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A Tale of Two Multivariate GARCH Models"

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Do We Really Need Both BEKK and DCC?
A Tale of Two Multivariate GARCH Models*

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Abstract

The management and monitoring of very large portfolios of financial assets are routine for many individuals and organizations. The two most widely used models of conditional covariances and correlations in the class of multivariate GARCH models are BEKK and DCC. It is well known that BEKK suffers from the archetypal "curse of dimensionality", whereas DCC does not. It is argued in this paper that this is a misleading interpretation of the suitability of the two models for use in practice. The primary purpose of this paper is to analyze the similarities and dissimilarities between BEKK and DCC, both with and without targeting, on the basis of the structural derivation of the models, the availability of analytical forms for the sufficient conditions for existence of moments, sufficient conditions for consistency and asymptotic normality of the appropriate estimators, and computational tractability for ultra large numbers of financial assets. Based on theoretical considerations, the paper sheds light on how to discriminate between BEKK and DCC in practical applications.

Keywords: Conditional correlations, conditional covariances, diagonal models, scalar models, targeting, asymptotic theory.

JEL Classification: C32, G11, G17, G32.

1. Introduction

The management and monitoring of very large portfolios of financial assets are routine for many individuals and organizations. Consequently, a careful analysis, specification, estimation, and forecasting of financial asset returns dynamics, and the construction and evaluation of financial portfolios, are essential in the tool kit of any financial planner and analyst. Correlations are used to determine portfolios, with appropriate attention being given to hedging and asset specialization strategies, whereas variances and covariances are used to forecast Value-at-Risk (VaR) thresholds to satisfy the requirements of the Basel Accord. There are different models for different purposes, such as correlation models to create and evaluate a portfolio, and covariance models to forecast VaR on a daily basis for a given portfolio (see, for example, McAleer (2005)).The two most widely used models of conditional covariances and correlations are BEKK and DCC, as developed in Engle and Kroner (1995) and Engle (2002), respectively.

There are many similarities between BEKK and DCC. A scalar version of BEKK was compared with DCC, which is inherently scalar in practice, in Caporin and McAleer (2008). It was found empirically that scalar versions of the two models are very similar in forecasting conditional variances, covariances and correlations, which would suggest that they would also be similar in forecasting VaR thresholds and the consequent daily capital charges.

Accordingly, there are pertinent aspects regarding alternative versions of the two models that have not yet been addressed and clarified in the literature. First, we note that BEKK and DCC co-exist, despite one model being able to do virtually everything the other can do, thereby raising the pertinent question posed in the title of the paper. Second, we argue that BEKK is used to forecast conditional covariances, although it may also be used to forecast conditional correlations indirectly, and that DCC is used to forecast conditional correlations only, while its structure could easily be applied to forecast conditional covariances. Third, the inherent differences between BEKK and DCC do not seem to be widely known. This is particularly relevant as DCC is similar to a targeted scalar BEKK model as applied to the variance standardized residuals, and can thereby be interpreted as a conditional correlation matrix only because of the standardization. Fourth, both the structural and statistical differences and similarities between the two models have not previously been analyzed in the literature.

With respect to the first question, we note that Engle and Kroner (1995) is a widely cited paper, but most citations would seem to be of a theoretical rather than empirical nature. The model is an archetypical example of over-parameterization, thereby leading to the moniker "curse of dimensionality". Engle (2002) is also widely cited, but most citations would seem to be of an empirical rather than theoretical nature. The prevailing empirical wisdom would seem to be that DCC is preferred to BEKK because of the curse of dimensionality associated with the latter. It is argued in the paper that this is a misleading interpretation of the suitability of the two models to be used in practice.

A primary purpose of the paper is to shed some light on the similarities and differences between BEKK and DCC. The comparison commences from a theoretical perspective. A comparison of the two models considers several aspects which are generally associated with theoretical econometrics, but which are also fundamental in guaranteeing that the empirical applications, as well as their interpretation, are reliable. With this rationale, we first define targeting as an aid in estimating matrices associated with large numbers of financial assets, and then briefly discuss the use of targeting in estimating conditional covariance and correlation matrices in financial econometrics. We also consider the similarities and dissimilarities between BEKK and DCC, both with and without targeting; the analytical forms of the sufficient conditions for the existence of moments, sufficient conditions for consistency and asymptotic normality, computational tractability for ultra high numbers of financial assets, use of consistent two step estimation methods for the DCC model to enable it to be used sensibly in practical situations.

Based on the previous comparison, we are able to shed some light on the determination of whether BEKK or DCC is to be preferred in empirical applications. Nevertheless, the empirical comparison of BEKK and DCC specifications from small to large problem dimensions is not addressed in the current work, has been considered extensively in Caporin and McAleer (2010).

The remainder of the paper is organized as follows. Section 2 compares the BEKK and DCC specifications, defines the long run solution of conditional covariances (correlations), and defines the targeting of conditional covariance (correlation) models. Section 3 discusses the asymptotic results for BEKK and DCC. Some concluding comments and a link with empirical applications of the models are given in Section 4.

2. A Comparison of BEKK and DCC

The univariate models underlying DCC can be based on various conditional volatility specifications, such as the asymmetric GJR model of Glosten, Jagannathan and Runkle (1992) or the asymmetric/leverage EGARCH model of Nelson (1991). Similarly, both the BEKK and DCC structures might be generalized in alternative ways to introduce asymmetry, leverage and/or other stylized facts that are observed in financial returns variances and correlations.

However, as the primary focus of the paper is to compare the analytical and statistical performances of directly comparable BEKK and DCC models which are feasible under large cross-sectional dimensions, univariate and multivariate asymmetry and leverage are not considered. Moreover, forecasting comparisons of various versions of BEKK and DCC is left for further research (see also Caporin and McAleer (2008) for some results based on small scale models).

