KIER DISCUSSION PAPER SERIES

KYOTO INSTITUTE
OF
ECONOMIC RESEARCH

"Optimal taxation and constrained inefficiency
in an infinite-horizon economy with incomplete markets"

Piero Gottardi   Atsushi Kajii$_z$ Tomoyuki Nakajima

KYOTO UNIVERSITY

KYOTO, JAPAN

Optimal taxation and constrained inefficiency
in an infinite-horizon economy with incomplete markets[*]

Piero Gottardi[†]   Atsushi Kajii[‡]   Tomoyuki Nakajima[§]

May 8, 2011

Abstract

We study the dynamic Ramsey problem of finding optimal public debt and linear taxes on capital and labor income within a tractable infinite horizon model with incomplete markets. With zero public expenditure and debt, it is optimal to tax the risky labor income and subsidize capital, while a positive amount of public debt is welfare improving. A steady state optimality condition is derived which implies that the tax on capital is positive, when savings are sufficiently inelastic to returns. A calibration of our model to the US economy indicates positive optimal taxes and a small but positive optimal debt level.

Keywords: incomplete markets; Ramsey equilibrium; optimal taxation;
                optimal public debt; constrained inefficiency.

JEL Classification numbers: D52; D60; D90; E20; E62; H21; O40.

[†]European University Institute. Email: gottardi@unive.it.

[‡]Kyoto University. Email: kajii@kier.kyoto-u.ac.jp.

[§]Kyoto University and Canon Institute for Global Studies. Email: nakajima@kier.kyoto-u.ac.jp.

1 Introduction

In this paper we study how the government should tax capital and labor income and issue debt when individuals face uninsurable idiosyncratic shocks to their labor income. The purpose of this paper is to explore the basic principles for optimal linear taxation and debt finance in a dynamic incomplete markets model.

This question has been extensively studied in the complete-markets/representative-agent frameworks,[1] but it is still not well understood for the case in which asset markets are incomplete. For instance, it is often argued that capital should be taxed with incomplete markets, because capital tends to be 'over-accumulated' in such an economy provided that individuals are prudent.[2] But as clarified by Gottardi, Kajii and Nakajima (2009), whether or not capital should be taxed in an incomplete-markets economy has *nothing* to do with whether or not its equilibrium savings are larger than in the first-best allocation.

The Ramsey problem of finding the sequence of values of taxes and debt that allows to finance a given flow of government expenditure such that at the associated competitive equilibrium consumers welfare is maximal needs to be solved explicitly to properly understand the issue. The first one to pose this problem was Aiyagari (1995) in a seminal contribution, where he showed, in a specific environment, that if the tax rate on capital income is zero, capital accumulation will be unbounded.[3] He also showed, building on this fact, that the tax on capital is positive at an optimal steady state, where government expenditure is optimally chosen.[4] Some important progress has then been made by Aiyagari and McGrattan (1998), İmrohoroğlu (1998), Domeij and Heathcote (2004), and Conesa, Kitao and Krueger (2009), among others. Nevertheless, as far as we are aware of, none of these papers derives a solution of the complete dynamic Ramsey problem, since it is a very difficult task due to the 'curse of dimensionality' inherent in incomplete markets models. Additional simplifying

---

[1]See, among others, Stokey and Lucas (1983), Judd (1985), Chamley (1986), Zhu (1992), Chari, Christiano, and Kehoe (1994), Jones, Manuelli, and Rossi (1997).

[2]Ljungqvist and Sargent (2004, pp. 535-536) is an example of this type of argument: "(T)he optimal capital tax in a heterogeneous-agent model with incomplete insurance markets is actually positive, even in the long run. A positive capital tax is used to counter the tendency of such an economy to overaccumulate capital because of too much precautionary saving."

[3]This result relies, for instance, on the assumption that consumption and leisure are perfect substitutes. As shown by Marcet, Obiols-Homs, and Weil (2007), when consumption and leisure are imperfect substitutes, capital accumulation will be bounded even though the interest rate equals the time discount rate.

[4]This feature, which differs from the public finance problem considered here and in most of the literature, where public expenditure is exogenously given, has the important implication that at an optimal steady state the pre-tax rate of return on capital is equal to the discount factor.

assumptions are made to derive a solution, which are not innocuous at all. For instance, a typical assumption made in this literature is that the social planner merely maximizes the average welfare at the steady state. But then the solution to such a problem ignores the welfare gains/losses during the transition to the steady state, and can in principle be very different from the solution to the Ramsey problem which takes them into account, as we shall see in our analysis. A significant exception in this regard, among the papers listed above, is Domeij and Heathcote (2004), who analyze the transition but restrict fiscal policies to be such that tax rates are constant over time. As a consequence the optimal path of government debt, ensuring the optimal intertemporal allocation of the tax burden, cannot be investigated. More generally, the relationship between optimal taxes and optimal debt has not been properly analyzed in this literature.

We focus here on a highly stylized model of incomplete markets, in which the optimal taxation and debt problem can be analyzed in a tractable way. That is, we sacrifice the generality of the model for the completeness of the solution. As a result, we show in a very transparent fashion when and why capital and labor should be taxed/subsidized, and whether or not the government should borrow. We derive two general principles about the nature of optimal taxation under incomplete markets, which are somewhat different in nature from the findings of this literature. Furthermore, because of the tractability of the model, we obtain a numerical solution to the dynamic Ramsey problem of finding the optimal path of taxes and debt without the need to make ad hoc assumptions.

Now we shall outline our findings. Our model is an incomplete-markets version of the endogenous growth model studied by Jones and Manuelli (1990), and is closely related to Krebs (2003). Individuals have access to three types of assets: bonds, physical capital, and human capital. The first two assets are risk-free, but the accumulation of human capital is subject to idiosyncratic shocks. As in Aiyagari (1995) and the rest of the literature mentioned above, we restrict attention to linear taxes on labor and capital income.[5] Our model differs from the standard one in that there is no labor/leisure choice and the labor productivity of an individual is determined by his/her investment in human capital. Nevertheless, it shares some basic properties with the standard macroeconomic model of incomplete markets (that is, the so-called 'Bewley model'). First, the capital/labor ratio in the laissez-faire equilibrium without taxes is greater than the first-best level. Second, as shown in Section 3 of the paper, at the laissez-faire competitive equilibrium with no government purchases nor taxes a small reduction of capital improves welfare. Hence the capital/labor ratio is too high in

---

[5]This is for the sake of tractability and clarity. An important line of future research is to examine how robust our findings are when some non-linearities in taxes are allowed, in accord with the information available to the government over consumers' trades and characteristics, as in Kocherlakota (2005). Fukushima (2010) considers optimal non-linear taxation in the model of Conesa, Kitao and Krueger (2009).

this sense in our model, as it is in the two-period version of the Bewley model discussed by Davila, Hong, Krusell, and Ríos-Rull (2005).

We investigate the properties of the optimal taxes by considering first, in Section 4, the simpler case where the government expenditure is zero and its budget is balanced in each period. Thus the level of debt is fixed at zero by assumption. In this case, we show that, when the consumers' savings and portfolio choices exhibit standard comparative statics properties with respect to prices, *subsidizing* the interest income and *taxing* the wage income makes everyone better off, at the laissez-faire equilibrium. That is, the government should *increase* the capital/labor ratio. This might appear puzzling at first since as argued above the capital/labor ratio is inefficiently high, but there is a simple economic reason: the welfare will be improved if risky income is insured, so the first principle of optimal taxation is that the government should tax risky sources of income to subsidize less risky sources of income. In our model, it is the labor productivity which is risky, and taxing the wage income reduces the after-tax price of labor. Under the balanced budget of the government, the revenue from taxing the wage income is distributed back to the private sector by subsidizing the interest income. The benefit from this direct insurance effect outweighs a possible distortion of the capital/labor ratio.

Next we allow the government to borrow and lend and consider the case where public expenditure is nonzero, setting up the dynamic optimal taxation and debt problem in Section 5. In this problem, if the government wished to increase the steady state welfare only, the debt level should be large and negative. So one readily sees that the transition to the steady state must be important. We first show that when government's consumption is small enough, the government should borrow,[6] so a balanced budget is almost never optimal. To understand this result, recall that the optimal capital tax is negative in the case of a balanced budget without government purchases, and so the after-tax rate of return on physical capital is greater than its before-tax rate. In addition, because of risk aversion, the after-tax rate of return on human capital is greater than that on physical capital. Consequently, the average rate of return in the private sector is greater than that of the government, i.e., the private sector is "more efficient." Hence there will be a gain if the government borrows to reduce taxes at the margin to encourage the private sector to accumulate more wealth.[7]

How much should then the government borrow and tax at the optimum? To answer this ques-

---

[6]We assume that the amount of taxes paid by each individual in a given period depends only on his/her current labor and capital incomes. This makes our question here well defined. See Bassetto and Kocherlakota (2004) for more on this issue.

[7]In our model, in order to determine the optimal allocation of tax burdens, one also needs to take into account how the timing of taxes affects the saving rate. However, as we shall see, this effect vanishes when evaluated locally around the equilibrium obtained under the balanced budget restriction.

tion we derive a condition that the tax rates must satisfy at the steady state of the optimal tax equilibrium. Roughly speaking, the condition says that under the optimal tax and debt policy, the rate of return on government debt must be equated with the average return on savings earned in the private sector, after adjusting for the effect of public debt on the saving rate. Intuitively, this equality must hold for the tax burdens to be efficiently allocated intertemporally since at the margin, a transfer of wealth from the government to the private sector results in a direct effect of increasing private sector wealth as well as an indirect effect through a possible change in the saving rate.

The condition implies in particular, when the intertemporal elasticity of substitution is one and hence the saving rate is invariant with respect to the rates of return, that there should be no difference in the rates of return between the government and the private sector at the steady state of the optimal tax and debt equilibrium. Since the after-tax rate of return on human capital must be greater, as argued above, than that on physical capital, the parity of returns holds only when the tax rate on physical capital is strictly positive. Consequently, the optimal tax rate on physical capital is strictly positive in the long run. This observation reveals the second principle for optimal taxation: for efficient intertemporal allocation of tax burdens, the government should tax the riskless asset in the private sector to keep the returns of the government bond in parity with the average private sector returns.

Finally we calibrate our model to the U.S. economy in Section 6. Thanks to the tractability of our model, we are able to obtain a numerical solution to the optimal taxation problem in a relatively easy way. The parameter for the idiosyncratic income risk is chosen based on the evidence provided by Meghir and Pistaferri (2004), and the rest of the parameters are set as in Chari, Christiano and Kehoe (1994), with a positive level of government consumption, equal to 18 percent of GDP. We find that the presence of idiosyncratic labor-income risks significantly affects the optimal tax rates and the optimal amount of the government debt. Under our baseline calibration with the coefficient of risk aversion equal to three, we find that both the optimal steady state tax on capital and that on labor are positive and significant, while the optimal level of debt is positive but close to zero. Also, all the adjustment in fiscal policy to reach the steady state is concentrated in one period, where tax rates are quite high in order to bring down the debt ratio to its steady state level. Also, we find that the welfare gain of adopting the Ramsey policy amounts to a permanent increase in consumption of all individuals by 0.85 percent. We emphasize that if we only compare the steady state welfare level, the welfare gain from adopting the Ramsey policy is much higher (8.7 percent), as the substantial welfare loss that occurs during the transition to the steady state is ignored. This clearly reveals the importance of taking the transition into account in the analysis.

Our finding regarding the optimal debt level are quite sensitive to the degree of relative risk aversion as well as to the magnitude of the uninsurable risk. For instance, the steady state debt-output ratio is about -100 percent when the coefficient of risk aversion is one, and it is about 200 percent when the coefficient of risk aversion is 9. Moreover, it is negative and large when there is no idiosyncratic risk, as in the complete market case.

2 Model economy

In this section we describe the model economy. Time is discrete and indexed by $t = 0, 1, 2, \ldots$. The economy consists of a government, a continuum of individuals, and perfectly competitive firms that produce a single, homogeneous product. There is no aggregate risk in the model and so all the aggregate variables are non-random. Consequently, the market clearing prices are non-random throughout. The government needs to purchase an exogenously given amount of output in each period, which is financed by issuing debt and collecting taxes. Regarding taxes, we restrict attention to linear taxes on wages and interests. Negative taxes (i.e., subsidies) are also allowed. In this section we describe a competitive equilibrium associated with a given fiscal policy. The optimal fiscal policy is discussed later.

2.1 Firms

In each period, a single commodity is produced by perfectly competitive firms, using physical and human capital as inputs. All firms have identical production technology, described by a Cobb-Douglas production function:

$$y = F(k, h) = A k^{\alpha} h^{1-\alpha}, \tag{1}$$

where y is the level of output, A is a constant, k is the input of physical capital, and h is the input of human capital. In particular, there is no productivity shock in the technology.

Let K_{t-1} and H_{t-1} denote, respectively, the aggregate stock of physical and human capital at the beginning of period t. Market clearing requires that the quantities of the factors demanded by the firms equal to these values. Hence, the aggregate amount of output produced in period t is therefore given by

$$Y_t = F(K_{t-1}, H_{t-1}) = A K_{t-1}^{\alpha} H_{t-1}^{1-\alpha}.$$

The profit maximization condition with market clearing implies that the before-tax rental rate of physical capital in period t equals the marginal product of physical capital in that period, $F_{k,t}$, where

$$F_{k,t} \equiv \frac{\partial F(K_{t-1}, H_{t-1})}{\partial K_{t-1}}$$

Similarly, the before-tax wage rate per efficiency unit of labor in period t is the marginal product of human capital:

$$F_{h,t} \equiv \frac{\partial F(K_{t-1}, H_{t-1})}{\partial H_{t-1}}$$

2.2 Individuals

There is a continuum of individuals. In every period, individuals consume the consumption good, supply one unit of raw labor inelastically, and invest in three kinds of assets: risk-free bond, physical capital, and human capital. The level of human capital of each individual determines the "efficiency units" of his/her labor.

Each individual $i \in [0, 1]$ has Epstein-Zin-Weil preferences over random sequences of consumption, which are defined recursively by

$$u_{i,t} = \left\{ (1-\beta)(c_{i,t})^{1-\frac{1}{\psi}} + \beta \left[E_t(u_{i,t+1})^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{\psi}{\psi-1}} , \quad t = 0, 1, ... \tag{2}$$

where $u_{i,t}$ is the level of utility of individual i in period t, E_t is the conditional expectation operator at time t, $c_{i,t}$ is his/her consumption in period t, $\beta \in (0, 1)$ is the discount factor, ψ is the elasticity of intertemporal substitution, and γ is the coefficient of relative risk aversion.

Let $b_{i,t-1}$, $k_{i,t-1}$ and $h_{i,t-1}$ denote, respectively, the quantities of risk-free bond, physical capital, and human capital that individual i holds at the end of period $t-1$. To capture the idea that labor income is subject to uninsurable idiosyncratic shocks, we assume that the human capital of individual i is affected by a random shock parameter, $\theta_{i,t}$, at the beginning of each period t. We assume that $\theta_{i,t}$, $i \in [0, 1]$, $t = 1, ...$, are identically and independently distributed across individuals and across periods, with unit mean. Thus the actual amount of human capital of individual i available at the beginning of each period t is equal to $\theta_{i,t}h_{i,t-1}$. We further assume that the law of large number applies, so that the aggregate stock of human capital at the beginning of period t is not random: that is, the following relation holds with probability one:

$$\int_0^1 \theta_{i,t}h_{i,t-1} \, di = \int_0^1 h_{i,t-1} \, di = H_{t-1}.$$

We suppose that both physical and human capital are accumulated by investing output after the private shock has been observed; that is, the amount of capital is determined first by the time t shock and then by depreciation, and new investment takes place. Let $\iota_{k,i,t}$ and $\iota_{h,i,t}$ denote, respectively, the investment in physical and human capital of individual i in period t. Then the two types of capital evolve as, for $t = 1, 2, ...$,

$$k_{i,t} = \iota_{k,i,t} + (1 - \delta_k)k_{i,t-1} \tag{3}$$

$$h_{i,t} = \iota_{h,i,t} + (1 - \delta_h)\theta_{i,t}h_{i,t-1} \tag{4}$$

where δ_k and δ_h are the depreciation rates of physical and human capital, respectively.

