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A REDUCTION OF THE TARGET OF THE JOHNSON HOMOMORPHISMS OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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ABSTRACT. Let F_n be a free group of rank n and F_n^N the quotient group of F_n by a subgroup $[\Gamma_n(3), \Gamma_n(3)][[\Gamma_n(2), \Gamma_n(2)], \Gamma_n(2)]$, where $\Gamma_n(k)$ denotes the k-th subgroup of the lower central series of the free group F_n. In this paper, we determine the group structure of the graded quotients of the lower central series of the group F_n^N by using a generalized Chen's integration in free groups. Then we apply it to the study of the Johnson homomorphisms of the automorphism group of F_n. In particular, under taking a reduction of the target of the Johnson homomorphism induced from a quotient map $F_n \to F_n^N$, we see that there appear only two irreducible components, the Morita obstruction $S^k H_{\mathbf{Q}}$ and the Schur-Weyl module of type $H_{\mathbf{Q}}^{[k-2,1^2]}$, in the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbf{Q}} = \tau'_k \otimes \mathrm{id}_{\mathbf{Q}}$ for $k \geq 5$ and $n \geq k+2$.

1. Introduction

Let F_n be a free group of rank $n \geq 2$, and let $\mathrm{Aut}\, F_n$ be the automorphism group of F_n. Let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ denote the natural homomorphism induced from the abelianization $F_n \to H$ of F_n. The kernel of ρ is called the IA-automorphism group of F_n, denoted by IA_n. The group IA_n reflects much of the richness and complexity of the structure of $\mathrm{Aut}\, F_n$ and plays important roles on various studies of $\mathrm{Aut}\, F_n$.

Although the study of the IA-automorphism group has a long history, the combinatorial group structure of IA_n is still quite complicated. In 1935, Magnus [14] obtained finitely many generators of IA_n. Nielsen [21] showed that IA_2 coincides with the inner automorphism group of F_2; hence, it is isomorphic to F_2. In general, however, any presentation for IA_n is not known. Krstić and McCool [13] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is also not known whether IA_n is finitely presentable or not.

The purpose of our research is to clarify the group structure of IA_n. In particular, we are interested in determining the graded quotients of the Johnson filtration of $\mathrm{Aut}\, F_n$. The Johnson filtration is a descending central series

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

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consisting of normal subgroups of $\mathrm{Aut}\,F_n$. Then a homomorphism

$$\tilde{\tau}_k : \mathcal{A}_n(k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

is defined by $\tilde{\tau}_k(\sigma) = (x \mapsto x^{-1}x^{\sigma})$ for each $k \geq 1$. The map $\tilde{\tau}_k$ induces a homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

from the k-th graded quotient of the Johnson filtration. Both $\tilde{\tau}_k$ and τ_k are called the k-th Johnson homomorphisms of the automorphism group of a free group. In particular, τ_k is a $\mathrm{GL}(n,\mathbf{Z})$-equivariant injective homomorphism. (For the details, see Subsection 2.5.) The study of the Johnson homomorphisms was originally begun in 1980 by D. Johnson [10] who determined the abelianization of the Torelli subgroup of a mapping class group of a surface in [11]. Recently, the study of the Johnson filtration and the Johnson homomorphisms of $\mathrm{Aut}\,F_n$ achieved good progress through the work of many authors, for example, [7], [12], [18], [19], [20], [24] and [26].

Through the images of the Johnson homomorphisms, we can study IA_n using infinitely many pieces of a free abelian group of finite rank. They are regarded as one-by-one approximations of IA_n, and to clarify the structure of them plays an important role in various studies of IA_n. In this paper, we are interested in determining the $\mathrm{GL}(n,\mathbf{Z})$-module structure of the cokernel of the rational Johnson homomorphisms $\tau_{k,\mathbf{Q}} = \tau_k \otimes \mathrm{id}_{\mathbf{Q}}$. Now, for $1 \leq k \leq 3$, the cokernel of $\tau_{k,\mathbf{Q}}$ is completely determined. (See [1], [24] and [26] for $k = 1$, 2 and 3, respectively.) Recently, Morita [19, 20] showed that for each $k \geq 2$, there appears the symmetric tensor product $S^k H_{\mathbf{Q}}$ of $H_{\mathbf{Q}} := H_{\mathbf{Z}} \otimes \mathbf{Q}$ in the irreducible decomposition of $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$ using trace maps. The modules $S^k H_{\mathbf{Q}}$ are the first obstructions for the surjectivity of the Johnson homomorphisms, discovered by Morita. We call them the Morita obstructions. In general, however, it is quite a hard problem to determine $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$. Even its $\mathbf{Q}$-dimension is not calculated for $k \geq 4$. One reason for the difficulty is that we cannot study the image of the Johnson homomorphisms directly since there is little information for generators of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$.

To avoid this difficulty, we consider the lower central series $\mathcal{A}'_n(1) = \mathrm{IA}_n$, $\mathcal{A}'_n(2)$, $\ldots$ of IA_n. Since the Johnson filtration is central, $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for $k \geq 1$. It was conjectured that $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis, who showed that $\mathcal{A}'_2(k) = \mathcal{A}_2(k)$ for each $k \geq 1$ and $\mathcal{A}'_3(3) = \mathcal{A}_3(3)$ in [1]. Now, we have $\mathcal{A}'_n(2) = \mathcal{A}_n(2)$ due to Cohen-Pakianathan [3, 4], Farb [5] and Kawazumi [12]. (See (3) below.) Furthermore, $\mathcal{A}'_n(3)$ has at most a finite index in $\mathcal{A}_n(3)$ due to Pettet [24]. It is, however, also difficult to determine whether $\mathcal{A}'_n(k)$ coincides with $\mathcal{A}_n(k)$ or not.

For each $k \geq 1$, set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. We can also define the Johnson homomorphisms

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

by an argument similar to that in the definition of τ_k. In general, we can consider $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$ as a $\mathrm{GL}(n,\mathbf{Z})$-equivariant submodule in $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$. Namely, by studying the structure of $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$, we obtain an upper bound on $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$. Furthermore the most important thing is that since IA_n is finitely generated by the Magnus generators, each $\mathrm{gr}^k(\mathcal{A}'_n)$ is also finitely generated by commutators of weight k among them. Therefore, it is more accessible to study the cokernel of τ'_k than that of τ_k. Now, it is known that $\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) = \mathrm{Coker}(\tau_{k,\mathbf{Q}})$ for $1 \leq k \leq 3$. In our previous paper [28], we determined the $\mathrm{GL}(n,\mathbf{Z})$-module structure of $\mathrm{Coker}(\tau'_{4,\mathbf{Q}})$ for $n \geq 6$. However, to determine the structure of $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ is still complicated in general.

One of the main purposes of the paper is to consider a reduction of the target of the Johnson homomorphism τ'_k. More precisely, let F_n^N be the quotient group of F_n by the subgroup $[\Gamma_n(3),\Gamma_n(3)][[\Gamma(2),\Gamma_n(2)],\Gamma_n(2)]$. If we let $\Gamma_n^N(k)$ be the lower central series of F_n^N and set $\mathcal{L}_n^N(k) := \Gamma_n^N(k)/\Gamma_n^N(k+1)$, we have a natural map

$$H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k+1).$$

In this paper, we consider the composition

$$\tau'_{k,N} : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k+1)$$

of τ'_k and the natural projection above. The map $\tau'_{k,N}$ is a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism. Then we show

Theorem 1 (= Theorem 5.3)**.** *For $k \geq 5$ and $n \geq k+2$,*

$$\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}}) = S^k H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[k-2,1^2]},$$

where $H_{\mathbf{Q}}^{[k-2,1^2]}$ denotes the Schur-Weyl module of $H_{\mathbf{Q}}$ corresponding to the partition $[k-2,1^2]$ of k.

This shows that $H_{\mathbf{Q}}^{[k-2,1^2]}$ also appears in the irreducible decomposition of $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ for $n \geq k+2$. This work is an analogue and a certain extension of our previous work [27] in which we were concerned with the Johnson homomorphisms of the automorphism group of a free metabelian group. In particular, we showed that there appears only the Morita obstruction in the cokernel of it.

The reason why we consider the quotient group F_n^N is that the structure of the graded quotients $\mathcal{L}_n^N(k)$ of the lower central series of F_n^N is easier to handle than that of the other quotient group of F_n, for example $F_n/[\Gamma_n(3),\Gamma_n(3)]$, except for a free metabelian group. In general, although to give an irreducible decomposition of $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ is difficult, considering such a reduction of the target of the Johnson homomorphism τ'_k, we can find a new obstruction for the surjectivity of $\tau'_{k,\mathbf{Q}}$.

Before showing Theorem 1, we have to determine the group structure of each $\mathcal{L}_n^N(k)$ for $k \geq 6$. The other purpose of the paper is to show

Theorem 2 (= Theorem 4.1 and Corollary 4.1)**.** *For $n \geq 6$, each of $\mathcal{L}_n^N(k)$ is a free abelian group with*

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{L}_n^N(k)) = (k-1)\binom{k+n-2}{k} + \frac{1}{2}n(n-1)(k-3)\binom{n+k-4}{k-2}.$$

In general, it is easy to show that each $\mathcal{L}_n^N(k)$ is a finitely generated abelian group. Hence the difficult part is to show that $\mathcal{L}_n^N(k)$ is free and to determine its rank. To do this, we introduce a certain integration

$$I_j(f,w;a_1,\ldots,a_n) := \int_{l_w(a_1,\ldots,a_n)} f(t)dt_j$$

in Section 3. This is a generalization of Chen's integration in free groups introduced by K. T. Chen who determined the group structure of the graded quotients of the lower central series of a free metabelian group in [2].

This paper consists of six sections. In Section 2, we recall the associated Lie algebra of a group, the IA-automorphism group and the Johnson homomorphisms. In Section 3, we introduce a generalization of Chen's integration in free groups, and study some properties. In Section 4, we determine the group structure of the graded quotient $\mathcal{L}_n^N(k)$ of the lower central series of F_n^N. Finally, in Section 5, we determine the cokernel of $(\tau'_{k,N})_{\mathbf{Q}}$.

2. Preliminaries

In this section, we recall the definition and some properties of the associated Lie algebra of a group G, the IA-automorphism group of a free group and the Johnson homomorphisms of $\mathrm{Aut}\,F_n$.