In this section we introduce the most relevant specifications which could be considered when fitting BEKK and DCC conditional covariance (and correlation) models to real data. We present the mean specification, the two conditional covariance and correlation models, and discuss the issues associated with the respective model structures and asymptotic properties. Within this section, we also highlight those issues that have not yet been addressed critically in the literature.

We will make use of the following operators below: $\circ$ denotes the Hadamard, or element-by-element, product; *dg(a)* is a diagonal matrix, with scalar *a* along the main diagonal; *diag(A)* is a vector formed from the elements of the main diagonal of *A*; I^{t-1} is the information set to time *t-1*; and **i** is a vector composed of unit elements.

2.1 Mean specification, curse of dimensionality and the concept of "targeting"

In order to make a fair comparison of models for the conditional second-order moments, we assume that the mean dynamics are common across all possible specifications, and are adequately captured by an un-specified conditional model. As a result, the mean innovations (or residuals) will be identically distributed according to a multivariate density with conditional covariance matrix, Σ_t, and possibly dependent on a set of parameters, θ

(including, for instance, degrees of freedom or coefficients driving the distribution asymmetry).

Let x_t denote a *k*-dimensional vector of financial variables (returns), μ_t represent the expected mean of x_t obtained from a conditional mean model, and ε_t the mean innovation vector, as follows:

$$x_t \mid I^{t-1} \sim D\left(\mu_t, \Sigma_t\right),$$
$$x_t - \mu_t = \varepsilon_t \mid I^{t-1} \sim D\left(0, \Sigma_t\right). \qquad (1)$$

In the following, we do not consider the effects of different mean specifications. The mean could be fixed at sample values, or could be based on a variety of time series models. The relevant issue is that, for each pair of covariance models we compare, the mean models are identical.

Two definitions are given below in order to emphasize the approach taken in the paper:

Definition 1: *The long run solution of a conditional covariance (correlation) model is given by the unconditional expectation of the dynamic conditional covariance (correlation).*

We present below two illustrative examples based on the simplest GARCH and BEKK models. These examples will be also used in the following[1].

Example 1: GARCH(1,1)

Consider the simple GARCH(1,1) model for asset *i*:

$$\sigma_{i,t}^2 = \omega_i + \alpha_i \varepsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2, \quad i = 1, \ldots, k, \qquad (2)$$

where, under normality, $\sigma_{i,t}^2$ is the asset *i* conditional variance. It can be easily shown that the unconditional variance (the long run solution) of the model is given by:

[1] These examples are intended to be illustrative of the arguments, rather than exhaustive.

$$\bar{\sigma}_i^2 = \omega_i \left(1 - \alpha_i - \beta_i\right)^{-1}. \blacksquare \tag{3}$$

Example 2: Scalar BEKK

Consider the Scalar BEKK model of Ding and Engle (2001), which is given as

$$\Sigma_t = CC' + \alpha \varepsilon_{t-1} \varepsilon_{t-1}' + \beta \Sigma_{t-1}. \tag{4}$$

The unconditional covariance matrix of the model is

$$E\left[\Sigma_t\right] = E\left[CC' + \alpha \varepsilon_{t-1}\varepsilon_{t-1}' + \beta \Sigma_{t-1}\right] = CC' + \alpha E\left[\varepsilon_{t-1}\varepsilon_{t-1}'\right] + \beta E\left[\Sigma_{t-1}\right],$$

$$E\left[\varepsilon_{t-1}\varepsilon_{t-1}'\right] = E\left[\Sigma_{t-1}\right] = E\left[\Sigma_t\right],\ E\left[\Sigma_t\right] = \bar{\Sigma} = CC'\left(1 - \alpha - \beta\right)^{-1}. \blacksquare \tag{5}$$

The long-run solution for the BEKK model can be derived easily by mimicking the approach traditionally adopted for the GARCH(1,1) model.

Two topics are frequently discussed in the financial econometrics literature regarding covariance/correlation model estimation: the “curse of dimensionality” and “targeting”. The first issue is perceived as the most serious problem in covariance modelling, while the second could be considered as a tool for disentangling the serious problem.

It is known that many fully parameterized conditional covariance models have the number of parameters that increase at an order greater than the number of assets, otherwise known as the “curse of dimensionality”. For example, the most general BEKK model of Engle and Kroner (1995) has parameters increasing with order $O(k^4)$, its commonly used specification increases with order $O(k^2)$, the VECH model parameter number is of order $O(k^4)$, and the fully parameterized DCC model of Engle (2002)[2] increases with order $O(k^2)$ (further comments regarding the numbers of parameters and the cited models will be discussed in the following subsections).

In order to control the growth in the number of parameters, several restricted specifications have been proposed in the literature, such as the scalar and diagonal models presented in Ding

[2] See equations (24) and (25) in Engle (2002).

and Engle (2001), the block structured specifications suggested by Billio, Caporin and Gobbo (2006), Billio and Caporin (2009), Asai, Caporin and McAleer (2009), Bonato, Caporin and Ranaldo (2009), and the parameter restrictions inspired by spatial econometrics concept introduced in Caporin and Paruolo (2009). However, restrictions generally operate on the parameters driving the dynamics, while little can be done regarding the model intercepts which include $O(k^2)$ parameters in both the conditional covariance and correlation models. This still exposes the models to the curse of dimensionality (to the best of our knowledge, Caporin and Paruolo (2009) is the only paper proposing parameter restrictions on the model intercepts).

The "targeting" constraint then becomes useful because it imposes a structure on the model intercept based on sample information. Within "targeting", the constants in the dynamic equations are structured in order to make explicit the long run solution, which is then fixed using a consistent (sample) estimator. As a result, the number of parameters to be estimated by maximizing a conditional log-likelihood function can be reduced substantially. Although targeting could be rigorously applied only to BEKK, in practice it has been inappropriately used only for DCC (we discuss below the appropriateness of targeting in the DCC model). Targeting can be very useful computationally when the number of financial assets is large (say, $k > 20$), but can become essential when the number of assets is large (such as $k > 100$) or very large ($k >> 100$).