We assume that idiosyncratic shocks $\theta_{i,t}$ are the only sources of uncertainty, hence there is no aggregate uncertainty in the economy. Consequently, the market returns of production factors are not random. Both labor and capital income are subject to linear taxes. It follows that the risk-free bond and physical capital are perfect substitutes and therefore the after-tax rate of return of the physical capital must be equal to the risk free rate in equilibrium. Let $r_{k,t}$ denote the after-tax rental rate of physical capital (and hence the risk-free rate), and w_t the after-tax wage rate. The after-tax gross rates of return on the two types of capital are given in equilibrium by:

$$R_{k,t} = 1 - \delta_k + r_{k,t},$$

$$R_{h,t} = 1 - \delta_h + w_t.$$

Then the flow budget constraint of individual i is written as, for $t = 1, 2, ...,$

$$c_{i,t} + \iota_{k,i,t} + (1 - \delta_k)\, k_{i,t-1} + \iota_{h,i,t} + (1 - \delta_h)\, \theta_{i,t} h_{i,t-1} + b_{i,t} = R_{k,t} k_{i,t-1} + R_{h,t} \theta_{i,t} h_{i,t-1} + R_{k,t} b_{i,t-1} \quad (5)$$

Individuals may borrow so that $b_{i,t}$ can be negative, but the holdings of capital are non-negative: $k_{i,t} \geq 0$ and $h_{i,t} \geq 0$ are required for all periods and under all contingencies.

Let $x_{i,t}$ be the total wealth of individual i at the beginning of period t after the time t shock $\theta_{i,t}$ has been realized: that is,

$$x_{i,t} \equiv R_{k,t}(k_{i,t-1} + b_{i,t-1}) + R_{h,t} \theta_{i,t} h_{i,t-1}$$

The amount of borrowing is restricted by the natural debt limit, that prevents consumers from engaging in Ponzi schemes and in this environment (where the only source of future income is the revenue from the consumers' accumulated human and physical capital) takes the following form:

$$x_{i,t+1} \geq 0, \quad (6)$$

for all periods and at all contingencies.

To sum up, given the initial wealth $x_{i,0} > 0$ and a sequence of prices $\{r_{k,t}, w_t\}_{t=0}^{\infty}$, each individual i maximizes the lifetime utility $u_{i,0}$, defined by (2) subject to the flow budget constraints (7) and the debt limit (6).

This optimization problem can be complex in principle, since individual's choice variables depend on the history of shocks and individuals have different histories of shocks. But thanks to the specification of the utility function in (2) and in particular its homotheticity property, as well as the facts that the current wealth is the discounted present value of the future individual income stream and that shocks are permanent, there is a tractable characterization of the utility maximizing choices, which we shall summarize below.

First, equations (3)-(5) can be combined together to obtain:

$$x_{i,t+1} = (1 - \eta_{c,i,t}) \left\{ R_{k,t+1}(1 - \eta_{h,i,t}) + R_{h,t+1}\theta_{i,t+1}\eta_{h,i,t} \right\} x_{i,t} \tag{7}$$

where

$$\eta_{c,i,t} \equiv \frac{c_{i,t}}{x_{i,t}}$$

$$\eta_{h,i,t} \equiv \frac{h_{i,t}}{b_{i,t} + k_{i,t} + h_{i,t}}$$

with initial condition $x_{i,0} > 0$. That is, the optimization problem can be equivalently written as a problem of choosing a sequence of the rate of consumption out of his/her wealth, $\eta_{c,i,t}$, and the portfolio between the human capital and the riskless assets (physical capital and risk-free bond), $(\eta_{h,i,t}, 1 - \eta_{h,i,t})$ per unit of investment for every $t = 0, 1, ...$, given $x_{i,0}$. By construction, the original choice values are given iteratively starting with $x_{i,0}$ from the following equations, $t = 0, 1, ...$:

$$c_{i,t} = \eta_{c,t}x_{i,t}$$

$$k_{i,t} + b_{i,t} = (1 - \eta_{c,t})(1 - \eta_{h,t})x_{i,t} \tag{8}$$

$$h_{i,t} = (1 - \eta_{c,t})\eta_{h,t}x_{i,t}$$

As is well known,[8] the optimal choice of the portfolio in this type of utility maximization problem is reduced to a static problem which is independent of all the other choice variables. Specifically, define the certainty-equivalent rate of return ρ associated with the after-tax rental rate r_k, after-tax wage rate w, as follows:

$$\rho(r_k, w, \eta_h) \equiv \left\{ E \left((1 - \delta_k + r_k)(1 - \eta_h) + \theta(1 - \delta_h + w)\eta_h \right)^{1-\gamma} \right\}^{\frac{1}{1-\gamma}}. \tag{9}$$

It can be shown that at any time period, for any level of initial wealth hence for any individual, an optimal portfolio is given by a solution to the following maximization problem given the prevailing rates r_k and w:

$$\max_{\eta_h' \geq 0} \rho(r_k, w, \eta_h'). \tag{10}$$

Since $\rho(r, w, \eta_h)$ is strictly concave in η_h, the solution to this maximization problem is unique if it exists. Given this, it is straightforward to verify that we obtain the following simple characterization of utility maximization:

Lemma 1. *Given a sequence of prices, $\{r_{k,t}, w_t\}_{t=0}^{\infty}$, for any individual i, a utility maximizing sequence of portfolio and rate of consumption are characterized by the following rule: for the portfolio,*

---

[8]See, for instance, Epstein and Zin (1991), and Angeletos (2007).

at any time $t = 1, ...,$

$$\eta_{h,t} = \arg\max_{\eta'_h \geq 0} \rho(r_{k,t+1}, w_{t+1}, \eta'_h), \tag{11}$$

and for the rate of consumption,

$$\eta_{c,t} = \left\{ 1 + \sum_{s=0}^{\infty} \prod_{j=0}^{s} \left(\beta^\psi \bar{\rho}_{t+1+j}^{\psi-1} \right) \right\}^{-1}, \tag{12}$$

where $\bar{\rho}_{t+1}$, $t = 0, ...,$ *denotes the optimized certainty-equivalent rate of return between periods* t *and* $t + 1$ *which is*

$$\bar{\rho}_{t+1} \equiv \max_{\eta'_h \geq 0} \rho(r_{k,t+1}, w_{t+1}, \eta'_h), \tag{13}$$

Moreover, the time t *utility level is given by*

$$u_{i,t} = v_t x_{i,t},$$

where

$$v_t^{\psi-1} = (1 - \beta)^\psi + \beta^\psi \rho_{t+1}^{\psi-1} v_{t+1}^{\psi-1} \tag{14}$$

and hence

$$v_t = (1 - \beta)^{\frac{\psi}{\psi-1}} \left\{ 1 + \sum_{s=0}^{\infty} \prod_{j=0}^{s} \left(\beta^\psi \bar{\rho}_{t+1+j}^{\psi-1} \right) \right\}^{\frac{1}{\psi-1}}. \tag{15}$$

Notice in particular that since the right hand sides of (11) and (12) are independent of index i, this result implies that all the individuals in the economy choose the same rate of consumption, $\eta_{c,t}$, and the same portfolio, $\eta_{h,t}$, in each period in equilibrium. The differences across individuals appear in the level of utility, but notice that the level of utility is the level of wealth of the individual multiplied by a common constant v_t.[9] Thus in particular, the expected level of utility of a consumer at any date t is simply his expected level of wealth multiplied by v_t, hence to determine his welfare we only need to find the parameter v_t and his expected wealth.

Note that from (12) and (15) time t utility per wealth v_t and time t consumption share $\eta_{c,t}$ are related as

$$\eta_{c,t} = (1 - \beta)^\psi v_t^{1-\psi}. \tag{16}$$

Finally, we will need to know how the aggregate supplies of the two capitals change as the environment changes to study the effects of government policies. Lemma 1 says that the ratio of the aggregate supplies is determined by a solution to the maximization problem (10). Since ρ is a

---

[9]These properties depend on the fact that, as noticed above, consumers have identical homothetic preferences and their income is given by the revenue from their accumulated wealth, so that a representative consumer exists.

concave function of η_h, an interior solution for (10) is characterized by the first order condition. So define $\Phi : \mathbb{R}_+^3 \rightarrow \mathbb{R}$ by the rule:

$$\Phi(r_k, w, \eta_h) \equiv E\Big[\big\{(1 - \delta_k + r_k)(1 - \eta_h) + \theta(1 - \delta_h + w)\eta_h\big\}^{-\gamma} \tag{17}$$
$$\times \big\{\theta(1 - \delta_h + w) - (1 - \delta_k + r_k)\big\}\Big].$$

Then $\Phi(r_k, w, \eta_h) = 0$ corresponds to the first order condition. Especially for comparative static exercises, we will be concerned with the signs of the derivatives of Φ, which in general depends on the property of the shock variable. We shall assume the following throughout the analysis.

Assumption 1. *For the function $\Phi : \mathbb{R}_+^3 \rightarrow \mathbb{R}$ defined in equation (17), The derivatives of Φ at $\Phi = 0$ have the following signs:*

$$\frac{\partial \Phi}{\partial r_k} < 0, \qquad \frac{\partial \Phi}{\partial w} > 0, \qquad and \qquad \frac{\partial \Phi}{\partial \eta_h} < 0.$$

In fact $\frac{\partial \Phi}{\partial \eta_h} < 0$ readily follows from the concavity of the certainty equivalent function, so the main part of this assumption is $\frac{\partial \Phi}{\partial r_k} < 0$ and $\frac{\partial \Phi}{\partial w} > 0$. It can be shown that these are satisfied when $\gamma \leq 1$. When $\gamma > 1$, they hold under appropriate restrictions on the distribution of θ_i.

These conditions ensure that the consumers' optimal portfolio choice obtains at an interior solution for (10) exists and displays the 'normal' comparative statics properties: $\partial \eta_{h,t}/\partial r_{k,t+1} < 0$ and $\partial \eta_{h,t}/\partial w_{t+1} > 0$.

2.3 Government

The government purchases an exogenously given amount of output, G_t, in each period t. It is financed by collecting taxes and issuing debt. Following the common assumption in the literature, we assume that government purchases do not yield utility to individuals.

Let B_{t-1} be the government debt outstanding at the beginning of period t. Denote by $\tau_{k,t}$ and $\tau_{h,t}$ the effective tax rates on the returns of physical and human capital at time t, respectively. For a given sequence of aggregate stocks, $\{K_t, H_t\}_{t=0}^{\infty}$, the flow budget constraint of the government in period t is given by

$$B_t + \tau_{k,t} F_{k,t} K_{t-1} + \tau_{h,t} F_{h,t} H_{t-1} = G_t + R_{k,t} B_{t-1}, \tag{18}$$

where the initial stock of debt, B_{-1}, is given with $B_{-1} = \int_0^1 b_{i,-1}\, di$.

A fiscal policy $\{\tau_{k,t}, \tau_{h,t}, B_t\}_{t=0}^{\infty}$ is said to be feasible (under $\{K_t, H_t\}_{t=0}^{\infty}$) if the flow budget constraint (18) is satisfied for every $t = 0, 1, ...,$ and

$$\lim_{t \to \infty} \left(\prod_{j=1}^{t} R_{k,j}^{-1} \right) B_t = 0. \tag{19}$$

2.4 Competitive equilibrium

The initial conditions of the economy are $\{b_{i,-1},\ k_{i,-1},\ h_{i,-1},\ \theta_{i,0} : i \in [0,1]\}$, $K_{-1} = \int_0^1 k_{i,-1}\, di$, $B_{-1} = \int_0^1 b_{i,-1}\, di$, and $H_{-1} = \int_0^1 h_{i,-1}\, di$. An allocation is a collection of stochastic processes $\{c_{i,t}, x_{i,t},\ b_{i,t},\ k_{i,t},\ h_{i,t} : i \in [0,1]\}_{t=0}^\infty$, where for each i, $\{c_{i,t},\ x_{i,t},\ k_{i,t},\ h_{i,t}\}_{t=0}^\infty$ are stochastic processes adapted to the filtration generated by the process of idiosyncratic shocks $\{\theta_{i,t}\}_{t=0}^\infty$.

Given the initial conditions and a sequence of government purchases, $\{G_t\}_{t=0}^\infty$, a *competitive equilibrium* is defined by a price system $\{r_{k,t}, w_t\}_{t=0}^\infty$, a fiscal policy $\{\tau_{k,t},\ \tau_{h,t},\ B_t\}_{t=0}^\infty$, and an allocation $\{c_{i,t},\ x_{i,t},\ b_{i,t},\ k_{i,t},\ h_{i,t} : i \in [0,1]\}_{t=0}^\infty$ such that: (a) for each $i \in [0,1]$ $\{c_{i,t},\ x_{i,t},\ k_{i,t}, h_{i,t}\}_{t=0}^\infty$ solves the utility maximization problem, given the price system; (b) firms maximize profits (and the prices faced by consumers reflect taxes); that is,

$$r_{k,t} = (1 - \tau_{k,t})F_k(K_{t-1}, H_{t-1}), \quad \text{and} \quad w_t = (1 - \tau_{h,t})F_h(K_{t-1}, H_{t-1}),$$

for all $t \geq 0$, where

$$K_{t-1} = \int_0^1 k_{i,t-1}\, di,$$

$$H_{t-1} = \int_0^1 \theta_{i,t} h_{i,t-1}\, di = \int_0^1 h_{i,t-1}\, di\ ;$$

(c) all markets clear:

$$C_t + G_t + K_t + H_t = (1 - \delta_k)K_{t-1} + (1 - \delta_h)H_{t-1} + F(K_{t-1}, H_{t-1}), \tag{20}$$

$$B_t = \int_0^1 b_{i,t}\, di \tag{21}$$

where $C_t = \int_0^1 c_{i,t}\, di$; and (d) the government policy is feasible, that is (18) and (19) hold.