2.1. **Notation and conventions.**

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group $\mathrm{Aut}\,G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\,G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For any $\mathbf{Z}$-module M, we denote by $M \otimes_{\mathbf{Z}} \mathbf{Q}$ the symbol obtained by attaching a subscript $\mathbf{Q}$ to M, such as $M_{\mathbf{Q}}$ or $M^{\mathbf{Q}}$. Similarly, for any $\mathbf{Z}$-linear map $f : A \to B$, the induced $\mathbf{Q}$-linear map $A_{\mathbf{Q}} \to B_{\mathbf{Q}}$ is denoted by $f_{\mathbf{Q}}$ or $f^{\mathbf{Q}}$.
- For each $k \geq 1$, and any partition λ of k, we denote by H^λ the Schur-Weyl module of H corresponding to the partition λ of k. For example, the modules $H^{[k]}$ and $H^{[1^k]}$ are the symmetric product $S^k H$ and the exterior product $\Lambda^k H$, respectively. (For details, see [6].)
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. **Associated Lie algebra of a group.**

Let G be a group, and let $\Gamma_G(k)$ be the k-th term of the lower central series of G defined by

$$\Gamma_G(1) := G, \quad \Gamma_G(k) := [\Gamma_G(k-1), G], \ \ k \geq 2.$$

For each $k \geq 1$, set $\mathcal{L}_G(k) := \Gamma_G(k)/\Gamma_G(k+1)$ and

$$\mathcal{L}_G := \bigoplus_{k \geq 1} \mathcal{L}_G(k).$$

Then $\mathcal{L}_G$ has a graded Lie algebra structure induced from the commutator bracket on G. We call $\mathcal{L}_G$ the associated Lie algebra of a group G. Clearly, the correspondence from G to $\mathcal{L}_G$ is a covariant functor from the category of groups to that of graded Lie algebras. In particular, if $f : G_1 \to G_2$ is a surjective group homomorphism, the induced homomorphism $f_* : \mathcal{L}_{G_1} \to \mathcal{L}_{G_2}$ is also surjective.

For any $g_1, \ldots, g_k \in G$, a commutator of weight k among the components $g_1, \ldots, g_k$ of the type

$$[[\cdots[[g_1, g_2], g_3], \cdots], g_k]$$

with all of its brackets to the left of all the elements occurring is called a simple k-fold commutator, denoted by $[g_1, g_2, \ldots, g_k]$. In general, if G is generated by $g_1, \ldots, g_n$, then for each $k \geq 1$, $\mathcal{L}_G(k)$ is generated by (the coset classes of) the simple k-fold commutators

$$[g_{i_1}, g_{i_2}, \ldots, g_{i_k}], \quad i_j \in \{1, \ldots, n\}.$$

For details, see [15] for example.

Next we consider the case where G is a free group F_n on $x_1, \ldots, x_n$. For simplicity, we write $\Gamma_n(k)$, $\mathcal{L}_n(k)$ and $\mathcal{L}_n$ for $\Gamma_G(k)$, $\mathcal{L}_G(k)$ and $\mathcal{L}_G$, respectively. The associated Lie algebra $\mathcal{L}_n$ is called the free Lie algebra generated by H. (See [25] for basic materials concerning the free Lie algebra.) It is classically well known due to Witt [29] that for each $k \geq 1$, the graded quotient $\mathcal{L}_n(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank

$$r_n(k) := \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}, \tag{1}$$

where μ is the Möbius function.

Now, we denote by F_n^M the quotient group of F_n by a subgroup $[\Gamma_n(2), \Gamma_n(2)]$. The group F_n^M is called a free metabelian group of rank n. For simplicity, we write $\Gamma_n^M(k)$, $\mathcal{L}_n^M(k)$ and $\mathcal{L}_n^M$ for $\Gamma_{F_n^M}(k)$, $\mathcal{L}_{F_n^M}(k)$ and $\mathcal{L}_{F_n^M}$, respectively. The associated Lie algebra $\mathcal{L}_n^M$ is called the free metabelian algebra generated by H, or the Chen Lie algebra. By a remarkable work by Chen [2], it is known that for each $k \geq 1$ the graded quotient $\mathcal{L}_n^M(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank

$$r_n^M(k) := (k-1) \binom{n+k-2}{k} \tag{2}$$

with basis

$$\{[x_{i_1}, x_{i_2}, \ldots, x_{i_k}] \,|\, i_1 > i_2 \leq i_3 \leq \cdots \leq i_k\}.$$

Let F_n^N be the quotient group of F_n by the subgroup $[\Gamma_n(3), \Gamma_n(3)][[\Gamma(2), \Gamma_n(2)], \Gamma_n(2)]$. For simplicity, we write $\Gamma_n^N(k)$, $\mathcal{L}_n^N(k)$ and $\mathcal{L}_n^N$ for $\Gamma_{F_n^N}(k)$, $\mathcal{L}_{F_n^N}(k)$ and $\mathcal{L}_{F_n^N}$, respectively. In Section 4, we determine the rank of $\mathcal{L}_n^N(k)$ for each $k \geq 1$.

2.3. **Hall basis.** Here, we recall the Hall basis of $\mathcal{L}_n(k)$ for each $k \geq 1$. In [8], P. Hall introduced basic commutators of F_n and showed that those of weight k form a basis of $\mathcal{L}_n(k)$. Now, it is called the Hall basis of $\mathcal{L}_n(k)$. (For details for the basic commutators, see [9] and [25] for example.) In this paper, we consider a fixed sequence of basic commutators of F_n beginning with

$$x_1 < x_2 < \cdots < x_n < [x_2, x_1] < [x_3, x_1] < [x_3, x_2] < \cdots < [x_n, x_{n-1}] < \cdots,$$

where the ordering among $[x_i, x_j]$ is defined by the lexicographic ordering.

Let $c_{l,1} < \cdots < c_{l,m_l}$ be the basic commutators of weight l. If w is a product of basic commutators of weight $\geq l$, and if we apply the Hall's correcting process to w, then for each $k \geq l$, w is rewritten as a form

$$w = c_{l,1}^{e_{l,1}} \cdots c_{l,m_l}^{e_{l,m_l}} \cdots c_{k,1}^{e_{k,1}} \cdots c_{k,m_k}^{e_{k,m_k}} w',$$

where w' is a product of commutators $[u_1, u_2, \ldots, u_r]$ in $\Gamma_n(k+1)$ and each element u_i of the component is in $\Gamma_n(l)$. (For details for the correcting process, see [9].)

In particular, from the above we see that for each $k \geq 1$, any element $w \in F_n$ is uniquely written as a form

$$w \equiv c_{1,1}^{e_{1,1}} \cdots c_{1,n}^{e_{1,n}} \cdots c_{k,1}^{e_{k,1}} \cdots c_{k,m_k}^{e_{k,m_k}} \pmod{\Gamma_n(k+1)}$$

for some $e_{i,m_i} \in \mathbf{Z}$. We call it the mod-$\Gamma_n(k+1)$ normal form of w.

For any $k \geq 2$, the basic commutators which do not belong to $[\Gamma_n(2), \Gamma_n(2)]$ are $[x_{i_1}, x_{i_2}, \ldots, x_{i_k}]$ for $i_1 > i_2 \leq i_3 \leq \cdots \leq i_k$.

2.4. **IA-automorphism group.** Let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced from the abelianization $F_n \to H$ of F_n. In this paper we identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H as a free abelian group induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of ρ is called the IA-automorphism group of F_n. Magnus [14] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : \begin{cases} x_i & \mapsto {x_j}^{-1} x_i x_j, \\ x_t & \mapsto x_t \qquad (t \neq i) \end{cases}$$

for distinct i, $j \in \{1, 2, \ldots, n\}$ and

$$K_{ijl} : \begin{cases} x_i & \mapsto x_i x_j x_l {x_j}^{-1} {x_l}^{-1}, \\ x_t & \mapsto x_t \qquad (t \neq i) \end{cases}$$

for distinct i, j, $l \in \{1, 2, \ldots, n\}$ such that $j > l$.

Recently, Cohen-Pakianathan [3, 4], Farb [5] and Kawazumi [12] independently showed that the abelianization of IA_n is a free abelian group, and the Magnus generators above induce a basis of it. More precisely, they showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{3}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module where $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ denotes the dual group of H.

2.5. **Johnson homomorphisms.** In this subsection, we recall the Johnson homomorphisms of the automorphism group of a free group. To begin with, we recall a descending filtration of $\mathrm{Aut}\, F_n$ called the Johnson filtration. For $k \geq 0$, the action of $\mathrm{Aut}\, F_n$ on each nilpotent quotient $F_n/\Gamma_n(k+1)$ of F_n induces a homomorphism

$$\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

We denote the kernel of it by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, with $\mathcal{A}_n(1) = \mathrm{IA}_n$. (See [1] for details.) It is called the Johnson filtration of $\mathrm{Aut}\, F_n$. For each $k \geq 1$, the group $\mathrm{Aut}\, F_n$ acts on $\mathcal{A}_n(k)$ by conjugation, and it naturally induces an action of $\mathrm{GL}(n, \mathbf{Z})$ on $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$. The graded sum $\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ has a graded Lie algebra structure induced from the commutator bracket on IA_n.

In order to study the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n)$ for each $k \geq 1$, we consider the Johnson homomorphisms of $\mathrm{Aut}\, F_n$ as follows. For each $k \geq 1$, define a homomorphism $\tilde{\tau}_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \mapsto (x \mapsto x^{-1} x^{\sigma}), \quad x \in H.$$

Then the kernel of $\tilde{\tau}_k$ is just $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

The homomorphisms $\tilde{\tau}_k$ and τ_k are called the k-th Johnson homomorphisms of $\mathrm{Aut}\, F_n$. It is easily seen that each τ_k is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphism. For the Magnus generators of IA_n, their images by τ_1 are given by

$$(4) \qquad \tau_1(K_{ij}) = x_i^* \otimes [x_i, x_j], \quad \tau_1(K_{ijl}) = x_i^* \otimes [x_j, x_l].$$

Furthermore, we remark that $\tau_1 \,(= \tau_1')$ is just the abelianization of IA_n. (See [3, 4, 5, 12].)

Let $\mathrm{Der}\,(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree k part of $\mathrm{Der}\,(\mathcal{L}_n)$ is considered as $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, and we identify them in this paper. Then the sum of the Johnson homomorphisms

$$\tau := \bigoplus_{k \geq 1} \tau_k : \mathrm{gr}(\mathcal{A}_n) \to \mathrm{Der}\,(\mathcal{L}_n)$$

is a graded Lie algebra homomorphism. In fact, if we denote by $\partial \xi$ the element of $\mathrm{Der}\,(\mathcal{L}_n)$ corresponding to an element $\xi \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$, and write the action of $\partial \xi$ on $X \in \mathcal{L}_n$ as $X^{\partial \xi}$, then we have

$$\tau_{k+l}([\sigma, \sigma']) = \tau_k(\sigma)^{\partial \tau_l(\sigma')} - \tau_l(\sigma')^{\partial \tau_k(\sigma)}$$

for any $\sigma \in \mathcal{A}_n(k)$ and $\sigma' \in \mathcal{A}_n(l)$. This formula is very useful for calculating the image of the Johnson homomorphism inductively.