Differently from Engle (2002), we define the "targeting" constraint in a more structured way as follows:

Definition 2: *A conditional covariance (correlation) model is "targeted" if and only if the following two conditions are satisfied:*

i) the intercept is an explicit function of the long run covariance (correlation);

ii) the long run covariance (correlation) solution is replaced by a consistent estimator based on the sample covariance (correlation).

Note that condition i) implicitly requires the long run solution of the covariance (correlation) model to be equal to the long run covariance (correlation), and ensures that the long run solution does not depend on any of the parameters driving the model dynamics. Thus, targeting should be distinguished from the imposition of parametric restrictions. Furthermore, condition ii) implies the use of all the available sample data in constructing a consistent estimator of the observed long run covariance (correlation).

The definition of targeting excludes estimating the long run matrices using latent variables. Such exclusion is essential because estimation of latent variables in the conditional volatility literature does not ensure by construction the consistency of the estimator used for the sample covariance (correlation).

Further to the previous remarks, and contrary to what is purported to have been proved in the literature, consistency and asymptotic normality of the estimated parameters of any version of DCC has not yet been established (see the Appendix).

The following examples clarify these aspects.

Example 1 (continued)

In the GARCH(1,1) model of equation (2), there are three parameters to estimate for each asset, namely ω_i, α_i and β_i, i = 1,2, …, k. We know that the long run solution of the model is $\bar{\sigma}_i^2 = \left(1-\alpha_i-\beta_i\right)^{-1}\omega_i$. This result could be used to make explicit the long run variance in the GARCH equation by replacing the conditional variance constant, ω_i, with an alternative expression:

$$\sigma_{i,t}^2 = \left(1-\alpha_i-\beta_i\right)\bar{\sigma}_i^2 + \alpha_i\varepsilon_{i,t-1}^2 + \beta_i\sigma_{i,t-1}^2, \qquad (6)$$

where the long run variance becomes a parameter to be estimated.

Equation (6) is equivalent to the standard GARCH(1,1) model as there are three parameters to be estimated, namely α_i, β_i and $\bar{\sigma}_i^2$. The model is targeted if, in this alternative representation, $\bar{\sigma}_i^2$ is matched with the sample information, in which case there would only be two parameters to be estimated by maximum likelihood, namely α_i and β_i. Therefore, we may estimate the long run variance using the sample variance estimator of $\bar{\sigma}_i^2$, and substitute this into the model, thereby reducing the number of parameters to be estimated (namely, 3 parameters in equation (2) but only 2 parameters in equation (6)). Consistency of the other parameters is not influenced as the sample variance can be consistently estimated. ■

Example 2 (continued)

In a similar manner to Example 1, we can replace the intercept of the BEKK[3] model of equation (4) using the long run solution in (5), thereby obtaining

$$\Sigma_t = \bar{\Sigma}\left(1-\alpha-\beta\right)+\alpha\varepsilon_{t-1}\varepsilon_{t-1}{}'+\beta\Sigma_{t-1}\,. \tag{7}$$

Equation (7), without additional constraints, has two parameters associated with the dynamics and $k(k+1)/2$ in the intercept, $\bar{\Sigma}$ (the parameters in the long run covariance). Targeting implies the use of a sample covariance estimator for $\bar{\Sigma}$ and the maximization of the likelihood function with respect to the parameters α and β (maximization is made conditionally on the estimates of the long run covariance). The introduction of targeting reduces the number of intercept parameters, thereby making estimation feasible, even for large cross-sectional dimensions. However, the model will still be computationally complicated for large *k* because the likelihood evaluation of the model in (7) requires the inversion of a covariance matrix of dimension *k*. ■

Although targeting can be computationally useful in terms of reducing, sometimes dramatically, the number of parameters to be estimated by maximum likelihood, it requires care in terms of the sample estimator that is used. If targeting were to use an inconsistent estimator to reduce the number of parameters, as is typical in the dynamic correlation literature, the resulting estimators will also be inconsistent.

2.2 BEKK models

Engle and Kroner (1995) introduced the BEKK class of multivariate GARCH models. The specification they proposed was sufficiently general to allow the inclusion of special factor structures (see Bauwens et al., 2006). In this paper, we consider the simplest BEKK specification with all orders set to 1:

$$\Sigma_t = CC' + A\varepsilon_{t-1}\varepsilon_{t-1}{}'A' + B\Sigma_{t-1}B'\ , \tag{8}$$

[3] Originally, the use of targeting in multivariate GARCH started from VECH representations, of which BEKK is a special case. Similar examples could obviously be designed for VECH, but we refer here to BEKK in order to introduce notation and results that are used subsequently.

where A and B are $k \times k$ parameter matrices (not necessarily symmetric) and C is a lower triangular parameter matrix. The fully parameterized model includes $2.5k^2+0.5k$ parameters. The conditional covariance matrices are positive definite by construction, and the conditional variances are positive, regardless of the parameter signs. Covariance stationarity of the BEKK model is discussed in Engle and Kroner (1995), together with a representation that is more general than the one given in (8), but which does not seem to have been used in empirical applications.

Fully parameterized BEKK models are feasible only for small values of k, typically less than 10. In order to make the model feasible for large cross-sectional dimensions, two restricted parameterizations have been proposed in Ding and Engle (2001), namely the diagonal and scalar specifications.

