/K) \to \mathrm{Gal}(K'/K) \to 1.$$

Since the restriction $\mathrm{Gal}(F/K'_{2m_1}) \to \mathrm{Gal}(K'_{\pi',2n_1+1}/K')$ is an isomorphism,

$$\begin{aligned}
\mathrm{Gal}(F/K') &= \mathrm{Gal}(F/K'_{\pi',2n_1+1}) \times \mathrm{Gal}(F/K'_{2m_1})\\
&\cong \mathrm{Gal}(F/K'_{\pi',2n_1+1}) \times {k'}^\times \times \big(U_{K'}^{(1)}/U_{K'}^{(2n_1+1)}\big).
\end{aligned}$$

Let σ be a generator of $\mathrm{Gal}(F/K_{\pi',2n_1+1})$, and δ_0 be a generator of ${k'}^\times$.

We prove that the exact sequence $(\diamondsuit)$ does not split. We assume there is a lift $\iota \in \mathrm{Gal}(F/K)$ of ι_0 such that $\iota^2 = 1$. By multiplying ι by an element of

$\mathrm{Gal}(F/K'_{\pi',2n_1+1}) \subset \mathrm{Gal}(F/K')$, we may assume that $\iota \in I(F/K)$. Let $P(F/K)$ be the wild ramification subgroup of $I(F/K)$, and $I^{\mathrm{t}}(F/K)$ be the tame quotient group of $I(F/K)$. Let $\bar{\iota}$ be the image of ι in $I^{\mathrm{t}}(F/K)$. If $\bar{\iota} \neq 1$, we multiply ι by the element $\delta_0^{(q-1)/2}$ of $k'^{\times} \subset \mathrm{Gal}(F/K'_{2m_1})$. Then we have $\iota \in P(F/K)$, but this contradicts that $p \neq 2$. Thus we have proved the claim.

For any lift $\iota \in \mathrm{Gal}(F/K)$, we have $\iota^2 \in \mathrm{Gal}(F/K')$. Since the exact sequence ($\diamondsuit$) does not split and $p \neq 2$, multiplying ι by an element of $\mathrm{Gal}(F/K')$, we may assume that $\iota^2 = \delta_0$ and $\iota \in I(F/K)$. We fix this lift ι in the sequel.

We consider a decomposition

$$U_{K'}^{(1)}/U_{K'}^{(2n_1+1)} = U_{2n_1+1,+} \times U_{2n_1+1,-}$$

of abelian groups such that $\iota_0(\gamma_1) = \gamma_1$ for $\gamma_1 \in U_{2n_1+1,+}$ and $\iota_0(\gamma_2) = \gamma_2^{-1}$ for $\gamma_2 \in U_{2n_1+1,-}$. There is an exact sequence

$$1 \to U_K^{(1)}/U_K^{(n_1+1)} \to U_{K'}^{(1)}/U_{K'}^{(2n_1+1)} \to U_{K'}^{(1)}/U_{K'}^{(2n_1+1)},$$

where the first map is induced from a natural inclusion and the second map is induced from a map

$$U_{K'}^{(1)} \to U_{K'}^{(1)};\ g \mapsto \iota_0(g)g^{-1}.$$

Then, by the above exact sequence, we see that

$$U_{2n_1+1,+} \cong U_K^{(1)}/U_K^{(n_1+1)},\ U_{2n_1+1,-} \cong U_{K'}^{(1)}/\bigl(U_K^{(1)}U_{K'}^{(2n_1+1)}\bigr)$$

and $|U_{2n_1+1,+}| = |U_{2n_1+1,-}| = q^{n_1}$.

We can prove that, under an identification

$$\mathrm{Gal}(F/K'_{2m_1}) \cong \mathrm{Gal}(K'_{\pi',2n_1+1}/K') \cong k'^{\times} \times U_{2n_1+1,+} \times U_{2n_1+1,-},$$

we have

$$\iota^{-1}\delta\iota = \delta,\ \iota^{-1}\gamma_1\iota = \gamma_1 \text{ and } \iota^{-1}\gamma_2\iota = \gamma_2^{-1}$$

for $\delta \in k'^{\times}$, $\gamma_1 \in U_{2n_1+1,+}$ and $\gamma_2 \in U_{2n_1+1,-}$.