Recall that by Lemma 1, for any equilibrium allocation there is an associated sequence of $\{\eta_{ct}, \eta_{h_t}\}_{t=0}^\infty$, which is common across all the individuals. For this reason, the aggregate dynamics of a competitive equilibrium can be succinctly summarized by the average wealth and the sequence $\{\eta_{ct}, \eta_{h_t}\}_{t=0}^\infty$ as follows. Let X_t denote the average amount of wealth at the beginning of period t:

$$X_t \equiv \int_0^1 x_{i,t}\, di$$

Then X_t evolves as

$$X_{t+1} = R_{x,t+1}(1 - \eta_{c,t})X_t,\ \ t = 0, 1, 2, ... \tag{22}$$

where $R_{x,t+1}$ is the equilibrium average rate of return of individual portfolios: for $t = 0, 1, 2, ...$

$$R_{x,t+1} \equiv R_{k,t+1}(1 - \eta_{h,t}) + R_{h,t+1}\eta_{h,t},$$

$$= [1 - \delta_k + (1 - \tau_{k,t})F_{k,t}] (1 - \eta_{h,t}) + [1 - \delta_h + (1 - \tau_{h,t})F_{h,t}] \eta_{h,t}.$$

The aggregate amounts of consumption, physical capital and human capital are given, respectively, as follows: for $t = 0, 1, 2, ...$,

$$C_t = \eta_{c,t} X_t,$$

$$K_t = (1 - \eta_{c,t})(1 - \eta_{h,t}) X_t - B_t,$$

$$H_t = (1 - \eta_{c,t}) \eta_{h,t} X_t.$$

Finally, the average utility level is given for $t = 0, 1, 2, ...$ by

$$U_t = \int_0^1 u_{i,t} \, di = v_t X_t.$$

2.5 Benchmark equilibrium with no taxes

As a benchmark, let us consider the case in which the government does not purchase goods, and does not issue debt nor impose any taxes:

$$G_t = B_t = \tau_{k,t} = \tau_{h,t} = 0, \quad \text{for all } t \geq 0, \quad \text{and} \quad b_{i,-1} = 0, \quad \text{for all } i \in [0, 1]. \tag{23}$$

In this case, the competitive equilibrium has a very simple structure. The aggregate economy is always on a balanced growth path, although each individual's consumption fluctuates stochastically over time.

To see these, first notice that in the benchmark economy with (23), from Lemma 1, $\eta_{h,t}$ must maximize the certainty equivalent function ρ where the rates must be consistent with profit maximization and market clearing. This means that if we set $r_{k,t} = F_k(1 - \eta_{h,t}, \eta_{h,t})$ and $w_t = F_h(1 - \eta_{h,t}, \eta_{h,t})$, the first-order condition for maximization of ρ must be met: that is, for every $t = 0, 1, 2, ...$,

$$\Phi\left[F_k(1 - \eta_{h,t}, \eta_{h,t}), F_h(1 - \eta_{h,t}, \eta_{h,t}), \eta_{h,t}\right] = 0, \tag{24}$$

where Φ is given in (17). Note that $F_k(1, 0) = F_h(0, 1) = 0$ and $F_k(0, 1) = F_h(1, 0) = +\infty$, and that $\lim_{\eta_h \to 0} F_h(1 - \eta_h, \eta_h) h = \lim_{\eta_h \to 1} F_k(1 - \eta_h, \eta_h)(1 - \eta_h) = 0$. Furthermore, under Assumption 1, $\frac{\partial}{\partial \eta_h} \Phi\left[F_k(1 - \eta_h, \eta_h), F_h(1 - \eta_h, \eta_h), \eta_h\right] < 0$ whenever $\Phi = 0$, so it follows there exists a unique $\hat{\eta}_h \in (0, 1)$ that satisfies (24). The uniqueness implies that $\eta_{h,t} = \hat{\eta}_h$ must hold for every t.

Set $\hat{F}_k = F_k(1 - \hat{\eta}_h, \hat{\eta}_h)$, $\hat{F}_h = F_h(1 - \hat{\eta}_h, \hat{\eta}_h)$, and let $\hat{\rho}$ to be the associated certainty-equivalent rate of return:

$$\hat{\rho} = \rho\left(\hat{F}_k, \hat{F}_h, \hat{\eta}_h\right). \tag{25}$$

The argument above together with Lemma 1 yields the following characterization result.[10]

---

[10]Krebs (2003) derived analogous properties in a similar environment.

Proposition 2. *Suppose that Assumption 1 holds, and consider the benchmark economy, satisfying (23). Let $\hat{\eta}_h \in (0,1)$ be the solution to (24), and $\hat{\rho}$ be the associated certainty-equivalent rate of return defined in (25). Then if*

$$\beta^\psi \hat{\rho}^{\psi-1} < 1,$$

a unique competitive equilibrium of the benchmark economy exists, generated by $\hat{\eta}_h$ and $\hat{\eta}_c \equiv 1 - \beta^\psi \hat{\rho}^{\psi-1}$, which are common across i, through (7) and (8). Thus the aggregate variables C_t, K_t, H_t, and X_t all grow at the same rate g_x, which is given by

$$\hat{g}_x \equiv (1 - \hat{\eta}_c)\hat{R}_x$$

where

$$\hat{R}_x \equiv (1 - \delta_k + \hat{F}_k)(1 - \hat{\eta}_h) + (1 - \delta_h + \hat{F}_h)\hat{\eta}_h$$

The level of utility of each individual evolves as

$$u_{i,t} = \hat{v} x_{i,t}$$

where

$$\hat{v} \equiv \left[\frac{(1 - \beta)^\psi}{1 - \beta^\psi \hat{\rho}^{\psi-1}} \right]^{\frac{1}{\psi-1}}.$$

By the uniqueness property established in Proposition 2 it follows that any (interior) competitive equilibrium must be of the form above. In what follows we refer to this equilibrium of the benchmark economy without government purchases or taxes as the *benchmark equilibrium*, and the value of a variable in the benchmark equilibrium is denoted by a hat (ˆ) over the variable.

3 Constrained inefficiency

In this section we consider the allocation attained at the benchmark equilibrium obtained in Section 2.5 and demonstrate that, if a social planner can force individuals to invest less in physical capital and more in human capital, then all individuals are made better off. In this sense, the competitive equilibrium of our benchmark economy is constrained inefficient, as it exhibits over-accumulation of physical capital.[11]

Suppose that a social planner can directly pick a deterministic sequence $\{\eta_{h,t}\}_{t=0}^{\infty}$ of portfolio compositions. Consider then a hypothetical situation where each consumer is constrained to choose

---

[11]The constrained inefficiency analysis here is closely related to the one in the previous work by Davila, Hong, Krusell, and Ríos-Rull (2005).

$\{\eta_{h,t}\}_{t=0}^{\infty}$, that is, $k_{i,t}$ and $h_{i,t}$ must be chosen to satisfy

$$\eta_{h,t} = \frac{h_{i,t}}{k_{i,t} + h_{i,t}}, \qquad t = 0, 1, 2, \ldots, \tag{26}$$

but all the other variables are determined as in the benchmark equilibrium, namely, by the utility maximization, profit maximization, and market clearing conditions when there are no government purchases and no taxes as in (23).

Formally, define an $\{\eta_{h,t}\}_{t=0}^{\infty}$-*constrained competitive equilibrium* as an allocation $\{c_{i,t}, k_{i,t}, h_{i,t},$ $x_{i,t} : i \in [0,1]\}_{t=0}^{\infty}$ and a price system $\{r_{k,t}, w_t\}_{t=0}^{\infty}$ such that (a) for each i, given $\{r_{k,t}, w_t\}_{t=1}^{\infty}$ and $\{\eta_{h,t}\}_{t=0}^{\infty}$, $\{c_{i,t}, k_{i,t}, h_{i,t}, x_{i,t}\}$ solves the utility maximization problem of individual i with the additional constraint given by (26); (b) prices satisfy $r_{k,t} = F_k(K_{t-1}, H_{t-1})$ and $w_t = F_h(K_{t-1}, H_{t-1})$, where $K_{t-1} = \int_0^1 k_{i,t-1}\, di$ and $H_{t-1} = \int_0^1 h_{i,t-1}\, di$; and (c) all markets clear.

The benchmark equilibrium is by definition an $\{\hat{\eta}_{h,t}\}_{t=0}^{\infty}$-constrained equilibrium. We say that a competitive equilibrium is *constrained efficient* if there exists no sequence $\{\eta'_{h,t}\}_{t=0}^{\infty}$ such that all individuals are better off in the $\{\eta'_{h,t}\}_{t=0}^{\infty}$-constrained equilibrium than in the $\{\hat{\eta}_{h,t}\}_{t=0}^{\infty}$-constrained equilibrium.

Given $\{\eta_{h,t}\}_{t=0}^{\infty}$, the associated constrained equilibrium can be constructed as follows, using Lemma 1. First, market clearing and profit maximization imply

$$r_{k,t+1} = F_k(1 - \eta_{h,t}, \eta_{h,t}), \quad \text{and} \quad w_{t+1} = F_h(1 - \eta_{h,t}, \eta_{h,t}). \tag{27}$$

Next, the certainty-equivalent rate of return between t and $t+1$ is given by

$$\rho_{t+1} = \rho(r_{k,t+1}, w_{t+1}, \eta_{h,t}),$$
$$= \rho\left(F_k(1 - \eta_{h,t}, \eta_{h,t}), F_h(1 - \eta_{h,t}, \eta_{h,t}), \eta_{h,t}\right), \tag{28}$$

where $\rho(r_k, w, \eta_h)$ is defined in (9). In our environment, where $\{\eta_{h,t}\}_{t=0}^{\infty}$ is fixed exogenously, the consumers' problem of choosing $\{\eta_{c,t}\}_{t=0}^{\infty}$ is the same as in the competitive equilibrium. Therefore, a $\{\eta_{h,t}\}_{t=0}^{\infty}$-constrained equilibrium is characterized as in the next proposition.

Proposition 3. *Consider the benchmark economy satisfying (23). Given $\{\eta_{h,t}\}_{t=0}^{\infty}$, let $\{\rho_{t+1}\}_{t=0}^{\infty}$ be given as in (28). If the associated infinite sum, $\sum_{t=0}^{\infty} \prod_{j=0}^{t} \left(\beta^{\psi} \rho_{j+1}^{\psi-1}\right)$, is well defined and takes a finite value, then a unique associated constrained equilibrium exists: the prices and the certainty equivalent rates of return are determined by (27)-(28), and all the other endogenous variables are determined as in Section 2.4.*

For each i and t, consider infinitesimal changes from the portfolio choices $\hat{\eta}_h$ at the benchmark equilibrium characterized in Proposition 2:

$$\eta_{h,t} = \hat{\eta}_h + d\eta_{h,t},$$

where $d\eta_{h,,t}$ may differ across periods and so $\eta_{h,t}$ may be time dependent although $\hat{\eta}_h$ is not.

Here we ask whether or not this infinitesimal changes in consumers' portfolios, $\{d\eta_{h,t}\}$, can make all individuals better off. To answer this question, first notice that as shown in Lemma 1 each individual's lifetime utility in period 0 is given by

$$u_{i,0} = v_0 x_{i,0}$$

where

$$v_0 = (1-\beta)^{\frac{\psi}{\psi-1}} \left\{ 1 + \sum_{t=0}^{\infty} \prod_{j=0}^{t} \left(\beta^{\psi} \rho_{j+1}^{\psi-1} \right) \right\}^{\frac{1}{\psi-1}}.$$

where ρ_{j+1} is the certainty-equivalent rate of return between periods j and $j+1$, which is common for all individuals and given by the expression in (28).

On this basis we can make a few observations. First of all, the lifetime utility of each individual monotonically increases with the realized certainty-equivalent rate of return in each period: that is,

$$\frac{\partial v_0}{\partial \rho_{t+1}} > 0, \qquad \text{for all } t \geq 0. \tag{29}$$

Therefore, a sufficient condition for every individual i's welfare to increase is that the certainty-equivalent rate of return, ρ_{t+1}, increases for all t. Secondly, for each t, ρ_{t+1} only depends on $\eta_{h,t}$, as we see from (28). Thirdly, evaluated at $\eta_{h,t} = \hat{\eta}_h$, $\eta_{h,t}$ does not have any first-order effect on ρ_{t+1}, since $\hat{\eta}_h$ is maximizing the certainty equivalent in the individual utility maximization (the envelope property):

$$\left. \frac{\partial \rho(r_{k,t+1}, w_{t+1}, \eta_{h,t})}{\partial \eta_{h,t}} \right|_{\eta_{h,t}=\hat{\eta}_h} = 0.$$

Thus, $d\eta_{h,t}$ affects ρ_{t+1} only through its effect on the equilibrium prices, $r_{k,t+1}$ and w_{t+1}. The next proposition shows that an increase in $\eta_{h,t}$ from the equilibrium value $\hat{\eta}_h$, $d\eta_{h,t} > 0$, makes all individuals better off.[12]

Proposition 4. *The competitive equilibrium of the benchmark economy is constrained inefficient. Reducing the proportion of investment in physical capital improves the welfare of all individuals: for all $i \in [0,1]$ and all $t \geq 0$,*

$$\left. \frac{dv_{i,0}}{d\eta_{h,t}} \right|_{\eta_{h,t}=\hat{\eta}_h} > 0.$$

That is, the benchmark economy exhibits over-accumulation of physical capital in the constrained-efficiency sense.

---

[12]The fact that the benchmark equilibrium exhibits time independence is not important for this result. It can be readily checked even if it were time dependent, the rest of the argument goes through.

Proof. It is sufficient to show that $\frac{d\rho_{t+1}}{d\eta_{h,t}} > 0$, evaluated at $\eta_{h,t} = \hat{\eta}_h$. From equation (9) we get

$\rho_{t+1} = \left(E[R_{x,i,t+1}^{1-\gamma}] \right)^{\frac{1}{1-\gamma}}$, where $R_{x,i,t+1} = (1 - \delta_k + r_{k,t+1})(1 - \eta_{h,t}) + \theta_{i,t+1}(1 - \delta_h + w_{t+1})\eta_{h,t}$. It follows that

$$\frac{d\rho_{t+1}}{d\eta_{h,t}}\bigg|_{\eta_{h,t}=\hat{\eta}_{h,t}} = (\hat{\rho}_{t+1})^\gamma \cdot E\left[(R_{x,i,t+1})^{-\gamma} \cdot \left\{ \frac{dr_{k,t+1}}{d\eta_{h,t}}(1 - \hat{\eta}_{h,t}) + \theta_{i,t+1}\frac{dw_{t+1}}{d\eta_{h,t}}\hat{\eta}_{h,t} \right\} \right]$$

Since $r_{k,t+1} = F_k(1 - \eta_{h,t}, \eta_{h,t})$ and $w_{t+1} = F_h(1 - \eta_{h,t}, \eta_{h,t})$ for any $\eta_{h,t}$, from the homogeneity of the production function we have, for any $\eta_{h,t}$,

$$\frac{dr_{k,t+1}}{d\eta_{h,t}}(1 - \eta_{h,t}) + \frac{dw_{t+1}}{d\eta_{h,t}}\eta_{h,t} = 0.$$

Hence we obtain

$$\frac{d\rho_{t+1}}{d\eta_{h,t}}\bigg|_{\eta_{h,t}=\hat{\eta}_h} = (\hat{\rho}_{t+1})^\gamma \cdot E\left[(R_{x,i,t+1})^{-\gamma} \cdot (\theta_{i,t+1} - 1) \right] \cdot \hat{\eta}_h \frac{dw_{t+1}}{d\eta_{h,t}}.$$

Under the assumption that $E(\theta_{i,t+1}) = 1$,

$$E\left[(R_{x,i,t+1})^{-\gamma} \cdot (\theta_{i,t+1} - 1) \right] = \text{Cov}\left[(R_{x,i,t+1})^{-\gamma}, (\theta_{i,t+1} - 1) \right] < 0.$$

Since $w_{t+1} = F_h(1 - \eta_{h,t}, \eta_{h,t})$, so $\frac{dw_{t+1}}{d\eta_{h,t}} < 0$ holds. Therefore,

$$\frac{d\rho_{t+1}}{d\eta_{h,t}}\bigg|_{\eta_{h,t}=\hat{\eta}_{h,t}} > 0.$$

$\square$

This proposition shows that the ratio of physical capital to human capital is too high in the benchmark equilibrium: taking the structure of the asset markets as given, reducing the investment ratio in physical capital is welfare improving. This should not be confused with the simple observation that the physical-human capital ratio is larger than that in the complete market setting. Indeed, as shown by Hong, Davila, Krusell, and Ríos-Rull (2005), and Gottardi, Kajii, and Nakajima (2009), for some specifications of the structure of the uncertainty incomplete-markets economies exhibit a higher level of physical capital than when markets are complete, but still a welfare improvement can be attained by increasing the level of physical capital.