In the scalar BEKK model, the parameter matrices A and B in (8) are replaced by $A=diag(\alpha^{1/2})$ and $B=diag(\beta^{1/2})$, leading to the following specification:

$$\Sigma_t = CC' + \alpha\varepsilon_{t-1}\varepsilon_{t-1}' + \beta\Sigma_{t-1}. \tag{9}$$

In the diagonal specification, the parameter matrices A and B are set to be diagonal as $A=dg(a)$ and $B=dg(b)$, so that the model has the following structure:

$$\Sigma_t = CC' + dg(a)\varepsilon_{t-1}\varepsilon_{t-1}'dg(a) + dg(b)\Sigma_{t-1}dg(b) = CC' + (aa')\circ\varepsilon_{t-1}\varepsilon_{t-1}' + (bb')\circ\Sigma_{t-1}. \tag{10}$$

An additional representation of a BEKK-type model may be based on the Hadamard matrix product:

$$\Sigma_t = CC' + A\circ\varepsilon_{t-1}\varepsilon_{t-1}' + B\circ\Sigma_{t-1}. \tag{11}$$

In this case, the parameter matrices A and B must be symmetric and positive definite, and the number of parameters is still $O(k^2)$. Generally, (11) is not estimated directly, but rather by imposing a structure for A and B to ensure positive definiteness (by making A and B equal to the product of triangular matrices, as in the case of the intercept). Positive definiteness of conditional covariance matrices is guaranteed, by construction (see Ding and Engle, 2001).

Finally, we note that the diagonal specification in (10) is a restricted parameterization of the BEKK model in equation (8) and also of the Hadamard BEKK model in equation (11). Similarly, the scalar BEKK model in (9) can also be obtained from (11) by setting $A = \alpha \mathbf{ii}'$ and $B = \beta \mathbf{ii}'$. We also highlight that imposing diagonal A and B matrices in (11) is not considered as this restriction induces constant covariances.

Although it is not necessary to do so, BEKK can be specified with targeting. The introduction of this feature may require appropriate constraints to be imposed at the estimation step in order to guarantee that the covariance matrices are positive definite. As argued in Definition 2, the targeting constraints require two elements: a modification in the model structure, and matching some of the model parameters with appropriate sample estimators.

Define the long-run covariance matrix $E\left[\varepsilon_t \varepsilon_t'\right] = \overline{\Sigma}$, which can be consistently estimated by the corresponding sample estimator. The BEKK equations may be redefined, as follows:

Scalar BEKK with targeting: $$\Sigma_t = \overline{\Sigma} + \alpha\left(\varepsilon_{t-1}\varepsilon_{t-1}' - \overline{\Sigma}\right) + \beta\left(\Sigma_{t-1} - \overline{\Sigma}\right); \tag{12}$$

Diagonal BEKK with targeting: $$\Sigma_t = \overline{\Sigma} + \left(aa'\right) \circ \left(\varepsilon_{t-1}\varepsilon_{t-1}' - \overline{\Sigma}\right) + \left(bb'\right) \circ \left(\Sigma_{t-1} - \overline{\Sigma}\right); \tag{13}$$

BEKK with targeting: $$\Sigma_t = \overline{\Sigma} + A\left(\varepsilon_{t-1}\varepsilon_{t-1}' - \overline{\Sigma}\right)A' + B\left(\Sigma_{t-1} - \overline{\Sigma}\right)B'; \tag{14}$$

Hadamard BEKK with targeting: $$\Sigma_t = \overline{\Sigma} + A \circ \left(\varepsilon_{t-1}\varepsilon_{t-1}' - \overline{\Sigma}\right) + B \circ \left(\Sigma_{t-1} - \overline{\Sigma}\right). \tag{15}$$

In all of these specifications, it follows that $E\left[\Sigma_t\right] = \overline{\Sigma}$, as $E\left[\varepsilon_{t-1}\varepsilon_{t-1}'\right] = \overline{\Sigma}$ and $E\left[\Sigma_{t-1}\right] = \overline{\Sigma}$. However, positive definiteness of the conditional covariance matrices must be imposed at the estimation step by constraining the matrix of intercepts in the model, otherwise the estimates cannot be interpreted as covariance matrices.

For the BEKK, diagonal BEKK and Hadamard BEKK models, we can guarantee positive definiteness of the conditional covariance matrices by imposing positive definiteness of $\overline{\Sigma} - A\overline{\Sigma}A' - B\overline{\Sigma}B'$, $\overline{\Sigma} - dg\left(a\right)\overline{\Sigma}dg\left(a\right) - dg\left(b\right)\overline{\Sigma}dg\left(b\right)$ and $\overline{\Sigma} - A \circ \overline{\Sigma} - B \circ \overline{\Sigma}$, respectively. In the scalar case, the inequality constraint $\alpha + \beta < 1$ imposes positive definiteness of the conditional covariances.

Although the constraints may seem to be quite simple, their computational complexity is extremely relevant, in particular, when the cross-sectional dimension is simply moderate rather than high. In fact, imposing positive definiteness of the intercepts of (13), (14) or (15) results in a set of highly non-linear constraints on the parameters. One way of imposing positive definiteness is through the imposition of positivity of the eigenvalues of the intercepts. However, such a constraint is non-linear in the parameters and is extremely complicated, except for the scalar case.

In addition, covariance stationarity constraints should be taken into account. These are generally simple in restricted specifications, as shown by Engle and Kroner (1995). However, in fully parameterized cases, these additional constraints significantly increase the computational complexity of the model.

Other than for the scalar case, it is clear that imposing positive definiteness and covariance stationarity for various versions of BEKK is extremely complicated when there is more than one asset. The different degrees of complexity of several BEKK models can also be drawn from Table 1, where columns 3 to 5 report the numbers of parameters for each specification in a general representation, and also for the cases *k=10* and *k=100*.

2.3 DCC models

The Dynamic Conditional Correlation (DCC) model was introduced by Engle (2002) as a generalization of the Constant Conditional Correlation (CCC) model of Bollerslev (1990). In this case, the focus is on the separate modeling of the conditional variances and conditional correlations.

The covariance matrix is decomposed as follows:

$$\Sigma_t = D_t R_t D_t \,, \tag{16}$$

$$D_t = diag\left(\sigma_{1,t}, \sigma_{2,t}, \ldots \sigma_{k,t}\right), \tag{17}$$

$$R_t = \bar{Q}_t^{-1/2} Q_t \bar{Q}_t^{-1/2}, \qquad \bar{Q}_t = dg\left(Q_t\right), \tag{18}$$

where D_t includes the conditional volatilities which are modeled by a set of univariate GARCH equations (see Bollerslev (1990) and Engle (2002)). The dynamic correlation matrix, R_t , is not explicitly driven by a dynamic equation, but is derived from a standardization of a different matrix Q_t which has a dynamic structure. The form of Q_t determines the model complexity and the feasibility in large cross-sectional dimensions.

