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SIMPLE EXACT BAYESIAN METHODS FOR INCORPORATING DIRECTIONAL PRIOR INFORMATION BASED ON A GENERALIZED χ-DISTRIBUTION

HISASHI NOMA

Department of Biostatistics
Kyoto University School of Public Health
Yoshida Konoe-cho, Sakyo-ku
Kyoto 606-8501, Japan
e-mail: nomahi@bstat.mbox.media.kyoto-u.ac.jp

Abstract

In many clinical and epidemiological studies, the prior knowledge or belief regarding treatment effect is clearly directional, i.e., pointing to protective effects or to harmful effects. Although recent developments in Bayesian computations such as the Markov Chain Monte Carlo methods have enabled to implement flexible modeling and inference, they involve complicated techniques and require additional special softwares. In this article, we develop exact Bayesian methods that can be conducted by simple concepts and computations. We consider a simple normal-approximated likelihood model and some class of skewed prior distributions. We introduce a generalized χ-distribution, which constructs a conjugate family for the normal likelihood model, and show that it can be interpreted as a generalized model of the commonly-used normal prior model. We also show that the generalized χ-distribution is derived as a posterior distribution by a gamma-prior model. In addition, we present simple exact computational methods for Bayesian inference based on the generalized χ and gamma prior models. An application to an

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epidemiological study on the association of residential wire codes and magnetic fields with childhood leukemia is provided.

1. Introduction

Recent developments of Bayesian methodology have enabled flexible modeling involving nonconjugate models for various problems in clinical and epidemiologic researches (Carlin and Louis [5], Rothman et al. [12], Spiegelhalter et al. [15]). Most of these methodologies have been based on the recent advances in Bayesian computational techniques, such as the Markov Chain Monte Carlo (MCMC) (Gamerman and Lopes [6]). In many clinical and epidemiological studies, often the prior information is directional, i.e., the prior knowledge or belief is clearly pointing to protective effects or to harmful effects. In Section 2, we introduce an epidemiological study for investigating the association of residential wire codes and magnetic fields with childhood leukemia (Savitz et al. [13]) as a motivating example. In such cases, asymmetric priors are appropriate for expressing the prior knowledge adequately, and it would be sometimes essential for subjective Bayesians (Greenland [7, 8]). In most cases, expressing the directional information on priors cannot be treated within classical conjugate model families. The MCMC methods enable to implement them without any restrictions on prior models, however, they require highly technical computational knowledge to the practitioners of data analysis.

Recently, simple alternative practical methodologies for these settings have been developed. Greenland [7] developed approximate Bayesian methods for risk and survival regression analyses under nonconjugate prior models using data augmented priors (Bedrick et al. [3, 4]). Also, Greenland [8] proposed the data augmented prior methods for 2×2 tables and stratified analyses using a flexible log-F prior. The data augmented prior methods do not need special softwares, only require standard frequentist packages. In addition, owing to the simplicity of the computations, the data augmented prior methods can be a useful tool for checking simulation validity of the MCMC methods (Greenland [7]).

In this article, we consider another simple approach for the same purpose. We adopt the classical asymptotic normal approximation for the likelihood model, and construct priors using a flexible parametric class of continuous distributions. We introduce a generalized χ-distribution, which constructs a conjugate class of the commonly-used normal likelihood model, and show that the exact computation of the distribution summaries can be simply implemented. Also, it can be interpreted as a generalized class of the well-known conjugate normal prior model of the normal likelihood model. Furthermore, we reveal that the generalized χ-distribution is derived from a gamma-prior model for the normal likelihood model, and exact Bayesian inference by the gamma-normal model can be conducted with the simple exact computations. In addition to the technical and computational simplicities, the proposed methods have advantages that exact expressions of distribution summaries and graphical displays can be obtained simply and explicitly. These merits are especially important for prior elicitations and intuitive interpretations of the results.