For $1 \leq k \leq 4$, the irreducible decomposition of the cokernel of the rational Johnson homomorphism $\tau_{k,\mathbf{Q}}$ and the rank of $\mathrm{gr}^k(\mathcal{A}_n)$ are obtained as follows:

k	$\mathrm{Coker}(\tau_{k,\mathbf{Q}})$	$\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{A}_n))$	
1	0	$n^2(n-1)/2$	Andreadakis [1]
2	$S^2 H_{\mathbf{Q}}$	$n(n+1)(2n^2-2n-3)/6$	Pettet [24]
3	$S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}}$	$n(3n^4-7n^2-8)/12$	Satoh [26]

In general, however, to determine the structure of the image and the cokernel of τ_k is quite difficult.

Let $\mathcal{A}_n'(k)$ be the lower central series of IA_n with $\mathcal{A}_n'(1) = \mathrm{IA}_n$. Since the Johnson filtration is central, $\mathcal{A}_n'(k) \subset \mathcal{A}_n(k)$ for each $k \geq 1$. Set $\mathrm{gr}^k(\mathcal{A}_n') := \mathcal{A}_n'(k)/\mathcal{A}_n'(k+1)$ and $\mathrm{gr}(\mathcal{A}_n') := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n')$. Then $\mathrm{gr}(\mathcal{A}_n')$ is also a graded Lie algebra induced from the commutator bracket on IA_n, and $\mathrm{GL}(n, \mathbf{Z})$ naturally acts on each of $\mathrm{gr}^k(\mathcal{A}_n')$. Moreover, since IA_n is finitely generated by the Magnus generators K_{ij} and K_{ijl}, each $\mathrm{gr}^k(\mathcal{A}_n')$ is also finitely generated by the simple k-fold commutators among the components K_{ij} and K_{ijl}.

A restriction of $\tilde{\tau}_k$ to $\mathcal{A}_n'(k)$ induces a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\tau_k' : \mathrm{gr}^k(\mathcal{A}_n') \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1),$$

and the sum

$$\tau' := \bigoplus_{k \geq 1} \tau_k' : \mathrm{gr}(\mathcal{A}_n') \to \mathrm{Der}\,(\mathcal{L}_n)$$

is also a graded Lie algebra homomorphism. Furthermore, we have

$$\tau'_{k+l}([\sigma,\sigma']) = \tau'_k(\sigma)^{\partial\tau_l(\sigma')} - \tau'_l(\sigma')^{\partial\tau_k(\sigma)}$$

for any $\sigma \in \mathcal{A}'_n(k)$ and $\sigma' \in \mathcal{A}'_n(l)$. Using this formula recursively, we can easily compute $\tau'_k(\sigma)$ for any $\sigma \in \mathcal{A}'_n(k)$ from (4). We should remark that, in general, it is not known whether τ'_k is injective or not. In this paper, we study the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbf{Q}} = \tau'_k \otimes \mathrm{id}_{\mathbf{Q}}$.

3. A generalization of Chen's integration in free groups

In this section, we introduce a generalization of Chen's integration in free groups which is used to determine the structure of the graded quotients $\mathcal{L}_n^N(k)$ in Section 4.

Given the free group F_n generated by $x_1,\ldots,x_n$, denote by $\mathbf{E}$ the vector space over the real field $\mathbf{R}$ with basis $x_1,\ldots,x_n$ and $[x_i,x_j]$ for $1\le j<i\le n$. A Euclidean metric is introduced into $\mathbf{E}$ by taking $x_1,\ldots,x_n$ and $[x_i,x_j]$ as an orthonormal basis. Then $\mathbf{E}$ is a Euclidean $n(n+1)/2$-space. The orthonormal basis induces a Cartesian coordinate system in $\mathbf{E}$. We call the coordinates corresponding to x_i and $[x_i,x_j]$ the t_i-coordinates and the $t_{i,j}$-coordinates, respectively.

Let Ω_n be the set of words among the letters $x_1,\ldots,x_n$. A quotient set of Ω_n by an equivalence relation induced from $x_i^e x_i^{-e}=1$ for $e=\pm 1$ forms the free group F_n. For any word $w = x_{i_1}^{e_1}x_{i_2}^{e_2}\cdots x_{i_m}^{e_m}$ with $e_k=\pm 1$, and any integers $a_1,\ldots,a_n\in\mathbf{Z}$, we define points $P_s\in\mathbf{E}$ for $0\le s\le m$ by

$$P_0 := \mathbf{0},$$
$$P_s := P_{s-1} + e_s t_{i_s} + \sum_{i_s<j}\Big\{\Big(a_j + \sum_{\substack{1\le l\le s-1\\ i_l=j}} e_l\Big)e_s t_{j,i_s}\Big\}$$

for $1\le s\le m$. Let $\overline{P_sP_{s+1}}$ be the path from P_s to P_{s+1} defined by a segment, and let $l_w(a_1,\ldots,a_n)$ be the polygonal path whose successive vertices are $P_0,P_1,\ldots,P_m$.

Lemma 3.1. *As in the notation above, the vertex P_m depends only on the integers $a_1,\ldots,a_n$ and the equivalence class of w in Ω_n.*

Proof. For $w = ax_i^e x_i^{-e}b$, where $a,b\in\Omega_n$ and $e=\pm1$, set $a = x_{i_1}^{e_1}x_{i_2}^{e_2}\cdots x_{i_{m'}}^{e_{m'}}$. If $e=1$, we have

$$P_{m'+1} = P_{m'} + t_i + \sum_{i<j}\Big\{\Big(a_j + \sum_{\substack{1\le l\le m'\\ i_l=j}} e_l\Big)t_{j,i}\Big\},$$
$$P_{m'+2} = P_{m'+1} - t_i + \sum_{i<j}\Big\{\Big(a_j + \sum_{\substack{1\le l\le m'\\ i_l=j}} e_l\Big)\cdot(-1)t_{j,i}\Big\} = P_{m'},$$
$$P_s = P_{s-2},\quad s\ge m'+3.$$

By an argument similar to the above, we obtain the required result for $e=-1$. □

We denote P_m above by $P_w(a_1,\ldots,a_n)$ for $w\in F_n$. In particular, $P_1(a_1,\ldots,a_n) = \mathbf{0}$. It is clear that if $w = x_1^{w_1}x_2^{w_2}\cdots x_n^{w_n}$ in $H_1(F_n,\mathbf{Z})$ then the t_i-coordinate of

$P_w(a_1, \ldots, a_n)$ is w_i for $1 \leq i \leq n$. If $w \in \Gamma_n(2)$, $P_w(a_1, \ldots, a_n)$ also does not depend on $a_1, \ldots, a_n$. More precisely, we have

Lemma 3.2. *As in the notation above, if $w \in \Gamma_n(2)$ and*

$$w = [x_2, x_1]^{w_{2,1}} \cdots [x_n, x_{n-1}]^{w_{n,n-1}} \in \mathcal{L}_n(2),$$

then the $t_{i,j}$-coordinate of $P_w(a_1, \ldots, a_n)$ is $w_{i,j}$.

Proof. Set $w = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_m}^{e_m}$, and take points $P_0, \ldots, P_m$ as above. For each $1 \leq s \leq m$, since the $t_{i,j}$-coordinate increase

$$\delta_{ji_s}\Big(a_i + \sum_{\substack{1 \leq r \leq s-1 \\ i_r = i}} e_r\Big)e_s$$

as a point moves from P_{s-1} to P_s where δ denotes the Kronecker delta, the $t_{i,j}$-coordinate of P_m is

$$\begin{aligned} &\sum_{1 \leq s \leq m} \delta_{j,i_s}\Big(a_i e_s + \sum_{\substack{1 \leq r \leq s-1 \\ i_r = i}} e_r e_s\Big) \\ &\qquad = a_i \sum_{1 \leq s \leq m} \delta_{j,i_s} e_s + \sum_{1 \leq s \leq m} \delta_{j,i_s} \sum_{\substack{1 \leq r \leq s-1 \\ i_r = i}} e_r e_s. \end{aligned}$$

The first term is equal to zero since $w \in \Gamma_n(2)$. By considering rewriting w as the mod-$\Gamma_n(3)$ normal form using the correcting process, we verify that the second term is nothing but $w_{i,j}$. This completes the proof of Lemma 3.2. □

Corollary 3.1. *If $w \in \Gamma_n(3)$, $P_w(a_1, \ldots, a_n) = \mathbf{0}$.*

For any $P \in \mathbf{E}$, the translation function on $\mathbf{E}$ defined by

$$t \mapsto t + P$$

is denoted by T_P. By the definition of $l_w(a_1, \ldots, a_n)$, we see

Lemma 3.3. *For $u, v \in \Omega_n$, $a_1, \ldots, a_n \in \mathbf{Z}$ and $u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ in $H_1(F_n, \mathbf{Z})$,*

$$l_{uv}(a_1, \ldots, a_n) = l_u(a_1, \ldots, a_n) \cdot T_{P_u(a_1, \ldots, a_n)}(l_v(a_1 + u_1, \ldots, a_n + u_n)).$$

Next, for any $w \in \Omega_n$, $a_1, \ldots, a_n \in \mathbf{Z}$ and a continuous real-valued function $f : \mathbf{E} \to \mathbf{R}$, we define integrations by

$$I_j(f, w; a_1, \ldots, a_n) := \int_{l_w(a_1, \ldots, a_n)} f(t) dt_j.$$

Observing the proof of Lemma 3.1, we see that the integration $I_j(f, w; a_1, \ldots, a_n)$ depends only on f, $a_1, \ldots, a_n$ and the equivalence class of w in Ω_n. Hence, from now on, we always consider $I_j(f, w; a_1, \ldots, a_n)$ for $w \in F_n$. We remark that if $f : \mathbf{E} \to \mathbf{R}$ does not depend on the coordinates $t_{i,j}$ for any $1 \leq j < i \leq n$, then the integration $I_j(f, w; a_1, \ldots, a_n)$ coincides with Chen's original integration $I_j(\bar{f}, w)$ for each $1 \leq j \leq n$, where $\bar{f}$ is the restriction of f to the subspace $\mathbf{E}'$ of $\mathbf{E}$ generated by the basis $x_1, \ldots, x_n$. In the following, if there is no confusion, we always write f for $\bar{f}$ for simplicity.