Proposition 4 has a simple intuition nevertheless: in the benchmark incomplete-market equilibrium individuals are exposed to too much risk. By changing the portfolio of individuals, $\eta_{h,t}$, marginally from its equilibrium level, because of the envelope property the planner only affects their welfare through the effect on market clearing prices, $r_{k,t+1}$ and w_{t+1}. Now suppose that the planner increases $\eta_{h,t}$ from the equilibrium level, thus raising the rental rate $r_{k,t+1}$ and reducing the wage

rate w_{t+1}. Note that the labor income of each individual is subject to uninsurable idiosyncratic risk, and his/her capital income is not. It follows that such a change in the factor prices reduces the amount of risk that each individual faces. In this sense, reducing the investment in physical capital effectively provides some insurance. This is why increasing $\eta_{h,t}$ from the equilibrium level achieves a Pareto improvement in our model economy.

Notice that a similar constrained inefficiency result also obtains in a more common environment where there is no human capital accumulation and the labor productivity of each individual follows an exogenously specified stochastic process (see, for instance, Gottardi, Kajii, and Nakajima (2009)).

4   (Locally) Optimal Taxation

The previous section has shown that at a competitive equilibrium of the economy described with no government activity (expenditure, taxation or debt), the accumulation of physical capital relative to that of human capital is inefficiently too high. Does this mean that around a competitive equilibrium the government should tax physical capital (interest income) and subsidize human capital (wage income) to improve agents' welfare?

The answer to this question depends on what specific kind of fiscal instruments are available to the policy maker. In this section we consider the relatively simple case where the government has balanced budget at all times, thus there is no public debt:

$$B_t = 0, \quad \text{for all } t, \qquad \text{and} \qquad b_{i,-1} = 0, \quad \text{for all } i \in [0,1]$$

In this case we show that the answer to the above question turns out to be the opposite: the government should subsidize capital and tax labor.

For the sake of comparison with the previous section, let us start with the case without government purchases:

$$G_t = 0, \qquad \text{for all } t.$$

The balanced budget requirement implies that

$$\tau_{k,t} F_{k,t} K_{t-1} + \tau_{h,t} F_{h,t} H_{t-1} = 0, \qquad \text{for all } t.$$

where $F_{k,t} = F_k(K_{t-1}, H_{t-1})$ and $F_{h,t} = F_h(K_{t-1}, H_{t-1})$. Given the Cobb-Douglas specification of the technology (1),

$$\frac{F_{k,t} K_{t-1}}{F_{h,t} H_{t-1}} = \frac{\alpha}{1-\alpha}$$

holds in equilibrium at any t. Therefore the government budget constraint may be replaced by

$$\tau_{h,t} = -\frac{\alpha}{1-\alpha} \tau_{k,t}, \qquad \text{for all } t.$$

A competitive equilibrium with balanced budget of the government and no public purchases is characterized as in Section 2.4 under the additional conditions that $G_t = B_t = 0$ and $\alpha \tau_{k,t} + (1 - \alpha)\tau_{h,t} = 0$ for all t. Evidently when, in addition, $\tau_{k,t} = \tau_{h,t} = 0$ for all t, this is also the benchmark equilibrium of Section 2.5.

Consider then this equilibrium and examine whether or not changing $\tau_{k,t}$ from a zero level is welfare improving for each t. For any pair of tax rates, (τ_k, τ_h), the associated utility maximizing portfolio $\eta_h(\tau_k, \tau_h)$ is defined implicitly as the solution of

$$\Phi\left[(1 - \tau_k)F_k(1 - \eta_h, \eta_h), (1 - \tau_h)F_h(1 - \eta_h, \eta_h), \eta_h\right] = 0, \tag{30}$$

where the function Φ was defined in (17). Using this function $\eta_h(\tau_k, \tau_h)$, the certainty-equivalent rate of return specified in (9) can also be written as a function of (τ_k, τ_h):

$$\rho(\tau_k, \tau_h) \equiv \rho\left[r_k(\tau_k, \tau_h), w(\tau_k, \tau_h)), \eta_h(\tau_k, \tau_h)\right], \tag{31}$$

where

$$r_k(\tau_k, \tau_h) \equiv (1 - \tau_k)F_k\left[1 - \eta_h(\tau_k, \tau_h), \eta_h(\tau_k, \tau_h)\right],$$
$$w(\tau_k, \tau_h) \equiv (1 - \tau_h)F_h\left[1 - \eta_h(\tau_k, \tau_h), \eta_h(\tau_k, \tau_h)\right].$$

Under the balanced budget requirement, $\alpha \tau_k + (1 - \alpha)\tau_h = 0$, the effect of τ_k on the after tax return r_k at $\tau \equiv (\tau_k, \tau_h) = 0$ is given by

$$\left.\frac{dr_k}{d\tau_k}\right|_{\tau=0} = -F_k + (-F_{kk} + F_{kh})\left(\frac{\partial \eta_h}{\partial \tau_k} - \frac{\alpha}{1-\alpha}\frac{\partial \eta_h}{\partial \tau_h}\right).$$

where all derivatives are evaluated at $\tau = 0$. The first-term $(-F_k)$ is the direct effect of the change in the capital-income tax on the after tax return, the second term is the indirect effect due to the change in prices induced by the change in the agents' portfolio choice. The following lemma (whose proof is in the Appendix) shows that the direct effect dominates the indirect effect.

Lemma 5. *Suppose that Assumption 1 holds. Then*

$$\left.\frac{dr_k}{d\tau_k}\right|_{\tau=0} < 0.$$

We can then define a map yielding the equilibrium value of the coefficient v_0 appearing in the expression of the each individual's lifetime utility, multiplied by $x_{i,0}$, as a function of $\{(\tau_{k,t}, \tau_{h,t})\}_{t=0}^\infty$:

$$v_0\left((\tau_{k,t}, \tau_{h,t})\}_{t=0}^\infty\right) \equiv (1 - \beta)^{\frac{\psi}{\psi-1}}\left\{1 + \sum_{t=0}^\infty \prod_{j=0}^t \left(\beta^\psi \rho(\tau_{k,1+j}, \tau_{h,1+j})^{\psi-1}\right)\right\}^{\frac{1}{\psi-1}},$$

where $\rho(\tau_k, \tau_h)$ is defined in (31). Taxing physical capital is welfare improving if this function is increasing in $\tau_{k,t}$ while $\tau_{h,t}$ is determined to meet the government budget constraint. Interestingly enough, the following proposition establishes that this is indeed the case and hence that subsidizing capital makes everyone better off.

Proposition 6. *Suppose that $B_t = G_t = 0$ for all t and $b_{i,-1} = 0$ for all $i \in [0,1]$. Suppose also that Assumption 1 holds. Then, for all t,*

$$\left(\frac{\partial v_0}{\partial \tau_{k,t}} + \frac{\partial v_0}{\partial \tau_{h,t}} \frac{d\tau_h}{d\tau_k} \right) \bigg|_{\tau=0} < 0,$$

where $d\tau_h/d\tau_k = -\alpha/(1-\alpha)$. Therefore, any sequence of taxes $\{\tau_{k,t}, \tau_{h,t}\}_{t=0}^{\infty}$ satisfying budget balance, $\alpha \tau_{k,t} + (1-\alpha)\tau_{h,t} = 0$, for all t, and such that capital is subsidized, $\tau_{k,t} < 0$, and hence labor is taxed, $\tau_{h,t} = -\alpha\tau_{k,t}/(1-\alpha) > 0$ for every t, improves the welfare of all individuals over the benchmark equilibrium, if these taxes are small enough.

Proof. In the proof here, unless otherwise stated, all derivatives are evaluated at the benchmark equilibrium, with zero taxes, $\tau_k = \tau_h = 0$. Since at such equilibrium the consumption rate $\eta_{c,t}$ and the portfolio choice $\eta_{h,t}$ and hence prices are t-invariant, the argument is the same for every t, hence we shall omit t. Also to simplify the notation, we write $d/d\tau_k$ to denote $\partial/\partial\tau_k - \alpha/(1-\alpha)\partial/\partial\tau_h$, i.e., $d/d\tau_k$ takes into account the induced change in τ_h via the government budget constraint. Since the lifetime utility is increasing in ρ_t for each t, it suffices to show that $d\rho/d\tau_k < 0$. Note first that the following relationship holds between the changes in the before tax returns,

$$\begin{aligned}
\frac{dF_h}{d\tau_k} &= (F_{hh} - F_{hk})\frac{d\eta_h}{d\tau_k} \\
&= (F_{kk} - F_{kh})\frac{1-\eta_h}{\eta_h}\frac{d\eta_h}{d\tau_k} \\
&= -\frac{1-\eta_h}{\eta_h}\frac{dF_k}{d\tau_k}
\end{aligned}$$

where we have used the property $-F_{hk} + F_{hh} = (F_{kk} - F_{kh})(1-\eta_h)/\eta_h$ obtained from the Euler equation. Then, the marginal effect of the tax on the after tax returns, evaluated at $\tau_k = \tau_h = 0$, is:

$$\begin{aligned}
\frac{dw}{d\tau_k} &= \frac{F_k(1 - \hat{\eta}_h)}{F_h\hat{\eta}_h}w + \frac{dF_h}{d\tau_k} \\
&= F_k\frac{1 - \hat{\eta}_h}{\hat{\eta}_h} + (F_{kk} - F_{kh})\frac{1 - \hat{\eta}_h}{\hat{\eta}_h}\frac{d\eta_h}{d\tau_k} \\
&= -\frac{1 - \hat{\eta}_h}{\hat{\eta}_h}\left(\frac{dF_k}{d\tau_k} - F_k\right) \\
&= -\frac{1 - \hat{\eta}_h}{\hat{\eta}_h}\frac{dr_k}{d\tau_k},
\end{aligned} \tag{32}$$

so that, by Lemma 5, we have $\frac{dw}{d\tau_k} > 0$. Using the envelope property, we obtain

$$\frac{d\rho}{d\tau_k} = \hat{\rho}^{\gamma} E \left[\hat{R}_{x,i,\theta_i}^{-\gamma} \left\{ (1 - \hat{\eta}_h) \frac{dr_k}{d\tau_k} + \theta_i \hat{\eta}_h \frac{dw}{d\tau_k} \right\} \right]$$

where $\hat{R}_{x,i,\theta_i} \equiv (1 - \delta_k + \hat{r}_k)(1 - \hat{\eta}_h) + (1 - \delta_h + \hat{w})\theta_i \hat{\eta}_h$. Using (32) yields

$$\frac{d\rho}{d\tau_k} = \hat{\rho}^{\gamma} E \left[\hat{R}_{x,\theta}^{-\gamma}(\theta - 1) \right] \hat{\eta}_h \frac{dw}{d\tau_k} < 0,$$

where the inequality follows from the fact that $E[R_{x,\theta}^{-\gamma}(\theta - 1)] < 0$ and $dw/d\tau_k > 0$. $\qquad\square$

The intuition for this result is again very simple. Suppose the government changes marginally the tax rates, τ_k and τ_h from a zero level, under the balanced budget constraint. Due the the envelope property, such a change in the tax rates only affects the utility of each individual through its effects on the after-tax factor prices, r_k and w. As we have discussed in the previous section, individuals are exposed to too much risk in their labor income in the benchmark, incomplete-market equilibrium. In the constrained inefficiency result, we have shown that reducing the investment in physical capital allows to lower this risk by increasing r_k and decreasing w. Here, the planner uses linear taxes rather than directly controlling the portfolios of individuals. By taxing risky labor income and using the total revenue of this tax to subsidize the riskless return on capital, that is by setting $\tau_h > 0$ and $\tau_k < 0$, the government can reduce the individual exposure to idiosyncratic risk. These taxes also affect the agents' portfolio choice, by increasing the investment in physical capital relative to that in human capital and hence decreasing the before tax return of physical capital. As shown in Proposition 6 the first effect prevails over the second one so that the overall effect consists again in an increase in the (after-tax) return r_k and a decrease in w. Thus, Proposition 4 and Proposition 6 are perfectly consistent with each other. They show how, with different instruments available to the planner, a welfare improvement can be attained by reducing the return of the risky factor and increasing that of the riskless one.

Note that Proposition 6 only characterizes the properties of optimal taxes in a neighborhood of zero, and does not guarantee that the globally optimal tax rate on physical capital is indeed negative in the environment considered, where government consumption and debt are zero. A sufficient condition for this to hold is that the map $\rho(\tau_k, -\alpha/(1 - \alpha)\tau_k)$ defined in (31), where we used the government budget constraint to substitute for τ_h, has a unique local maximum:[13]

Assumption 2. *The function $\rho(\tau_k, -\alpha/(1 - \alpha)\tau_k)$ has a unique local maximum.*

---

[13]This property is satisfied in all the numerical examples considered in the rest of the paper.

When $B_t = G_t = 0$ for all t, the optimal tax rates $(\tau_{k,t}, \tau_{h,t})_{t=1}^{\infty}$ are obtained by maximizing $\rho(\tau_k, \tau_h)$ in (31) subject to the balanced-budget constraint $\alpha\tau_{k,t} + (1-\alpha)\tau_{h,t} = 0$. The solution is naturally time-invariant and under Assumption 2 we can then say that the optimal τ_k is negative.

5 Optimal taxation and debt

In this section we examine the case where the government can also borrow or lend and study the dynamic Ramsey problem where the optimal path of taxes and public debt are determined. With complete markets, Judd (1985) and Chamley (1986) have shown that the optimal tax rate on physical capital is zero in the steady state. In addition, Jones, Manuelli and Rossi (1997) have found that when there is human capital accumulation in such a model, the optimal tax rates on physical and human capital are both equal to zero in the steady state. Hence, with positive government purchases, the optimal level of debt in the steady state is negative.

In this section we investigate how the structure of optimal taxation changes under incomplete asset markets. We have shown in the previous section that, with incomplete markets, it is in general beneficial for the government to tax labor and physical capital even if government purchases are zero. In this section we present two theoretical results on the properties of the solution of the Ramsey problem, before turning to the numerical analysis of this solution in the next section. First, accumulating government debt improves welfare as long as government purchases are small enough. Second, the steady state return on government debt must equal the average rate of return of private consumers' portfolios, after adjusting for the effect of public debt on the consumers' savings rate. Hence, provided the latter effect is not too large, the optimal taxation on physical capital is strictly positive, whatever the level of government purchases.

5.1 Ramsey problem

The dynamic Ramsey problem consists in maximizing consumers' welfare at a competitive equilibrium, as defined in Section 2.4, across all fiscal policies $\{\tau_{k,t}, \tau_{h,t}, B_t\}_{t=0}^{\infty}$ satisfying (18) and (19), for a given exogenous sequence of government purchases, $\{G_t\}_{t=0}^{\infty}$.

To investigate the solutions of this problem, it is convenient to normalize aggregate variables in terms of the total wealth X_t, for each t

$$k_t \equiv \frac{K_t}{X_t}, \quad h_t \equiv \frac{H_t}{X_t}, \quad b_t \equiv \frac{B_t}{X_t}, \quad g_t \equiv \frac{G_t}{X_{t-1}}.$$

The government's flow budget constraint can be rewritten as follows, using (22), in terms of these

normalized variables

$$g_t + (1 - \delta_k + r_{k,t})b_{t-1} = (1 - \eta_{c,t-1})R_{x,t}b_t + F(k_{t-1}, h_{t-1}) - r_{k,t}k_{t-1} - w_t h_{t-1}$$

A competitive equilibrium can then be equivalently defined in terms of these normalized variables, for a given sequence of normalized government purchases, $\{g_t\}_{t=0}^{\infty}$.