2. The Motivating Example: Residential Magnetic Fields and Childhood Leukemia

Rothman et al. [12] (Chapter 18) illustrated Bayesian methods for epidemiological data analyses through a case-control study of residential wire codes and childhood leukemia. Table 1 shows the case-control data from Savitz et al. [13], which was the first widely published work to report an association between household wiring and leukemia, where the cases with the disease were linked with higher levels of magnetic fields. 3 cases and 5 controls had estimated average fields above a 3 milligause (mG) cutpoint, and 33 cases and 193 controls had those below 3 mG. The estimated odds-ratio (OR) between the two exposure levels is $3\ (193)/5\ (33) = 3.51,$ and the estimated variance of log OR is $(1/3 + 1/33 + 1/5 + 1/193) = 0.569.$ The Wald-type approximate 95% confidence limits are $\exp[\log(3.51) \pm 1.96(0.569)^{1/2}] = 0.80,\ 15.4.$

Table 1. Case-control data on residential magnetic fields and childhood leukemia (Savitz et al. [13])

	Case	Control	
> 3mG	3	5	
≤ 3mG	33	193	Odds-ratio = 3.51
Total	36	198	

$y = \text{log odds-ratio} = \log(3.51) = 1.26$

$\sigma^2 = \text{estimated variance} = 0.569$

95% Wald confidence limits $= \exp\{1.26 \pm 1.96(0.569)^{1/2}\} = 0.80,\ 15.4$

After this study, several epidemiologic researches also investigated the relationships between household wiring and leukemia. However, these results are not consistent; some of these studies replicated the positive associations, but others reported non-positive results (Greenland et al. [9]). In addition, the laboratory evidence and the mechanistic evidence failed to support a relationship between the low-level magnetic fields and changes in biological function or disease status (World Health Organization [16]). Thus, the evidence on the association between household wiring and leukemia is not strong enough to be considered causal. Based on the current knowledge, reasonable prior distributions for OR would be those supporting no association $(\mathrm{OR} = 1)$ or some harmful effects of household wiring $(\mathrm{OR} > 1)$.

For the log-transformed OR, a normal-normal model is widely adopted, in which normal priors are employed for the mean of normally distributed estimators of the log-transformed OR (e.g., Ashby et al. [2]). For the dataset in Table 1, Rothman et al. [12] also considered a conjugate normal prior, which might represent a skeptical prior that supports no association. However, this symmetric prior would be inappropriate, because the current knowledge clearly suggests one particular direction for the effect of interest; harmful effects are more supportive than preventive effects for household wiring on the disease, suggesting asymmetric priors. Greenland [7] also discussed the practical limitation of the normal prior and relevance of introducing skewness to the prior for a similar study.

3. A Generalized χ-distribution and Bayesian Analysis

3.1. Definition

In the first, we define a parametric family of skewed probability distributions that generalize the χ-distribution:

$$g_0(\theta|k) = \frac{1}{2^{k/2-1}\Gamma(k/2)}\theta^{k-1}\exp\left(-\frac{\theta^2}{2}\right), \quad (\theta > 0),$$

where k is the shape parameter $(k = 1, 2, \ldots)$ and $\Gamma(z)$ is the gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

The χ-distribution is a continuous distribution of the square root of χ^2-random variables. Also, it becomes the half-normal distribution when $k = 1$. Here, we introduce three parameters μ, τ, and λ $(-\infty < \mu < \infty,\ \tau > 0,\ -\infty < \lambda < \infty)$, which change the shape, location, and scale of the distribution, and generalize it.

Definition (A generalized χ-distribution)**.** We define a family of probability distributions as a *generalized* χ*-distribution* whose density functions are expressed as

$$g(\theta|k,\ \mu,\ \tau,\ \lambda) = \frac{\dfrac{1}{2^{k/2-1}\Gamma(k/2)}(\theta - \lambda)^{k-1}\exp\left\{-\dfrac{1}{2}\left(\dfrac{\theta-\mu}{\tau}\right)^2\right\}}{\displaystyle\int_\lambda^\infty \dfrac{1}{2^{k/2-1}\Gamma(k/2)}(\nu - \lambda)^{k-1}\exp\left\{-\dfrac{1}{2}\left(\dfrac{\nu-\mu}{\tau}\right)^2\right\}d\nu},\quad (\theta > \lambda)$$

$$= \frac{1}{C(k,\ \mu,\ \tau,\ \lambda)}\frac{1}{\sqrt{2\pi\tau^2}}(\theta - \lambda)^{k-1}\exp\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right), \tag{1}$$

where $C(k,\ \mu,\ \tau,\ \lambda)$ is the standardizing constant,

$$C(k,\ \mu,\ \tau,\ \lambda) = \int_\lambda^\infty \frac{1}{\sqrt{2\pi\tau^2}}(\nu - \lambda)^{k-1}\exp\left(-\frac{(\nu-\mu)^2}{2\tau^2}\right)d\nu.$$