Here we recall a few properties of Chen's integration. For any continuous real-valued function $f, g : \mathbf{E}' \to \mathbf{R}$, and $u, v, w \in F_n$, we have

$$\begin{aligned}
&I_j(1, w) = w_j \quad \text{where } w = x_1^{w_1} \cdots x_n^{w_n} \in H_1(F_n, \mathbf{Z}),\\
&I_j(\alpha f + \beta g, w) = \alpha I_j(f, w) + \beta I_j(g, w), \quad \alpha, \beta \in \mathbf{R},\\
&I_j(f, uv) = I_j(f, u) + I_j(f \circ T'_u, v),\\
&I_j(f, u^{-1}) = -I_j(f \circ T'_{u^{-1}}, u).
\end{aligned}$$

Here T'_u denotes the translation function on $\mathbf{E}'$ defined by

$$t' \mapsto t' + u_1 t_1 + \cdots + u_n t_n, \quad u = x_1^{u_1} \cdots x_n^{u_n} \in H_1(F_n, \mathbf{Z}).$$

(See [2] for basic materials concerning Chen's integration.)

Now, we consider some properties of the integration $I_j(f, w; a_1, \ldots, a_n)$. By the linearity of the integration, we have

$$I_j(\alpha f + \beta g, w; a_1, \ldots, a_n) = \alpha I_j(f, w; a_1, \ldots, a_n) + \beta I_j(g, w; a_1, \ldots, a_n)$$

for any $\alpha, \beta \in \mathbf{R}$.

Lemma 3.4. *For $u, v \in F_n$, $a_1, \ldots, a_n \in \mathbf{Z}$, if $u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ in $H_1(F_n, \mathbf{Z})$,*

$$\begin{aligned}
&I_j(f, uv; a_1, \ldots, a_n)\\
&\qquad = I_j(f, u; a_1, \ldots, a_n) + I_j(f \circ T_{P_u(a_1,\ldots,a_n)}, v; a_1 + u_1, \ldots, a_n + u_n).
\end{aligned}$$

Proof. From Lemma 3.3, we see that

$$\begin{aligned}
&I_j(f, uv; a_1, \ldots, a_n)\\
&\qquad = \int_{l_{uv}(a_1,\ldots,a_n)} f(t) dt_j\\
&\qquad = \int_{l_u(a_1,\ldots,a_n) \cdot T_{P_u(a_1,\ldots,a_n)}(l_v(a_1+u_1,\ldots,a_n+u_n))} f(t) dt_j\\
&\qquad = \int_{l_u(a_1,\ldots,a_n)} f(t) dt_j + \int_{T_{P_u(a_1,\ldots,a_n)}(l_v(a_1+u_1,\ldots,a_n+u_n))} f(t) dt_j.
\end{aligned}$$

In the second term, if we consider the transformation of variables from t to $t - P_u(a_1, \ldots, a_n)$, we have

$$\int_{T_{P_u(a_1,\ldots,a_n)}(l_v(a_1+u_1,\ldots,a_n+u_n))} f(t) dt_j = \int_{l_v(a_1+u_1,\ldots,a_n+u_n)} f \circ T_{P_u(a_1,\ldots,a_n)}(t) dt_j.$$

Hence,

$$\begin{aligned}
&I_j(f, uv; a_1, \ldots, a_n)\\
&\qquad = I_j(f, u; a_1, \ldots, a_n) + I_j(f \circ T_{P_u(a_1,\ldots,a_n)}, v; a_1 + u_1, \ldots, a_n + u_n).
\end{aligned}$$

This completes the proof of Lemma 3.4. □

As a corollary, we obtain

Corollary 3.2. *For any $a_1, \ldots, a_n \in \mathbf{Z}$, $u \in F_n$ such that $u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} \in H_1(F_n, \mathbf{Z})$, and a real-valued function f on $\mathbf{E}$, we have:*

(1) $I_j(f, 1; a_1, \ldots, a_n) = 0$.
(2) $I_j(f, u^{-1}; a_1, \ldots, a_n) = -I_j(f \circ T_{P_{u^{-1}}(a_1,\ldots,a_n)}, u; a_1 - u_1, \ldots, a_n - u_n)$.

(3) *Furthermore, if* $v \in F_n$ *and* $v = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} \in H_1(F_n, \mathbf{Z})$,

$$\begin{aligned} I_j(f,&[u,v];a_1,\ldots,a_n) \\ &= I_j(f,u;a_1,\ldots,a_n) + I_j(f \circ T_{P_u(a_1,\ldots,a_n)}, v; a_1+u_1,\ldots,a_n+u_n) \\ &\quad - I_j(f \circ T_{P_{uvu^{-1}}(a_1,\ldots,a_n)}, u; a_1+v_1,\ldots,a_n+v_n) \\ &\quad - I_j(f \circ T_{P_{[u,v]}(a_1,\ldots,a_n)}, v; a_1,\ldots,a_n). \end{aligned}$$

Let $\mathbf{R}[t]$ be the commutative polynomial ring over $\mathbf{R}$ among indeterminates t_i for $1 \leq i \leq n$ and $t_{i,j}$ for $1 \leq j < i \leq n$. Each element of $\mathbf{R}[t]$ is regarded as a real-valued function on $\mathbf{E}$ in a usual way. We consider the polynomial ring $\mathbf{R}[t_1,\ldots,t_n]$ as a subring of $\mathbf{R}[t]$. For any $f \in \mathbf{R}[t]$, we denote by $\deg(f)$ the degree of f.

Here we give some examples of calculations of the integrations. Clearly, for any $w \in F_n$, $I_j(1,w;a_1,\ldots,a_n) = I_j(1,w)$ is the sum of the exponents of those x_j which occur in w.

Lemma 3.5. (1) *For any* $p > q$,

$$I_j(t_i,[x_p,x_q];a_1,\ldots,a_n) = \begin{cases} \delta_{jq}, & i = p, \\ -\delta_{jp}, & i = q, \\ 0, & i \neq p,q. \end{cases}$$

(2) *For any* $x \in \Gamma_n(3)$, $I_j(t_i,x;a_1,\ldots,a_n) = 0$.

Proof. For part (1), let us consider the case where $i = p$. From (3) of Corollary 3.2, we have

$$\begin{aligned} I_j(t_i,&[x_i,x_q];a_1,\ldots,a_n) \\ &= I_j(t_i,x_i;a_1,\ldots,a_n) + I_j(t_i+1,x_q;a_1,\ldots,a_i+1,\ldots,a_n) \\ &\quad - I_j(t_i,x_i;a_1,\ldots,a_q+1,\ldots,a_n) - I_j(t_i,x_q;a_1,\ldots,a_n) \\ &= I_j(t_i,x_i) + I_j(t_i+1,x_q) - I_j(t_i,x_i) - I_j(t_i,x_q) \\ &= I_j(1,x_q) = \delta_{jq}. \end{aligned}$$

By an argument similar to the above, we obtain the other cases. The calculations are left to the reader for exercises.

For part (2), let us consider an element $[y,z] \in \Gamma_n(3)$ for $y \in \Gamma_n(2)$ and $z \in F_n$ such that $z = x_1^{z_1} \cdots x_n^{z_n} \in H_1(F_n,\mathbf{Z})$. Then, from (3) of Corollary 3.2, we see that

$$\begin{aligned} I_j(t_i,&[y,z];a_1,\ldots,a_n) \\ &= I_j(t_i,y;a_1,\ldots,a_n) - I_j(t_i+z_i,y;a_1+z_1,\ldots,a_n+z_n) \\ &= I_j(t_i,y) - I_j(t_i+z_i,y) = -z_i I_j(1,y) \\ &= 0. \end{aligned}$$

Since $\Gamma_n(3)$ is generated by those elements $[y,z]$, we obtain the required result from Lemma 3.4. This completes the proof of Lemma 3.5. □

The following theorem is essentially due to Chen [2].

Theorem 3.1 (Chen [2])**.** *Let* $k \geq 2$ *and* $f \in \mathbf{R}[t_1,\ldots,t_n]$.

(1) *If* $w \in [\Gamma_n(2),\Gamma_n(2)]$, *then* $I_j(f,w;a_1,\ldots,a_n) = 0$.

(2) *If* $w = [x_{i_1}, x_{i_2}, \dots, x_{i_k}]$ *and* $\deg(f) \le k-1$, *then*

$$I_j(f, w; a_1, \dots, a_n) = \begin{cases} (-1)^{k-1}\alpha_1, & j = i_1, \\ (-1)^k \alpha_2, & j = i_2, \\ 0, & j \neq i_1, i_2, \end{cases}$$

where

$$\alpha_1 = \frac{\partial^{k-1} f}{\partial t_{i_2} \partial t_{i_3} \cdots \partial t_{i_k}}, \quad \alpha_2 = \frac{\partial^{k-1} f}{\partial t_{i_1} \partial t_{i_3} \cdots \partial t_{i_k}}.$$

Next, we consider a certain modification of (2) of the theorem above.

Lemma 3.6. *Let* $k \ge 5$ *and* $w = [x_{i_1}, \dots, x_{i_{k-2}}, [x_{i_{k-1}}, x_{i_k}]]$, $i_{k-1} > i_k$, *and let* $f \in \mathbf{R}[t]$ *such that*

$$f = g + g_{2,1}t_{2,1} + \cdots + g_{n,n-1}t_{n,n-1}$$

for some $g, g_{i,j} \in \mathbf{R}[t_1, \dots, t_n]$. *Then*

$$I_j(f, w; a_1, \dots, a_n) = -I_j\Big(\frac{\partial f}{\partial t_{i_{k-1}, i_k}}, w'; a_1, \dots, a_n\Big),$$

where $w' = [x_{i_1}, \dots, x_{i_{k-2}}]$.

Proof. Using (3) of Corollary 3.2, we obtain

$$\begin{aligned} I_j&(g, w; a_1, \dots, a_n) \\ &= I_j(g, w'; a_1, \dots, a_n) + I_j(g \circ T_{P_{w'}(a_1, \dots, a_n)}, [x_{i_{k-1}}, x_{i_k}]; a_1, \dots, a_n) \\ &\quad - I_j(g \circ T_{P_{w'[x_{i_{k-1}}, x_{i_k}]w'^{-1}}(a_1, \dots, a_n)}, w'; a_1, \dots, a_n) \\ &\quad - I_j(g \circ T_{P_w(a_1, \dots, a_n)}, [x_{i_{k-1}}, x_{i_k}]; a_1, \dots, a_n). \end{aligned}$$

Since w' and $w \in \Gamma_n(3)$, we have

$$P_{w'}(a_1, \dots, a_n) = P_w(a_1, \dots, a_n) = \mathbf{0}$$

and

$$P_{w'[x_{i_{k-1}}, x_{i_k}]w'^{-1}}(a_1, \dots, a_n) = P_{[x_{i_{k-1}}, x_{i_k}]}(a_1, \dots, a_n).$$

Since $g \in \mathbf{R}[t_1, \dots, t_n]$, we see that

$$g \circ T_{P_{w'}(a_1, \dots, a_n)} = g \circ T_{P_{w'[x_{i_{k-1}}, x_{i_k}]w'^{-1}}(a_1, \dots, a_n)} = g \circ T_{P_w(a_1, \dots, a_n)} = g.$$

Hence, $I_j(g, w; a_1, \dots, a_n) = 0$.