Since $K'_{\pi',2n_1+1}$ is not a normal extension of K, we have $\iota^{-1}\sigma\iota \neq \sigma$. We put $K'' = K'_{\pi',2n_1+1}K'_{-\pi',2n_1+1}$. Then σ^2 is a generator of $\mathrm{Gal}(F/K'')$, and ι determines an automorphism of K''. So we have $\iota^{-1}\sigma^2\iota = \sigma^2$. Since $\sigma^{-1}\iota^{-1}\sigma\iota$ is an element of $\mathrm{Gal}(F/K')$ of order 2 and fixes K_{2m_1}, it is $\delta_0^{(q-1)/2}$. Hence we have

$$(\star) \qquad \iota^{-1}\sigma\iota = \sigma\delta_0^{(q-1)/2}.$$

Considering $\chi|_{I_{K'}}$ as a character of

$$I(F/K') \cong k'^{\times} \times U_{2n_1+1,+} \times U_{2n_1+1,-},$$

we write $\chi = \omega^s \cdot \chi_1 \cdot \chi_2$, where ω is the Teichmüller character, s is an integer, and χ_1 and χ_2 are characters of $U_{2n_1+1,+}$ and $U_{2n_1+1,-}$ respectively. The condition χ does not extend to I_K is equivalent to that $\chi \neq \chi^{\iota}$ on $I_{K'}$, and it is further equivalent to that $\chi_2^2 \neq 1$.

Now we have $[F_0 : \mathbb{Q}_p] = 2m_0m_1$. We take bases $e_{i,1}$, $e_{i,2}$ of D_i over E for $1 \leq i \leq 2m_0m_1$ so that

$$\begin{aligned}
&\iota e_{i,1} = e_{i,2}, && \delta e_{i,1} = \omega^s(\delta)e_{i,1}, && \gamma_1 e_{i,1} = \chi_1(\gamma_1)e_{i,1}, && \gamma_2 e_{i,1} = \chi_2(\gamma_2)e_{i,1},\\
&\iota e_{i,2} = \omega^s(\delta_0)e_{i,1}, && \delta e_{i,2} = \omega^s(\delta)e_{i,2}, && \gamma_1 e_{i,2} = \chi_1(\gamma_1)e_{i,2}, && \gamma_2 e_{i,2} = \chi_2(\gamma_2)^{-1}e_{i,2}
\end{aligned}$$

for $\delta \in k'^{\times}$, $\gamma_1 \in U_{n_1,+}$ and $\gamma_2 \in U_{n_1,-}$.

Since σ takes D_i to D_{i+m_0}, as in the unramified case, we may assume that $\sigma e_{i,1} = e_{i+m_0,1}$. Then we have that $\sigma e_{i,2} = (-1)^s e_{i+m_0,2}$ by $(\star)$.

Since ϕ takes D_i to D_{i+1} and commutes with the action of $I(F/K)$, we have that

$$\phi(e_{i,1}) = \frac{1}{\alpha_{i+1}} e_{i+1,1},\ \phi(e_{i,2}) = \frac{1}{\alpha_{i+1}} e_{i+1,2}$$

for some $\alpha_{i+1} \in E^\times$ for all i. Further, since ϕ commutes with the action of σ, we have $\alpha_i = \alpha_{i+m_0}$ for all i. Replacing $e_{i,1}$ and $e_{i,2}$ by their scalar multiples, we may further assume that $\alpha_i = 1$ for $2 \leq i \leq m_0$.

Let e_1, e_2 be a basis of D over $F_0 \otimes_{\mathbb{Q}_p} E$ determined by $(e_{i,1})_i$, $(e_{i,2})_i$ under the isomorphism $D \xrightarrow{\sim} \prod_i D_i$. Then $\sigma e_1 = e_1$ and $\sigma e_2 = (-1)^s e_2$.