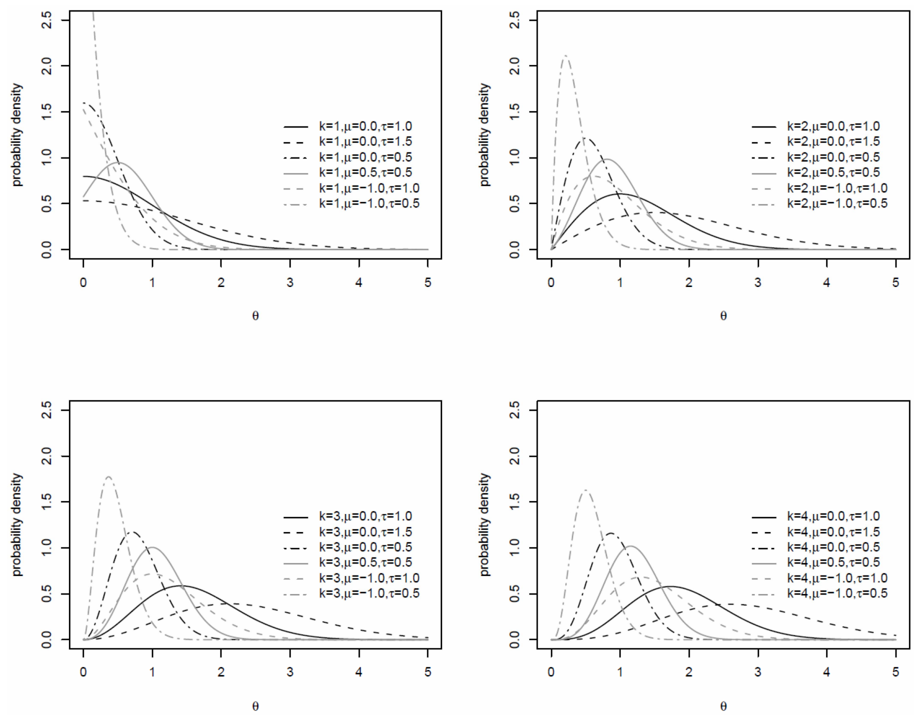

Figure 1. Some illustrative examples of the generalized χ-distribution (1). For all examples, λ is set to 0. In each panel, the combinations of μ and τ are fixed, and k is varied 1 to 4.

In Figure 1, we illustrated some examples of the generalized χ-distribution. For all panels, λ is set to 0, and combinations of μ and τ is unvaried. Each panel corresponds to $k = 1, 2, 3,$ and 4, respectively. The black curve of each panel is the standard χ-distribution with shape parameter k $(\mu = \lambda = 0, \tau = 1)$. When $k = 1$, the distribution corresponds to a truncated normal distribution of $N(\mu, \tau^2)$, truncated at λ. As $k \geq 2$, the distribution becomes a right-skewed unimodal distribution with support on (λ, ∞). On the whole, the location of the distribution shifts to right, the dispersion gets larger, and the skewness is milder when k becomes larger under the other parameters are fixed. This trend accords to that of the standard χ-distribution. Also, under a fixed k, λ, and τ, the shape of the distribution gets to sharp as μ becomes smaller (especially when $\mu < \lambda$). The shape also becomes sharp as τ gets to smaller under the other parameters unvaried.