By an argument similar to the above, for any $p > q$, we see that

$$\begin{aligned} I_j&(g_{p,q}t_{p,q}, w; a_1, \dots, a_n) \\ &= I_j(g_{p,q}t_{p,q}, w'; a_1, \dots, a_n) + I_j(g_{p,q}t_{p,q}, [x_{i_{k-1}}, x_{i_k}]; a_1, \dots, a_n) \\ &\quad - I_j(g_{p,q}(t_{p,q} + \delta_{(p,q),(i_{k-1}, i_k)}), w'; a_1, \dots, a_n) \\ &\quad - I_j(g_{p,q}t_{p,q}, [x_{i_{k-1}}, x_{i_k}]; a_1, \dots, a_n) \\ &= -\delta_{(p,q),(i_{k-1}, i_k)} I_j(g_{p,q}, w'; a_1, \dots, a_n). \end{aligned}$$

This completes the proof of Lemma 3.6. $\square$

From Theorem 3.1 and Lemma 3.6, we obtain

Proposition 3.1. *Let* $k \geq 5$ *and* $w = [x_{i_1}, \dots, x_{i_{k-2}}, [x_{i_{k-1}}, x_{i_k}]]$, $i_{k-1} > i_k$, *and let* $f \in \mathbf{R}[t]$ *such that* $\deg(f) \leq k-2$ *and*

$$f = g + g_{2,1}t_{2,1} + \cdots + g_{n,n-1}t_{n,n-1}$$

for some $g, g_{i,j} \in \mathbf{R}[t_1, \dots, t_n]$. *Then*

$$I_j(f, w; a_1, \dots, a_n) = \begin{cases} (-1)^{k-1}\beta_1, & j = i_1, \\ (-1)^k \beta_2, & j = i_2, \\ 0, & j \neq i_1, i_2, \end{cases}$$

where

$$\beta_1 = \frac{\partial^{k-2} f}{\partial t_{i_{k-1},i_k} \partial t_{i_2} \partial t_{i_3} \cdots \partial t_{i_{k-2}}}, \quad \beta_2 = \frac{\partial^{k-2} f}{\partial t_{i_{k-1},i_k} \partial t_{i_1} \partial t_{i_3} \cdots \partial t_{i_{k-2}}}.$$

Corollary 3.3. *Using the same notation as that in Proposition* 3.1, *we have:*

(1) *If* $\deg(f) \leq k-3$ *and* $f = g + g_{2,1}t_{2,1} + \cdots + g_{n,n-1}t_{n,n-1}$ *for some* $g, g_{i,j} \in \mathbf{R}[t_1, \dots, t_n]$, *then* $I_j(f, w; a_1, \dots, a_n) = 0$.

(2) $I_j(t_{j_1} t_{j_2} \cdots t_{j_{k-3}} t_{p,q}, w; a_1, \dots, a_n) \neq 0$ *if and only if*
   (i) $(p, q) = (i_{k-1}.i_k)$,
   (ii) $t_{j_1} \cdots t_{j_{k-3}} t_j = t_{i_1} \cdots t_{i_{k-2}}$,
   (iii) $j = i_1$ *or* $j = i_2$.

4. The structure of the graded quotients $\mathcal{L}_n^N(k)$

In this section, we determine the group structure of the graded quotient $\mathcal{L}_n^N(k)$ of the lower central series of F_n^N. Set $K = [\Gamma_n(3), \Gamma_n(3)][[\Gamma_n(2), \Gamma_n(2)], \Gamma_n(2)]$. If $k \leq 5$, we have $\mathcal{L}_n^N(k) \cong \mathcal{L}_n(k)$. Hence there is nothing to do anymore in this case. Consider a surjective homomorphism

$$\iota_k : \mathcal{L}_n^N(k) \to \mathcal{L}_n^M(k)$$

of abelian groups induced from a natural map $F_n^N \to F_n^M$. Since $\mathcal{L}_n^M(k)$ is a free abelian group due to Chen [2], if we denote by $\mathcal{K}_n(k)$ the kernel of ι_k, we have

$$\mathcal{L}_n^N(k) \cong \mathcal{K}_n(k) \oplus \mathcal{L}_n^M(k).$$

Hence it suffices to determine the group structure of $\mathcal{K}_n(k)$ for $k \geq 6$.

First, we have the natural isomorphisms

$$\mathcal{L}_n^N(k) \cong \Gamma_n(k)K \big/ \Gamma_n(k+1)K,$$

$$\mathcal{L}_n^M(k) \cong \Gamma_n(k)[\Gamma_n(2), \Gamma_n(2)] \big/ \Gamma_n(k+1)[\Gamma_n(2), \Gamma_n(2)].$$

In general, for a group F and its normal subgroups G, H' and K' such that H' is a subgroup of G, we have a natural isomorphism

$$GK'/H'K' \cong G \big/ H'(G \cap K'). \tag{5}$$

Using (5), we see that

$$\mathcal{L}_n^N(k) \cong \Gamma_n(k) \big/ \Gamma_n(k+1)(\Gamma_n(k) \cap K),$$

$$\mathcal{L}_n^M(k) \cong \Gamma_n(k) \big/ \Gamma_n(k+1)(\Gamma_n(k) \cap [\Gamma_n(2), \Gamma_n(2)]).$$

Under these isomorphisms above, we verify that

$$\begin{aligned}\mathcal{K}_n(k) &\cong \Gamma_n(k+1)(\Gamma_n(k)\cap[\Gamma_n(2),\Gamma_n(2)])\Big/\Gamma_n(k+1)(\Gamma_n(k)\cap K)\\ &\cong \Gamma_n(k)\cap[\Gamma_n(2),\Gamma_n(2)]\Big/(\Gamma_n(k)\cap K)(\Gamma_n(k+1)\cap[\Gamma_n(2),\Gamma_n(2)])\\ &\cong (\Gamma_n(k)\cap[\Gamma_n(2),\Gamma_n(2)])K\Big/(\Gamma_n(k+1)\cap[\Gamma_n(2),\Gamma_n(2)])K\end{aligned}$$

by using (5).

To determine the structure of $\mathcal{K}_n(k)$, we prepare a descending series of subgroups of F_n. For $k \geq 3$, denote by $\Theta_n(k)$ the subset of F_n which consists of elements w such that

$$I_j(f, w; a_1, \ldots, a_n) = 0, \quad 1 \leq j \leq n$$

for any $a_1, \ldots, a_n \in \mathbf{Z}$ and any $f \in \mathbf{R}[t]$ such that

$$\text{(6)} \qquad \deg(f) \leq k-3, \quad f = g + g_{2,1}t_{2,1} + \cdots + g_{n,n-1}t_{n,n-1}$$

for some $g, g_{i,j} \in \mathbf{R}[t_1, \ldots, t_n]$. Then we have

$$\Theta_n(3) \supset \Theta_n(4) \supset \Theta_n(5) \supset \cdots.$$

Since $I_j(1, w; a_1, \ldots, a_n) = I_j(1, w)$ is the sum of the exponents of those x_j which occur in w, we see that $\Theta_n(3) = \Gamma_n(2)$. By Lemma 3.4 and (2) of Corollary 3.2, $\Theta_n(k)$ is a subgroup of F_n for each $k \geq 3$. Furthermore, by (3) of Corollary 3.2, each of $\Theta_n(k)$ contains $[\Gamma_n(3), \Gamma_n(3)]$. Here we show that each of $\Theta_n(k)$ is a normal subgroup of F_n. First, we consider

Lemma 4.1. $\Theta_n(4) \subset \Gamma_n(3)$.

Proof. For any $w \in \Theta_n(4)$, since $w \in \Gamma_n(2)$, considering the mod-$\Gamma_n(3)$ normal form of w, we have

$$w = [x_2, x_1]^{w_{2,1}} \cdots [x_n, x_{n-1}]^{w_{n,n-1}}\gamma$$

for some $w_{i,j} \in \mathbf{Z}$ and $\gamma \in \Gamma_n(3)$. For any $1 \leq j < i \leq n$, from Lemmas 3.4 and 3.5, we see that

$$\begin{aligned}I_j(t_i, w; a_1, \ldots, a_n) &= I_j(t_i, [x_2, x_1]^{w_{2,1}} \cdots [x_n, x_{n-1}]^{w_{n,n-1}}; a_1, \ldots, a_n)\\ &\qquad + I_j(t_i, \gamma; a_1, \ldots, a_n)\\ &= \sum_{r>s} w_{r,s} I_j(t_i, [x_r, x_s]; a_1, \ldots, a_n)\\ &= w_{i,j} = 0.\end{aligned}$$

This shows $w = \gamma \in \Gamma_n(3)$. This completes the proof of Lemma 4.1. □

For any $w \in \Theta_n(k)$, $u \in F_n$ and $f \in \mathbf{R}[t]$ satisfying (6), we have

$$\begin{aligned}&I_j(f, uwu^{-1}; a_1, \ldots, a_n)\\ &\quad = I_j(f, u; a_1, \ldots, a_n) + I_j(f \circ T_{P_u(a_1,\ldots,a_n)}, w; a_1 + u_1, \ldots, a_n + u_n)\\ &\qquad - I_j(f \circ T_{P_{uwu^{-1}}(a_1,\ldots,a_n)}, u; a_1, \ldots, a_n)\\ &\quad = 0\end{aligned}$$

since $uwu^{-1} \in \Gamma_n(3)$. Therefore $\Theta_n(k)$ is a normal subgroup of F_n.

Lemma 4.2. *For* $k \geq 3$, $[[\Gamma_n(2), \Gamma_n(2)], \Gamma_n(2)] \subset \Theta_n(k)$.

