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FILTERED MODULES CORRESPONDING TO
POTENTIALLY SEMI-STABLE REPRESENTATIONS

NAOKI IMAI

Abstract. We classify the filtered modules with coefficients corresponding to
two-dimensional potentially semi-stable $p$-adic representations of the absolute
Galois groups of $p$-adic fields under the assumptions that $p$ is odd and the
coefficients are large enough.

Introduction

Let $p$ be an odd prime number, and let $K$ be a $p$-adic field. The absolute
Galois group of $K$ is denoted by $G_K$. By the fundamental theorem of Colmez
and Fontaine [CF], there exists a correspondence between potentially semi-stable
$p$-adic representations and admissible filtered $(\phi, N)$-modules with Galois action.
The aim of this paper is the classification of the admissible filtered $(\phi, N)$-modules
with Galois action corresponding to two-dimensional potentially semi-stable $p$-adic
representations of $G_K$ with coefficients in a $p$-adic field $E$.

If $K = \mathbb{Q}_p$ and $E = \mathbb{Q}_p$, the classification is given in [FM, Appendix A] under the
assumption that $p \geq 5$. If $K = \mathbb{Q}_p$ and $E$ is general, these filtered $(\phi, N)$-modules
are studied in [BM] and [Sav], and the classification is given by Ghate and Mézard
in [GM] under the assumptions that $p$ is odd and $E$ is large enough. In this paper,
we generalize the results of [GM] to the case where $K$ is a general $p$-adic field.

In the case where $K$ is a general $p$-adic field, filtrations are determined by many
weights and many elements of $\mathbb{P}^1(E)$. In fact we need $[K : \mathbb{Q}_p]$ elements of $\mathbb{P}^1(E)$ to
parametrize two-dimensional potentially semi-stable $p$-adic representations. These
elements of $\mathbb{P}^1(E)$ play a role similar to Fontaine-Mazur’s $\mathcal{E}$-invariants.

After writing of this paper, the author has known that there is preceding research
[Do] on this subject by Dousmanis. The author does not claim priority, but there
are some differences. In [Do], a classification is given by Frobenius action, and in
this paper, we give a classification by Galois action. Let $F$ be a finite extension
of $K$. A potentially semi-stable representation $\rho$ is said to be $F$-semi-stable, if
the restriction of $\rho$ to the absolute Galois group of $F$ is semi-stable. In [Do], a
classification of $F$-semi-stable representations is given for a general finite Galois
extension $F$ of $K$. In this paper, we give a class of finite Galois extensions of $K$
such that any potentially semi-stable representation is $F$-semi-stable for a field $F$
in this class, and give a classification of $F$-semi-stable representations and a more
explicit description of Galois action of $\text{Gal}(F/K)$ for $F$ in this class, assuming
$p \neq 2$. This difference is conspicuous in the supercuspidal case. Let $F_0$ be the
maximal unramified extension of $\mathbb{Q}_p$ contained in $F$. In [Do, 5.3], it is proved that
$\text{Gal}(F/K)$-action on a filtered $(\phi, N)$-$(F_0 \otimes_{\mathbb{Q}_p} E)$-module comes from a $\text{Gal}(F/K)$-
action on the two-dimensional $E$-vector space in the supercuspidal case. In this
paper, we study the $\text{Gal}(F/K)$-action explicitly by using a structure of $\text{Gal}(F/K)$,
of course, assuming $F$ is in some class. Then, in this paper, we first fix a large enough coefficient field, and do not extend it in the classification.

This paper is clearly influenced by the paper [GM], and we owe a lot of arguments to [GM]. We mention it here, and do not repeat it each times in the sequel.

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**Notation.** Throughout this paper, we use the following notation. Let $p$ be an odd prime number, and $\mathbb{C}_p$ be the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$. Let $K$ be a $p$-adic field. We consider $K$ as a subfield of $\mathbb{C}_p$. The residue field of $K$ is denoted by $k$, whose cardinality is $q$. Let $K_0$ be the maximal unramified extension of $\mathbb{Q}_p$ contained in $K$. For any $p$-adic field $L$, the absolute Galois group of $L$ is denoted by $G_L$, the inertia subgroup of $G_L$ is denoted by $I_L$, the Weil group of $L$ is denoted by $W_L$, the ring of integers of $L$ is denoted by $\mathcal{O}_L$ and the unique maximal ideal of $\mathcal{O}_L$ is denoted by $p_L$. For a Galois extension $L$ of $K$, the inertia subgroup of $\text{Gal}(L/K)$ is denoted by $I(L/K)$. Let $v_p$ be the valuations of $p$-adic fields normalized by $v_p(p) = 1$.

1. **Filtered $(\phi, N)$-modules**

Let $E$ be a $p$-adic field. We consider a two-dimensional $p$-adic representation $V$ of $G_K$ over $E$, which is denoted by $\rho : G_K \to GL(V)$. As in [Fon], we can construct $K_0$-algebra $B_{st}$ with a Frobenius endomorphism, a monodromy operator and Galois action. Further, we can define a decreasing filtration on $K \otimes_{K_0} B_{st}$. Let $F$ be a finite Galois extension of $K$, and $F_0$ be the maximal unramified extension of $\mathbb{Q}_p$ contained in $F$. Then we have $B_{st}^{G_F} = F_0$. The $p$-adic representation $\rho$ is called $F$-semi-stable if and only if the dimension of $D_{st,F}(V) = (B_{st} \otimes_{\mathbb{Q}_p} V)^{G_F}$ over $F_0$ is equal to the dimension of $V$ over $\mathbb{Q}_p$. If $\rho$ is $F$-semi-stable for some finite Galois extension $F$ of $K$, we say that $\rho$ is potentially semi-stable representation.

Potentially semi-stable representations are Hodge-Tate. To fix a convention, we recall the definition of the Hodge-Tate weights. For $i \in \mathbb{Z}$, we put

$$D^i_{HT}(V) = (\mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)^{G_K}.$$  

Here and in the following, $(i)$ means $i$ times twists by the $p$-adic cyclotomic character of $G_K$. Then there is a $G_K$-equivariant isomorphism

$$\bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(-i) \otimes_K D^i_{HT}(V) \cong \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$$

of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} E)$-modules. The Hodge-Tate weights of the representation $V$ are the integers $i$ such that $D^i_{HT}(V) \neq 0$, with multiplicities $\dim_E(D^i_{HT}(V))$.

Next, we recall the definition of the filtered $(\phi, N, \text{Gal}(F/K), E)$-modules. A filtered $(\phi, N, \text{Gal}(F/K), E)$-module is a finite free $(F_0 \otimes_{\mathbb{Q}_p} E)$-module $D$ endowed with

- the Frobenius endomorphism: an $F_0$-semi-linear, $E$-linear, bijective map $\phi : D \to D$,
- the monodromy operator: an $(F_0 \otimes_{\mathbb{Q}_p} E)$-linear, nilpotent endomorphism $N : D \to D$ that satisfies $N\phi = p\phi N$, 

• the Galois action: an $F_0$-semi-linear, $E$-linear action of $\text{Gal}(F/K)$ that commutes with the action of $\phi$ and $N$,
• the filtration: a decreasing filtration $(\text{Fil}^i D_{F})_{i \in \mathbb{Z}}$ of $(F \otimes_{\mathbb{Q}_p} E)$-submodules of $D_F = F \otimes_{F_0} D$ that are stable under the action of $\text{Gal}(F/K)$ and satisfy

$\text{Fil}^i D_F = D_F$ for $i \ll 0$ and $\text{Fil}^i D_F = 0$ for $i \gg 0$.

Let $D$ be a filtered $(\phi, N, \text{Gal}(F/K), E)$-module. Then, by forgetting the $E$-module structure, $D$ is also a filtered $(\phi, N, \text{Gal}(F/K), \mathbb{Q}_p)$-module. We put $d = \dim_{F_0} D$. Then $\bigwedge^d_{F_0} D$ is a filtered $(\phi, N, \text{Gal}(F/K), \mathbb{Q}_p)$-module of dimension 1 over $F_0$. We put

$$t_H(D) = \max \{ i \in \mathbb{Z} \mid \text{Fil}^i (F \otimes_{F_0} \bigwedge^d_{F_0} D) \neq 0 \}, \quad t_N(D) = v_p(\lambda)$$

where $\lambda$ is an element of $F_0^*$ that satisfies $\phi(x) = \lambda x$ for a non-zero element $x$ of $\bigwedge^d_{F_0} D$. We say that $D$ is admissible if it satisfies the following two conditions:

• $t_H(D) = t_N(D)$,
• For any $F_0$-submodule $D'$ of $D$ that is stable by $\phi$ and $N$, we have $t_H(D') \leq t_N(D')$, where $D'_F \subset D_F$ is equipped with the induced filtration.

By [BM, Proposition 3.1.1.5], we may replace the above second condition by the following condition:

• For any $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodule $D'$ of $D$ that is stable by $\phi$ and $N$, we have $t_H(D') \leq t_N(D')$, where $D'_F \subset D_F$ is equipped with the induced filtration.

Let $k_0$ be a non-negative integer. By the results of [CF], there is an equivalence of categories between the category of two-dimensional $F$-semi-stable representations of $G_K$ over $E$ with Hodge-Tate weights in $\{0, \ldots, k_0\}$ and the category of admissible filtered $(\phi, N, \text{Gal}(F/K), E)$-modules of rank 2 over $F_0 \otimes_{\mathbb{Q}_p} E$ such that $\text{Fil}^{-k_0} (D_F) = D_F$ and $\text{Fil}^1 (D_F) = 0$. This equivalence of categories is given by the functor $D_{st,F}$ defined above. The aim of this paper is the classification of the objects of later categories under the assumption that $E$ is large enough.

2. Preliminaries

Let $\rho : G_K \to GL(V)$ be a two-dimensional potentially-semi-stable representation over $E$. We assume that $\rho$ is $F$-semi-stable, and put $D = D_{st,F}(V)$. We recall the definition of Weil-Deligne representation associated to $\rho$. Now we have $W_K/W_F = \text{Gal}(F/K)$. Let $m_\rho$ be the degree of the field extension $K_0$ over $\mathbb{Q}_p$. We define an $F_0$-linear action of $g \in W_K$ on $D$ by $(g \mod W_F) \circ \phi^{-m_\alpha(a)}$, where the image of $g$ in $\text{Gal}({\overline{k}}/k)$ is the $\alpha(g)$-th power of the $g$-th power Frobenius map.