The condition $t_{\mathrm{H}}(D) = t_{\mathrm{N}}(D)$ is equivalent to that

$$(R) \qquad 2[K : K_0]\, v_p(\alpha_1) = \sum_j (k_{j,1} + k_{j,2}).$$

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is $\mathrm{Gal}(F/K)$-invariant. By $\iota(a_j e_1 + b_j e_2) = (a_j e_1 + b_j e_2)$, we get $\iota(a_j) = b_j$ and $\iota(b_j)\omega^s(\delta_0) = a_j$. So $a_j \in E_j^\times$ if and only if $b_j \in E_j^\times$.

Since $\big(a_j e_1 + \iota(a_j) e_2\big)$ is $\mathrm{Gal}(F/K)$-invariant, $\sigma(a_j) = a_j$ and $g \in I(F/K')$ acts on a_j by $\chi(g)^{-1}$. We prove that there are $x_{j,1}, x_{j,2} \in E_j$ such that

- a_j satisfies the above condition if and only if $a_j = a_{j,1} x_{j,1} + a_{j,2} x_{j,2}$ for some $a_{j,1}, a_{j,2} \in E$,
- for $a_{j,1}, a_{j,2} \in E$, we have $a_{j,1} x_{j,1} + a_{j,2} x_{j,2} \in E_j^\times$ if and only if $a_{j,1} \neq 0$ and $a_{j,2} \neq 0$.

By Lemma 2.3, we may replace E_j by $F \otimes_K E$. Then $\sigma(a_j) = a_j$ if and only if $a_j \in K'_{\pi',2n_1+1} \otimes_K E$. By Lemma 2.5, we get the claim. We put $x_j(a_{j,1}, a_{j,2}) = a_{j,1} x_{j,1} + a_{j,2} x_{j,2}$ and $x_j^\iota(a_{j,1}, a_{j,2}) = \iota\big(x_j(a_{j,1}, a_{j,2})\big)$. Then we have

$$\mathrm{Fil}_j^{-k_{j,1}} D_F = E_j\big(x_j(a_{j,1}, a_{j,2}) e_1 + x_j^\iota(a_{j,1}, a_{j,2}) e_2\big)$$

for $(a_{j,1}, a_{j,2}) \in \mathbb{P}^1(E)$.

The non-trivial (ϕ, N)-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodules of D are $D'_1 = (F_0 \otimes_{\mathbb{Q}_p} E)e_1$, $D'_2 = (F_0 \otimes_{\mathbb{Q}_p} E)e_2$ and $D'_{\mathfrak{L}} = (F_0 \otimes_{\mathbb{Q}_p} E)(e_1 - \mathfrak{L} e_2)$ for $\mathfrak{L} \in E^\times$. The condition $t_{\mathrm{H}}(D'_1) \leq t_{\mathrm{N}}(D'_1)$ is equivalent to that

$$[K : K_0]\, v_p(\alpha_1) \leq \sum_{a_{j,1} a_{j,2} = 0} \frac{k_{j,1} + k_{j,2}}{2} + \sum_{a_{j,1} a_{j,2} \neq 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2},$$

and this condition is automatically satisfied by the condition (R). The condition $t_{\mathrm{H}}(D'_2) \leq t_{\mathrm{N}}(D'_2)$ is also equivalent to the same condition. For $\mathfrak{L} \in E^\times$, the condition $t_{\mathrm{H}}(D'_{\mathfrak{L}}) \leq t_{\mathrm{N}}(D'_{\mathfrak{L}})$ is equivalent to that

$$(R_{\mathfrak{L}}) \quad [K : K_0]\, v_p(\alpha_1) \leq \sum_{a_{j,1} a_{j,2} = 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2} + \sum_{a_{j,1} a_{j,2} \neq 0} \Big\{ t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big) k_{j,1} + \Big(1 - t_j\big(\mathfrak{L}, (a_{j,1}, a_{j,2})\big)\Big) k_{j,2} \Big\},$$

where

$$t_j\big(\mathfrak{L},(a_{j,1},a_{j,2})\big)=\frac{\left|\left\{j_F : F\hookrightarrow E \;\middle|\; j_F\text{-component of } \frac{x_j^{\iota}(a_{j,1},a_{j,2})}{x_j(a_{j,1},a_{j,2})}\in E_j \text{ is } -\mathfrak{L}\right\}\right|}{[F:K]}.$$