As is standard in the literature, in order to rule out a trivial solution, we assume that the tax rates in the initial period are exogenously fixed:

$$\tau_{k,0} = \bar{\tau}_{k,0}, \qquad \text{and} \qquad \tau_{h,0} = \bar{\tau}_{h,0},$$

together with the initial conditions determining k_{-1}, h_{-1}, b_{-1}. It follows that $r_{k,0}$, w_0, and b_0 are predetermined as well.

The dynamic optimal taxation and debt problem consists so in the choice of a fiscal policy, implicitly defined by a sequence of normalized debt and net of tax prices $\{b_{t+1}, r_{k,t+1}, w_{t+1}\}_{t=0}^{\infty}$, as well as of all the other endogenous variables $\{\rho_{t+1}, \eta_{h,t}, \eta_{c,t}, R_{x,t+1}, k_t, h_t\}_{t=0}^{\infty}$ so as to maximize the consumers' equilibrium utility level

$$\max v_0 = (1 - \beta)^{\frac{\psi}{\psi-1}} \left\{ 1 + \sum_{t=0}^{\infty} \prod_{j=0}^{t} \left(\beta^{\psi} \rho_{1+j}^{\psi-1} \right) \right\}^{\frac{1}{\psi-1}} \tag{33}$$

subject to the equilibrium constraints which guarantee that the endogenous variables constitute a competitive equilibrium:

$$\eta_{h,t} = \arg\max_{\eta'_h} \rho(r_{k,t+1}, w_{t+1}, \eta'_h)$$

$$\rho_{t+1} = \max_{\eta'_h} \rho(r_{k,t+1}, w_{t+1}, \eta'_h)$$

$$R_{x,t+1} = (1 - \delta_k + r_{k,t+1})(1 - \eta_{h,t}) + (1 - \delta_h + w_{t+1})\eta_{h,t}$$

$$\eta_{c,t} = \left\{ 1 + \sum_{s=0}^{\infty} \prod_{j=0}^{s} \left(\beta^{\psi} \rho_{t+1+j}^{\psi-1} \right) \right\}^{-1}$$

$$k_t = (1 - \eta_{c,t})(1 - \eta_{h,t}) - b_t$$

$$h_t = (1 - \eta_{c,t})\eta_{h,t}$$

$$g_{t+1} + (1 - \delta_k + r_{k,t+1})b_t = (1 - \eta_{c,t})R_{x,t+1}b_{t+1} + F(k_t, h_t) - r_{k,t+1}k_t - w_{t+1}h_t$$

$$\lim_{t\to\infty} \left\{ \prod_{j=1}^{t} (1 - \delta_k + r_{k,j})^{-1} (1 - \eta_{c,j-1})R_{x,j} \right\} b_t = 0$$

given $b_{-1}, k_{-1}, h_{-1}, b_0, r_{k,0}, w_0$. We shall call this maximization problem the *Ramsey problem*, and the resulting equilibrium the *Ramsey equilibrium*.

It is convenient to divide the Ramsey problem into two steps. The first step is to find, for a given pair of sequences of debt and rates of consumption, optimal after-tax prices, together with portfolio choices and average returns such that the associated equilibrium conditions are satisfied. Then these conditionally optimal prices and other variables will be functions of the sequence of debt and consumption shares. The second step is then to find an optimal sequence of debt and consumption shares, taking into account these functional relations, as well as the rest of the equilibrium conditions.

The first step can then be expressed as the problem of maximizing the certainty equivalent function, given a pair of sequences $\{b_t, \eta_{c,t}\}_{t=0}^{\infty}$, for each fixed t:

$$\tilde{\rho}(b_t, b_{t+1}, \eta_{c,t}) \equiv \max_{\{r_{k,t+1}, w_{t+1}, \eta_{h,t}, R_{x,t+1}\}} \rho(r_{k,t+1}, w_{t+1}, \eta_{h,t}) \tag{34}$$

subject to

$$g_{t+1} + (1 - \delta_k + r_{k,t+1})b_t \tag{35}$$
$$= (1 - \eta_{c,t})R_{x,t+1}b_{t+1} + F\left[(1 - \eta_{c,t})(1 - \eta_{h,t}) - b_t, (1 - \eta_{c,t})\eta_{h,t}\right]$$
$$- r_{k,t+1}\left[(1 - \eta_{c,t})(1 - \eta_{h,t}) - b_t\right] - w_{t+1}(1 - \eta_{c,t})\eta_{h,t}$$

$$\eta_{h,t} = \arg\max_{\eta_h} \rho(r_{k,t+1}, w_{t+1}, \eta_h)$$

$$R_{x,t+1} = (1 - \delta_k + r_{k,t+1})(1 - \eta_{h,t}) + (1 - \delta_h + w_{t+1})\eta_{h,t}$$

Since there is a one-to-one relation between the rate of consumption $\eta_{c,t}$ and the utility parameter v_t when utility is maximized (see (16)), the second step can be written as the problem of choosing $\{b_{t+1}, \eta_{c,t+1}, v_{t+1}\}_{t=0}^{\infty}$ so as to maximize v_0 given b_0:

$$\max_{\{v_{t+1}, b_{t+1}, \eta_{c,t+1}\}_{t=0}^{\infty}} v_0 \tag{36}$$

subject to the remaining equilibrium conditions

$$v_t^{\psi-1} = (1 - \beta)^{\psi} + \beta^{\psi} \rho_{t+1}^{\psi-1} v_{t+1}^{\psi-1} \tag{37}$$

$$\rho_{t+1} = \tilde{\rho}(b_t, b_{t+1}, \eta_{c,t}) \tag{38}$$

and (16).

Regarding the function $\tilde{\rho}(b, b', \eta_c)$ defined in (34), the following simple observation is useful.

Lemma 7. *Assume that $g_{t+1} = 0$. Consider the function $\tilde{\rho}(b, b', \eta_c)$ defined in (34). If $b = b' = 0$, the first order effect of v is zero everywhere: that is,*

$$\frac{\partial \tilde{\rho}}{\partial \eta_c} = 0, \qquad \text{if } b = b' = 0.$$

Proof. When $g_{t+1} = 0$ and $b_t = b_{t+1} = 0$, taking advantage of the homogeneity of F, the first equation in the constraints for the maximization problem (34) becomes $0 = F[(1 - \eta_{h,t}), \eta_{h,t}] - r_{k,t+1}(1 - \eta_{h,t}) - w_{t+1}\eta_{h,t}$. Thus the variable $\eta_{c,t}$ no longer appears in any of the constraints. So the value of the objective function remains unchanged when $\eta_{c,t}$ changes, thus establishing the result. $\square$

5.2 Desirability of government debt

Before discussing the solution to the Ramsey problem, we examine the welfare effects of issuing government debt, when markets are incomplete. We show that, starting from a zero level of government debt, increasing the amount of government debt is welfare improving as long as government purchases are small enough.

Consider the Ramsey problem under the additional restriction

$$b_t = g_t = 0, \qquad \text{for all } t,$$

that is, there is no government debt nor expenditure. The variables solving the Ramsey problem under these conditions, as argued in Section 4, are time invariant. Let us denote them with the superscript o, that is, $v^o, \rho^o, R_x^o, F_k^o$ etc. They satisfy the following conditions

$$v^o \equiv \left[\frac{(1 - \beta)^\psi}{1 - \beta^\psi (\rho^o)^{\psi-1}} \right]^{\frac{1}{\psi-1}}$$

$$\eta_c^o = (1 - \beta)^\psi (v^o)^{1-\psi}$$

where $\rho^o = \tilde{\rho}(0, 0, \eta_c^o)$.

We investigate whether allowing for an arbitrarily small (positive or negative) level of debt at only one date yields a welfare improvement. Consider the Ramsey problem under the alternative restriction

$$b_{T+1} = \bar{b}_{T+1}, \qquad \text{and } b_t = 0 \text{ for all } t \neq T+1, \, g_t = 0, \qquad \text{for all } t. \tag{39}$$

for given $\bar{b}_{T+1}$. Denote the variables solving this problem as $v_t(\bar{b}_{T+1})$, $\rho_t(\bar{b}_{T+1})$, etc.. Thus, we have from (14)

$$v_t(\bar{b}_{T+1}) = \left\{ (1 - \beta)^\psi + \beta^\psi \rho_{t+1}(\bar{b}_{T+1})^{\psi-1} v_{t+1}(\bar{b}_{T+1})^{\psi-1} \right\}^{\frac{1}{\psi-1}} \tag{40}$$

where

$$\rho_{t+1}(\bar{b}_{T+1}) = \tilde{\rho}(b_t, b_{t+1}, \eta_{c,t}(\bar{b}_{T+1}))$$

$$\eta_{c,t}(\bar{b}_{T+1}) = (1 - \beta)^\psi \left(v_t(\bar{b}_{T+1}) \right)^{1-\psi}$$

25

with $b_t = 0$ for all $t \neq T + 1$. Notice that

$$v_t(\bar{b}_{T+1}) = v^o, \quad \forall t \geq T + 2, \tag{41}$$

$$\rho_t(\bar{b}_{T+1}) = \rho^o, \quad \forall t \neq T + 1, T + 2$$

where the second equality follows from Lemma 7 and the first one from (15).

The next proposition states that having a positive amount of debt, $\bar{b}_{T+1} > 0$, is welfare improving as long as government purchases are sufficiently small. Its proof is in the Appendix.

Proposition 8. *Suppose that $g_t = g$ for all t, and that Assumptions 1 and 2 hold. Consider the optimal tax equilibrium under the balanced budget requirement: $b_t = 0$ for all t. Then increasing $\bar{b}_{T+1}$ from zero for a given period $T + 1$ improves the lifetime utility of all individuals if g is sufficiently small.*

To obtain an intuition for this result note, in the light of equation (33), that whether or not increasing $\bar{b}_{T+1}$ from zero is welfare improving depends on how this change affects the equilibrium values of $\{\rho_{t+1}\}_{t=0}^{\infty}$. As shown in the previous section, in an optimal tax equilibrium, for any pair of sequences $\{b_t, \eta_{c,t}\}_{t=0}^{\infty}$ we have $\rho_{t+1} = \tilde{\rho}(b_t, b_{t+1}, \eta_{c,t})$, with the map $\tilde{\rho}(.)$ obtained as solution of problem (34). In addition, when $b_t = b_{t+1} = 0$ by Lemma 7 $\tilde{\rho}$ is locally independent of $\eta_{c,t}$. This greatly simplifies the argument, as we only have to look at the partial derivatives of $\tilde{\rho}(b_T, b_{T+1}, \eta_{c,T})$ and $\tilde{\rho}(b_{T+1}, b_{T+2}, \eta_{c,T+1})$ with respect to $\bar{b}_{T+1}$ evaluated at $b_T = \bar{b}_{T+1} = b_{T+2} = 0$ and the optimal tax equilibrium value with $b_t = g_t = 0$ for all t, $\eta_{c,T} = \eta_{c,T+1} = \eta_c^o$. Let us denote those derivatives by ρ_2^o and ρ_1^o, respectively.

The derivative ρ_2^o measures the benefit in $T + 1$ of increasing the government debt, primarily due to the associated tax cut, while ρ_1^o measures the cost of the increase in taxes in period $T + 2$ that is required to redeem b_{T+1}. The value of $\bar{b}_{T+1}$ enters the constraints of problem (34) via the government budget constraint (35), which by the homogeneity of $F(k, h)$ can be rewritten when $g_t = 0$ as

$$(1 - \delta_k + F_k)b_t = (1 - \eta_{c,t})R_{x,t+1}b_{t+1}$$
$$+ (F_{k,t+1} - r_{k,t+1})(1 - \eta_{c,t})(1 - \eta_{h,t}) + (F_{h,t+1} - w_{t+1})(1 - \eta_{c,t})\eta_{h,t}$$

where $F_{k,t+1} = F_k((1 - \eta_{c,t})(1 - \eta_{h,t}) - b_t, (1 - \eta_{c,t})\eta_{h,t})$, and $F_{h,t+1} = F_h((1 - \eta_{c,t})(1 - \eta_{h,t}) - b_t, (1 - \eta_{c,t})\eta_{h,t})$. Hence we see that the benefit of the (marginal) increase in $\bar{b}_{T+1}$, ρ_2^o, is proportional to the after-tax average rate of return of individual portfolios, R_x^o, at the optimal tax equilibrium with $b_t = g_t = 0$ for all t. This is natural because R_x^o is the average rate that individuals earn using the proceeds from the tax cut in $T+1$. The cost incurred in period $T+2$ when the debt $\bar{b}_{T+1}$ is repaid,

ρ_1^o, is then proportional to the (before-tax) rate of return on government debt, $1 - \delta_k + F_k^o$. Whether or not increasing $\bar{b}_{T+1}$ is beneficial depends on the comparison between these two terms. In fact, we show in the proof that the benefit of increasing $\bar{b}_{T+1}$ dominates over its cost if and only if

$$R_x^o > 1 - \delta_k + F_k^o. \tag{42}$$

The reason why (42) holds is simple. As discussed in Section 4, under Assumptions 1 and 2 the optimal tax rates when $b_t = g_t = 0$ for all t satisfy $\tau_k^o < 0$ and $\tau_h^o > 0$. Note also that, since the investment in the human capital is risky, the after-tax rate of return on human capital is greater than the after-tax rate of return on physical capital. Hence we have

$$\begin{aligned} R_x^o &= [1 - \delta_k + (1 - \tau_k^o)F_k^o] (1 - \eta_h^o) + [1 - \delta_h + (1 - \tau_h^o)F_h^o] \eta_h^o \\ &> 1 - \delta_k + (1 - \tau_k^o)F_k^o \\ &> 1 - \delta_k + F_k^o \end{aligned}$$

establishing (42) and showing that increasing the amount of government debt in one period is welfare improving. The argument can then be extended by continuity to the case where $g_t = g$ for all t for g sufficiently small.

Notice that the desirability of public debt we find is not due to the provision of liquidity emphasized by Aiyagari McGrattan (1998), as in our set-up the borrowing constraint never binds.

5.3   Ramsey steady state

We focus here on the properties of the Ramsey equilibrium, obtained as a solution of (33) at a steady state, i.e., along a balanced growth path.[14]

Consider the optimal choice of b_{t+1}, that is, the (normalized) amount of government debt issued in period $t+1$, keeping b_s fixed for all $s \neq t+1$, determined in the second-step problem (36). Look at problem (34) and consider the effects of changing b_{t+1} at an optimal solution. Note first that b_{t+1} does not affect $\{\rho_j\}_{j=t+3}^\infty$ and hence neither v_s for $s \geq t+2$ since, as we see in (15), v_t is determined by the future values of ρ_t. The direct effect of b_{t+1} is to change ρ_{t+1} and ρ_{t+2} and hence v_t and v_{t+1} and also $\eta_{c,t}$ and $\eta_{c,t+1}$. By the envelope theorem v_s for $s \leq t-1$, in particular v_0, are only affected by the change in b_{t+1} because v_t and v_{t+1} are affected. The effects of an increase in b_{t+1} are then summarized as follows. (i) First, it increases ρ_{t+1} because an increase in the government debt

---

[14]A presumption here is that a solution to the Ramsey problem converges to a steady state. We do not have a formal proof for this, but in all the numerical results reported later, such a convergence takes place in just one period. Note also that we use the terms "steady state" and "balanced growth path" interchangeably given the fact that a steady state for the normalized variables corresponds to a balanced growth path of the economy.

issued in period $t+1$ implies a reduction in the tax rates in that period. This is a benefit of the increase in b_{t+1}; (ii) Second, it reduces ρ_{t+2} because the increase in b_{t+1} requires an increase in the tax rates in the next period, $t+2$. This is a cost of the increase in b_{t+1}. (iii) Finally, the change in b_{t+1} may also affect the saving rates η_c in periods $t+1$ and $t+2$. A change in the saving rate also affects the certainty-equivalent rates of return. The first two effects are analogous to those found in the previous section, the third one arises here, where debt levels may be nonzero.