We assume that $F_0 \subset E$. According to an isomorphism

$$F_0 \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma_i : F_0 \rightarrow E} E; a \otimes b \mapsto \sigma_i(a)b,$$

we have a decomposition

$$D \xrightarrow{\sim} \prod_{\sigma_i : F_0 \rightarrow E} D_i.$$ 

Here and in the sequel, $\sigma_i$ is an embedding determined by the $(-i)$-th power of the $p$-th power Frobenius map for $1 \leq i \leq [F_0 : \mathbb{Q}_p]$. Then $D_i$, with an induced action of $W_K$ and an induced monodromy operator, defines a Weil-Deligne representation.
The isomorphism class of this Weil-Deligne representation is independent of choice of $F$ and $\sigma_i$ (cf. [BM, Lemme 2.2.1.2]), and is, by definition, the Weil-Deligne representation $\text{WD}(\rho)$ attached to $\rho$.

We note that, in the above decomposition of $D_\chi$, the Frobenius endomorphism $\phi$ induce $E$-linear isomorphism $\phi : D_i \simto D_{i+1}$. Naturally, we consider a suffix $i$ modulo $\{F_0 : Q_p\}$, and we often use such conventions in the sequel.

A Galois type $\tau$ of degree 2 is an equivalence class of representations $\tau : I_K \to G L_2(\Q_p)$ with open kernel that extend to representations of $W_K$. We say that an two-dimensional potentially semi-stable representation $\rho$ has Galois type $\tau$ if $\text{WD}(\rho)|_{I_K} \simeq \tau$. The potentially semi-stable representation $\rho$ is $F$-semi-stable if and only if $\tau|_{I_F}$ is trivial.

For a group $G$, an element $g \in G$, a normal subgroup $H$ of $G$ and a character $\chi : H \to \Q_p^\times$, we define a character $\chi^g : H \to \Q_p^\times$ by $\chi^g(h) = \chi(ghg^{-1})$ for $h \in H$.

**Lemma 2.1.** Let $\tau$ be a Galois type of degree 2. Then $\tau$ has one of the following forms:

1. $\tau \simeq \chi_1|_{I_K} \oplus \chi_2|_{I_K}$, where $\chi_1, \chi_2$ are characters of $W_K$ finite on $I_K$,
2. $\tau \simeq \text{Ind}_{W_K'}{\chi}|_{I_K} = \chi|_{I_K} \oplus \chi^\sigma|_{I_K}$, where $K'$ is the unramified quadratic extension of $K$, $\chi$ is a character of $W_{K'}$ that is finite on $I_{K'}$ and does not extend to $W_K$, and $\sigma \in W_K$ is a lift of the generator of $\text{Gal}(K'/K)$,
3. $\tau \simeq \text{Ind}_{W_K'}{\chi}|_{I_K}$, where $K'$ is a ramified quadratic extension of $K$, and $\chi$ is a character of $W_{K'}$ such that $\chi$ is finite on $I_{K'}$ and $\chi|_{I_K}$ does not extend to $I_K$.

**Proof.** This is a classical lemma, but we briefly recall a proof.

We extend $\tau$ to a representation of $W_K$, which is denoted by $\bar{\tau}$. If $\bar{\tau}$ is reducible, we are in the case (1), so we may assume that $\bar{\tau}$ is irreducible.

First, we treat the case where $\tau$ is reducible. In this case, $\tau \simeq \chi \oplus \chi'$ for some characters $\chi, \chi'$ of $I_K$. By irreducibility of $\bar{\tau}$, we have $\chi' = \chi^\sigma$. Then $\bar{\tau}|_{W_{K'}}$ is already reducible for the unramified quadratic extension $K'$ of $K$. So we are in the case (2).

Next, we treat the case where $\tau$ is irreducible. Let $I_K^w$ be the wild inertia subgroup of $I_K$. Then $\tau|_{I_K^w}$ is reducible, because a dimension of an irreducible representation of a $p$-group is a power of $p$ and $p \neq 2$. Then $\bar{\tau}|_{W_{K'}}$ is already reducible for a ramified quadratic extension $K'$ of $K$. So we are in the case (3). \hfill $\Box$

To avoid the problem of the rationality, we assume that $E$ is a Galois extension over $Q_p$, $F \subset E$ and the following:

For all $p$-adic fields $K'$ such that $K \subset K' \subset F$ and $[K' : K] \leq 2$, and for all characters $\chi$ of $W_{K'}$, that are trivial on $I_F$, the restrictions $\chi|_{I_{K'}}$ factor through $E^\times$.

For example, if $E$ contains the $|I(F/K)|$-th roots of unity, then this condition is satisfied.

In the sequel, let $\rho : G_K \to GL(V)$ be a two-dimensional potentially semi-stable representation over $E$ with Hodge-Tate weight in $\{0, \ldots, k_0\}$, and $\tau$ be its Galois type.

**Lemma 2.2.** (cf. [GM, Lemme 2.3]) If $\rho$ is not potentially crystalline, then $\tau$ is a scalar.
Therefore, there are following three possibilities:

- Special or Steinberg case: \( N \neq 0 \) and \( \tau \) is a scalar.
- Principal series case: \( N = 0 \) and \( \tau \) is as in (1) of Lemma 2.1.
- Supercuspidal case: \( N = 0 \) and \( \tau \) is as in (2) or (3) of Lemma 2.1.

Next, we study the structure of the filtrations. We assume \( \rho \) is \( F \)-semi-stable, and take the corresponding filtered \(( \phi, N, \Gal(F/K), E)\)-module \( D \). We have a decomposition

\[
F \otimes_{\Q_p} E \xrightarrow{\sim} \prod_{j_F:F \twoheadrightarrow E} E = \prod_{j:F \twoheadrightarrow E} \left( \prod_{j_K:F \twoheadrightarrow E, j_F|_K = j} E \right) = \prod_{j:K \twoheadrightarrow E} E_j,
\]

where \( j_F \) and \( j \) are \( \Q_p \)-embeddings and we put

\[
E_j = \prod_{j_F:F \twoheadrightarrow E, j_F|_K = j} E.
\]

According to the above decomposition, we have decompositions

\[
D_F \cong \prod_{j:K \twoheadrightarrow E} D_{F,j} \text{ and } \Fil^1 D_F \cong \prod_{j:K \twoheadrightarrow E} \Fil^1 D_{F,j}.
\]

Because \( \Fil^1 D_F \) is \( \Gal(F/K)\)-stable, \( \Fil^1 D_F \) is free over \( E_j \). We take integers \( 0 \leq k_{j,1} \leq k_{j,2} \leq k_0 \) such that

\[
D_{F,j} = \Fil^{j_{k,2}} D_F \supseteq \Fil^{j_{k,1}} D_F = \Fil^{j_{k,1}} D_F \supseteq \Fil^{j_{k,1}} D_F = 0.
\]

Then the Hodge-Tate weights of \( \rho \) are \( \bigcup_{j:K \twoheadrightarrow E} \{ k_{j,1}, k_{j,2} \} \).

We are going to prepare some lemmas.

**Lemma 2.3.** There is a \( \Gal(F/K) \)-equivariant isomorphism

\[
F \otimes_K E \cong E_j
\]

of \( E \)-algebra.

**Proof.** Let \( j_0 \) be a natural inclusion \( K \subset E \). Take an extension \( j_K : E \xrightarrow{\sim} E \) of \( j : K \hookrightarrow E \). Then a \( \Gal(F/K) \)-equivariant isomorphism

\[
\prod_{j_F:F \twoheadrightarrow E, j_F|_K = j_0} E \xrightarrow{\sim} \prod_{j_F:F \twoheadrightarrow E, j_F|_K = j} E
\]

of \( E \)-algebra is given by sending \( j_F \)-components to \(( j_F \circ j_F \)-components. \( \square \)

**Lemma 2.4.** If \( k_{j,1} < k_{j,2} \), then \( \Fil^{-k_{j,1}} D_F \subset D_{F,j} \) is spanned by a Galois invariant element over \( E_j \).

**Proof.** A generator of \( \Fil^{-k_{j,1}} D_F \) over \( E_j \) generates an \( E_j^\times \)-torsor with \( \Gal(F/K) \)-action. An \( E_j^\times \)-torsor with \( \Gal(F/K) \)-action is trivial, if \( H^1(\Gal(F/K), E_j^\times) = 0 \). So it suffices to show that \( H^1(\Gal(F/K), E_j^\times) = 0 \). By Lemma 2.3, \( E_j^\times \) is isomorphic to \( (F \otimes_K E)^\times \), and it is further isomorphic to \( \Ind_{\id_F}^{\Gal(F/K)} E^\times \). By Shapiro’s lemma, \( H^1(\Gal(F/K), \Ind_{\id_F}^{\Gal(F/K)} E^\times) = H^1(\{\id_F\}, E^\times) = 0 \). \( \square \)

**Lemma 2.5.** Let \( K', M \) be \( p \)-adic fields such that \( K \subset K' \subset M \subset F \) and \( M \) is a Galois extension of \( K' \). Let \( \chi : \Gal(M/K') \to E^\times \) be a character. We put \( m = [K' : K] \). Then there exist \( x_1, \ldots, x_m \in M \otimes_K E \) that satisfy the followings:
• For $x \in M \otimes_K E$, we have $gx = (1 \otimes \chi(g)^{-1})x$ for all $g \in \text{Gal}(M/K')$ if and only if $x = \sum_{i=1}^{m}(1 \otimes a_i)x_i$ for $a_i \in E$.

• For $a_i \in E$, we have $\sum_{i=1}^{m}(1 \otimes a_i)x_i \in (M \otimes_K E)^{\times}$ if and only if $a_i \neq 0$ for all $i$.