Several specifications have been suggested for Q_t. We present here the most simple specifications, which can be matched with the BEKK cases illustrated in equations (8)-(11). The DCC model (or Hadamard DCC) is given in Engle (2002) as:

$$Q_t = S + A \circ \left(D_{t-1}^{-1} \varepsilon_{t-1} \varepsilon_{t-1}' D_{t-1}^{-1} - S \right) + B \circ \left(Q_{t-1} - S \right), \tag{19}$$

where A and B are symmetric parameter matrices and S is a referred to as a long run correlation matrix.[4] As distinct from standard practice, we maintain explicitly in the model the dependence on the conditional variances. This model has parameter numbers of order $O(k^2)$, meaning that the model is affected by the curse of dimensionality. Notably, the model has been proposed in the literature directly with a targeting-like constraint, thereby highlighting the presence of a long run component.

The Hadamard DCC (HDCC) model without targeting has the following structure:

$$Q_t = CC' + A \circ D_{t-1}^{-1} \varepsilon_{t-1} \varepsilon_{t-1}' D_{t-1}^{-1} + B \circ Q_{t-1}, \tag{20}$$

where C is a lower triangular matrix whose diagonal elements are constrained in order to ensure CC' is a correlation matrix. Considering now the alternative specifications for Q_t, we first highlight that diagonal specifications cannot be used for the GDCC model, as can be used for the Hadamard BEKK.

An alternative fully parameterized model, the Generalized DCC (GDCC) specification, is considered in Cappiello, Engle and Sheppard (2006). The dynamic equation driving the conditional correlation matrix is given for the cases with and without targeting, respectively, as follows:

$$Q_t = S + A \left(D_{t-1}^{-1} \varepsilon_{t-1} \varepsilon_{t-1}' D_{t-1}^{-1} - S \right) A' + B \left(Q_{t-1} - S \right) B', \tag{21}$$

$$Q_t = CC' + A D_{t-1}^{-1} \varepsilon_{t-1} \varepsilon_{t-1}' D_{t-1}^{-1} A' + B Q_{t-1} B', \tag{22}$$

[4] Ding and Engle (2001) introduce a similar specification for the dynamic covariances.

where A and B are parameter matrices (not necessarily symmetric), while S and C are as in equations (19) and (20). The GDCC model has parameter numbers increasing with order $O(k^2)$, as in the case of the Hadamard DCC. However, despite the introduction of the so-called correlation targeting, the two models, Hadamard DCC and Generalized DCC, are infeasible with large cross sectional dimensions because the parameter matrices A and B in both models include $O(k^2)$ parameters.

Two major restricted specifications may be considered, namely the diagonal and scalar models. Notably, as in the BEKK model, the scalar representation is a special case of both the HDCC and GDCC models, while the diagonal specification of GDCC may be associated with a restricted HDCC model.

We report the alternative DCC specifications in the following equations:

- Scalar DCC with targeting: $$Q_t = S + \alpha\left(D_{t-1}^{-1}\varepsilon_{t-1}\varepsilon_{t-1}{}' D_{t-1}^{-1} - S\right) + \beta\left(Q_{t-1} - S\right); \quad (23)$$

- Scalar DCC without targeting: $$Q_t = CC' + \alpha D_{t-1}^{-1}\varepsilon_{t-1}\varepsilon_{t-1}{}' D_{t-1}^{-1} + \beta Q_{t-1}; \quad (24)$$

- Diagonal DCC with targeting: $$Q_t = S + \left(aa'\right)\circ\left(D_{t-1}^{-1}\varepsilon_{t-1}\varepsilon_{t-1}{}' D_{t-1}^{-1} - S\right) + \left(bb\right)\circ\left(Q_{t-1} - S\right); \quad (25)$$

- Diagonal DCC without targeting: $$Q_t = CC' + \left(aa'\right)\circ\left(D_{t-1}^{-1}\varepsilon_{t-1}\varepsilon_{t-1}{}' D_{t-1}^{-1}\right) + \left(bb\right)\circ Q_{t-1}, \quad (26)$$

where α and β are scalars, a and b are vectors, while C and S are as in equations (19) and (20), respectively.

The most frequently estimated version of DCC (put simply, the DCC model) is what we will call the scalar DCC model, for purposes of strict comparability with its scalar BEKK counterpart. Note that the models without targeting require the joint estimation of all the parameters, including the intercept. As a result, if targeting is excluded, all models are affected by the curse of dimensionality. This is shown in Table 1, where the parameter dimension for DCC without targeting is comparable to that of the standard BEKK models.

However, we note that imposing targeting in all DCC specifications is counterintuitive since the Q_t are then standardized to obtain dynamic conditional correlations. A so-called targeting was included in DCC as a tool for the reduction of the number of parameters, given the suggestion of estimating the S matrix by the sample correlation matrix, so that A and B can then be estimated by maximum likelihood, conditional on the value assigned to S.