Proof. Since $[[\Gamma_n(2), \Gamma_n(2)], \Gamma_n(2)]$ is normally generated by

$$\{[x, y, z] \mid x,\ y,\ z \in \Gamma_n(2)\}$$

in F_n, and since $\Theta_n(k)$ is a normal subgroup of F_n, it suffices to show that $[x, y, z] \in \Theta_n(k)$ for $x, y, z \in \Gamma_n(2)$. For any $f \in \mathbf{R}[t]$ satisfying (6), using (3) of Corollary 3.2, we have

$$\begin{aligned}
&I_j(f, [x, y, z]; a_1, \dots, a_n) \\
&\qquad = I_j(f, [x, y]; a_1, \dots, a_n) + I_j(f \circ T_{P_{[x,y]}(a_1,\dots,a_n)}, z; a_1, \dots, a_n) \\
&\qquad\quad - I_j(f \circ T_{P_{[x,y]z[y,x]}(a_1,\dots,a_n)}, [x, y]; a_1, \dots, a_n) \\
&\qquad\quad - I_j(f \circ T_{P_{[x,y,z]}(a_1,\dots,a_n)}, z; a_1, \dots, a_n) \\
&\qquad = I_j(f - f \circ T_{P_{[x,y]z[y,x]}(a_1,\dots,a_n)}, [x, y]; a_1, \dots, a_n).
\end{aligned}$$

On the other hand, if

$$z = [x_2, x_1]^{z_{2,1}} \cdots [x_n, x_{n-1}]^{z_{n,n-1}} \in \mathcal{L}_n(3)$$

for $z_{i,j} \in \mathbf{Z}$, we have

$$P_{[x,y]z[y,x]}(a_1, \dots, a_n) = z_{2,1}t_{2,1} + \cdots + z_{n,n-1}t_{n,n-1}.$$

Hence if we set

$$\begin{aligned}
g &:= f - f \circ T_{P_{[x,y]z[y,x]}(a_1,\dots,a_n)} \\
&= z_{2,1}g_{2,1} + \cdots + z_{n,n-1}g_{n,n-1} \in \mathbf{R}[t_1, \dots, t_n],
\end{aligned}$$

then $I_j(g, [x, y]; a_1, \dots, a_n) = I_j(g, [x, y]) = 0$ since Chen's integration $I_j(g, \cdot)$ vanishes on $[\Gamma_n(2), \Gamma_n(2)]$ in general. This completes the proof of Lemma 4.2. □

Lemma 4.3. *For $k \geq 5$, $[\Gamma_n(k-2), \Gamma_n(2)] \subset \Theta_n(k)$.*

Proof. The group $[\Gamma_n(k-2), \Gamma_n(2)]$ is generated by $[u, v]$ for $u \in \Gamma_n(k-2)$ and $v \in \Gamma_n(2)$. For any $f \in \mathbf{R}[t]$ satisfying (6), by an argument similar to that in Lemma 4.2, we see that

$$I_j(f, [u, v]; a_1, \dots, a_n) = I_j(g, u; a_1, \dots, a_n) = I_j(g, u)$$

for some $g \in \mathbf{R}[t_1, \dots, t_n]$ and $\deg(g) \leq k-4$. On the other hand, from Chen's result, we see that $I_j(g, u) = 0$ since $u \in \Gamma_n(k-2)$. (See pages 150–151 in [2].) This completes the proof of Lemma 4.3. □

Lemma 4.4. *For any $k \geq 5$ and $w \in [\Gamma_n(2), \Gamma_n(2)]$, there exists some $r \geq 1$ and $e_1, \dots, e_r \in \mathbf{Z}$ such that*

$$w \equiv c_1^{e_1} \cdots c_r^{e_r} \pmod{[\Gamma_n(k-2), \Gamma_n(2)]},$$

where $c_1 < \cdots < c_r$ are the basic commutators of F_n which belong to $[\Gamma_n(2), \Gamma_n(2)]$.

Proof. In general, for any $y, z \in \Gamma_n(2)$, there exist some $y', z' \in \Gamma_n(k-2)$, and $d_{i,j}, d'_{i,j} \in \mathbf{Z}$ for $2 \leq i \leq k-1$ and $1 \leq j \leq m_i$ such that

$$y = c_{2,1}^{d_{2,1}} \cdots c_{k-1,m_{k-1}}^{d_{k-1,m_{k-1}}} y', \quad z = c_{2,1}^{d'_{2,1}} \cdots c_{k-1,m'_{k-1}}^{d'_{k-1,m_{k-1}}} z'.$$

Hence,

$$[y, z] \equiv [c_{2,1}^{d_{2,1}} \cdots c_{k-1,m_{k-1}}^{d_{k-1,m_{k-1}}}, c_{2,1}^{d'_{2,1}} \cdots c_{k-1,m'_{k-1}}^{d'_{k-1,m_{k-1}}}] \pmod{[\Gamma_n(k-2), \Gamma_n(2)]}.$$

Since $[\Gamma_n(2), \Gamma_n(2)]$ is generated by $[y, z]$ for $y, z \in \Gamma_n(2)$, we see that any $w \in [\Gamma_n(2), \Gamma_n(2)]$ can be written as

$$w \equiv \bar{c}_1^{e'_1} \cdots \bar{c}_s^{e'_s} \pmod{[\Gamma_n(k-2), \Gamma_n(2)]},$$

where the $\bar{c}_i$ are the basic commutators in $\Gamma_n(2)$.

Then if we apply Hall's correcting process to $w' := \bar{c}_1^{e'_1} \cdots \bar{c}_s^{e'_s}$ to obtain the mod-$\Gamma_n(2k-4)$ normal form, we have

$$w' = c_1^{e_1} \cdots c_r^{e_r} \gamma,$$

where all the c_i belong to $[\Gamma_n(2), \Gamma_n(2)]$, and γ is a product of the commutators $[u_1, u_2, \ldots, u_t] \in \Gamma_n(2k-4)$ and each element u_i of the component is in $\Gamma_n(2)$. Since such commutators belong to $[\Gamma_n(k-2), \Gamma_n(2)]$, so does γ. This completes the proof of Lemma 4.4. □

Lemma 4.5. *For $k \geq 5$, $\Gamma_n(k) \cap [\Gamma_n(2), \Gamma_n(2)] \subset [\Gamma_n(k-2), \Gamma_n(2)][\Gamma_n(3), \Gamma_n(3)]$.*

Proof. For any $w \in \Gamma_n(k) \cap [\Gamma_n(2), \Gamma_n(2)]$, we see

$$w \equiv c_1^{e_1} \cdots c_r^{e_r} \pmod{[\Gamma_n(k-2), \Gamma_n(2)]}$$

for basic commutators $c_1 < \cdots < c_r$ of F_n which belong to $[\Gamma_n(2), \Gamma_n(2)]$ from Lemma 4.4. Since $w \in \Gamma_n(k)$, we may assume the weight of c_i is greater than $k-1$ for each $1 \leq i \leq r$. On the other hand, such basic commutators belong to $[\Gamma_n(k-2), \Gamma_n(2)]$ or $[\Gamma_n(3), \Gamma_n(3)]$. This completes the proof of Lemma 4.5. □

From Lemmas 4.3 and 4.5, we see that for each $k \geq 5$,

$$\Gamma_n(k) \cap [\Gamma_n(2), \Gamma_n(2)] \subset \Theta_n(k).$$

Using this, we can determine the group structure of $\mathcal{K}_n(k)$. Set

$$\mathfrak{E} := \{[x_{i_1}, \ldots, x_{i_{k-2}}, [x_{i_{k-1}}, x_{i_k}]] \mid i_1 > i_2 \leq i_3 \leq \cdots \leq i_{k-2},\ i_{k-1} > i_k\}.$$

Theorem 4.1. *For $k \geq 6$, $\mathcal{K}_n(k)$ is a free abelian group with basis $\mathfrak{E}$.*

Proof. For any $x \in \Gamma_n(k) \cap [\Gamma_n(2), \Gamma_n(2)]$, we have

$$x = c_1^{e_1} \cdots c_r^{e_r} x'$$

for some basic commutators $c_1 < \cdots < c_r$ of weight k, and $x' \in \Gamma_n(k+1)$. Since $x \in [\Gamma_n(2), \Gamma_n(2)]$, observing the image of x by the natural map $\mathcal{L}_n(k) \to \mathcal{L}_n^M(k)$, we may assume that $c_i \in [\Gamma_n(2), \Gamma_n(2)]$ for $1 \leq i \leq r$. Hence $x' \in [\Gamma_n(2), \Gamma_n(2)]$, and each of the c_i belongs to $[\Gamma_n(3), \Gamma_n(3)]$, $[[\Gamma_n(2), \Gamma_n(2)], \Gamma_n(2)]$ or $\mathfrak{E}$ since $k \geq 6$. This shows that $\mathfrak{E}$ generates $\mathcal{K}_n(k)$. Set

$$y := \prod_{i_1 > i_2 \leq \cdots \leq i_{k-2}} \prod_{i_{k-1} > i_k} [x_{i_1}, \ldots, x_{i_{k-2}}, [x_{i_{k-1}}, x_{i_k}]]^{b_{i_1, \ldots, i_k}} \in \Gamma_n(k) \cap [\Gamma_n(2), \Gamma_n(2)]$$

for $b_{i_1, \ldots, i_k} \in \mathbf{Z}$, and suppose $y = 1 \in \mathcal{K}_n(k)$.

Now, for any $j_1 > j_2 \leq j_3 \leq \cdots \leq j_{k-2}$ and $j_{k-1} > j_k$, consider

$$g := t_{j_2} \cdots t_{j_{k-2}} t_{j_{k-1}, j_k} \in \mathbf{R}[t].$$

Since $\deg(g) = k-2$ and $x \in \Theta_n(k+1)$, for any $a_1, \ldots, a_n$, we have

$$0 = I_{j_1}(g, x; a_1, \ldots, a_n) = (-1)^{k-1} b_{j_1, \ldots, j_k} \frac{\partial^{k-3}(t_{j_2} \cdots t_{j_{k-2}})}{\partial t_{j_2} \cdots \partial t_{j_{k-2}}}$$

from Proposition 3.1. Since

$$\frac{\partial^{k-3}(t_{j_2}\cdots t_{j_{k-2}})}{\partial t_{j_2}\cdots\partial t_{j_{k-2}}}\neq 0,$$

we obtain $b_{j_1,\dots,j_k}=0$. This shows that $\mathfrak{E}$ is linearly independent. This completes the proof of Theorem 4.1. □

Corollary 4.1. *For $k\geq 6$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{K}_n(k))=\frac{1}{2}n(n-1)(k-3)\binom{n+k-4}{k-2}$$

and

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{L}_n^N(k))=(k-1)\binom{k+n-2}{k}+\frac{1}{2}n(n-1)(k-3)\binom{n+k-4}{k-2}.$$

5. An application to the study of the Johnson homomorphisms

In this section, we consider a reduction of the target of the Johnson homomorphism τ_k' to $H^*\otimes_{\mathbf{Z}}\mathcal{L}_n^N(k+1)$. Let

$$\tau_{k,N}':\mathrm{gr}^k(\mathcal{A}_n')\to H^*\otimes_{\mathbf{Z}}\mathcal{L}_n^N(k+1)$$

be the composition of τ_k' and the natural projection $H^*\otimes_{\mathbf{Z}}\mathcal{L}_n(k+1)\to H^*\otimes_{\mathbf{Z}}\mathcal{L}_n^N(k+1)$. It is easily seen that $\tau_{k,N}'$ is a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism.