**Proof.** We have a decomposition

$$M \otimes_K E \xrightarrow{\sim} \prod_{j_M : M \to E} E = \prod_{j_M : M \to E} \left( \prod_{j_M : M \to E, j_M | j'} E \right) = \prod_{j_M : M \to E} E_{j'},$$

where $j_M$ and $j'$ are $K$-embeddings and we put

$$E_{j'} = \prod_{j_M : M \to E, j_M | j'} E.$$

Let $(x_{j'})_{j'} \in \prod_{j', K' \to E} E_{j'}$ be the image of $x$ under the above isomorphism. Then, $gx = (1 \otimes \chi(g)^{-1})x$ for all $g \in \text{Gal}(M/K')$ if and only if $gx_{j'} = \chi(g)^{-1}x_{j'}$ for all $g \in \text{Gal}(M/K')$ and all $j' : K' \to E$. Further, $x \in (M \otimes_K E)^{\times}$ if and only if $x_{j'} \in E_{j'}^{\times}$ for all $j'$. As in the proof of Lemma 2.3, we can show there is a $\text{Gal}(M/K')$-equivariant isomorphism $M \otimes_K E \xrightarrow{\sim} E_{j'}$ of $E$-algebra. So, to prove this Lemma, it suffices to treat the case where $m = 1$.

We assume that $m = 1$. Take $\alpha \in M$ such that $g(\alpha)$ for $g \in \text{Gal}(M/K)$ form a basis of $M$ over $K$. Then $x \in M \otimes_K E$ can be written uniquely as

$$\sum_{g \in \text{Gal}(M/K)} g(\alpha) \otimes a_g$$

for $a_g \in E$. If $hx = (1 \otimes \chi(h)^{-1})x$ for all $h \in \text{Gal}(M/K)$, we have $a_{i,h^{-1}g} = \chi^{-1}(h)a_g$ for all $g, h \in \text{Gal}(M/K)$. By putting $a_1 = a_{\text{id}_M}$, we have

$$x = (1 \otimes a_1) \sum_{g \in \text{Gal}(M/K)} g(\alpha) \otimes \chi(g).$$

It suffices to put $x_1 = \sum_{g \in \text{Gal}(M/K)} g(\alpha) \otimes (\chi(g)).$ \hfill \square

### 3. Classification

#### 3.1. Special or Steinberg case.** In this case, $\tau \simeq \chi|_{I_K} \otimes \chi|_{I_K}$ for some character $\chi$ of $W_K$ that is finite on $I_K$, and there exists a totally ramified cyclic extension $F$ of $K$ such that $\chi|_{I_F}$ is trivial. So we may assume that $\rho$ is $F$-semi-stable, and $\chi$ determine the action of $\text{Gal}(F/K)$ on $D$, which is again denoted by $\chi$.

Since $N\phi = p\phi N$, we have that $\text{Ker} N$ is $\phi$-stable and free of rank 1 over $F_0 \otimes_{\mathbb{Q}_p} E$. So we can take a basis $e_1, e_2$ of $D$ over $F_0 \otimes_{\mathbb{Q}_p} E$ such that $N(e_1) = e_2$ and $N(e_2) = 0$. Again by $N\phi = p\phi N$, we must have $\phi(e_1) = \frac{p}{\phi} e_1 + \gamma e_2$ and $\phi(e_2) = \frac{1}{\phi} e_2$ with $\alpha \in (F_0 \otimes_{\mathbb{Q}_p} E)^{\times}$ and $\gamma \in F_0 \otimes_{\mathbb{Q}_p} E$. Modifying $e_1$ by a scalar multiple of $e_2$, we may assume $\gamma = 0$. Let $(\alpha_{t})_{t} \in \prod_{\sigma : F_0 \to E} E$ be the image of $\alpha$ under the isomorphism

$$F_0 \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma : F_0 \to E} E.$$
Then, by calculations, we have
\[ t_H(D) = -[E : K] \sum_{j : K \to E} (k_{j,1} + k_{j,2}), \]
\[ t_N(D) = [E : F_0] \left( m_0 - 2 \sum_i v_p(\alpha_i) \right). \]

So the condition \( t_H(D) = t_N(D) \) is equivalent to that
\[ 2[K : K_0] \sum_i v_p(\alpha_i) = \sum_j (k_{j,1} + k_{j,2} + 1). \]

For \( j : K \to E \) satisfying \( k_{j,1} < k_{j,2} \), by Lemma 2.4, we take \( a_j, b_j \in E_j \) such that \( \text{Fil}_{j}^{-k_{j,1}} D_F = E_j(a_j, e_1 + b_j, e_2) \), and \( (a_j, e_1 + b_j, e_2) \) is \( \text{Gal}(F/K) \)-invariant. We note that \( a_j = 0 \) or \( a_j \in E_x^* \) and that \( b_j = 0 \) or \( b_j \in E^* \).

The only non-trivial \((\phi, N)\)-stable \((F_0 \otimes_{Q_p} E)\)-submodule of \( D = (F_0 \otimes_{Q_p} E)e_2 \). By calculations, we have
\[ t_H(D') = -[E : K] \left( \sum_{a_j = 0} k_{j,1} + \sum_{a_j \neq 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2} \right), \]
\[ t_N(D') = -[E : F_0] \sum_i v_p(\alpha_i). \]

So the condition \( t_H(D') \leq t_N(D') \) is equivalent to that
\[ [K : K_0] \sum v_p(\alpha_i) \leq \sum_{a_j = 0} k_{j,1} + \sum_{a_j \neq 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}. \]

Since \((a_j, e_1 + b_j, e_2)\) is \( \text{Gal}(F/K) \)-invariant, \( g \in \text{Gal}(F/K) \) acts on \( a_j \) and \( b_j \) by \( \chi(g)^{-1} \). By Lemma 2.3 and Lemma 2.5, there is \( x_1 \in E_j \) such that \( a_j = a_j'x_1 \) and \( b_j = b_j'^2x_1 \) for \( a_j', b_j' \in E \). Then, for \( j \) such that \( a_j \neq 0 \),
\[ \text{Fil}_{j}^{-k_{j,1}} D_F = E_j(a_j', x_1 e_1 + b_j', x_1 e_2) = E_j(e_1 - \xi e_2) \]
for \( \xi \in E \).

**Proposition 3.1.** We assume that \( N \neq 0 \). Then \( \tau \simeq \chi|_{I_K} \otimes \chi|_{I_K} \) for some character \( \chi \) of \( W_K \) that is finite on \( I_K \). If we take a totally ramified cyclic extension \( F \) of \( K \) such that \( \chi \) is trivial on \( I_F \), then \( D = (F_0 \otimes_{Q_p} E)e_1 + (F_0 \otimes_{Q_p} E)e_2 \) with
\[ N(e_1) = e_2, \ N(e_2) = 0, \ \phi(e_1) = \frac{p}{\alpha} e_1, \ \phi(e_2) = \frac{1}{\alpha} e_2 \]
for \( \alpha \in (F_0 \otimes_{Q_p} E)^\times \),
\[ ge_1 = \chi(g)e_1, \ ge_2 = \chi(g)e_2 \]
for \( g \in \text{Gal}(F/K) \) and
\[ \text{Fil}_{j}^{-k_{j,1}} D_F = \begin{cases} E_j e_2 & \text{if } j \in I_1, \\ E_j(e_1 - \xi e_2) & \text{for } \xi \in E & \text{if } j \in I_2 \end{cases} \]
for \( j \) such that \( k_{j,1} < k_{j,2} \), where
\[ 2[K : K_0] \sum v_p(\alpha_i) = \sum_j (k_{j,1} + k_{j,2} + 1), \]
and $I_1, I_2$ are any disjoint sets such that $I_1 \cup I_2 = \{j \mid k_{j,1} < k_{j,2}\}$ and
\[
[K : K_0] \sum_i v_p(\alpha_i) \leq \sum_{j \in I_1} k_{j,1} + \sum_{j \in I_2} k_{j,2} + \sum_{j, k_{j,1} = k_{j,2}} k_{j,2}.
\]

3.2. Principal series case. In this case, $\tau \simeq \chi_1|_F \oplus \chi_2|_F$ and $N = 0$. We can take a totally ramified abelian extension $F$ of $K$ such that $\chi_1|_F$ and $\chi_2|_F$ are trivial. Then $\chi_1$ and $\chi_2$ determine the action of $\text{Gal}(F/K)$ on $D$, which is again denoted by the same symbols.

3.2.1. Irreducible case. First, we assume that $\chi_1|_F = \chi_2|_F$ and $D$ has no non-trivial $\phi$-stable $(F_0 \otimes \mathbb{Q}_p) E$-submodule. In this case, we say that $\phi$ is irreducible. If not, we say that $\phi$ is reducible. We put $\chi = \chi_1$.

Take bases $e_{i,1}, e_{i,2}$ of $D$, over $E$ for $1 \leq i \leq m_0$ so that
\[
\phi(e_{i,1}) = ae_{2,1} + ce_{2,2}, \quad \phi(e_{i,2}) = be_{2,1} + de_{2,2}
\]
for $a, b, c, d \in E$, and
\[
\phi(e_{i,1}) = e_{i+1,1}, \quad \phi(e_{i,2}) = e_{i+1,2}
\]
for $2 \leq i \leq m_0$. Let $e_{1,2}$ be a basis of $D$ over $F_0 \otimes \mathbb{Q}_p E$ determined by $(e_{i,1})$, $(e_{i,2})$, under the isomorphism $D \sim \prod D_i$. We will use the same notation in the classification of other cases.

Since $\phi$ is irreducible, $b \neq 0$ and $c \neq 0$. Modifying $e_{1,1}$ by a scalar multiple of $e_{1,2}$, we may assume $d = 0$. If $X^2 - aX - bc$ is reducible in $E[X]$, by replacing the bases, we can see that $\phi$ is reducible. This is a contradiction. So $X^2 - aX - bc$ is irreducible in $E[X]$.

Conversely, we suppose that $a, b, c \in E$ are given, $d = 0$, and $X^2 - aX - bc$ is irreducible in $E[X]$. Then the above description determines an endomorphism $\phi$. We prove that this endomorphism $\phi$ is irreducible. If $\phi$ is reducible, there are $A_i \in GL_2(E)$ such that
\[
A_i^{-1} \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} A_1, \ A_3^{-1} A_2, \ A_4^{-1} A_3, \ldots, \ A_1^{-1} A_{m_0}
\]
are all upper triangular matrices. Then, multiplying these matrices together, we have that $A_1^{-1} \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} A_1$ is an upper triangular matrix. This contradicts that $X^2 - aX - bc$ is irreducible in $E[X]$.