The term "generalized χ-distribution" has also been used in Arnold and Lin [1]. Although there are some similarities between the distribution derived in Arnold and Lin [1] and (1), we discuss here from a substantially different viewpoint, the following conjugate property and computational utilities in Bayesian analysis. Besides, in this article, we consider the right-skewed version of the generalized χ-distribution consistently, but a transformation to a left-skewed version is straightforward and the same results of the following sections are also hold.

3.2. Conjugate property for the normal likelihood model

Let θ be the parameter of interest and Y be its consistent and asymptotic normal estimator, such as the maximum likelihood estimator. Consider the normal likelihood model for Bayesian analysis, often constructed by the Wald-type asymptotic approximation of the estimator $Y \mid \theta \sim N(\theta, \sigma^2)$, for example, in the case-control example in Section 2, θ corresponds to the log-transformed OR and Y corresponds to the unconditional maximum likelihood estimate of it. Here, σ^2 is assumed to be known and set to be its valid estimate. The likelihood model is a quite simplified but reasonable model in many practical situations, and has been widely adopted in medical studies (e.g., Spiegelhalter et al. [15]).

When we consider the generalized χ-distribution as the prior distribution for θ, the following result is derived.

Proposition 1. *Consider the normal likelihood model* $Y \sim N(\theta, \sigma^2)$, *and the prior distribution of* θ *as the generalized* χ*-distribution* (1). *The probability density function of the posterior distribution is obtained as*

$$\tilde{g}(\theta \mid y, k, \mu, \tau, \lambda) = \frac{1}{C(k, \tilde{\mu}, \tilde{\tau}, \lambda)} \frac{1}{\sqrt{2\pi\tilde{\tau}^2}} (\theta - \lambda)^{k-1} \exp\left(-\frac{(\theta - \tilde{\mu})^2}{2\tilde{\tau}^2}\right), \quad (\theta > \lambda),$$

where

$$\tilde{\mu} = \frac{\tau^2 y + \sigma^2 \mu}{\tau^2 + \sigma^2}, \qquad \tilde{\tau}^2 = \frac{\tau^2 \sigma^2}{\tau^2 + \sigma^2}.$$

Note that the posterior distribution belongs to the same family of the prior distribution. Therefore, the generalized χ-distribution (1) is a conjugate parametric family of the normal likelihood model.

The proposition is derived naturally, because the kernel of the generalized χ-distribution involves the kernel of the normal likelihood function. Therefore, the normal kernel part of the posterior distribution becomes the same as the well-known normal-normal model. As a special case, when $k = 1$ and $\lambda = -\infty$, the prior distribution corresponds to a non-truncated normal distribution $N(\mu, \tau^2)$. Thus, the generalized χ-prior model includes the commonly used normal-prior model.

3.3. Computations

3.3.1. Standardizing constant

The integration $C(k, \mu, \tau, \lambda)$ can be solved by a simple algorithm. First, a tractable expression for $C(k, \mu, \tau, \lambda)$ can be obtained by substituting $\eta = -(v - \mu)/\tau$,

$$C(k, \mu, \tau, \lambda) = \int_a^\infty \{\eta\tau - (\lambda - \mu)\}^{k-1}\phi(\eta)\,d\eta,$$

where $a = -(\lambda - \mu)/\tau$ and $\phi(z)$ the density function of $N(0, 1)$. Thus, the integration can be regarded as an expectation of the integrated function by the truncated standard normal distribution (truncated on $[a, \infty)$). This yields the following formula by the binomial expansion (Jawitz [10]):

$$C(k, \mu, \tau, \lambda) = \sum_{r=0}^{k-1} \binom{k-1}{r} \tau^r \{-(\lambda - \mu)\}^{k-r-1} R_r, \tag{2}$$

where

$$R_r = \int_a^\infty \eta^r \phi(\eta)\,d\eta, \qquad (r = 0, 1, 2, \ldots). \tag{3}$$

Since R_r is the rth moment of the truncated standard normal distribution, $R_0 = 1 - \Phi(a)$ and $R_1 = \phi(a)$, where $\Phi(z)$ is the cumulative distribution function of $\mathrm{N}(0, 1)$. Also, using integration by parts, a recursive rule of R_r is obtained as:

$$R_r = \phi(a)a^{r-1} + (r-1)R_{r-2}. \tag{4}$$

Then, for any k of a positive integer, $R_0, R_1, \ldots, R_{k-1}$ can be computed using the recursive expression (4). Using the calculated $R_r s$, $C(k, \mu, \tau, \lambda)$ can be directly calculated via (2).