Optimality requires, of course, that the marginal effect of a change in b_{t+1} on v_t and v_{t+1} must be zero. The next proposition (whose proof is in the Appendix) shows that, at the steady state, this condition is characterized by a relatively simple equation.

Proposition 9. *In the steady state of the Ramsey equilibrium the following condition holds:*

$$R_x = (1 - \delta_k + F_k) \left[1 - (1 - \psi)\beta^\psi \tilde{\rho}^{\psi-2} \tilde{\rho}_{\eta_c} \eta_c \right]^{-1} \tag{43}$$

with $\rho_{\eta_c} \equiv \frac{\partial \tilde{\rho}}{\partial \eta_c}$, *where* $\tilde{\rho}$ *is the function defined in (34).*

An intuition for this result is given as follows. As shown in the proof, at the steady state, the first-order condition for an optimal value of b_{t+1}, at any t, becomes

$$\rho_2 + \frac{\beta^\psi \tilde{\rho}^{\psi-1}}{1 - \beta^\psi \tilde{\rho}^{\psi-2}(1-\psi)\rho_{\eta_c}\eta_c} \rho_1 = 0, \tag{44}$$

where $\rho_2 \equiv \frac{\partial}{\partial b'} \tilde{\rho}(b, b', \eta_c)$, $\rho_1 \equiv \frac{\partial}{\partial b} \tilde{\rho}(b, b', \eta_c)$, both evaluated at $b' = b$. This equation shows the three effects on ρ_t and hence on v_t of a change in the value of b_{t+1} which were described in the previous paragraph. The first term, ρ_2, represents the beneficial effect on ρ_t and hence also on v_t of an increase in b_{t+1} (effect i); ρ_1 in the second term describes its cost (effect ii), while $(1 - \beta^\psi \tilde{\rho}^{\psi-2}(1-\psi)\rho_{\eta_c}\eta_c)^{-1}$ is the effect due to the change in the saving rate (effect iii). Finally, the factor $\beta^\psi \tilde{\rho}^{\psi-1}$ in the second term represents the effective discount factor (see (37)).

It then follows from the definition of $\tilde{\rho}(b, b', \eta_c)$ in the first-step problem (34) that at the steady state

$$\frac{\rho_2}{\rho_1} = -\frac{(1 - \eta_c)R_x}{1 - \delta_k + F_k}. \tag{45}$$

Also, at the steady state the savings rate satisfies $1 - \eta_c = \beta^\psi \tilde{\rho}^{\psi-1}$. Then, combining (44) and (45) yields (43).

Condition (45) allows us to relate the beneficial effect (i) and the costly effect (ii) previously identified to, respectively, R_x, the average rate of return earned in the private sector, and $1 - \delta_k + F_k$, the rate of return for the government (that is, the before-tax rate on the risk free asset). Thus, the steady-state condition (43) says that, after adjusting for the change in the saving rate, the rates of return earned by the private sector and by the government should be equal at the steady state.

Notice that in the special case where $\psi = 1$ (33) becomes

$$\ln v_0 = \sum_{t=0}^{\infty} \beta^{t+1} \ln \rho_{t+1} \tag{46}$$

and $\eta_{c,t}$ is a constant irrespective of the sequence of market rates:

$$\eta_{c,t} = 1 - \beta$$

So in this case there is no effect due to a change in the saving rate, so (43) reduces to the equality between the average private rate and the before tax return on physical capital. More importantly, this equation implies that the steady-state tax rate on physical capital is positive in our incomplete-market economy.

Corollary 10. *Consider the case of $\psi = 1$. Then, in the steady state of the Ramsey equilibrium, the following condition holds:*

$$R_x = 1 - \delta_k + F_k \tag{47}$$

It implies that the optimal tax rate on physical capital at the steady state is positive:

$$\tau_k > 0$$

Proof. Condition (47) immediately follows from (43) since $\rho_{\eta_c} = 0$ when $\psi = 1$. To see that $\tau_k > 0$, notice that because of *risk aversion*, the rate of return on human capital must be greater than the rate of return of the risk-free assets. That is,

$$1 - \delta_k + r_k < R_x < 1 - \delta_h + w$$

This inequality, together with (47), implies then that

$$\tau_k > 0.$$

$\square$

Corollary 10 can be directly related to the previous results obtained by Judd (1985), Chamley (1986), Jones, Manuelli, and Rossi (1997), among others, which show that with complete markets the optimal tax rate on physical capital at the steady state is zero. In our setup, if shocks to human capital of individuals were perfectly insurable (at fair prices) so that human capital were also, effectively, a riskless asset, all three assets (risk-free bonds, physical capital, and human capital) would yield the same rate of return, that is,

$$1 - \delta_k + r_k = 1 - \delta_h + w = R_x$$

It would then follow from (47) that

$$\tau_k = 0.$$

Aiyagari (1995) also found that the optimal tax on capital is positive at a steady state when markets are incomplete. His model however is such that τ_k must be positive for a steady state where the level of government expenditure is optimally chosen to exist. In our model in contrast the optimal value of τ_k is primarily determined by the comparison of costs and benefits of varying the level of public debt.

6 Numerical result

In this section we calibrate our model based on some empirical evidence on the U.S. economy, and examine how market incompleteness affects the structure of Ramsey taxation and debt.

6.1 Baseline calibration

Suppose that $\theta_{i,t} \in \{1 + \bar{\theta}, 1 - \bar{\theta}\}$, each occurring with equal probability. Also, suppose that the normalized amount of government purchases is constant over time, $g_t = g$ for all t. Then the set of parameters of our model economy is given by $\{\beta, \psi, \gamma, A, \alpha, \delta_k, \delta_h, g, \bar{\theta}\}$. The baseline values for these parameters are set as follows. First, we set $\psi = 1$ and $\gamma = 3$, that is, the intertemporal elasticity of substitution is unity, and the coefficient of relative risk aversion is three. Second, the capital share of income is set to 0.36, and the depreciation rates of physical and human capital are both 0.06: $\alpha = 0.36$ and $\delta_k = \delta_h = 0.06$. The values for the remaining parameters and for the fiscal policy $\tau_{k,t} = \tau_{h,t} = \tau$ and $b_t = b$ for all t are then set so that, at a balanced growth path, the following features of the U.S. economy are replicated: (i) government purchases are 18 percent of GDP; (ii) government debt is 51 percent of GDP; (iii) the capital-output ratio is 2.7; (iv) the growth rate of GDP is 1.6 percent; (v) the variance of the permanent shock to individual labor earnings is 0.0313. The first four facts are based on Chari, Christiano and Kehoe (1994), the last one on Meghir and Pistaferri (2004).[15] The baseline parameter values are summarized in Table 1.

6.2 Results

Table 2 shows the tax rates and the government debt at the steady state of the Ramsey equilibrium. For reference, the first column shows the corresponding values when the parameter values of the economy and the fiscal policy are set at the baseline levels reported in Table 1, with a (uniform) tax

---

[15]It is also consistent with the evidence reported by Storesletten, Telmer and Yaron (2004).

rate of 19.95 percent yielding a debt-to-output ratio of 51 percent and a growth rate of 1.6 percent, which are as we said the values calibrated to the U.S. economy. The second column describes the corresponding values at the Ramsey steady state, that is with the optimal fiscal policy, still for the baseline calibration of the economy. The third column reports the values at a Ramsey steady state when government purchases are zero.

As shown in Corollary 10, the tax rate on capital at the Ramsey steady state is strictly positive. Quantitatively, we see the optimal value of the tax is a non trivial amount even when there are no government purchases. The wage tax rate at the Ramsey steady state is also positive. As shown in Proposition 6, taxing the wage income is beneficial because it allows to reduce the idiosyncratic risk that individuals face. Finally, we see that the level of government debt at the Ramsey steady state is close to zero (0.19 percent of GDP). To better understand this finding, it is useful to compare it with the value obtained in the last column, when $g = 0$. In this case we see that the debt-to-output ratio at the Ramsey steady state is positive and fairly large (202.6 per cent of GDP). This is in accord with the results of Proposition 8 showing the benefits of issuing debt when government purchases are close to zero. In contrast, when g is positive the optimal debt level at the steady state with complete markets is negative, as recalled in Section 5. Hence with incomplete markets these two forces push in opposite direction, which explains the much lower, but still non negative optimal level of debt reported in the second column.

The tractable nature of the model considered allows us to find also the transitional dynamics of the Ramsey equilibrium and not just the steady state. This dynamics is illustrated in Figure 1, which plots the debt-to-output ratio, b_t/y_t, and the two tax rates, $\tau_{k,t}$ and $\tau_{h,t}$, in the Ramsey equilibrium, that is at a solution of (33) when the initial condition for the fiscal policy parameters sets their values $b_0, \tau_{k,0}, \tau_{h,0}$ at a level equal to their baseline levels of Table 1. We see from the figure that the dynamics turns out to be quite simple. The adjustment in the fiscal policy only lasts one period, in which we see a spike in both tax rates which allows to bring down the debt ratio at date 1, b_1/y_1, to its Ramsey steady state level. After this first period also the tax rates are set at their new steady state level and the economy reaches immediately the steady-state for the Ramsey equilibrium and stays on the balanced growth path afterwards. It proves then to be optimal that all the adjustment in the fiscal policy is concentrated in one period, to minimize distortions over time. The fact then that the transition to the new steady state of the equilibrium variables is immediate clearly depends on the specific features of the economy considered, in particular its technology.

How much benefits do individuals in our economy obtain by moving from the baseline policy to the Ramsey policy? This benefit can be measured by the rate of permanent increase in consumption of each individual that makes him/her indifferent between the two policies. Note that, as can be

seen in Lemma 1, this rate is the same for all consumers and is given by the ratio of the values of v_0 under the two policies in comparison. Table 3 shows the result. When we only compare the steady state associated with the baseline policy and the Ramsey policy, the welfare gain of adopting the Ramsey policy amounts to an increase of about 8.7 percent in each individual's consumption. But this number ignores the cost of transition, where the significant increase in taxes takes place. When the transition is taken into account, the gain gets substantially smaller, 0.85 percent, which is nevertheless a significant amount.

6.3   Sensitivity analysis

In this section we conduct some sensitivity analysis. Specifically, we examine how the debt-to-output ratio and the tax rates at the Ramsey steady state vary under different values for the risk aversion γ, the intertemporal elasticity of substitution ψ, and the idiosyncratic risk $\bar{\theta}$. For the purpose of normalization, when we change the values of these parameters, we adjust the value of the discount factor β so that the steady-state growth rate under the baseline policy continues to be equal to 1.6 percent.

Figure 2 plots the results for the changes in risk aversion. We see that the steady-state debt-to-output ratio is very sensitive to the choice of the degree of risk aversion. It is about -100 percent when $\gamma = 1$, and about 200 percent when $\gamma = 9$. Regarding the tax rates, the risk aversion coefficient affects the capital tax rate τ_k much more than the labor tax rate τ_h.

Figure 3 then shows how the Ramsey steady state is affected by the magnitude of the idiosyncratic risk. Again, the steady-state debt-to-output ratio varies a lot. It is negative and large (-200 percent) when there is no idiosyncratic risk ($\text{std}(\theta) = 0$), in accord with the findings mentioned above from the complete market literature. The debt ratio then gets larger when the risk increases, reaching a zero level when $\text{std}(\theta)$ is near its baseline level, 0.1585, and a positive level of about 60 percent when $\text{std}(\theta) = 0.2$. The two tax rates are very similar when the amount of idiosyncratic risk is moderate ($\text{std}(\theta) < 0.1$), but when it gets large ($\text{std}(\theta) > 0.1$), the labor tax rate τ_h becomes much less sensitive to the change in the amount of idiosyncratic risk. Most of the increase in the steady state level of debt is then financed with an increase in τ_k.

In contrast to risk aversion and the amount of idiosyncratic risk, the value of the intertemporal elasticity of substitution does not affect the Ramsey steady state much, as illustrated in Figure 4. Any effect that the intertemporal elasticity of substitution has on the Ramsey steady state is offset by the change in the value of β needed for the normalization purpose (to keep the growth rate under the baseline policy unchanged).

7 Conclusion

In this paper we have developed a tractable infinite horizon model and examined the optimal tax of labor and capital as well as the optimal path of debt when there are uninsurable idiosyncratic shocks to the labor income. Our results can be summarized by the two general principles for public policy in the presence of idiosyncratic income risks. That is, (i) design taxes so as to increase the degree of insurance provided against the idiosyncratic risks; (ii) allocate the tax burdens efficiently over time. In the environment considered, the first principle calls for taxing risky labor income, the second for issuing debt and taxing capital income.

For the sake of tractability and clarity, we have made a number of simplifying assumptions in this paper. As we noted in Introduction, the Ramsey problem is difficult to solve in general, so we contend that the benefit from deriving an explicit solution exceeds the cost of loss of generality. We readily admit however that our results might be sensitive to these assumptions, and it is important to examine the robustness of our findings in more general environments.

We conclude by briefly discussing two generalizations of the analysis. The first one concerns our present focus on linear taxes. This is in line with most of the literature on Ramsey taxation. Allowing for non-linear taxes clearly expands the set of tax equilibria, at the same time raises the issue of what is the information available to the government on consumers' trades, that determines which forms of non-linearity of taxes can be implemented. With taxes exhibiting quite strong forms of non-linearity the first best can be attained, however their informational requirements are quite strong. Thus we speculate that our findings would have some counterpart when taxes exhibiting more limited forms of non-linearity are available for the government.

The second extension regards the role of income distribution. In our model, each individual's lifetime utility is given by $u_{i,0} = v_0 x_{i,0}$, where $x_{i,0}$ is his/her wealth in period 0 and v_0 is a constant common across all individuals. The optimal tax and debt policy is the one that allows to attain the highest possible value for this common constant v_0, hence the income inequality has no effect on the optimal policy. In more general environments taxes also change the income distribution across the households, and their welfare effects may be different for different types of consumers. It is then an important task to generalize the analysis to address this important question.

References

[1] Árpád Ábrahám, and Eva Cárceles-Poveda. 2010. "Endogenous trading constraints with incomplete asset markets." *Journal of Economic Theory*, 145, 974-1004.

[2] Aiyagari, S. Rao. 1995. "Optimal capital income taxation with incomplete markets and borrowing constraints." *Journal of Political Economy*, 103, 1158-1175.

[3] Aiyagari, S. Rao, and Ellen R. McGrattan. 1998. "The Optimum Quantity of Debt." *Journal of Monetary Economics*, 42, 447-469.

[4] Angeletos, George-Marios. 2007. "Uninsured idiosyncratic investment risk and aggregate saving." *Review of Economic Dynamics*, 10 (1), 1-30.

[5] Bassetto, Marco, and Narayana Kocherlakota. 2004. "On the irrelevance of government debt when taxes are distortionary." *Journal of Monetary Economics*, 51, 299-304.

[6] Chamley, Christophe. 1986. "Optimal taxation of capital income in general equilibrium with infinite lives." *Econometrica*, 54, 607-622.