As above, the endomorphism $\phi$ is given by $a, b, c \in E$ such that $X^2 - aX - bc$ is reducible in $E[X]$. Now, by calculation, we have
\[
t_F(D) = -[E : K] \sum_{j : \text{Fil}_j D_F} (k_{j,1} + k_{j,2}),
\]
\[
t_N(D) = [E : F_0] v_p(bc).
\]
So the condition $t_F(D) = t_N(D)$ is equivalent to that
\[
-[K : K_0] v_p(bc) = \sum_j (k_{j,1} + k_{j,2}).
\]
Since $\phi$ is irreducible, $D$ has no non-trivial $(\phi, N)$-stable $(F_0 \otimes \mathbb{Q}_p) E$-submodule. So there is no condition on the filtrations. For $j$ such that $k_{j,1} < k_{j,2}$, by Lemma 2.3, Lemma 2.4 and Lemma 2.5, we have
\[
\text{Fil}_j^{-k_{j,1}} D_F = E_j(a_i e_1 + b_j e_2)
\]
for \((a_j, b_j) \in \mathbb{P}^1(E)\).

By studies of the other cases, \(\phi\) is irreducible only if \(N = 0\) and \(\tau \simeq \chi|_{I_K} \oplus \chi|_{I_K}\) for some character \(\chi\) of \(W_K\) that is finite on \(I_K\).

**Proposition 3.2.** We assume that \(\phi\) is irreducible. Then \(N = 0\) and \(\tau \simeq \chi|_{I_K} \oplus \chi|_{I_K}\) for some character \(\chi\) of \(W_K\) that is finite on \(I_K\). If we take a totally ramified cyclic extension \(F\) of \(K\) such that \(\chi\) is trivial on \(I_F\), then \(D = (F_0 \otimes_{Q_p} E)e_1 \oplus (F_0 \otimes_{Q_p} E)e_2\) with

\[
\phi(e_{1,i}) = a(e_{2,1} + ce_{2,2}), \quad \phi(e_{1,2}) = be_{2,1}
\]

for \(a, b \in E^\times\) such that \(X^2 - aX - bc\) is irreducible in \(E[X]\),

\[
\phi(e_{1,i}) = e_{i+1,1}, \quad \phi(e_{1,2}) = e_{i+1,2}
\]

for \(2 \leq i \leq m_0\),

\[
eg[K : K_0] v_p(bc) = \sum_j (k_{j,1} + k_{j,2}).
\]

3.2.2. *Non-split reducible case.* If \(D\) has two or more non-trivial \(\phi\)-stable \((F_0 \otimes_{Q_p} E)\)-submodules, we say that \(\phi\) is split. If not, we say that \(\phi\) is non-split. We assume that \(\chi_1|_{I_K} = \chi_2|_{I_K}\) and that \(\phi\) is non-split and reducible. We put \(\chi = \chi_1\).

Since \(\phi\) is reducible, we can take bases \(e_{i,1}, e_{i,2}\) of \(D_i\) over \(E\) and \(a_i, b_i, d_i \in E\) for all \(i\) such that

\[
\phi(e_{i,1}) = a_i e_{i+1,1}, \quad \phi(e_{i,2}) = b_i e_{i+1,1} + d_i e_{i+1,2}
\]

for all \(i\). Replacing the bases, we may assume that \(a_i = d_i = 1\) and \(b_i = 0\) for \(2 \leq i \leq n\). Since \(\phi\) is non-split, \(a_1 = d_1 \neq 0\) and \(b_1 \neq 0\). We put \(a = a_1\) and \(b = b_1\).

Conversely, we suppose that \(a, b \in E^\times\) are given. Then the above description determines an endomorphism \(\phi\). We prove that this endomorphism \(\phi\) is non-split. If \(\phi\) is split, there are \(A_i \in GL_2(E)\) such that

\[
A_{i+1}^{-1} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} A_i, \quad A_3^{-1} A_2, \quad A_3^{-1} A_2, \ldots, \quad A_1^{-1} A_m
\]

are all diagonal matrices. Then, multiplying these matrices together, we have that

\[
A_{i+1}^{-1} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} A_i \text{ is a diagonal matrix. This contradicts that } b \neq 0.
\]

As above, the endomorphism \(\phi\) is given by \(a, b \in E^\times\). The condition \(t_{\mathbb{H}}(D) = t_N(D)\) is equivalent to that

\[
-2[K : K_0] v_p(a) = \sum_j (k_{j,1} + k_{j,2}).
\]

Now we have bases \(e_{i,1}, e_{i,2}\) of \(D_i\) over \(E\) such that

\[
\phi(e_{1,1}) = ae_{2,1}, \quad \phi(e_{1,2}) = be_{2,1} + ae_{2,2}
\]

for \(a, b \in E^\times\), and

\[
\phi(e_{1,1}) = e_{i+1,1}, \quad \phi(e_{1,2}) = e_{i+1,2}
\]
for $2 \leq i \leq m_0$.

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\text{Fil}^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is Gal($F/K$)-invariant.

The only non-trivial $(\phi, N)$-stable $(F_0 \otimes_{Q_p} E)$-submodule of $D$ is $D'_1 = (F_0 \otimes_{Q_p} E)e_1$. The condition $t_H(D'_1) \leq t_N(D'_1)$ is equivalent to that

$$-[K : K_0] v_p(a) \leq \sum_{b_j = 0}^{k_{j,1} + 1} \sum_{b_j \neq 0}^{k_{j,1} + 1} \sum_{k_{j,1} = k_{j,2}} k_{j,2}.$$  

As in the special or Steinberg case, for $j$ such that $b_j \neq 0$,

$$\text{Fil}^{-k_{j,1}} D_F = E_j(-\sum_j e_1 + e_2),$$

for $\Sigma_j \in E$.

By studies of the other cases, $\phi$ is non-split reducible only if $N = 0$ and $\tau \simeq \chi |_{I_K} \oplus \chi |_{I_K}$ for some character $\chi$ of $W_K$ that is finite on $I_K$.

**Proposition 3.3.** We assume that $\phi$ is non-split reducible. Then $N = 0$ and $\tau \simeq \chi |_{I_K} \oplus \chi |_{I_K}$ for some character $\chi$ of $W_K$ that is finite on $I_K$. If we take a totally ramified cyclic extension $F$ of $K$ such that $\chi$ is trivial on $I_F$, then $D = (F_0 \otimes_{Q_p} E)e_1 \oplus (F_0 \otimes_{Q_p} E)e_2$ with

$$\phi(e_{1,1}) = a e_{2,1}, \; \phi(e_{1,2}) = b e_{2,1} + a e_{2,2}$$

for $a, b \in E^\times$,

$$\phi(e_{1,1}) = e_{i+1,1}, \; \phi(e_{1,2}) = e_{i+1,2}$$

for $2 \leq i \leq m_0$,  

$$g e_1 = \chi(g)e_1, \; g e_2 = \chi(g)e_2$$

for $g \in \text{Gal}(F/K)$ and

$$\text{Fil}^{-k_{j,1}} D_F = \begin{cases} E_j e_1 & \text{if } j \in I_1, \\ E_j(-\Sigma_j e_1 + e_2) & \text{for } \Sigma_j \in E \text{ if } j \in I_2 \\ \end{cases}$$

for $j$ such that $k_{j,1} < k_{j,2}$, where

$$-2[K : K_0] v_p(a) = \sum_j (k_{j,1} + k_{j,2}),$$

and $I_1, I_2$ are any disjoint sets such that $I_1 \cup I_2 = \{ j \mid k_{j,1} < k_{j,2} \}$ and

$$-[K : K_0] v_p(a) \leq \sum_{j \in I_1} k_{j,1} + \sum_{j \in I_2} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}.$$  

3.2.3. Split case. The remaining cases are the following two cases:

- $\chi |_{I_K} = \chi_2 |_{I_K}$ and $\phi$ is split.
- $\chi |_{I_K} \neq \chi_2 |_{I_K}$.

First, we assume that $\chi |_{I_K} \neq \chi_2 |_{I_K}$. Let $e_1, e_2$ be a basis of $D$ over $F_0 \otimes_{Q_p} E$ such that $\text{Gal}(F/K)$ acts on $e_1$ by $\chi_1$ and $e_2$ by $\chi_2$. We put

$$\phi(e_1) = \alpha e_1 + \gamma e_2, \; \phi(e_2) = \beta e_1 + \delta e_2,$$

where $\alpha, \beta, \gamma, \delta \in F_0 \otimes_{Q_p} E$. Since $\phi$ commutes with the action of $\text{Gal}(F/K)$ and $\chi |_{I_K} \neq \chi_2 |_{I_K}$, we have $\beta = \gamma = 0$. So, in the both cases, we may assume that $\phi$ is split.

We take bases $e_{i,1}, e_{i,2}$ of $D_i$ over $E$ so that

$$\phi(e_{1,1}) = a e_{2,1}, \; \phi(e_{1,2}) = b e_{2,2}$$
for some $a, b \in E^\times$ and
\[ \phi(e_{i,1}) = e_{i+1,1}, \quad \phi(e_{i,2}) = e_{i+1,2} \]
for $2 \leq i \leq m_0$. Let $e_{i,1}, e_{i,2}$ be a basis of $D$ over $F_0 \otimes_{\mathbb{Q}_p} E$ determined by $(e_{i,1}), (e_{i,2})$, under the isomorphism $D \xrightarrow{\sim} \prod D_i$.

Then the condition $t_H(D) = t_N(D)$ is equivalent to that
\begin{align*}
(S) \quad [K : K_0] v_p(ab) &= \sum_j (k_{j,1} + k_{j,2}).
\end{align*}

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\text{Fil}^{-k_{j,1}} D_F = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is $\text{Gal}(F/K)$-invariant.