3.3.2. Moments

The sth moment of the generalized χ-distribution $(s = 1, 2, \ldots)$ is expressed as

$$E[\theta^s] = \frac{1}{C(k, \mu, \tau, \lambda)} \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi\tau^2}} \theta^s (\theta - \lambda)^{k-1} \exp\left(-\frac{(\theta - \mu)^2}{2\tau^2} \right) d\theta.$$

As the computation in Section 3.3.1, this form can be arranged by the binomial expansion:

$$E[\theta^s] = \frac{1}{C(k, \mu, \tau, \lambda)} \sum_{u=0}^{k-1} \binom{k-1}{u} (-\lambda)^{k-u-1} J_{s+u},$$

where

$$J_{s+u} = \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi\tau^2}} \theta^{s+u} \exp\left(-\frac{(\theta - \mu)^2}{2\tau^2} \right) d\theta.$$

Using integration by parts, J_{s+u} is expressed as a function of R_r of (3),

$$J_{s+u} = \sum_{r=0}^{s+u} \binom{s+u}{r} \lambda^{s+u-r} \tau^r R_r.$$

Since R_r $(r = 1, 2, \ldots)$ can be obtained by the recursive rule (4), the posterior moment is computed via these formulae.

3.3.3. Mode

The mode can be obtained as the zero-point of the derivative of log-transformed the probability density function or λ if it is smaller than the edge point when $k = 1$. This is analytically obtained as

$$\max\left\{\lambda, \frac{1}{2}\{\lambda + \mu + \sqrt{(\lambda - \mu)^2 + 4\tau^2(k-1)}\}\right\}.$$

3.3.4. Cumulative distribution function and quantiles

The cumulative distribution function is expressed as

$$\Pr(\theta < \alpha) = \frac{1}{C(k, \mu, \tau, \lambda)} \int_{\lambda}^{\alpha} (\theta - \lambda)^{k-1} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{(\theta - \mu)^2}{2\tau^2} \right) d\theta.$$

The integration is identical to that of $C(k, \mu, \tau, \lambda)$ except for the region to be integrated. Thus, the method presented in Section 3.3.1 can be applied to the computation with small modifications. The quantiles are also explored by using the cumulative probabilities of some candidates. In Bayesian analysis, interval estimates for θ can be obtained using this result.

3.4. Derivation from the gamma-prior model

As another point of view, the generalized χ-distribution is derived as a posterior distribution of the normal likelihood model with gamma-prior distribution. Consider a three-parameter gamma distribution for the prior distribution of θ, whose density function is given by

$$h(\theta \mid \kappa, \omega, \xi) = \frac{\xi^{\kappa}}{\Gamma(\kappa)}(\theta - \omega)^{\kappa-1} e^{-\xi(\theta-\omega)}, \quad (\theta \geq \omega), \tag{5}$$

where κ is the shape parameter, that is assumed to be a positive integer as the χ-distribution (1). Also, ω is the location parameter and ξ is the scale parameter $(-\infty < \omega < \infty, \xi > 0)$. The gamma distribution is a well-investigated skewed continuous probability distribution, which can express the direction prior information flexibly. For the details of its properties, see for example, Johnson et al. [11].

Using the Bayes' theorem, the posterior density function is given by

$$\tilde{h}(\theta \mid y, \kappa, \omega, \xi) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}}(\theta - \omega)^{\kappa-1} \exp\left(-\frac{(\theta - y + \sigma^2\xi)^2}{2\sigma^2}\right)}{\int_{\omega}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}(\nu - \omega)^{\kappa-1} \exp\left(-\frac{(\nu - y + \sigma^2\xi)^2}{2\sigma^2}\right) d\nu}, \quad (\theta \geq \omega).$$

Note the functional form of the posterior density function is consistent to the generalized χ-distribution (1), i.e., the following results are derived.