As in the unramified case, we can prove that $t_j\big(\mathfrak{L},(a_{j,1},a_{j,2})\big)\leq 1/2$, using the condition that $\chi\neq\chi^{\iota}$ on $I_{K'}$. So the condition $(R_{\mathfrak{L}})$ is automatically satisfied by the condition (R).

Proposition 3.7. *We assume $\tau\simeq\mathrm{Ind}_{W_{K'}}^{W_K}(\chi)|_{I_K}$ for a ramified quadratic extension K' of K and a character χ of $W_{K'}$ such that $\chi|_{I_{K'}}$ is finite and does not extend to I_K. We take a uniformizer π' of K' and a totally ramified abelian extension L of K' such that χ is trivial on I_L, and take positive integers m_1 and n_1 so that L is contained in $K'_{2m_1}K'_{\pi',2n_1+1}$. We put $F=K'_{2m_1}K'_{\pi',2n_1+1}$. Then $N=0$ and $D=(F_0\otimes_{\mathbb{Q}_p}E)e_1\oplus(F_0\otimes_{\mathbb{Q}_p}E)e_2$ with*

$$\phi(e_{i,1})=\frac{1}{\alpha_1}e_{i+1,1},\qquad \phi(e_{i,2})=\frac{1}{\alpha_1}e_{i+1,2},\qquad \text{if } i\equiv 0 \pmod{m_0},$$
$$\phi(e_{i,1})=e_{i+1,1},\qquad \phi(e_{i,2})=e_{i+1,2},\qquad \text{if } i\not\equiv 0 \pmod{m_0}$$

for $\alpha_1\in E^{\times}$,

$$\sigma e_1=e_1,\qquad \iota e_1=e_2,\qquad ge_1=\big(1\otimes\chi(g)\big)e_1,$$
$$\sigma e_2=(-1)^s e_2,\qquad \iota e_2=\big(1\otimes\omega^s(\delta_0)\big)e_1,\qquad ge_2=\big(1\otimes\chi^{\sigma}(g)\big)e_2$$

for $s\in\mathbb{Z}$ and $g\in I(F/K')$ and, for j such that $k_{j,1}<k_{j,2}$,

$$\mathrm{Fil}_j^{-k_{j,1}}D_F=E_j\big(x_j(a_{j,1},a_{j,2})e_1+x_j^{\iota}(a_{j,1},a_{j,2})e_2\big)$$

for $(a_{j,1},a_{j,2})\in\mathbb{P}^1(E)$ where

$$2[K:K_0]\,v_p(\alpha_1)=\sum_j(k_{j,1}+k_{j,2}).$$

Here $\omega: k'\to\mathcal{O}_{K'}^{\times}$ is the Teichmüller character, and the definitions of σ,ι,δ_0 are in the above discussion.

References

[BM] C. Breuil, A. Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $\ell=p$, Duke Math. J. **115** (2002), no. 2, 205–310.

[CF] P. Colmez, J.-M. Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), no. 1, 1–43.

[Do] G. Dousmanis, *Rank two filtered* (φ,N)*-modules with Galois descent data and coefficients*, Trans. Amer. Math. Soc. **362** (2010), no. 7, 3883–3910.

[FM] J.-M. Fontaine, B. Mazur, *Geometric Galois representations*, Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), 41–78, Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995.

[Fon] J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque **223** (1994), 59–111.

[GM] E. Ghate, A. Mézard, *Filtered modules with coefficients*, Trans. Amer. Math. Soc. **361** (2009), no. 5, 2243–2261.

[Sav] D. Savitt, *On a Conjecture of Conrad, Diamond, and Taylor*, Duke Math. J. **128** (2005), no. 1, 141–197.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan

E-mail address: naoki@kurims.kyoto-u.ac.jp