Since $(a_j e_1 + b_j e_2)$ is $\text{Gal}(F/K)$-invariant, $g \in \text{Gal}(F/K)$ acts on $a_j$ and $b_j$ by $\chi_1 g^{-1}$ and $\chi_2 g^{-1}$ respectively. By Lemma 2.3 and Lemma 2.5, there are $x_1, x_2 \in E_j$ such that $a_j = a_j' x_1$ and $b_j = b_j' x_2$ for $a_j, b_j \in E$. Then, for $j$ such that $a_j \neq 0$ and $b_j \neq 0$, we have
\[ \text{Fil}^{-k_{j,1}} D_F = E_j(a_j' x_1 e_1 + b_j' x_2 e_2) = E_j(e_1 - \mathfrak{L}_j x_0 e_2) \]
for $\mathfrak{L}_j \in E^\times$, where we put $x_0 = x_1^{-1} x_2$.

If $a \neq b$, the non-trivial $(\phi, N)$-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodules of $D$ are $D'_1 = (F_0 \otimes_{\mathbb{Q}_p} E)e_1$ and $D'_2 = (F_0 \otimes_{\mathbb{Q}_p} E)e_2$. The condition $t_H(D'_1) \leq t_N(D'_1)$ is equivalent to that
\[ [K : K_0] v_p(a) \leq \sum_{k_{j,1} = 0} k_{j,1} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}. \]

The condition $t_H(D'_2) \leq t_N(D'_2)$ is equivalent to that
\[ [K : K_0] v_p(b) \leq \sum_{a_j = 0} k_{j,1} + \sum_{a_j \neq 0} k_{j,2}. \]

If $a = b$, the non-trivial $(\phi, N)$-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodules of $D$ are $D'_1$, $D'_2$ and $D'_{\mathfrak{L}} = (F_0 \otimes_{\mathbb{Q}_p} E)(e_1 - \mathfrak{L}_2 e_2)$ for $\mathfrak{L} \in E^\times$. For $\mathfrak{L} \in E^\times$, the condition $t_H(D'_\mathfrak{L}) \leq t_N(D'_\mathfrak{L})$ is equivalent to that
\begin{align*}
(S_{\mathfrak{L}}) \quad [K : K_0] v_p(a) &\leq \sum_{a_j = 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}
+ \sum_{a_j \neq 0} \left\{ t_j(\mathfrak{L}, \mathfrak{L}_j) k_{j,1} + (1 - t_j(\mathfrak{L}, \mathfrak{L}_j)) k_{j,2} \right\},
\end{align*}
where
\[ t_j(\mathfrak{L}, \mathfrak{L}_j) = \frac{|\{ j_F : F \hookrightarrow E | \text{jet-component of } \mathfrak{L}_j x_0 \in E_j \text{ is } \mathfrak{L} \}|}{[F : K]} \]
If $t_j(\mathfrak{L}, \mathfrak{L}_j) \leq 1/2$, the condition $(S_{\mathfrak{L}})$ is automatically satisfied by the condition $(S)$.

We assume that $t_j(\mathfrak{L}, \mathfrak{L}_j) > 1/2$. Then we have
\[ \frac{|\text{Ker} \chi_1 \chi_2^{-1} : \text{Gal}(F/K) \to \overline{\mathbb{Q}}_p^\times|}{[F : K]} > \frac{1}{2}, \]
because $\text{Gal}(F/K)$ act on $x_0$ by $\chi_1 \chi_2^{-1}$. This implies that $\chi_1 |_{K} = \chi_2 |_{K}$ and
\[ x_0 = (x_E)|_{j_F} \in \prod_{j_F : F \hookrightarrow E, j_F | K = j} E \]
for some $x_E \in E^\times$. Then $\mathfrak{L}_j x_E = \mathfrak{L}$ and $t_j(\mathfrak{L}, \mathfrak{L}_j) = 1$.

**Proposition 3.4.** We assume that $N = 0$ and $\phi$ is split reducible and $\tau \simeq \chi_1|_{I_K} \oplus \chi_2|_{I_K}$ for some character $\chi_1, \chi_2$ of $W_K$ that are finite on $I_K$. If we take a totally ramified cyclic extension $F$ of $K$ such that $\chi_1, \chi_2$ is trivial on $I_F$, then $D = (F_0 \otimes _{\mathbb{Q}_p} E)e_1 \oplus (F_0 \otimes _{\mathbb{Q}_p} E)e_2$ with

$$\phi(e_{1,1}) = ae_{2,1}, \ \phi(e_{1,2}) = be_{2,2}$$

for $a, b \in E^\times$ and

$$\phi(e_{i,1}) = e_{i+1,1}, \ \phi(e_{i,2}) = e_{i+1,2}$$

for $2 \leq i \leq m_0$ and

$$\text{Fil}_{F}^{k_{j,1}} D_{F} = \begin{cases} E_{j} e_{1} & \text{if } j \in I_1, \\ E_{j} e_{2} & \text{if } j \in I_2, \\ E_{j} (e_{1} - \mathfrak{L}_j x_{0} e_{2}) & \text{for } \mathfrak{L}_j \in E^\times \text{ if } j \in I_3 \end{cases}$$

for $j$ such that $k_{j,1} < k_{j,2}$, where

$$[K : K_0] v_p(ab) = \sum_{j} (k_{j,1} + k_{j,2}),$$

and $I_1, I_2, I_3$ are any disjoint sets such that $I_1 \cup I_2 \cup I_3 = \{ j \mid k_{j,1} < k_{j,2} \}$ and

$$[K : K_0] v_p(a) \leq \sum_{j \in I_1} k_{j,1} + \sum_{j \in I_2 \cup I_3} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2},$$

$$[K : K_0] v_p(b) \leq \sum_{j \in I_2} k_{j,1} + \sum_{j \in I_1 \cup I_3} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2},$$

and, if $a = b$ and $\chi_1|_{I_K} = \chi_2|_{I_K}$, further

$$[K : K_0] v_p(ab) \leq \sum_{j \in I_3, \mathfrak{L}_j x_E = \mathfrak{L}} k_{j,1} + \sum_{j \in I_3, \mathfrak{L}_j x_E \neq \mathfrak{L}} k_{j,2} + \sum_{j \in I_1 \cup I_2} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2}$$

for all $\mathfrak{L} \in E^\times$.

3.3. **Supercuspidal case.** In this case, $N = 0$ and $\tau \simeq \text{Int}_{W_{K'}}^W (\chi)|_{I_K}$ for a quadratic extension $K'$ of $K$ and a character $\chi$ of $W_{K'}$ that is finite on $I_{K'}$. Let $k'$ be the residue field of $K'$. We take a totally ramified abelian extension $L$ of $K'$ such that $\chi|_{I_L}$ is trivial.

For a uniformizer $\pi'$ of $K'$ and a positive integer $n$, let $K''_{n, n}$ be the Lubin-Tate extension of $K'$ generated by the $\pi'^n$-torsion points. For any $p$-adic field $M$ and a positive integer $n$, we put $U_M^{(n)} = 1 + p^n M$. Then we have

$$\Gal(K''_{n, n}/K') \cong (\mathcal{O}_{K'}/p^n K')^\times \cong k'^\times \times (U^{(1)}_{K'}/U^{(n)}_{K'}).$$

For any $p$-adic field $M$ and a positive integer $m$, let $M_m$ be the unramified extension of $M$ of degree $m$. 
3.3.1. Unramified case. We first treat the case in (2) of Lemma 2.1, where $K'$ is unramified over $K$ and $\chi$ does not extend to $W_K$. We take a uniformizer $\pi$ of $K$. This is also a uniformizer of $K'$. We take positive integers $m_1$ and $n_1$ so that $L$ is contained in $K'_m K'_{n_1}$, and put $F = K'_m K'_{n_1}$. Then $\rho$ is crystalline over $F$, and $F$ is a Galois extension of $K$.

We put $f(X) = \pi X + X^{q^2}$. For a positive integer $n$, let $f^{(n)}(X)$ be the $n$-th iterate of $f(X)$. We take a root $\theta$ of $f^{(n)}(X)$ in $K'_{n_1}$ that is not a root of $f^{(n_1-1)}(X)$. Then $K'_{n_1} = K'_{\theta}$. We can see that $K'_{\theta}$ is a totally ramified extension of $K$, and $F$ is an unramified extension of $K$ of degree $2m_1$. Now the restriction $\text{Gal}(F/K(\theta)) \to \text{Gal}(K'_{n_1}/K)$ is an isomorphism, and $\text{Gal}(F/K)$ is a semi-direct product of $\text{Gal}(F/K(\theta))$ by $\text{Gal}(K'_{n_1})$. We take a generator $\sigma$ of $\text{Gal}(F/K(\theta))$. Then the restriction $\sigma|_{K'}$ is the non-trivial element of $\text{Gal}(K'/K)$.

We consider a decomposition

$$U_K^{(1)} = U_{K'}^{(1)} / U_{K'}^{(n_1)}$$

of abelian groups such that $\sigma(\gamma_1) = \gamma_1$ for $\gamma_1 \in U_{n_1,+}$ and $\sigma(\gamma_2) = \gamma_2^{-1}$ for $\gamma_2 \in U_{n_1,-}$. There is an exact sequence

$$1 \to U_K^{(1)} / U_K^{(n_1)} \to U_{K'}^{(1)} / U_{K'}^{(n_1)} \to U_{K'}^{(1)} / U_{K'}^{(n_1)}$$

where the first map is induced from a natural inclusion and the second map is induced from a map

$$U_K^{(1)} \to U_{K'}^{(1)}; g \mapsto \sigma(g) g^{-1}.$$  

Then, by the above exact sequence, we see that

$$U_{n_1,+} \cong U_K^{(1)} / U_K^{(n_1)}, U_{n_1,-} \cong U_{K'}^{(1)} / (U_{K'}^{(1)} U_{K'}^{(n_1)})$$

and $|U_{n_1,+}| = |U_{n_1,-}| = q^{n_1-1}$.

Now, the restriction $\text{Gal}(F/K'_{n_1}) \to \text{Gal}(K'_{n_1}/K')$ is an isomorphism. Then we can prove that, under an identification

$$\text{Gal}(F/K'_{n_1}) \cong \text{Gal}(K'_{n_1}/K') \cong k'^\times \times U_{n_1,+} \times U_{n_1,-},$$

we have

$$\sigma^{-1} \delta \sigma = \delta^q, \quad \sigma^{-1} \gamma_1 \sigma = \gamma_1 \quad \text{and} \quad \sigma^{-1} \gamma_2 \sigma = \gamma_2^{-1}$$

for $\delta \in k'^\times, \gamma_1 \in U_{n_1,+}$ and $\gamma_2 \in U_{n_1,-}$.