In the following we study the cokernel of $(\tau_{k,N}')_{\mathbf{Q}}$ for $n\geq k+2$. In particular, we show that there is an obstruction $H_{\mathbf{Q}}^{[k-2,1^2]}$ for the surjectivity of $\tau_{k,\mathbf{Q}}'$, and that it also appears in $\mathrm{Coker}((\tau_{k,N}')_{\mathbf{Q}})$. Finally, we conclude that the $\mathrm{GL}(n,\mathbf{Z})$-irreducible decomposition of $\mathrm{Coker}((\tau_{k,N}')_{\mathbf{Q}})$ is $S^kH_{\mathbf{Q}}\oplus H_{\mathbf{Q}}^{[k-2,1^2]}$ for $n\geq k+2$.

5.1. **The image of τ_k'.** In the next subsection, we detect $H_{\mathbf{Q}}^{[k-2,1^2]}$ in $\mathrm{Coker}(\tau_{k,\mathbf{Q}}')$ using trace maps. To do this, we prepare a finitely generated submodule of $H^*\otimes_{\mathbf{Z}}\mathcal{L}_n(k+1)$ which contains $\mathrm{Im}(\tau_k')$. Let $V_n(k)$ be a submodule of $H^*\otimes_{\mathbf{Z}}\mathcal{L}_n(k+1)$ generated by

(A1): $x_i^*\otimes[A,B]$,
(A2): $x_i^*\otimes[A,B,C]$,
(A3): $x_i^*\otimes[x_{i_1},x_{i_2},x_{i_3},\dots,x_{i_{k+1}}]$,
(A4): $x_i^*\otimes[x_i,x_{i_2},\dots,x_{i_{k+1}}]-x_j^*\otimes[x_j,x_{i_3},\dots,x_{i_{k+1}},x_{i_2}]$,
(A5): $x_i^*\otimes[x_i,x_{i_2},\dots,x_{i_{k+1}}]-x_i^*\otimes[x_i,x_{i_3},\dots,x_{i_{k+1}},x_{i_2}]$,
(A6): $x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4},\dots,x_{i_{k+1}}]-x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_5}\dots,x_{i_{k+1}},x_{i_4}]$,
(A7): $x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4},x_{i_5},x_{i_6},\dots,x_{i_{k+1}}]$
$\quad-x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_5},x_{i_4},x_{i_6},\dots,x_{i_{k+1}}]$,
(A8): $x_i^*\otimes[x_i,x_{i_2},x_i,x_{i_4},\dots,x_{i_{k+1}}]-x_j^*\otimes[x_j,x_{i_2},x_i,x_{i_4},\dots,x_{i_{k+1}}]$
$\quad-x_j^*\otimes[x_i,x_{i_2},x_j,x_{i_4},\dots,x_{i_{k+1}}]$,

where A, B, C and the indices $1\leq i,j,i_l\leq n$ satisfy the conditions

(A1): $\mathrm{wt}(A)$, $\mathrm{wt}(B)\geq 3$ and $\mathrm{wt}(A)+\mathrm{wt}(B)=k+1$,
(A2): $\mathrm{wt}(A)$, $\mathrm{wt}(B)$, $\mathrm{wt}(C)\geq 2$ and $\mathrm{wt}(A)+\mathrm{wt}(B)+\mathrm{wt}(C)=k+1$,
(A3): $i\neq i_1,i_2,i_3$,
(A4): $i\neq i_2,i_3,j$ and $j\neq i_3,i_4$,
(A5): $i\neq i_2,i_3,i_4$,

(A6), (A7): $i \neq i_1, i_2$,
(A8): $i \neq j, i_2$ and $j \neq i_2$,

respectively. We do not consider **(A1)** and **(A2)** for $k < 5$. In this subsection, we use $\equiv$ for the equality in the quotient module of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ by $V_n(k)$. Then we show

Theorem 5.1. *For $k \geq 1$ and $n \geq 6$, $\mathrm{Im}(\tau'_k) \subset V_n(k)$.*

Before showing Theorem 5.1, we prepare

Lemma 5.1. *For any $n \geq 3$, we have:*

(1) *For any $i \neq i_1, i_2$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \dots, x_{i_{k+1}}] = x_i^* \otimes [x_i, x_{i_2}, x_{i_1}, x_{i_4}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_1}, x_{i_2}, x_{i_4}, \dots, x_{i_{k+1}}].$$

(2) *For any $i, j \neq i_1, i_2$ and $\sigma \in \mathfrak{S}_{k-2}$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{j_1}, \dots, x_{j_{k-2}}] \equiv x_j^* \otimes [x_{i_1}, x_{i_2}, x_j, x_{\sigma(j_1)} \dots, x_{\sigma(j_{k-2})}].$$

(3) *If $n \geq 6$, for any $i, j \neq i_2, i_3, i_4$ and $i \neq j$,*

$$x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}] \equiv x_j^* \otimes [x_j, x_{i_2}, \dots, x_{i_{k+1}}].$$

Proof of Lemma 5.1. Part (1) is immediately obtained from the Jacobi identity

$$[x_{i_1}, x_{i_2}, x_i] = [x_i, x_{i_2}, x_{i_1}] - [x_i, x_{i_1}, x_{i_2}].$$

For part (2), if $j = i$, it is obtained from **(A6)** and **(A7)**. If not, we have

$$\begin{aligned} x_i^* \otimes & [x_{i_1}, x_{i_2}, x_i, x_{j_1}, \dots, x_{j_{k-2}}] \\ & \overset{(1)}{=} x_i^* \otimes [x_i, x_{i_2}, x_{i_1}, x_{i_4}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_1}, x_{i_2}, x_{i_4}, \dots, x_{i_{k+1}}] \\ & \overset{\mathbf{(A4)}}{\equiv} x_j^* \otimes [x_j, x_{i_1}, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_2}] - x_j^* \otimes [x_j, x_{i_2}, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_1}] \\ & \overset{\mathbf{(A5)}}{\equiv} x_j^* \otimes [x_j, x_{i_2}, x_{i_1}, x_{i_4}, \dots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_1}, x_{i_2}, x_{i_4}, \dots, x_{i_{k+1}}] \\ & \overset{(1)}{=} x_j^* \otimes [x_{i_1}, x_{i_2}, x_j, x_{j_1}, \dots, x_{j_{k-2}}]. \end{aligned}$$

Hence we obtain part (2).

For part (3), we can take some $1 \leq k \leq n$ such that $k \neq i, j, i_2, i_3, i_4$. Then we see that

$$x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}] \equiv x_k^* \otimes [x_k, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}] \equiv x_j^* \otimes [x_j, x_{i_2}, \dots, x_{i_{k+1}}]$$

by **(A4)**. This completes of the proof of Lemma 5.1. □

Proof of Theorem 5.1. We prove this theorem by induction on k. For $k = 1$, since $\mathrm{gr}^1(\mathcal{A}_n) = \mathrm{IA}_n^{\mathrm{ab}}$ is generated by K_{ij} and K_{ijl}, it is clear from (4). Assume $k \geq 1$. Since

$$\tau' = \bigoplus_{k \geq 1} \tau'_k : \mathrm{gr}(\mathcal{A}'_n) \to \mathrm{Der}(\mathcal{L}_n)$$

is a Lie algebra homomorphism, it suffices to show that $[(\mathbf{A1}), \tau_1(K_{pq})], \dots, [(\mathbf{A8}), \tau_1(K_{pq})]$ and $[(\mathbf{A1}), \tau_1(K_{pqr})], \dots, [(\mathbf{A8}), \tau_1(K_{pqr})]$ belong to $V_n(k+1)$ for any p, q and r. We show this by direct computation. Here we give some examples of it.

Step I. $[(\mathbf{A1}), \tau_1(K_{pq})]$.

Observe

$$\begin{aligned}&[x_i^* \otimes [A,B], \tau_1(K_{pq})]\\&\quad = x_i^* \otimes [A^{\partial\tau_1(K_{pq})}, B] + x_i^* \otimes [A, B^{\partial\tau_1(K_{pq})}] - \delta_{i,p} x_p^* \otimes [[A,B], x_q]\\&\qquad - \delta_{i,q} x_p^* \otimes [x_p, [A,B]].\end{aligned}$$

By the Jacobi identity, we have

$$[[A,B],x_q] = -[[B,x_q],A] - [[x_q,A],B], \quad [x_p,[A,B]] = -[A,[B,x_p]] - [B,[x_p,A]].$$

Hence $[(\mathbf{A1}), \tau_1(K_{pq})] \in V_n(k+1)$. Similarly, we see $[(\mathbf{A1}), \tau_1(K_{pqr})] \in V_n(k+1)$.

Step II. $[(\mathbf{A2}), \tau_1(K_{pq})]$.

Observe

$$\begin{aligned}&[x_i^* \otimes [A,B,C], \tau_1(K_{pq})]\\&\quad = x_i^* \otimes [A^{\partial\tau_1(K_{pq})}, B, C] + x_i^* \otimes [A, B^{\partial\tau_1(K_{pq})}, C] + x_i^* \otimes [A, B, C^{\partial\tau_1(K_{pq})}]\\&\qquad - \delta_{i,p} x_p^* \otimes [[A,B,C], x_q] - \delta_{i,q} x_p^* \otimes [x_p, [A,B,C]].\end{aligned}$$

By the Jacobi identity, we have

$$\begin{aligned}[[A,B,C],x_q] &= -[[C,x_q],[A,B]] - [[x_q,[A,B]],C]\\&= [A,B,[C,x_q]] + [A,[B,x_q],C] + [B,[x_q,A],C],\\ [x_p,[A,B,C]] &= -[A,B,[C,x_p]] - [A,[B,x_p],C] - [B,[x_p,A],C].\end{aligned}$$

Hence $[(\mathbf{A2}), \tau_1(K_{pq})] \in V_n(k+1)$. Similarly, we see $[(\mathbf{A2}), \tau_1(K_{pqr})] \in V_n(k+1)$.

Step III. $[(\mathbf{A3}), \tau_1(K_{pq})]$.

In

$$\begin{aligned}&[(\mathbf{A3}), \tau_1(K_{pq})]\\&= \delta_{i_1,p} x_i^* \otimes [x_{i_1}, x_q, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}] + \delta_{i_2,p} x_i^* \otimes [x_{i_1}, [x_{i_2}, x_q], x_{i_3}, \dots, x_{i_{k+1}}]\\&\quad + \delta_{i_3,p} x_i^* \otimes [x_{i_1}, x_{i_2}, [x_{i_3}, x_q], x_{i_4}, \dots, x_{i_{k+1}}]\\&\quad + \sum_{l=4}^{k+1} \delta_{i_l,p} \underline{x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, \dots, x_{i_{l-1}}, [x_{i_l}, x_q], x_{i_{l+1}}, \dots, x_{i_{k+1}}]}_{①}\\&\quad - \delta_{i,p} \underline{x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}, x_q]}_{②} - \delta_{i,q} x_p^* \otimes [x_p, [x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}]],\end{aligned}$$

② $\equiv 0$ by **(A3)**. On the other hand, using the Jacobi identity

$$[X,[x_a,x_b]] = [X,x_a,x_b] - [X,x_b,x_a], \tag{7}$$

we see ① $\equiv 0$ by **(A3)**. If $q \neq i$, we see $[(\mathbf{A3}), \tau_1(K_{pq})] \equiv 0$ since all terms other than ① and ② in the equation above are of type **(A3)**. Hence, consider the case where $q = i$.