[7] Chari, V. V., Lawrence J. Christiano, and Patrick J. Kehoe. 1994. "Optimal fiscal policy in a business cycle model." *Journal of Political Economy*, 102, 617-652.

[8] Chari, V. V., and Patrick J. Kehoe. 1999. "Optimal fiscal and monetary policy." In John B. Taylor and Michael Woodford (eds.), *Handbook of Macroeconomics, Vol. 1C*, Amsterdam: North-Holland.

[9] Conesa, Juan Carlos, Sagiri Kitao, and Dirk Krueger. 2009. "Taxing capital? Not a bad idea after all!" *American Economic Review*, 99, 25-48.

[10] Constantinides, George M., and Darrell Duffie. 1996. "Asset pricing with heterogeneous consumers." *Journal of Political Economy*, 104 (2), 437-467.

[11] Davila, Julio, Jay H. Hong, Per Krusell, and José-Víctor Ríos-Rull. 2005. "Constrained efficiency in the neoclassical growth model with uninsurable idiosyncratic shocks." Mimeo.

[12] Epstein, Larry G., and Stanley E. Zin. 1991. "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis." *Journal of Political Economy*, 99, 263-286.

[13] Fukushima, Kenichi. 2010. "Quantifying the welfare gains from flexible dynamic income tax systems." Mimeo.

[14] Gottardi, Piero, Atsushi Kajii, and Tomoyuki Nakajima. 2009. "Constrained inefficiency and optimal taxation under uninsurable risks." Mimeo.

[15] İmrohoroğlu, Selahattin. 1998. "A quantitative analysis of capital income taxation." *International Economic Review*, 39, 307-328.

[16] Jones, Larry E., and Rodolfo E. Manuelli. 1990. "A convex model of equilibrium growth: Theory and policy implications," *Journal of Political Economy* 98, 1008-1038.

[17] Jones, Larry E., and Rodolfo E. Manuelli, and Peter E. Rossi. 1997. "On the optimal taxation of capital income." *Journal of Economic Theory*, 73, 93-117.

[18] Judd, Kenneth L. 1985. "Redistributive taxation in a simple perfect foresight model." *Journal of Public Economics*, 28, 59-83.

[19] Kocherlakota, Narayana R. 2005. "Zero expected wealth taxes: A Mirrlees approach to dynamic optimal taxation." *Econometrica*, 73, 1587-1621.

[20] Krebs, Tom. 2003. "Human capital risk and economic growth." *Quarterly Journal of Economcs*, 118, 709-744.

[21] Ljungqvist, Lars, and Thomas J. Sargent. 2004. *Recursive macroeconomic theory*, Cambridge, Mass: MIT Press, Second Edition.

[22] Lucas, Robert E., Jr., and Nancy L. Stokey. 1983. "Optimal monetary and fiscal policy in an economy without capital." *Journal of Monetary Economics*, 12, 55-94.

[23] Marcet, Albert, Francesc Obiols-Homs, and Philippe Weil. 2007. "Incomplete markets, labor supply and capital accumulation." *Journal of Monetary Economics*, 54, 2621-2635.

[24] Meghir, Costas, and Luigi Pistaferri. 2004. "Income variance dynamics and heterogeneity." *Econometrica*, 72 (1), 1-32.

[25] Storesletten, Kjetil, Chris I. Telmer, and Amir Yaron. 2004. "Cyclical dynamics in idiosyncratic labor market risk." *Journal of Political Economy*, 112 (3), 695-717.

[26] Weil, Philippe. 1990. "Nonexpected utility in macroeconomics." *Quarterly Journal of Economics*, 105, 29-42.

[27] Zhu, Xiaodong. 1992. "Optimal fiscal policy in a stochastic growth model." *Journal of Economic Theory*, 58, 250-289.

Table 1: Baseline parameter values

parameter	value	description
ψ	1	intertemporal elasticity of substitution
γ	3	risk aversion coefficient
A	0.315	coefficient in the production function
α	0.36	share of capital
δ_k	0.06	depreciation rate of physical capital
δ_h	0.06	depreciation rate of human capital
β	0.9511	discount factor
g	0.0256	government purchases as a fraction of total wealth
τ	0.1955	tax rate in the baseline policy $(\tau_{k,t} = \tau_{h,t} = \tau)$
$\bar{\theta}$	0.1585	idiosyncratic shock

Table 2: Steady states

	model notation	baseline	Ramsey	Ramsey with $g = 0$
capital tax rate (%)	τ_k	19.95	19.64	11.56
labor tax rate (%)	τ_h	19.95	14.88	4.99
debt-GDP ratio (%)	$\frac{B_{t-1}}{Y_t}$	51	0.19	202.6
growth rate (%)	$\frac{Y_{t+1}}{Y_t} - 1$	1.6	2.26	3.25

Table 3: Welfare gain of adopting the Ramsey policy

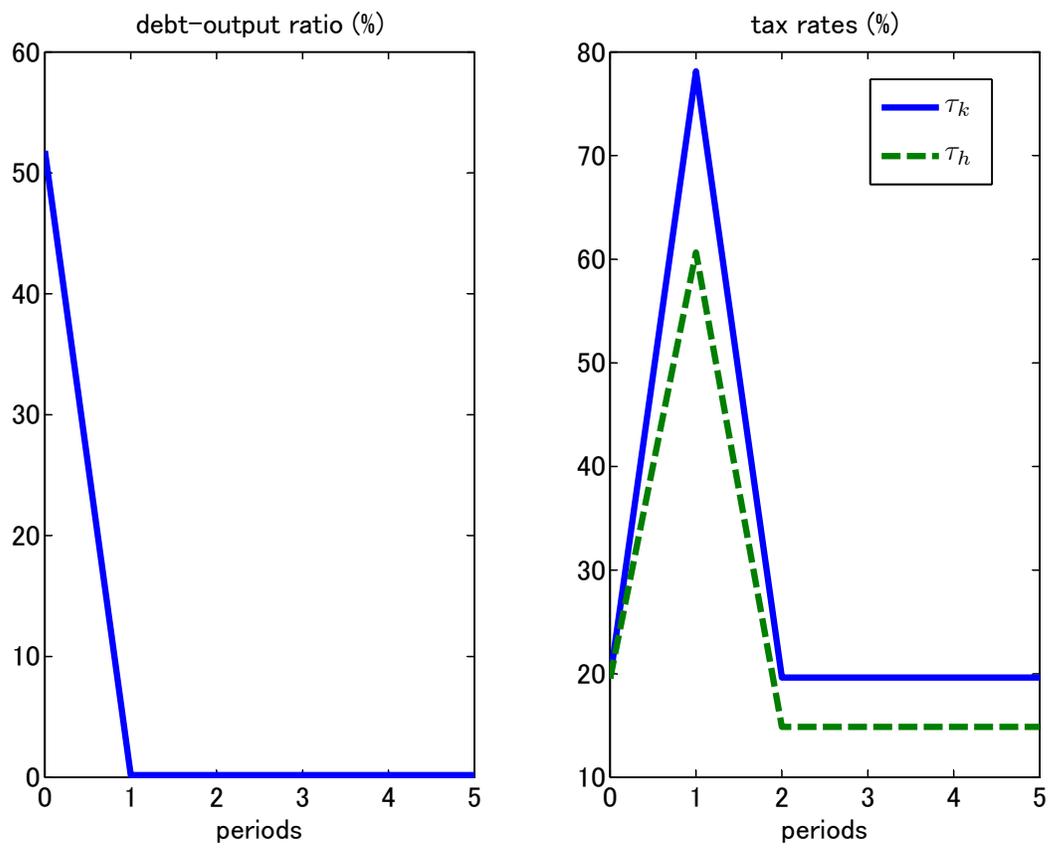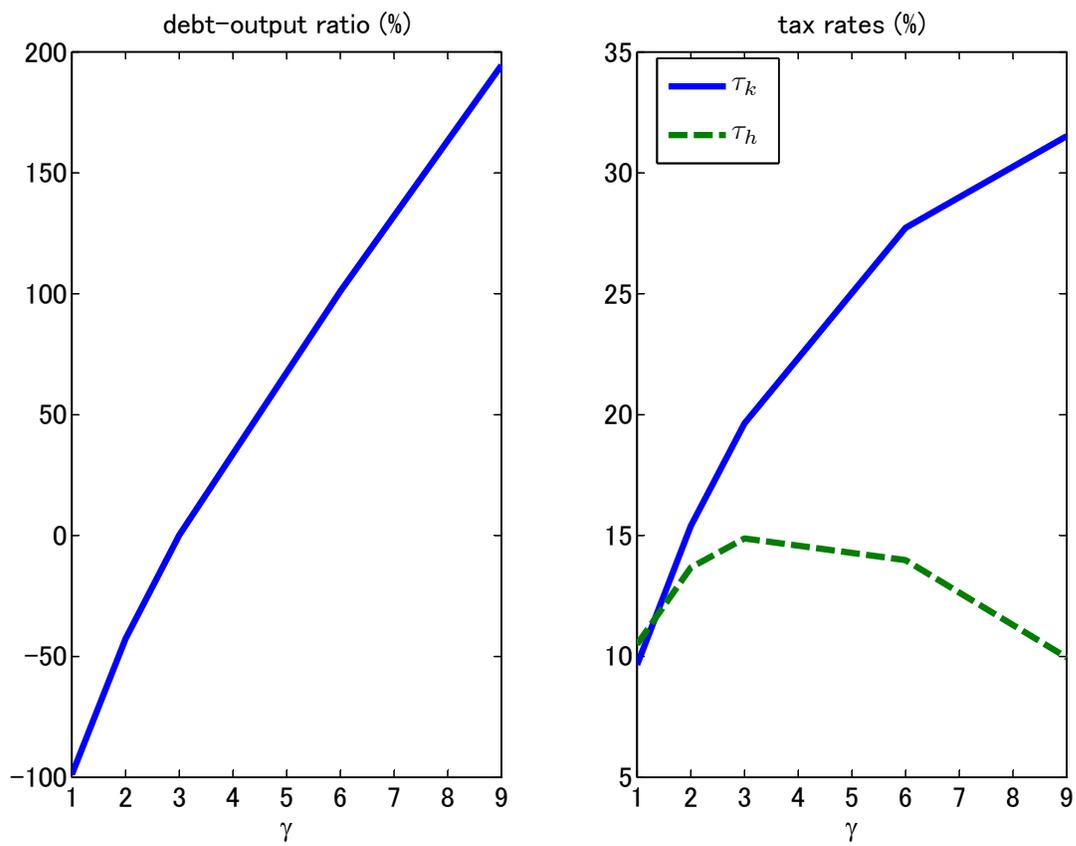| ignoring transition	considering transition
8.7320	0.8494

Figure 1: Transitional dynamics.

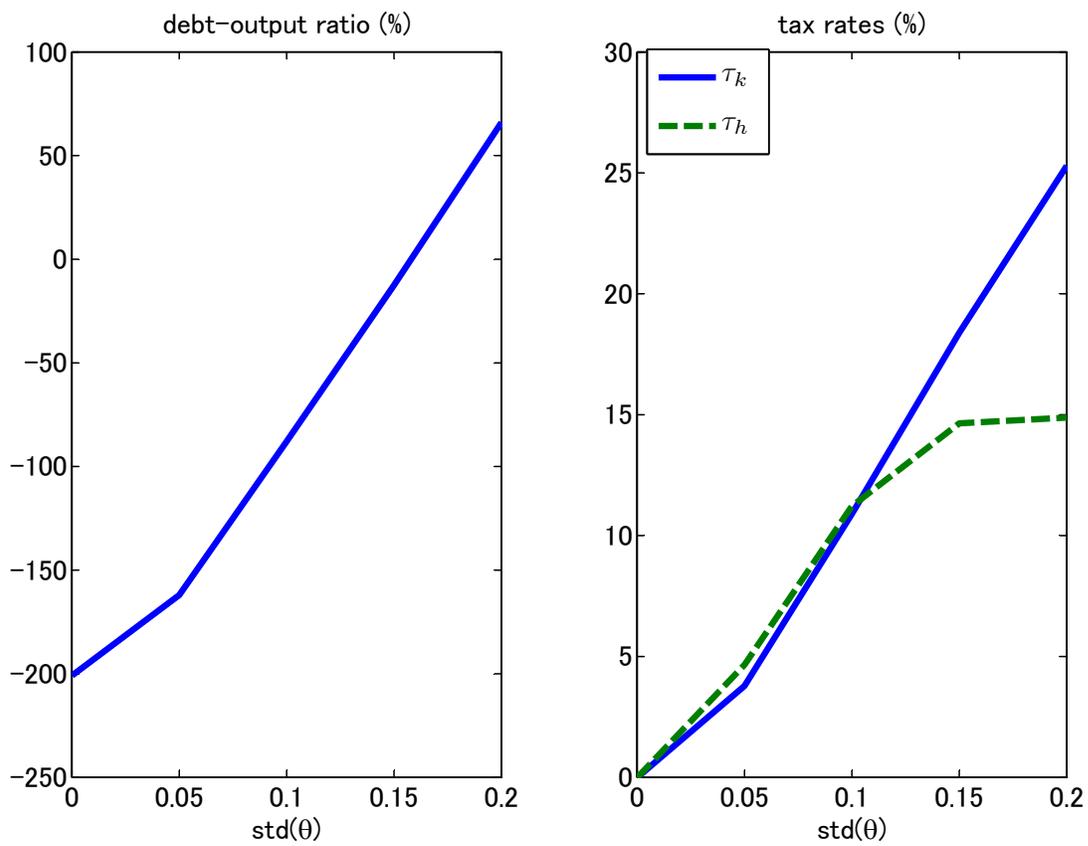Figure 2: Different values of risk aversion.

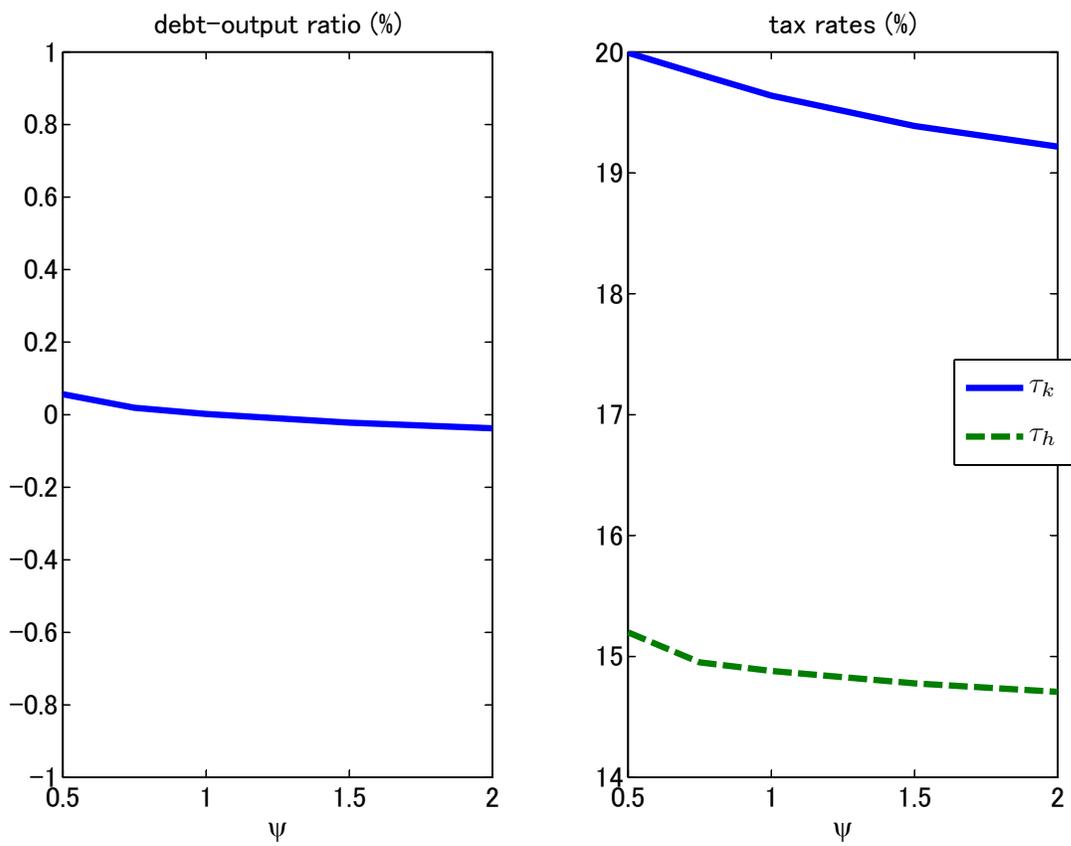Figure 3: Different values of the idiosyncratic risk.