Considering $\chi|_{I_K}$ as a character of $I(F/K) \cong k'^\times \times U_{n_1,+} \times U_{n_1,-}$, we write $\chi = \omega^s \cdot \chi_1 \cdot \chi_2$, where $\omega$ is the Teichmüller character, $s$ is an integer, and $\chi_1$ and $\chi_2$ are characters of $U_{n_1,+}$ and $U_{n_1,-}$ respectively. The condition that $\chi$ does not extend to $W_K$ is equivalent to that $\chi \not= \chi'^s$ on $W_{K'}$, and it is further equivalent to that $\chi \not= \chi'^s$ on $I_K$. This last condition is equivalent to that $s \not= 0 \mod q + 1$ or $\chi_2 \not= 1$.

Now we have $[F_0 : Q_{p_0}] = 2m_0m_1$. We take bases $e_{i,1}, e_{i,2}$ of $D_i$ over $E$ for $1 \leq i \leq 2m_0m_1$ so that

$$\delta e_{i,1} = \omega^s(\delta) e_{i,1}, \quad \gamma_1 e_{i,1} = \chi_1(\gamma_1) e_{i,1}, \quad \gamma_2 e_{i,1} = \chi_2(\gamma_2) e_{i,1},$$

$$\delta e_{i,2} = \omega^s(\delta) e_{i,2}, \quad \gamma_1 e_{i,2} = \chi_1(\gamma_1) e_{i,2}, \quad \gamma_2 e_{i,2} = \chi_2(\gamma_2)^{-1} e_{i,2}$$

for $\delta \in k'^\times, \gamma_1 \in U_{n_1,+}$ and $\gamma_2 \in U_{n_1,-}$. 
Remark 3.5. A normalization of bases here is different from that in [GM, 3.3.2]. We prefer that the action of $\delta$ on $e_{i,1}, e_{i,2}$ is the same form for all $i$. In stead of this, the action of $\sigma$ does not preserve lines generated by $e_1$ and $e_2$ as we see in the below.

Since $\sigma$ takes $D_i$ to $D_{i+m_0}$, we have that

$$\sigma e_{i,1} = a_{i+m_0} e_{i+m_0,2}, \quad \sigma e_{i,2} = b_{i+m_0} e_{i+m_0,1}$$

for some $a_{i+m_0}, b_{i+m_0} \in E^\times$ by $(\ast)$. Because $\sigma^{2m_1} = 1$, we see that

$$\prod_{i=1}^{m_1} (a_{i+2lm_0} - m_0 b_{i+2lm_0}) = 1$$

for all $i$. Replacing $e_{i,1}$ and $e_{i,2}$ by their scalar multiples, we may assume that

$$\sigma e_{i,1} = e_{i+m_0,2}, \quad \sigma e_{i,2} = e_{i+m_0,1}.$$

Since $\phi$ takes $D_i$ to $D_{i+1}$ and commutes with the action of $I(F/K)$, we have that

$$\phi(e_{i,1}) = \frac{1}{\alpha_{i+1}} e_{i+1,1}, \quad \phi(e_{i,2}) = \frac{1}{\beta_{i+1}} e_{i+1,2}$$

for some $\alpha_{i+1}, \beta_{i+1} \in E^\times$ for all $i$. Since $\phi$ commutes with the action of $\sigma$, we have $\alpha_i = \beta_{i+m_0}$ and $\beta_i = \alpha_{i+m_0}$ for all $i$. Replacing $e_{i,1}$ and $e_{i,2}$ by their scalar multiples, we may further assume that $\alpha_i = \beta_i = 1$ for $2 \leq i \leq m_0$.

Let $e_1, e_2$ be a basis of $D$ over $F_0 \otimes_{\mathbb{Q}_p} E$ determined by $(e_{i,1}), (e_{i,2})$, under the isomorphism $D \cong \prod D_i$. Then $\sigma e_1 = e_2$ and $\sigma e_2 = e_1$.

The condition $t_H(D) = t_N(D)$ is equivalent to that

$$(U) \quad [K : K_0] v_p(\alpha_i \beta_i) = \sum_j (k_{j,1} + k_{j,2}).$$

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\operatorname{Fil}^{j-1} D_F = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is Gal$(F/K)$-invariant. By $\sigma(a_j e_1 + b_j e_2) = (a_j e_1 + b_j e_2)$, we get $\sigma(a_j) = b_j$ and $\sigma(b_j) = a_j$. So $a_j \in E_j^\times$ if and only if $b_j \in E_j^\times$.

Since $(a_j e_1 + \sigma(a_j) e_2)$ is Gal$(F/K)$-invariant, $\sigma^2(a_j) = a_j$ and $g \in I(F/K)$ acts on $a_j$ by $\chi(g)^{-1}$. We prove that there are $x_{j,1}, x_{j,2} \in E_j$ such that

- $a_j$ satisfies the above condition if and only if $a_j = a_{j,1} x_{j,1} + a_{j,2} x_{j,2}$ for some $a_{j,1}, a_{j,2} \in E$,
- for $a_{j,1}, a_{j,2} \in E$, we have $a_{j,1} x_{j,1} + a_{j,2} x_{j,2} \in E_j^\times$ if and only if $a_{j,1} \neq 0$ and $a_{j,2} \neq 0$.

By Lemma 2.3, we may replace $E_j$ by $F \otimes_K E$. Then $\sigma^2(a_j) = a_j$ if and only if $a_j \in K_{n,1}^\times \otimes_K E$. By Lemma 2.5, we get the claim. We put $x_j(a_{j,1}, a_{j,2}) = a_{j,1} x_{j,1} + a_{j,2} x_{j,2}$ and $x_j^\sigma(a_{j,1}, a_{j,2}) = \sigma(x_j(a_{j,1}, a_{j,2}))$. Then we have

$$\operatorname{Fil}^{j-1} D_F = E_j(x_j(a_{j,1}, a_{j,2}) e_1 + x_j^\sigma(a_{j,1}, a_{j,2}) e_2)$$

for $(a_{j,1}, a_{j,2}) \in \mathbb{P}^1(E)$.

The non-trivial $(\phi, N)$-stable $(F_0 \otimes_{\mathbb{Q}_p} E)$-submodules of $D$ are $D_1' = (F_0 \otimes_{\mathbb{Q}_p} E)e_1$, $D_2' = (F_0 \otimes_{\mathbb{Q}_p} E)e_2$ and $D_\mathcal{E}' = (F_0 \otimes_{\mathbb{Q}_p} E)(e_1 - \mathcal{E} e_2)$ for $\mathcal{E} \in (F_0 \otimes_{\mathbb{Q}_p} E)^\times$ satisfying the following:
If $\mathfrak{L}$ corresponds to $(\mathfrak{L}_i)_i$ under the isomorphism

$$F_0 \otimes_{Q_p} E \cong \prod_{\sigma_i:F_0 \to E} E,$$

then $\mathfrak{L}_{i+1} = \frac{a_{i+1}}{b_{i+1}} \mathfrak{L}_i$ for all $i$.

The condition $t_H(D'_1) \leq t_N(D'_1)$ is equivalent to that

$$[K : K_0] v_p(\alpha_1) \leq \sum_{a_j, a_j \neq 0} \frac{k_{j,1} + k_{j,2}}{2} + \sum_{a_{j,1} a_{j,2} \neq 0} k_{j,2} + \sum_{k_{j,1} = k_{j,2}} k_{j,2},$$

the condition $t_H(D'_2) \leq t_N(D'_2)$ is equivalent to that

$$[K : K_0] v_p(\beta_1) \leq \sum_{a_{j,1} a_{j,2} = 0} k_{j,1} + k_{j,2} + \sum_{a_{j,1} a_{j,2} \neq 0} k_{j,2},$$

and the condition $t_H(D'_3) \leq t_N(D'_3)$ is equivalent to that

$$\langle U_\mathfrak{L} \rangle \quad [K : K_0] v_p(\beta_1) \leq \sum_{a_{j,1} a_{j,2} = 0} k_{j,1} + k_{j,2} + \sum_{a_{j,1} a_{j,2} \neq 0} k_{j,2} + \sum_{a_{j,1} a_{j,2} \neq 0} \left\{ t_j(\mathfrak{L}, (a_{j,1}, a_{j,2})) k_{j,1} + \left( 1 - t_j(\mathfrak{L}, (a_{j,1}, a_{j,2})) \right) k_{j,2} \right\},$$

where

$$t_j(\mathfrak{L}, (a_{j,1}, a_{j,2})) = \frac{\{ j_F : F \leftarrow E \mid j_F \text{-component of } \frac{x_j(\mathfrak{L}, (a_{j,1}, a_{j,2}))}{x_j(\mathfrak{L}, (a_{j,1}, a_{j,2}))} \in E_j \text{ is } -\mathfrak{L}_{j_F} \}}{[F : K]}.$$  

Here and in the sequel, $\mathfrak{L}_{j_F}$ is the $j_F$-component of $\mathfrak{L} \in F_0 \otimes_{Q_p} E \subset F \otimes_{Q_p} E$. If $t_j(\mathfrak{L}, (a_{j,1}, a_{j,2})) \leq 1/2$, the condition $(U_\mathfrak{L})$ is automatically satisfied by the condition $(U)$.  