Proposition 2. *Consider the normal likelihood model* $Y \sim N(\theta, \sigma^2)$, *and the prior distribution of* θ *as the gamma distribution* (5). *The posterior distribution is the generalized* χ*-distribution*, *whose parameters are* $k = \kappa$, $\mu = y - \sigma^2\xi$, $\tau = \sigma$, *and* $\lambda = \omega$.

The result implies that an exact expression of the posterior distribution for the gamma-normal model can be obtained via the simple computations presented in Section 3.3. Generalization to the left-skewed gamma-distribution is also straightforward. In practical uses, a gamma distribution can express various shapes flexibly, and has an advantage for simplicity to be elicited as a prior because the three parameters κ, ω, and ξ control its skewness, location, and scale, independently. However, the restriction of κ to be a positive integer is sometimes inflexible. As noted above, skewness of a gamma distribution is completely determined by the shape parameter κ. Within the restriction, the representability of the gamma distribution is somewhat limited. However, if the limitation is not a serious matter in practice, the gamma prior model can also be a useful tool for simple exact Bayesian analysis. In addition, the posterior distribution of the gamma-normal model can be easily applied to another normal likelihood model with simple computations, because the posterior distribution is a conjugate model for the normal likelihood.

4. Applications

We revisit the case-control study in Section 2 (Savitz et al. [13]). As described in Section 2, we adopt a prior distribution for the OR, that express asymmetric prior belief with mode of 1 and $\Pr(\mathrm{OR} < 1) : \Pr(\mathrm{OR} > 1) = 1 : 2.$ We here consider three-variations for the prior of the 95% probability interval for comparative purposes: (a) [0.81, 1.93], (b) [0.69, 3.25], and (c) [0.62, 4.55], here we denote a 95% probability interval as an interval between 2.5th and 97.5th percentiles of a probability distribution. The prior (a) is the same with the adopted prior under a similar situation of Greenland [7]. For each setting, we consider three priors for the log-transformed OR by the generalized χ-distribution, the gamma distribution, and the normal distribution, respectively.

(χ-a) A generalized χ-distribution with $k = 2$, $\mu = -0.69$, $\tau = 0.43$, and $\lambda = -0.27$.

(Γ-a) A gamma distribution with $\kappa = 3$, $\omega = -0.25$, and $\xi = 7.95$.

(N-a) A normal distribution with mean 0 and variance 0.22^2.

(χ-b) A generalized χ-distribution with $k = 2$, $\mu = -1.27$, $\tau = 0.78$, and $\lambda = -0.48$.

(Γ-b) A gamma distribution with $\kappa = 3$, $\omega = -0.45$, and $\xi = 4.43$.

(N-b) A normal distribution with mean 0 and variance 0.40^2.

(χ-c) A generalized χ-distribution with $k = 2$, $\mu = -1.62$, $\tau = 1.00$, and $\lambda = -0.62$.

(Γ-c) A gamma distribution with $\kappa = 3$, $\omega = -0.59$, and $\xi = 3.41$.

(N-c) A normal distribution with mean 0 and variance 0.52^2.

Table 2. Summary for the prior distributions of the nine settings for the Bayesian analyses of the leukemia study†

						Percentiles	
	Mode	Mean	SD	Skewness	Pr(OR >1)	2.5th	97.5th
(χ-a)	1.00	1.13	0.23	0.84	0.66	0.81	1.93
(Γ-a)	1.00	1.13	0.22	1.16	0.68	0.84	1.93
(N-a)	1.00	1.00	0.13	0.00	0.50	0.65	1.54
(χ-b)	1.00	1.25	0.41	0.84	0.66	0.69	3.25
(Γ-b)	1.00	1.25	0.39	1.16	0.68	0.73	3.25
(N-b)	1.00	1.00	0.40	0.00	0.50	0.46	2.17
(χ-c)	1.00	1.33	0.53	0.84	0.66	0.62	4.55
(Γ-c)	1.00	1.34	0.51	1.16	0.68	0.67	4.64
(N-c)	1.00	1.00	0.51	0.00	0.50	0.37	2.71

†The mode, mean, and percentiles are transformed to OR scale. The SD and skewness are presented in log OR scale.