Furthermore, given Definition 2, we can state that the DCC model cannot be targeted since the long-run solution of Q_t is not the unconditional correlation matrix, as shown by Aielli (2009). In fact, $Q_t \neq E\left[D_t^{-1}\varepsilon_t\varepsilon_t' D_t^{-1} \mid I^{t-1}\right]$, due to the standardization. The source of this common misunderstanding, as shown by Aielli (2009), is the interpretation of the DCC dynamic equation as a BEKK (or VECH) specification, where targeting is possible. As it is not a BEKK process, the unconditional expectation of the DCC innovations is not the matrix S. As a further consequence, point ii) of Definition 2 is not satisfied. In summary, the concept of targeting outlined in Definition 2 cannot be applied to the DCC specifications listed previously. For this reason, we suggest that such a concept be called "approximate" targeting.[5]

Aielli (2009) also shows that the sample correlation is an inconsistent estimator of S, thereby eliminating the advantage of targeting as a tool for controlling the curse of dimensionality for DCC models. The author also proposes a corrected or consistent DCC model (called cDCC), thereby resolving the lack of consistency associated with DCC. The model introduced by Aielli (2009) in its scalar version modifies only the correlation dynamics, as follows:

$$Q_t = S + \alpha\left(\bar{Q}_{t-1}^{1/2} D_{t-1}^{-1}\varepsilon_{t-1}\varepsilon_{t-1}' D_{t-1}^{-1}\bar{Q}_{t-1}^{1/2} - S\right) + \beta\left(Q_{t-1} - S\right). \tag{27}$$

where $\bar{Q}_t = dg\left(diag\left(Q_t\right)\right)$. In the cDCC model, S, is a symmetric positive definite matrix, with ones along the main diagonal, As S is neither a correlation matrix nor an unconditional correlation of the innovations, it cannot be replaced by the sample estimator of the correlations of $D_{t-1}^{-1}\varepsilon_{t-1}$. Therefore, targeting, as defined in Definition 2, cannot be obtained. Noting that S is the covariance of $\eta_t = \bar{Q}_{t-1}^{1/2} D_{t-1}^{-1}\varepsilon_{t-1}$, Aielli (2009) suggests an alternative estimation method based on a profile-likelihood approach which recovers the innovations, η_t, and replace S with the sample covariance estimator of η_t. Given that these innovations are not observable, we label this approach "implicit targeting".

[5] We are indebted to a referee for a very useful suggestion.

3. Asymptotic Theory

Several papers have purported to establish the consistency and asymptotic normality of the Quasi Maximum Likelihood Estimation (QMLE) of BEKK and DCC. Apart from two papers that have proved consistency and asymptotic normality of BEKK and VECH, albeit under high-order stated but parametrically untestable assumptions, the proofs for DCC have typically being based on unstated regularity conditions. When the regularity conditions have been stated, they are untestable or irrelevant for the stated purposes.

Both DCC and BEKK require the imposition of parameter constraints to ensure covariance stationarity. The constraints are discussed in Engle and Kroner (1995), and are valid for the Generalized DCC model of Engle (2002). Constraints for the scalar representations have a very simple structure, are identical for targeted BEKK and DCC (with approximate targeting), and are closely related to the constraints needed to achieve a positive variance for BEKK and positive definiteness of the conditional covariance (correlation) matrices in the two models.

For BEKK, Jeantheau (1998) proved consistency under the multivariate log-moment condition. However, the derivation of the log-moment condition requires the assumption of the existence of sixth-order moments, which cannot be parametrically tested. Using the consistency result proved in Jeantheau (1998), Comte and Lieberman (2003) established the asymptotic normality of the QMLE of BEKK under eighth-order moments which, though stated explicitly, cannot be parametrically tested. Finally, Hafner and Preminger (2009) proved asymptotic normality of the VECH model (which nests BEKK) of Engle and Kroner (1995) under the existence of sixth-order moments.

The consistency and asymptotic normality results for Scalar and Diagonal BEKK follow as special cases of the results given above, while those of Hadamard BEKK can be derived similarly by noting that Hadamard BEKK has a companion VECH representation with diagonal parameter matrices. The proofs in Jeantheau (1998), Comte and Lieberman (2003), and Hafner and Preminger (2009) can be generalized to include the BEKK representations where the long run solution of the model enters the intercept explicitly. In such cases, appropriate modifications of the regularity conditions are required. Therefore, the asymptotic theory for BEKK models has been established, albeit under untestable conditions from a parametric point of view. However, we note that moment conditions may be tested using nonparametric approaches, though such as discussion is not the purpose of the present paper.

Ling and McAleer (2003) prove consistency and asymptotic normality under the existence of sixth order moments for a CCC model with a general variance dynamic, subsequently called Extended CCC by He and Terasvirta (2004). More recently, Franq and Zakoian (2010) prove consistency and asymptotic normality of the Extended CCC model used by Ling and McAleer (2003) without imposing any assumption on the observable process, but requiring strict stationarity.

3.1. Do Asymptotic Results Exist for DCC?

The primary appeal of the DCC specification, at least in its scalar incarnation, is supposed to be its computational tractability for very large numbers of financial assets, with two step estimation reducing the computational complexity relative to systems maximum likelihood estimation. This presumption is appropriate if the following three conditions hold: (i) the model can be targeted; (ii) the two step estimators are consistent; and (iii) the number of parameters increases as a power function of the cross-sectional dimension, with an exponent less than or equal to 1.

Point (i), targeting, reduces by $0.5k(k-1)$ the number of parameters to be estimated by QMLE, given that it fixes part of the intercept. Point (ii) ensures that correct inferential procedures can be derived from the estimated parameters and the likelihood function. Finally, point (iii) controls for the parameters in the model dynamics. Conditions (i) and (iii) avoid the curse of dimensionality, while the inclusion of just one of the two previous points (either (i) or (iii)) makes the model feasible only for small dimensional systems (the full model parameters will increase at least with power $O(k^2)$).

Engle (2002) suggests the introduction of targeting (point (i)) and the use of scalar representations (point (iii)), and assumes that the standard regularity conditions yielding consistent and asymptotically normal QML two step estimators are satisfied (point (ii)).