To prove that $t_j(\mathfrak{L}, (a_{j,1}, a_{j,2})) \leq 1/2$, we assume that $t_j(\mathfrak{L}, (a_{j,1}, a_{j,2})) > 1/2$. We consider a decomposition

$$E_j = \prod_{j_F:F \to E, j_F|_{K=j}} E = \prod_{j_F:F \to E, j_F|_{K=j}} \left( \prod_{j_F:F \to E, j_F|_{K=j} \neq j} E \right).$$

Then there is $j_{F_0} : F_0 \to E$ such that $j_{F_0}|_K = j$ and

$$\left\{ j_F : F \leftarrow E \mid j_F|_{F_0} = j_{F_0} \text{ and } j_F \text{-component of } \frac{x_j(\mathfrak{L}, (a_{j,1}, a_{j,2}))}{x_j(\mathfrak{L}, (a_{j,1}, a_{j,2}))} \in E_j \text{ is } -\mathfrak{L}_{j_F} \right\}$$

is greater than 1/2. Here $\mathfrak{L}_{j_F}$ is independent of $j_F$ such that $j_F|_{F_0} = j_{F_0}$, because $\mathfrak{L} \in F_0 \otimes_{Q_p} E$. Then we have

$$\left| \text{Ker}\left( \chi(\chi^\sigma)^{-1} : l(F/K) \to \mathbb{U}_p^\sigma \right) \right|_{[F : F_0]} > \frac{1}{2},$$

because $l(F/K)$ act on $x_j(\mathfrak{L}, (a_{j,1}, a_{j,2}))/x_j(\mathfrak{L}, (a_{j,1}, a_{j,2}))$ by $\chi(\chi^\sigma)^{-1}$. This implies that $\chi|_{K^\sigma} = \chi^\sigma|_{K^\sigma}$, and contradicts the condition that $\chi$ does not extend to $W_K$. Thus we have proved that $t_j(\mathfrak{L}, (a_{j,1}, a_{j,2})) \leq 1/2.$
Proposition 3.6. We assume $\tau \simeq \Ind_{W_K}^{W_{K'}}(\chi)|_{I_K}$ for the unramified quadratic extension $K'$ of $K$ and a character $\chi$ of $W_{K'}$ that is finite on $I_{K'}$, and does not extend to $W_K$. We take a uniformizer $\pi$ of $K$ and a totally ramified abelian extension $L$ of $K'$ such that $\chi$ is trivial on $I_L$, and take positive integers $m_1$ and $n_1$ so that $L$ is contained in $K_{m_1}^{1',n_1}$. We put $F = K_{m_1}^{1',n_1}$, Then $N = 0$ and $D = (F_0 \otimes_{Q_p} E)e_1 \oplus (F_0 \otimes_{Q_p} E)e_2$ with

$\phi(e_{i,1}) = \frac{1}{\alpha_1} e_{i+1,1}$, $\phi(e_{i,2}) = \frac{1}{\beta_1} e_{i+1,2}$, if $i \equiv 0 \pmod{2m_0}$,

$\phi(e_{i,1}) = \frac{1}{\alpha_1} e_{i+1,1}$, $\phi(e_{i,2}) = \frac{1}{\alpha_1} e_{i+1,2}$, if $i \equiv m_0 \pmod{2m_0}$,

$\phi(e_{i,1}) = e_{i+1,1}$, $\phi(e_{i,2}) = e_{i+1,2}$, if $i \not\equiv 0 \pmod{m_0}$

for $\alpha_1, \beta_1 \in E^\times$,

$\sigma e_1 = e_2$, $\sigma e_2 = e_1$, $g e_1 = (1 \otimes \chi(g)) e_1$, $g e_2 = (1 \otimes \chi^\sigma(g)) e_2$

for $g \in I(F/K)$ and, for $j$ such that $k_{j,1} < k_{j,2}$,

$\Fil_j^{-k_{j,1}} D_{F} = E_j(x_j(a_{j,1}, a_{j,2}) e_1 + x_j^2(a_{j,1}, a_{j,2}) e_2)$

for $(a_{j,1}, a_{j,2}) \in \mathbb{P}^1(E)$ where

$[K : K_0] v_p(\alpha_1 \beta_1) = \sum_j (k_{j,1} + k_{j,2})$

and

$\sum_j k_{j,1} + \sum_{a_{j,1}, a_{j,2} = 0} k_{j,2} - k_{j,1} \leq [K : K_0] v_p(\alpha_1) \leq \sum_j k_{j,2} + \sum_{a_{j,1}, a_{j,2} = 0} k_{j,2} - k_{j,1} \frac{2}{2}$.

The definition of $\sigma$ is in the above discussion.

3.3.2. Ramified case. Next, we treat the case in (3) of Lemma 2.1, where $K'$ is ramified over $K$ and $\chi|_{I_K}$, does not extend to $I_K$.

Let $\eta$ be the non-trivial element of $\Gal(K'/K)$. We take a uniformizer $\pi'$ of $K'$ such that $\eta_0(\pi') = -\pi'$. Then we have $(K_{m_1}^{1',n_1})' = K_{m_1}'$, for a positive integer $n$ and any lift $\pi \in G_K$ of $\eta_0$. So $K_{m_1}' = K_{m_1}'$ is a Galois extension of $K$. By the class field theory, the abelian extensions $K_{m_1}'$ and $K_{m_1}'$ of $K'$ correspond to $(\pi') \times (1 + p_{K'}^n)$ and $(-\pi') \times (1 + p_{K'}^n)$, respectively. Then the abelian extension $K_{m_1}' = K_{m_1}'$, of $K'$ corresponds to $(\pi')^2 \times (1 + p_{K'}^n)$. So we see that $K_{m_1}' = K_{m_1}'$.

We take positive integers $m_1$ and $n_1$ so that $L$ is contained in $K_{m_1}^{1,2n_1+1}$, and put $F = K_{m_1}^{2n_1}$. Then $F$ is a Galois extension of $K$, and $\rho$ is crystalline over $F$ because $\tau|_F$ is trivial.

We consider an exact sequence

$1 \to \Gal(F/K') \to \Gal(F/K) \to \Gal(K'/K) \to 1$.

Since the restriction $\Gal(F/K_{m_1}) \to \Gal(K_{m_1}^{1,2n_1+1})$ is an isomorphism,

$\Gal(F/K') = \Gal(F/K_{m_1}^{1,2n_1+1}) \times \Gal(F/K_{m_1}^{2n_1}) \cong \Gal(F/K_{m_1}^{1,2n_1+1}) \times k'^\times \times (U_{K'})^2\times U_{K'}(2n_1+1)$.

Let $\sigma$ be a generator of $\Gal(F/K_{m_1}^{1,2n_1+1})$, and $\delta_0$ be a generator of $k'^\times$.

We prove that the exact sequence $(\varnothing)$ does not split. We assume there is a lift $\iota \in \Gal(F/K)$ of $\iota_0$ such that $\iota^2 = 1$. By multiplying $\iota$ by an element of
\( \text{Gal}(F/K') \subset \text{Gal}(F/K) \), we may assume that \( \iota \in I(F/K) \). Let \( P(F/K) \)
be the wild ramification subgroup of \( I(F/K) \), and \( P'(F/K) \) be the tame quotient

group of \( I(F/K) \). Let \( \iota \) be the image of \( \iota \) in \( P'(F/K) \). If \( \iota \neq 1 \), we multiply \( \iota \) by the element \( \delta_{0}^{(q-1)/2} \) of \( k^x \subset \text{Gal}(F/K') \). Then we have \( \iota \in P(F/K) \), but this

contradicts that \( p \neq 2 \). Thus we have proved the claim.

For any lift \( \iota \in \text{Gal}(F/K) \), we have \( \iota^2 \in \text{Gal}(F/K') \). Since the exact sequence

\( \langle \rangle \) does not split and \( p \neq 2 \), multiplying \( \iota \) by an element of \( \text{Gal}(F/K') \), we may

assume that \( \iota^2 = \delta_0 \) and \( \iota \in I(F/K) \). We fix this lift \( \iota \) in the sequel.

We consider a decomposition

\[
U_{K'}^{(1)}/U_{K'}^{(2n_1+1)} = U_{2n_1+1,+} \times U_{2n_1+1,-}
\]

of abelian groups such that \( \iota_0(\gamma_1) = \gamma_1 \) for \( \gamma_1 \in U_{2n_1+1,+} \) and \( \iota_0(\gamma_2) = \gamma_2^{-1} \) for \( \gamma_2 \in U_{2n_1+1,-} \). There is an exact sequence

\[
1 \rightarrow U_{K'}^{(1)}/U_{K'}^{(n_1+1)} \rightarrow U_{K'}^{(1)}/U_{K'}^{(2n_1+1)} \rightarrow U_{K'}^{(1)}/U_{K'}^{(2n_1+1)},
\]

where the first map is induced from a natural inclusion and the second map is induced from a map

\[
U_{K'}^{(1)} \rightarrow U_{K'}^{(1)}: g \mapsto \iota_0(g)g^{-1}.
\]

Then, by the above exact sequence, we see that

\[
U_{2n_1+1,+} \cong U_{K'}^{(1)}/U_{K'}^{(n_1+1)}, \quad U_{2n_1+1,-} \cong U_{K'}^{(1)}/(U_{K'}^{(1)}U_{K'}^{(2n_1+1)})
\]

and \( |U_{2n_1+1,+}| = |U_{2n_1+1,-}| = q^{n_1} \).

We can prove that, under an identification

\[
\text{Gal}(F/K'_{2m_1}) \cong \text{Gal}(K'_{\pi,2n_1+1}/K') \cong k^x \times U_{2n_1+1,+} \times U_{2n_1+1,-},
\]

we have

\[
\iota^{-1}\delta \iota = \delta, \quad \iota^{-1}\gamma_1 \iota = \gamma_1 \quad \text{and} \quad \iota^{-1}\gamma_2 \iota = \gamma_2^{-1}
\]

for \( \delta \in k^x \), \( \gamma_1 \in U_{2n_1+1,+} \) and \( \gamma_2 \in U_{2n_1+1,-} \).

Since \( K'_{\pi,2n_1+1} \) is not a normal extension of \( K \), we have \( \iota^{-1}\sigma \iota \neq \sigma \). We put

\( K'' = K'_{\pi,2n_1+1}K'_{-\pi,2n_1+1} \). Then \( \sigma^2 \) is a generator of \( \text{Gal}(F/K'') \), and \( \iota \) determines an automorphism of \( K'' \). So we have \( \iota^{-1}\sigma^2 \iota = \sigma^2 \). Since \( \iota^{-1}\sigma \iota \) is an element of \( \text{Gal}(F/K') \) of order 2 and fixes \( K_{2m_1} \), it is \( \delta_0^{(q-1)/2} \). Hence we have

\[
(*) \quad \iota^{-1}\sigma \iota = \sigma\delta_0^{(q-1)/2}.
\]

Considering \( \chi |_{K'} \) as a character of

\[
I(F/K') \cong k^x \times U_{2n_1+1,+} \times U_{2n_1+1,-},
\]

we write \( \chi = \omega^s \cdot \chi_1 \cdot \chi_2 \), where \( \omega \) is the Teichmüller character, \( s \) is an integer, and \( \chi_1 \) and \( \chi_2 \) are characters of \( U_{2n_1+1,+} \) and \( U_{2n_1+1,-} \) respectively. The condition \( \chi \) does not extend to \( I_K \) is equivalent to that \( \chi \neq \chi^s \) on \( I_K \), and it is further equivalent to that \( \chi^2 \neq 1 \).