Note the means of the normal priors are set to 0, and the variances are set to those have 95% probability intervals with same widths to the settings (a), (b), and (c). Table 2 presents the summaries of the above prior distributions. The generalized χ-priors represent the considered features precisely. Although the gamma priors have small differences from the settings, they also give roughly good approximations of the prior beliefs. Also, Figure 2 presents graphical displays for demonstrating relative heights of the prior (dashed lines), likelihood (dot-dashed lines), and posterior density functions (solid lines) for the nine examples. The height of likelihood function is scaled to have area 1 under the curve.

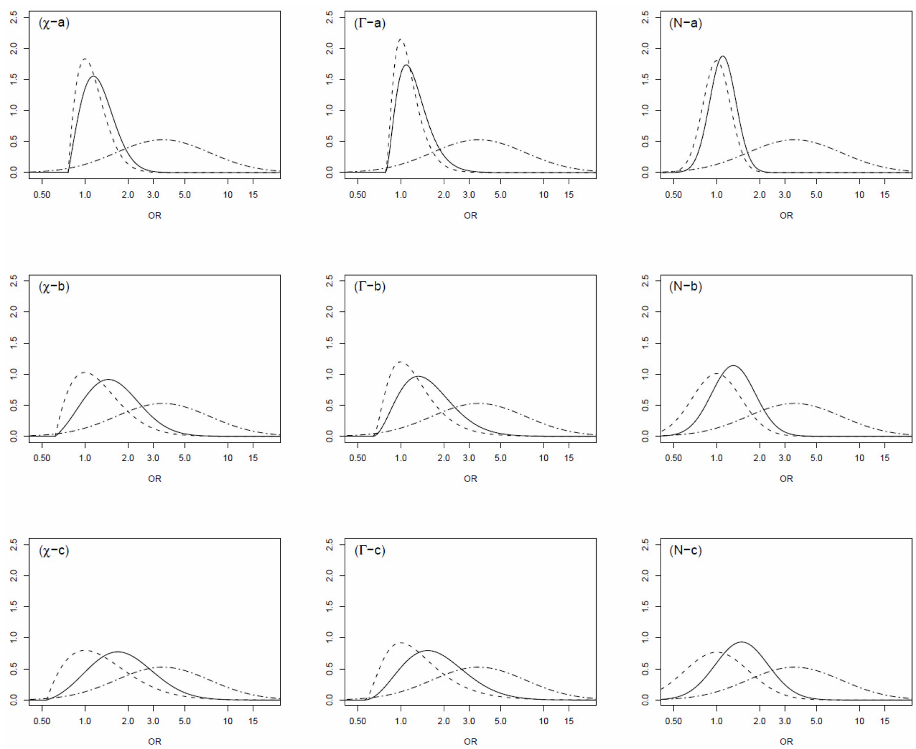

Figure 2. Graphical displays of relative heights for prior density (dashed lines), likelihood (dot-dashed lines), and posterior density functions (solid lines) for the nine examples. The height of likelihood function is scaled to have area 1 under the curve.

Table 3 provides summaries of the posterior distributions. In addition, as a current standard method of Bayesian analyses, we considered the logistic regression model:

$$\mathrm{logit}(\Pr(U = 1 \mid x)) = \beta_0 + \beta_1 x, \qquad (6)$$

where U denotes the response binary variable and x, exposure indicator. Since the binomial likelihood model (6) does not involve the asymptotic approximation, this model is exact for the case-control sampling model. We compared the results of the proposed methods with those of the binomial likelihood model assuming the same priors, for evaluating the operational characteristics of the two methods. Therefore, we adopted the nine priors in Table 2 for β_1, which is interpreted as the log-transformed OR, and a non-informative proper prior $N(0, 10^{10})$ for β_0. Based on 500,000 draws after burn-in via the Metropolis-Hasting algorithm (Gamerman and Lopes [6]), approximate summaries of posterior distributions for β_1 are presented in Table 3.