However, Aielli (2009) proved that the two step estimation of DCC models with targeting is inconsistent. In fact, Aielli (2009) showed that the sample correlation estimator is an *inconsistent* estimator of the S matrix appearing in the DCC intercept. As a result, the parameters driving the dynamics cannot be consistently estimated by Quasi Maximum Likelihood (QML), conditional on an inconsistent estimator of S. Moreover, the long run solution of the model is not equal to matrix *S*, as we previously discussed, and cannot be estimated with a sample estimator. These, in turn, eliminate the targeting constraint in point (i) and, as a consequence, makes the parameter number at least of order $O(k^2)$. Furthermore, this also affects the consistency of the QML estimates of the other parameters, as well as their asymptotic distribution, thereby eliminating point (ii). Consequently, all the purported proofs for models with targeting, as presented in Engle (2002) and Engle and Sheppard (2001), must be reconsidered.

The need to introduce an inappropriately called long run solution matrix, *S*, into the estimation step of QML makes DCC (even in the scalar case) inconsistent with its primary intended purpose, namely the computational tractability for large cross sections of assets.

As noted above, Aielli (2009) suggested a correction to the DCC model to resolve the previous inconsistency between the unconditional expectations. We stress that, however, the new model proposed does not allow targeting, as given in Definition 2. Furthermore, the asymptotic results are not fully reported (the author presumes the existence of regularity conditions without actually stating them). It is worth mentioning that Aielli's (2009) model was used in Engle, Sheppard and Shephard (2008), and Engle and Kelly (2009), under the assumption that it included targeting, which is not possible by construction.

Aielli's (2009) results preclude the estimation of DCC with targeting, but this does not affect the DCC specifications without targeting. Hence, the asymptotic properties are still unknown. Clearly, despite the possibility of estimating DCC models in a single step, the curse of dimensionality will always be present as the intercept includes $0.5k(k-1)$ parameters in the long run correlation matrix.

In summary, the purported asymptotic theory for DCC models has simply been stated without formal proofs of the conditions required for the results to hold, and without checking any of the assumptions underlying the general results in Newey and McFadden (1994).

We further note that the structure of the DCC model of Engle (2002) inherently creates a limitation to the development of an asymptotic theory for the model, mainly due to the non-standard dynamic nature of Q_t (it is not the conditional expectation of its innovation, $Q_t \neq E\left[D_t^{-1}\varepsilon_t\varepsilon_t' D_t^{-1} \mid I^{t-1}\right]$). As the cDCC model of Aielli (2009) restores the equality $Q_t = E\left[\bar{Q}_t^{1/2} D_t^{-1}\varepsilon_t\varepsilon_t' D_t^{-1}\bar{Q}_t^{1/2} \mid I^{t-1}\right]$, it is possible that asymptotic results for the cDCC will be feasible.

3.2 Consistent Estimation of Correlations for BEKK

McAleer et al. (2008) showed that scalar BEKK and diagonal BEKK could be derived as a multivariate extension of the vector random coefficient autoregression (RCA) model of Tsay (1987) (see Nicholls and Quinn (1982) for a statistical analysis of random coefficient models). However, BEKK and Hadamard BEKK cannot be derived using the RCA approach.

Caporin and McAleer (2008) showed that a theoretical relation could be derived to compare scalar DCC and BEKK, with and without targeting. They suggested the derivation of conditional correlations from alternative BEKK representations, and referred to the derived model as Indirect DCC (which, despite its name, is not a different model but rather a bi-product of BEKK).

As there is presently no consistency result for DCC when estimated by QML, the theorem below will represent a first contribution to the area. Its advantage will be clarified in the following:

Theorem 1: *The indirect DCC conditional correlations derived from BEKK representations are consistent for the true conditional correlations.*

Proof: The conditional covariance matrix satisfies the decomposition, $\Sigma_t = D_t R_t D_t$. If the dynamic covariances have been estimated by a BEKK model, with or without targeting, they are consistent. The matrices, D_t, contain the conditional volatilities along the main diagonal. In turn, these may be obtained as part of the conditional variance matrix, Σ_t, or from a different univariate or multivariate GARCH model. In all cases, they will include consistent estimates of the conditional volatilities, as given by the results for an appropriate BEKK model, or in Bougerol and Picard (1992), Kristensen and Linton (2004) (for univariate models), and Ling and McAleer (2003) (for VARMA-GARCH specifications). Therefore, the indirect conditional correlations, $R_t = D_t^{-1}\Sigma_t D_t^{-1}$, are given by the product of consistent estimators of the conditional covariance matrices and conditional standard deviations, and are hence consistent.■

The theorem shows how BEKK may be used to obtain consistent estimates of the conditional correlation matrix. The BEKK model may also be used to derive starting values for a full system estimation of DCC models by QML. In this case, the intercept may be calibrated as the sample mean of indirect conditional correlations, while the DCC parameters may be calibrated at the corresponding parameters in a given BEKK model.

An empirical example showing the indirect derivation of dynamic conditional correlations from scalar BEKK estimates is given in Caporin and McAleer (2008).

4. Concluding Remarks

BEKK and DCC are the two most widely used models of conditional covariances and correlations, as developed in Engle and Kroner (1995) and Engle (2002), respectively, in the multivariate GARCH class. Although the two models are similar in many respects, the literature has not yet addressed some critical issues pertaining to these models, namely: clarification of the reasons for BEKK and DCC to co-exist when one model can do virtually everything the other model can do, namely the determination as to why DCC is used to forecast conditional correlations rather than conditional covariances, and why BEKK is used to forecast conditional covariances rather than conditional correlations; examination of the inherent differences between BEKK and DCC; and comparisons of both structural and statistical differences and similarities between the two models.

The primary purpose of the paper has been to examine these issues. For this purpose, we highlighted that BEKK possessed asymptotic properties under untestable moment conditions, whereas the asymptotic properties of DCC have simply been stated under a set of untestable regularity conditions. In addition, we clarified the concept of targeting as a tool for reducing the curse of dimensionality associated with multivariate conditional covariance models. Finally, we provided a result which demonstrated that BEKK could be used to obtain consistent estimates of dynamic conditional correlations, with a direct link to the Indirect DCC model suggested in Caporin and McAleer (2008).

In summary, the paper demonstrated that, from a theoretical perspective, the optimal model for estimating conditional covariances (and thereby also conditional correlations) was the Scalar BEKK model, regardless of whether targeting was used.