Now we have \( [F_0 : Q_p] = 2m_0m_1 \). We take bases \( e_{i,1}, e_{i,2} \) of \( D_i \) over \( E \) for \( 1 \leq i \leq 2m_0m_1 \) so that

\[
\nu e_{i,1} = e_{i,2}, \quad \delta e_{i,1} = \omega^s(\delta) e_{i,1}, \quad \gamma_1 e_{i,1} = \chi_1(\gamma_1) e_{i,1}, \quad \gamma_2 e_{i,1} = \chi_2(\gamma_2) e_{i,1},
\]

\[
\nu e_{i,2} = \omega^s(\delta) e_{i,1}, \quad \delta e_{i,2} = \omega^s(\delta) e_{i,2}, \quad \gamma_1 e_{i,2} = \chi_1(\gamma_1) e_{i,2}, \quad \gamma_2 e_{i,2} = \chi_2(\gamma_2)^{-1} e_{i,2}
\]

for \( \delta \in k^x \), \( \gamma_1 \in U_{n_1,+} \) and \( \gamma_2 \in U_{n_1,-} \).
Since $\sigma$ takes $D_t$ to $D_{t+m_0}$, as in the unramified case, we may assume that $\sigma e_{i,1} = e_{i+m_0,1}$. Then we have that $\sigma e_{i,2} = (-1)^i e_{i+m_0,2}$ by ($*$).

Since $\phi$ takes $D_t$ to $D_{t+1}$ and commutes with the action of $I(F/K)$, we have that

$$
\phi(e_{i,1}) = \frac{1}{\alpha_{i+1}} e_{i+1,1}, \quad \phi(e_{i,2}) = \frac{1}{\alpha_{i+1}} e_{i+1,2}
$$

for some $\alpha_{i+1} \in E^\times$ for all $i$. Further, since $\phi$ commutes with the action of $\sigma$, we have $\alpha_i = \alpha_{i+m_0}$ for all $i$. Replacing $e_{i,1}$ and $e_{i,2}$ by their scalar multiples, we may further assume that $\alpha_i = 1$ for $2 \leq i \leq m_0$.

Let $e_1, e_2$ be a basis of $D$ over $F_0 \otimes_{Q_p} E$ determined by $(e_{1,1}), (e_{2,1})$ under the isomorphism $D \cong \prod D_t$. Then $\sigma e_1 = e_1$ and $\sigma e_2 = (-1)^i e_2$.

The condition $t_H(D) = t_N(D)$ is equivalent to that

$$
(\mathcal{R}) \quad 2[K : K_0] v_p(\alpha_1) = \sum_j (k_{j,1} + k_{j,2}).
$$

For $j : K \hookrightarrow E$ satisfying $k_{j,1} < k_{j,2}$, by Lemma 2.4, we take $a_j, b_j \in E_j$ such that $\text{Fil}^{-k_{j,1}} D_E = E_j(a_j e_1 + b_j e_2)$, and $(a_j e_1 + b_j e_2)$ is $\text{Gal}(F/K)$-invariant. By $\nu(a_j e_1 + b_j e_2) = (a_j e_1 + b_j e_2)$, we get $\nu(a_j) = b_j$ and $\nu(b_j) \omega^*(\delta_0) = a_j$. So $a_j \in E_j^\times$ if and only if $b_j \in E_j^\times$.

Since $(a_j e_1 + \nu(a_j) e_2)$ is $\text{Gal}(F/K)$-invariant, $\sigma(a_j) = a_j$ and $g \in I(F/K')$ acts on $a_j$ by $\chi(g)^{-1}$. We prove that there are $x_{j,1}, x_{j,2} \in E_j$ such that

- $a_j$ satisfies the above condition if and only if $a_j = a_{j,1} x_{j,1} + a_{j,2} x_{j,2}$ for some $a_{j,1}, a_{j,2} \in E$,
- for $a_{j,1}, a_{j,2} \in E$, we have $a_{j,1} x_{j,1} + a_{j,2} x_{j,2} \in E_j^\times$ if and only if $a_{j,1} \neq 0$ and $a_{j,2} \neq 0$.

By Lemma 2.3, we may replace $E_j$ by $F \otimes_K E$. Then $\sigma(a_j) = a_j$ if and only if $a_j \in K'_{\{2n+1\}} \otimes_K E$. By Lemma 2.5, we get the claim. We put $x_j(a_{j,1}, a_{j,2}) = a_{j,1} x_{j,1} + a_{j,2} x_{j,2}$ and $x_j(a_{j,1}, a_{j,2}) = \nu(x_j(a_{j,1}, a_{j,2}))$. Then we have

$$
\text{Fil}^{-k_{j,1}} D_F = E_j(x_j(a_{j,1}, a_{j,2}) e_1 + x_j(a_{j,1}, a_{j,2}) e_2)
$$

for $(a_{j,1}, a_{j,2}) \in \mathcal{P}(E)$.

The non-trivial $(\phi, N)$-stable $(F_0 \otimes_{Q_p} E)$-submodules of $D$ are $D_1' = (F_0 \otimes_{Q_p} E)e_1$, $D_2' = (F_0 \otimes_{Q_p} E)e_2$ and $D_2'' = (F_0 \otimes_{Q_p} E)(e_1 - \Sigma e_2)$ for $\Sigma \in E^\times$. The condition $t_H(D_1') \leq t_N(D_1')$ is equivalent to that

$$
[K : K_0] v_p(\alpha_1) \leq \sum_{a_{j,1}, a_{j,2} = 0} \frac{k_{j,1} + k_{j,2}}{2} + \sum_{a_{j,1} a_{j,2} \neq 0} k_{j,2} + \sum_{a_{j,1} = k_{j,2}} k_{j,2},
$$

and this condition is automatically satisfied by the condition $(\mathcal{R})$. The condition $t_H(D_2') \leq t_N(D_2')$ is also equivalent to the same condition. For $\Sigma \in E^\times$, the condition $t_H(D_2'') \leq t_N(D_2'')$ is equivalent to that

$$
(\mathcal{R}_\Sigma) \quad [K : K_0] v_p(\alpha_1) \leq \sum_{a_{j,1}, a_{j,2} = 0} k_{j,2} + \sum_{a_{j,1} = k_{j,2}} k_{j,2}
$$

$$
+ \sum_{a_{j,1} a_{j,2} \neq 0} \left\{ t_j(\Sigma, (a_{j,1}, a_{j,2})) k_{j,1} + \left(1 - t_j(\Sigma, (a_{j,1}, a_{j,2}))\right) k_{j,2} \right\},
$$
where

$$t_j(\mathcal{L}, (a_{j,1}, a_{j,2})) = \frac{\{ j_F : F \hookrightarrow E \mid j_F\text{-component of } \mathcal{E}_{j}(a_{j,1}, a_{j,2})^{\prime} \subset E_j \text{ is } -\mathcal{L} \}}{|F : K|}.$$ 

As in the unramified case, we can prove that $t_j(\mathcal{L}, (a_{j,1}, a_{j,2})) \leq 1/2$, using the condition that $\chi \not\equiv \chi^s$ on $I_{K'}$. So the condition $(R_\mathcal{L}^j)$ is automatically satisfied by the condition $(R)$.

Proposition 3.7. We assume $\tau \simeq \text{Ind}_{W_{K'}}^{W_K}(\chi)|_{I_K}$ for a ramified quadratic extension $K'$ of $K$ and a character $\chi$ of $W_K$, such that $\chi|_{I_{K'}}$ is finite and does not extend to $I_K$. We take a uniformizer $\pi'$ of $K'$ and a totally ramified abelian extension $L$ of $K'$ such that $\chi$ is trivial on $I_L$, and take positive integers $m_1$ and $n_1$ so that $L$ is contained in $K_{2n_1}^2 K_{2n_1+1}^2$. We put $F = K_{2n_1}^2 K_{2n_1+1}^2$. Then $N = 0$ and $D = (F_0 \otimes_{Q_p} E)e_1 \oplus (F_0 \otimes_{Q_p} E)e_2$ with

$$\phi(e_{i,1}) = \frac{1}{\alpha_1} e_{i+1,1}, \quad \phi(e_{i,2}) = \frac{1}{\alpha_1} e_{i+1,2}, \quad \text{if } i \equiv 0 \pmod{m_0},$$

$$\phi(e_{i,1}) = e_{i+1,1}, \quad \phi(e_{i,2}) = e_{i+1,2}, \quad \text{if } i \not\equiv 0 \pmod{m_0}$$

for $\alpha_1 \in E^\times$, $\sigma e_1 = e_1$, $\iota e_1 = e_2$, $ge_1 = (1 \otimes \chi(g))e_1$, $ge_2 = (1 \otimes \chi^s(g))e_2$ for $s \in \mathbb{Z}$ and $g \in I(F/K')$ and, for $j$ such that $k_{j,1} < k_{j,2}$,

$$\text{Fil}_j^{k_{j,1}} D_F = E_j(\mathcal{E}_j(a_{j,1}, a_{j,2})e_1 + \mathcal{E}_j(a_{j,1}, a_{j,2})e_2)$$

for $(a_{j,1}, a_{j,2}) \in \mathbb{P}^1(E)$ where

$$2[K : K_0] v_p(\alpha_1) = \sum_j (k_{j,1} + k_{j,2}).$$

Here $\omega : k' \to \mathcal{O}_K^\times$ is the Teichmüller character, and the definitions of $\sigma, \iota, \delta_0$ are in the above discussion.

References


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