Table 3. Posterior summaries of the nine prior models for the leukemia case-control study†

						Percentiles	
	Mode	Mean	SD	Skewness	Pr(OR > 1)	2.5th	97.5th
Exact posterior summaries by the normal likelihood model							
(χ-a)	1.15	1.25	0.25	0.59	0.80	0.84	2.20
(Γ-a)	1.09	1.25	0.25	0.94	0.80	0.86	2.27
(N-a)	1.11	1.11	0.21	0.00	0.68	0.73	1.68
(χ-b)	1.46	1.60	0.42	0.41	0.86	0.77	3.93
(Γ-b)	1.33	1.56	0.42	0.65	0.86	0.80	4.01
(N-b)	1.31	1.31	0.35	0.00	0.78	0.66	2.60
(χ-c)	1.70	1.83	0.50	0.31	0.89	0.75	5.19
(Γ-c)	1.54	1.78	0.50	0.51	0.88	0.77	5.25
(N-c)	1.50	1.50	0.43	0.00	0.83	0.65	3.46
Posterior summaries under the logistic model (6) computed by MCMC							
(χ-a)	1.17	1.23	0.25	0.63	0.78	0.83	2.18
(Γ-a)	1.07	1.23	0.25	0.98	0.78	0.86	2.23
(N-a)	1.09	1.09	0.22	0.01	0.65	0.71	1.66
(χ-b)	1.39	1.55	0.43	0.43	0.84	0.75	3.90
(Γ-b)	1.34	1.51	0.42	0.68	0.83	0.78	3.94
(N-b)	1.23	1.26	0.37	–0.03	0.74	0.61	2.57
(χ-c)	1.60	1.76	0.51	0.32	0.86	0.71	5.12
(Γ-c)	1.48	1.72	0.51	0.53	0.86	0.75	5.19
(N-c)	1.43	1.45	0.45	–0.04	0.79	0.58	3.41

†The mode, mean, and percentiles are transformed to OR scale. The SD and skewness are presented in log OR scale.

The obtained posterior distributions for the generalized χ-prior and gamma-prior models are clearly indicate directional briefs after observed the result. In comparison, skewness of the gamma prior models are greater than those of the generalized χ-prior models, because the approximation errors of the gamma-priors caused by the restriction of κ. Also, due to the same reason, widths of the 95% probability intervals of the gamma-priors are slightly greater. Besides, the posterior distributions of the normal prior models have narrower credible intervals, regardless of the same widths of the prior probability intervals. Furthermore, the 95% credible interval of the generalized χ-prior and gamma-prior models are wider than those of the priors, and the posterior variances are larger than those of the priors for settings (b) and (c). This result cannot occur for the normal-prior model. Also, the prior precisions are properly reflected in the posterior distributions, compared with the (a) ~ (c) prior models.

Lastly, the exact posterior summaries and those of the logistic model (b) are nearly accorded, although a bit differences are remained. There is a consistent trend for the differences of the locations of modes, means, and 95% credible intervals. Also, the skewness of the proposed methods is a bit milder than that of the latter results. However, the magnitudes are quite small. As a whole, the differences of the two results are so small that cannot influence the conclusions seriously.

5. Discussion

In this article, we provide an exact Bayesian analysis using a generalized χ-priors and gamma priors for the normal likelihood models, which can be easily implemented using standard softwares. The two families of parametric probability distributions can represent a broad class of continuous probability distributions with skewness. Our method is to allow the adaptation or elicitation of flexible asymmetric prior distributions with ease of computation. As such, it is especially useful for the common situations where one would like to elicit skewed priors, such as the example in Section 2.

Although the normal likelihood model is founded on the simplified assumption for σ^2 to be known, and asymptotic normality of Y, they would not raise serious problems for approximations, owing to the results of the comparisons with the MCMC results presented in Section 4. The differences would be caused by two reasons: the numerical error of the Monte Carlo integrations and the differences of the likelihood models. Since the number of samples of the MCMC is sufficiently large, magnitudes of the former numerical errors cannot be so large. Therefore, most of the differences would be come from the latter reason, but the magnitudes of the differences are quite small. The differences would not also influence the conclusions seriously under such practical situations.

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