Figure 4: Different values of the intertemporal elasticity of substitution.

8 Appendix

Proof of Lemma 5

In this proof, all derivatives are evaluated at the competitive equilibrium with zero taxes, $\tau_k = \tau_h = 0$, unless otherwise stated. First notice that

$$\frac{dr_k}{d\tau_k} = -F_k + (-F_{kk} + F_{kh})\frac{d\eta_h}{d\tau_k}. \tag{48}$$

Next, differentiating the function f in (30) and evaluating it at $\tau_k = 0$, we obtain

$$\left\{-f_r F_k + f_w \frac{\alpha}{1-\alpha} F_h\right\} d\tau_k + \{f_r(-F_{kk} + F_{kh}) + f_w(-F_{hk} + F_{hh}) + f_\eta\} d\eta_h = 0.$$

Note here that

$$\frac{\alpha}{1-\alpha} F_h = F_k \frac{1-\eta_h}{\eta_h},$$

$$-F_{hk} + F_{hh} = \frac{1-\eta_h}{\eta_h}(F_{kk} - F_{kh}).$$

It then follows that

$$\frac{d\eta_h}{d\tau_k} = \frac{F_k\left(-f_r + f_w \frac{1-\eta_h}{\eta_h}\right)}{(-F_{kk} + F_{kh})\left(-f_r + f_w \frac{1-\eta_h}{\eta_h}\right) - f_\eta}. \tag{49}$$

From (48) and (49), we obtain

$$\frac{dr_k}{d\tau_k} = \frac{F_k f_\eta}{(-F_{kk} + F_{kh})\left(-f_r + f_w \frac{1-\eta_h}{\eta_h}\right) - f_\eta} < 0.$$

The last inequality follows from the fact that $F_k > 0$, $f_\eta < 0$, $-F_{kk} + F_{kh} > 0$, $f_r < 0$ and $f_w > 0$.

Proof of Proposition 8

As shown in the text, (41) holds. Using then also (15) we see that to determine the effect of $\bar{b}_{T+1}$ on $v_0(\bar{b}_{T+1})$ it suffices to determine the effect of $\bar{b}_{T+1}$ on $v_T(\bar{b}_{T+1})$:

$$\frac{dv_0}{db_{T+1}} \gtreqless 0 \qquad \Longleftrightarrow \qquad \frac{dv_T}{db_{T+1}} \gtreqless 0$$

For this, let us first see how b_{T+1} affects $\eta_{c,T+1}$ and hence, given (38), v_{T+1}. Recall that

$$v_{T+1}(\bar{b}_{T+1}) = \left\{(1-\beta)^\psi + \beta^\psi \rho_{T+2}(\bar{b}_{T+1})^{\psi-1} v_{T+2}(\bar{b}_{T+1})^{\psi-1}\right\}^{\frac{1}{\psi-1}}, \tag{50}$$

$\rho_{T+2}(\bar{b}_{T+1}) = \tilde{\rho}(\bar{b}_{T+1}, 0, \eta_{c,T+1}(\bar{b}_{T+1}))$ and $\eta_{c,T+2}(\bar{b}_{T+1}) = \eta_c^o$. It follows from Lemma 7 that

$$\left.\frac{\partial \rho_{T+2}}{\partial \eta_{c,T+1}}\right|_{\bar{b}_{T+1}=0} = 0.$$

Differentiating v_{T+1} with respect to $\bar{b}_{T+1}$ in (50) and evaluating it at $\bar{b}_{T+1} = 0$ yields (since by (41) $\partial v_{T+2}/\partial \bar{b}_{T+1} = 0$)

$$\frac{dv_{T+1}}{d\bar{b}_{T+1}} = \beta^{\psi}(\rho^o)^{\psi-2}\rho_1^o v^o, \tag{51}$$

where $\rho_1^o \equiv \partial \tilde{\rho}(b, 0, \eta_c^o)/\partial b$ evaluated at $b = 0$.

Next, look at the equation analogous to (50) for date T:

$$v_T(\bar{b}_{T+1}) = \left\{ (1 - \beta)^{\psi} + \beta^{\psi} \left(\rho_{T+1}(\bar{b}_{T+1}) \right)^{\psi-1} v_{T+1}(\bar{b}_{T+1})^{\psi-1} \right\}^{\frac{1}{\psi-1}} \tag{52}$$

and differentiate v_T with respect to $\bar{b}_{T+1}$. This derivative, when evaluated at $\bar{b}_{T+1} = 0$, using (51) and the fact that at this value we have $\frac{\partial \rho_{T+1}}{\partial \eta_{c,T}} = 0$, equals

$$\frac{dv_T}{d\bar{b}_{T+1}} = \beta^{\psi}(\rho^o)^{\psi-2}(v^o)^{-1} \left[\rho_2^o + \beta^{\psi}(\rho^o)^{\psi-1}\rho_1^o \right],$$

where $\rho_2^o \equiv \partial \tilde{\rho}(0, b', \eta_c^o)/\partial b'$ evaluated at $b' = 0$.

Now remember the definition of the function $\tilde{\rho}(b, b', \eta_c)$ in problem (34). Let $\lambda(b, b', \eta_c)$ denote the Lagrange multiplier on the flow budget constraint for the government in that maximization problem. Let $\eta_h(b, b', \eta_c)$, $r_k(b, b', \eta_c)$, $w(b, b', \eta_c)$, and $R_x(b, b', \eta_c)$ denote its solution, and define

$$F_k(b, b', \eta_c) \equiv F_k[(1 - \eta_c)(1 - \eta_h(b, b', \eta_c)) - b, (1 - \eta_c)\eta_h(b, b', \eta_c)].$$

Then, using the envelope property, since b, b' only appear in constraint (35) of this problem,

$$\frac{\partial \tilde{\rho}}{\partial b} = -\lambda(b, b', \eta_c) \left[1 - \delta_k + F_k(b, b', \eta_c) \right],$$

$$\frac{\partial \tilde{\rho}}{\partial b'} = \lambda(b, b', \eta_c)(1 - \eta_c) R_x(b, b', \eta_c).$$

Hence, in the optimal tax equilibrium under the constraint $b_t = g_t = 0$ for all t, we have[16]

$$\rho_1^o = -\lambda^o(1 - \delta_k + F_k^o),$$

$$\rho_2^o = \lambda^o \beta^{\psi}(\rho^o)^{\psi-1} R_x^o,$$

since

$$\eta_c^o = 1 - \beta^{\psi}(\rho^o)^{\psi-1}.$$

Therefore, we obtain

$$\frac{dv_T}{d\bar{b}_{T+1}} = \xi \left[R_x^o - (1 - \delta_k + F_k^o) \right], \tag{53}$$

---

[16]To better understand the form of these expressions, notice that, as we see from (35), a marginal increase of $\bar{b}_{T+1}$ relaxes this constraint at $T + 1$ yielding a gain of $\lambda^o(1 - \eta_c) R_x$, while tightening this constraint at $T + 2$ with a loss of $\lambda o \beta^{\psi}(\rho o)^{\psi-1}(1 - \delta_k + F_k^o)$. Since $(1 - \eta_c) = \beta^{\psi}(\rho o)^{\psi-1}$, the comparison of these two reduce to the comparison between R_x and $(1 - \delta_k + F_k^o)$.

where ξ is a positive constant defined by

$$\xi \equiv \beta^\psi (\rho^o)^{\psi-1} \lambda^o \beta^\psi (\rho^o)^{\psi-1} > 0.$$

That $\xi > 0$ follows from the fact that $\lambda^o > 0$, which is shown at the end of this proof.

Under Assumptions 1 and 2, as argued in Section 4, we have

$$\tau_k^o < 0.$$

Because the investment in human capital is risky, its expected return must be greater than the rate of return on physical capital:

$$1 - \delta_k + (1 - \tau_k^o)F_k^o < 1 - \delta_h + (1 - \tau_h^o)F_h^o$$

It follows that

$$\begin{aligned}
R_x^o &= \left[1 - \delta_k + (1 - \tau_k^o)F_k^o\right](1 - \eta_h^o) + \left[1 - \delta_h + (1 - \tau_h^o)F_h^o\right]\eta_h^o \\
&> 1 - \delta_k + (1 - \tau_k^o)F_k^o \\
&> 1 - \delta_k + F_k^o
\end{aligned} \tag{54}$$

From (53) and (54) it follows that $\frac{dv_0}{d\bar{b}_{T+1}} > 0$ at $g = 0$. By continuity, this is also the case with a sufficiently small g.

It remains then to show that $\lambda^o > 0$. Consider problem (34) under the constraint $b_t = g_t = 0$ for all t. The problem can be written as

$$\max_{\{r_k, w, \eta_h\}} \rho(r_k, w, \eta_h)$$

subject to

$$F(1 - \eta_h, \eta_h) - r_k(1 - \eta_h) - w\eta_h = 0,$$

$$\Phi(r_k, w, \eta_h) = 0,$$

where $\Phi(r_k, w, \eta_h) = \partial\rho(r_k, w, \eta_h)/\partial\eta_h$ as defined in (17). Let λ^o and μ^o be the Lagrange multipliers for these two constraints, respectively. Then the first-order conditions are

$$\frac{\partial\rho^o}{\partial r_k} - (1 - \eta_h^o)\lambda^o + \mu^o\frac{\partial\Phi^o}{\partial r_k} = 0,$$

$$\frac{\partial\rho^o}{\partial w} - \eta_h^o\lambda^o + \mu^o\frac{\partial\Phi^o}{\partial r_k} = 0,$$

$$\mu^o\frac{\partial\Phi^o}{\partial\eta_h} + \lambda^o(-F_k^o + F_h^o + r_k^o - w^o) = 0.$$

where the superscript ^{o} indicates, as before, variables evaluated at a solution of the Ramsey problem under the constraint $b_t = g_t = 0$ for all t. In this case the government budget constraint can be rewritten as $(r_k^o - F_k^o)(1 - \eta_h^o) + (w^o - F_h^o)\eta_h^o = 0$. We have so

$$\frac{\partial \rho^o}{\partial r_k} = \lambda^o \left(1 - \eta_h^o + \frac{\frac{\partial \Phi^o}{\partial r_k}}{\frac{\partial \Phi^o}{\partial \eta_h}} \frac{r_k^o - F_k^o}{\eta_h^o}\right).$$

Here, note that: (i) $\rho_{r_k}^o > 0$, which follows from the definition of $\rho(r_k, w, \eta_h)$ and the fact that $\eta_h^o \in (0, 1)$; (ii) $\frac{\Phi_{r_k}^o}{\Phi_{\eta_h}^o} > 0$, under Assumption 1; and (iii) $r_k^o - F_k^o > 0$, under Assumption 2. Therefore, $\lambda^o > 0$.

Proof of Proposition 9

Define the Lagrangean for problem (36), using (38) and (16) to substitute for ρ_{t+1} and $\eta_{c,t}$, as

$$v_0 + \sum_{t=0}^{\infty} \lambda_t^v \left\{ (1 - \beta)^\psi + \beta^\psi \tilde{\rho}(b_t, b_{t+1}, (1 - \beta)^\psi v_t^{1-\psi})^{\psi-1} v_{t+1}^{\psi-1} - v_t^{\psi-1} \right\}.$$

The first-order condition with respect to b_{t+1} is

$$\lambda_t^v \beta^\psi \tilde{\rho}_{t+1}^{\psi-2} \tilde{\rho}_{2,t+1} v_{t+1}^{\psi-1} + \lambda_{t+1}^v \beta^\psi \tilde{\rho}_{t+2}^{\psi-2} \tilde{\rho}_{1,t+2} v_{t+2}^{\psi-1} = 0, \tag{55}$$

where $\tilde{\rho}_{t+1} \equiv \tilde{\rho}(b_t, b_{t+1}, \eta_{c,t})$, $\rho_{2,t+1} \equiv \partial \tilde{\rho}(b_t, b_{t+1}, \eta_{c,t})/\partial b_{t+1}$, and $\rho_{1,t+2} \equiv \partial \tilde{\rho}(b_{t+1}, b_{t+2}, \eta_{c,t+1})/\partial b_{t+1}$. The first-order condition for v_{t+1} is

$$\lambda_t^v \beta^\psi \tilde{\rho}_{t+1}^{\psi-1} v_{t+1}^{\psi-2} + \lambda_{t+1}^v \beta^\psi \tilde{\rho}_{t+2}^{\psi-2} \rho_{\eta c,t+2}(1-\beta)^\psi (1 - \psi) v_{t+1}^{-\psi} v_{t+2}^{\psi-1} - \lambda_{t+1}^v v_{t+1}^{\psi-2} = 0, \tag{56}$$

where $\rho_{\eta c,t+2} \equiv \partial \tilde{\rho}(b_{t+1}, b_{t+2}, \eta_{c,t+1})/\partial \eta_{c,t+1}$.

In a steady-state equilibrium, equation (55) becomes

$$\rho_2 + \frac{\lambda_{t+1}^v}{\lambda_t^v} \rho_1 = 0 \tag{57}$$

whereas equation (56) implies that

$$\frac{\lambda_{t+1}^v}{\lambda_t^v} = \beta^\psi \tilde{\rho}^{\psi-1} \left(1 - \beta^\psi \tilde{\rho}^{\psi-1}(1-\beta)^\psi (1 - \psi) \frac{\rho_{\eta c} v^{1-\psi}}{\tilde{\rho}}\right)^{-1} \tag{58}$$

Now notice that from the definition of $\tilde{\rho}(b, b', \eta_c)$ in (34), the derivative of $\tilde{\rho}$ with respect to b and b' is given by the derivative of the first constraint times the corresponding multiplier. Then it follows that in the steady-state equilibrium,

$$-\frac{\tilde{\rho}_1}{\tilde{\rho}_2} = \frac{1 - \delta_k + F_k}{(1 - \eta_c)R_x}$$
$$= \frac{1 - \delta_k + F_k}{\beta^\psi \tilde{\rho}^{\psi-1} R_x}, \tag{59}$$

44

where for the second equality, we used again (16), $\eta_c = (1-\beta)^\psi v^{1-\psi}$, and (38), $v^{\psi-1} = (1-\beta)^\psi + \beta^\psi \tilde{\rho}^{\psi-1} v^{\psi-1}$, from the constraints of (36).

Combining (57)-(59) and using again $\eta_c = (1-\beta)^\psi v^{1-\psi}$, yields

$$R_x = (1 - \delta_k + F_k) \left[1 - (1-\psi)\beta^\psi \tilde{\rho}^{\psi-2} \tilde{\rho}_{\eta_c} \eta_c \right]^{-1} \tag{60}$$

This completes the proof of Proposition 9.