An open question still remains, and refers to the appropriate use of the models from an empirical perspective. We believe this issue is a relevant topic to be discussed in a purely empirical setting (for further details, see Caporin and McAleer, 2010). Nevertheless, we note that, when focusing on the empirical use of models, the existence of formal proofs for asymptotic properties is not of central interest, even if the focus is on the feasibility of the models. In such a case, the availability of targeting, either consistent with Definition 2, or approximate, or even implicit, is of fundamental relevance. Both the BEKK and DCC specifications could be estimated with a form of targeting, thereby making both families feasible even for a large number of assets (perhaps after appropriate and strong constraints have been imposed).

However, we note that the inconsistency shown by Aielli (2010) would suggest the use of cDCC instead of the DCC model of Engle (2002) (see also Engle et al., 2008, and Engle and Kelly, 2009). Then, between BEKK and cDCC, the final choice should be based on model performance within the appropriate framework in which they are used (such as covariance or correlation forecasting, risk monitoring, or portfolio allocation, to cite the most relevant), with the series and sample data of interest. From an empirical point of view, it would not seem to be possible to provide an appropriate judgment a priori as to which of BEKK and cDCC is to be preferred.

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Table 1: Model dimension and asymptotic properties

Model	Targeting	Two step	Number of parameters			Consistency	Asymptotic Normality
			K assets	K=10	K=100		
Scalar BEKK	No	No	$0.5k(k+1)+2$	57	5052	Yes	Yes
Diagonal BEKK	No	No	$0.5k(k+1)+2k$	75	5250	Yes	Yes
BEKK	No	No	$0.5k(k+1)+2k^2$	255	25050	Yes	Yes
Hadamard BEKK	No	No	$1.5k(k+1)$	165	15150	Yes	Yes
Scalar BEKK	Yes	No	2	2	2	Yes	Yes
Diagonal BEKK	Yes	No	$2k$	20	200	Yes	Yes
BEKK	Yes	No	$2k^2$	200	20000	Yes	Yes
Hadamard BEKK	Yes	No	$k(k+1)$	110	10100	Yes	Yes
Scalar DCC	Yes	Yes	2	2	2	No	No
Diagonal DCC	Yes	Yes	$2k$	20	200	No	No
Generalized DCC	Yes	Yes	$2k^2$	200	20000	No	No
Hadamard DCC	Yes	Yes	$k(k+1)$	110	10100	No	No
Scalar DCC	No	Yes	$0.5k(k-1)+2$	47	4952	?	?
Diagonal DCC	No	Yes	$0.5k(k-1)+2k$	65	5150	?	?
Generalized DCC	No	Yes	$0.5k(k-1)+2k^2$	245	24950	?	?
Hadamard DCC	No	Yes	$0.5k(k-1)+k(k+1)$	155	15050	?	?

Note: For DCC models, the number of parameters does not include the univariate GARCH parameters, namely *3K* at a minimum, that are estimated for each asset. The number of parameters for DCC is higher if various asymmetric and leverage-based univariate models were to be used in conjunction with DCC.

APPENDIX: Review of the purported proofs of consistency and asymptotic normality of various DCC models

Engle and Sheppard (2001)	The authors assume, but do not verify, that the standard regularity conditions required for two step GMM to yield consistent and asymptotically normal estimators, as given, for example, in Newey and McFadden (1994), are satisfied for the DCC model. This ignores the fact that temporal dependence of correlations was not considered in Newey and McFadden (1994).
Ding and Engle (2001)	The authors discuss estimation of various multivariate conditional covariance (correlation) models, without discussing their statistical properties. Two diagnostic checks are presented, without establishing their statistical properties, and the tests are evaluated using Monte Carlo simulations.
Engle (2002)	The author assumes "reasonable regularity conditions" and "standard regularity conditions" (p. 342), without stating them, and refers to the theoretical results in Engle and Sheppard (2001) (see point 1) above).
Cappiello, Engle and Sheppard (2006)	The authors develop an extension of the DCC model to incorporate asymmetries, but do not establish the asymptotic properties of the estimators.
Aielli (2009)	The author makes the following statements: "We assume that QL regularities are satisfied. Basically, this requires correct specification and identification of the first two conditional moments …" (p.10); "a vector of QML estimators, then, it is consistent by assumption of QL regularities." (p. 11). Thus, the author assumes that the typical regularity conditions are satisfied, without stating them.
Engle, Shephard and Sheppard (2008)	The authors refer to Aielli's (2009) model and estimation method, but not to his purported proofs of consistency and asymptotic normality. Moreover, they do not refer to the main result in Aielli (2009), which is the inconsistency of QMLE for Engle's (2002) scalar DCC model. In addition, it seems that the authors have used Aielli's model with targeting, which is impossible, by construction. The authors purport to prove consistency and asymptotic normality. However, Theorem 1 for consistency is not a proof of consistency of the estimator of the appropriate parameter, while Theorem 2 for asymptotic normality actually assumes consistency of the estimator of the appropriate parameter (which was not proved in Theorem 1).
McAleer et al (2008)	In comparison with the purported proofs of consistency and asymptotic normality of the QML estimators of the DCC parameters in the literature, McAleer et al. (2008) develop a generalized autoregressive conditional correlation (GARCC) model when the standardized residuals for each asset follow a multivariate random coefficient autoregressive (RCA) process. The scalar and diagonal versions of BEKK are also shown to be special cases of a multivariate RCA process. As a multivariate generalization of Tsay's (1987) RCA model, GARCC provides a motivation for the conditional correlations to be time varying. Although GARCC is non-nested with respect to the DCC model, owing to different non-nested parametric restrictions being imposed in the two models, it is shown that two special cases of GARCC are virtually identical to the scalar and Hadamard versions of DCC. The analytical forms of the sufficient conditions for the existence of moments are derived, and the sufficient conditions for the asymptotic properties of the QML estimators are established.