Suppose $p = i_1$. If $i_3 \neq i_1$, we have

$$\begin{aligned}&[(\mathbf{A3}), \tau_1(K_{pq})]\\&\quad \equiv -x_i^* \otimes [x_i, x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}] + x_{i_1}^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_1}] \equiv 0\end{aligned}$$

by **(A4)**. If $i_3 = i_1$, using **(A5), (A6)** and **(A8)**, we have

$$\begin{aligned}
&[(\mathbf{A3}), \tau_1(K_{pq})] \\
&\equiv -x_i^* \otimes [x_i, x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}] \\
&\qquad + x_{i_1}^* \otimes [x_{i_1}, x_{i_2}, x_{i_1}, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_1}] \\
&\equiv -x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_1}] - x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_3}] \\
&\qquad + x_{i_1}^* \otimes [x_{i_1}, x_{i_2}, x_{i_1}, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_1}] \\
&\equiv 0.
\end{aligned}$$

Similarly, we see $[(\mathbf{A3}), \tau_1(K_{pq})] \equiv 0$ for $p = i_2$. Suppose $p = i_3$ and $p \neq i_1, i_2$. By **(A6)**, we have

$$\begin{aligned}
&[(\mathbf{A3}), \tau_1(K_{pq})] \\
&\equiv -x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}] + x_{i_3}^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_3}] \\
&\equiv 0.
\end{aligned}$$

Therefore we have $[(\mathbf{A3}), \tau_1(K_{pq})] \in V_n(k+1)$ for any cases. Similarly, we obtain $[(\mathbf{A3}), \tau_1(K_{pqr})] \in V_n(k+1)$.

Step IV. $[(\mathbf{A6}), \tau_1(K_{pq})]$.

In

$$\begin{aligned}
&[(\mathbf{A6}), \tau_1(K_{pq})] \\
&\equiv \delta_{i_1,p}(x_i^* \otimes [x_{i_1}, x_q, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_{i_1}, x_q, x_{i_2}, x_i, x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}]) \\
&+ \delta_{i_2,p}(x_i^* \otimes [x_{i_1}, [x_{i_2}, x_q], x_i, x_{i_4}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_{i_1}, [x_{i_2}, x_q], x_i, x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}]) \\
&+ \delta_{i,p}(\underline{x_i^* \otimes [x_{i_1}, x_{i_2}, [x_i, x_q], x_{i_4}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_{i_1}, x_{i_2}, [x_i, x_q], x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}]}_{①}) \\
&+ \delta_{i_4,p}(\underline{x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, [x_{i_4}, x_q], \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_5}, \ldots, x_{i_{k+1}}, [x_{i_4}, x_q]]}_{②}) \\
&+ \sum_{l=5}^{k+1} \delta_{i_l,p}(\underline{x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{l-1}}, [x_{i_l}, x_q], x_{i_{l+1}}, \ldots, x_{i_{k+1}}]} \\
&\qquad \underline{- x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_5}, \ldots, x_{i_{l-1}}, [x_{i_l}, x_q], x_{i_{l+1}}, \ldots, x_{i_{k+1}}, x_{i_4}]}_{③}) \\
&- \delta_{i,p}(\underline{x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}, x_q] - x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}, x_q]}_{④}) \\
&- \delta_{i,q}(x_p^* \otimes [x_p, [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}]] - x_p^* \otimes [x_p, [x_{i_1}, x_{i_2}, x_i, x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}]]),
\end{aligned}$$

we see ① $\equiv \cdots \equiv$ ④ $\equiv 0$ by (7) and (2) of Lemma 5.1. Furthermore, if $q \neq i$, $[(\mathbf{A6}), \tau_1(K_{pq})] \equiv 0$ since all terms other than ①, …, ④ are of type **(A3)**. Hence, we consider the case where $q = i$. In this case, $p \neq i$.

If $p \neq i_1, i_2$, it is clear $[(\mathbf{A6}), \tau_1(K_{pq})] \equiv 0$ by **(A3)**. Suppose $p = i_1$. Then,

$$\begin{aligned}
&[(\mathbf{A6}), \tau_1(K_{pq})] \\
&\equiv -x_i^* \otimes [x_i, x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}] + x_i^* \otimes [x_i, x_{i_1}, x_{i_2}, x_i, x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}] \\
&\qquad + x_{i_1}^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_1}] - x_{i_1}^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}, x_{i_1}] \\
&\equiv 0
\end{aligned}$$

by **(A4)**. Similarly, we see $[(\mathbf{A6}), \tau_1(K_{pq})] \equiv 0$ for $p = i_2$. Furthermore, by an argument similar to the above, we verify that $[(\mathbf{A6}), \tau_1(K_{pqr})]$, $[(\mathbf{A7}), \tau_1(K_{pq})]$ and $[(\mathbf{A7}), \tau_1(K_{pqr})] \in V_n(k+1)$.

Step V. $[(\mathbf{A5}), \tau_1(K_{pq})]$.

In

$$\begin{aligned}
&[(\mathbf{A5}), \tau_1(K_{pq})] \\
&= \delta_{i,p}(\underline{x_i^* \otimes [x_i, x_q, x_{i_2}, \dots, x_{i_{k+1}}]}_{\textcircled{1}} - \underline{x_i^* \otimes [x_i, x_q, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}]}_{\textcircled{2}}) \\
&\quad + \delta_{i_2,p}(x_i^* \otimes [x_i, [x_{i_2}, x_q], x_{i_3}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, [x_{i_2}, x_q]]) \\
&\quad + \delta_{i_3,p}(x_i^* \otimes [x_i, x_{i_2}, [x_{i_3}, x_q], x_{i_4}, \dots, x_{i_{k+1}}] \\
&\quad - x_i^* \otimes [x_i, [x_{i_3}, x_q], x_{i_4}, \dots, x_{i_{k+1}}, x_{i_2}]) \\
&\quad + \delta_{i_4,p}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, [x_{i_4}, x_q], \dots, x_{i_{k+1}}] \\
&\quad - x_i^* \otimes [x_i, x_{i_3}, [x_{i_4}, x_q], x_{i_5}, \dots, x_{i_{k+1}}, x_{i_2}]) \\
&\quad + \sum_{l=5}^{k+1} \delta_{i_l,p}(\underline{x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{l-1}}, [x_{i_l}, x_q], x_{i_{l+1}}, \dots, x_{i_{k+1}}]}_{\textcircled{3}} \\
&\qquad\qquad \underline{-x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{l-1}}, [x_{i_l}, x_q], x_{i_{l+1}}, \dots, x_{i_{k+1}}, x_{i_2}])}_{\textcircled{3}} \\
&\quad + \delta_{i,p}(\underline{-x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}, x_q]}_{\textcircled{1}} + \underline{x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}, x_q]}_{\textcircled{2}}) \\
&\quad + \delta_{i,q}(-x_p^* \otimes [x_p, [x_i, x_{i_2}, \dots, x_{i_{k+1}}]] + x_p^* \otimes [x_p, [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}]]),
\end{aligned}$$

$\textcircled{1} \equiv \textcircled{2} \equiv \textcircled{3} \equiv 0$ by (7) and **(A5)**. Furthermore, if $q \neq i$, we see $[(\mathbf{A5}), \tau_1(K_{pq})] \equiv 0$ similarly. Hence it suffices to consider the case where $q = i$. In this case, $p \neq i$. Then using (7) and **(A5)**, we see

$$\begin{aligned}
&[(\mathbf{A5}), \tau_1(K_{pq})] \\
&\equiv \delta_{i_2,p}(x_i^* \otimes [x_i, x_{i_2}, x_i, x_{i_3}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}, x_i] \\
&\qquad + x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_i, x_{i_2}]) \\
&\quad + \delta_{i_3,p}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_i, x_{i_4}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_2}, x_i, x_{i_3}, x_{i_4}, \dots, x_{i_{k+1}}] \\
&\qquad - x_i^* \otimes [x_i, x_{i_3}, x_i, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_2}]) \\
&\quad - \delta_{i_4,p}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_i, x_{i_4}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, x_i, x_{i_4}, x_{i_5}, \dots, x_{i_{k+1}}, x_{i_2}]) \\
&\quad + x_p^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}, x_p] - x_p^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}, x_p].
\end{aligned}$$

Since $n \geq 6$, there exist some $1 \leq j \leq n$ such that $j \neq i, i_2, i_3, i_4$. We fix it.

Case I. i_2, i_3 and i_4 are distinct.

If i_2, i_3 and i_4 are distinct, using **(A3)**, we have

$$\begin{aligned}
&[(\mathbf{A5}), \tau_1(K_{pq})] \\
&\equiv \delta_{i_2,p}(x_i^* \otimes [x_i, x_{i_2}, x_i, x_{i_3}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}, x_i] \\
&\qquad + x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_i, x_{i_2}] + x_{i_2}^* \otimes [x_i, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}]) \\
&\quad + \delta_{i_3,p}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_i, x_{i_4}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_2}, x_i, x_{i_3}, x_{i_4}, \dots, x_{i_{k+1}}] \\
&\qquad - x_i^* \otimes [x_i, x_{i_3}, x_i, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_2}] + x_{i_3}^* \otimes [x_i, x_{i_2}, x_{i_3}, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_3}] \\
&\qquad - x_{i_3}^* \otimes [x_i, x_{i_3}, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_2}, x_{i_3}]) \\
&\quad - \delta_{i_4,p}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, x_i, x_{i_4}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, x_i, x_{i_4}, x_{i_5}, \dots, x_{i_{k+1}}, x_{i_2}] \\
&\qquad + x_{i_4}^* \otimes [x_i, x_{i_3}, x_{i_4}, \dots, x_{i_{k+1}}, x_{i_2}, x_{i_4}]).
\end{aligned}$$

Then from **(A8)** and (3) of Lemma 5.1, the $\delta_{i_2,p}$-part is equal to

$$\begin{aligned}x_j^* \otimes [x_j, x_{i_2}, x_i, x_{i_3}, \dots, x_{i_{k+1}}] + x_j^* \otimes [x_i, x_{i_2}, x_j, x_{i_3}, \dots, x_{i_{k+1}}]\\ - x_j^* \otimes [x_j, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}, x_i] + x_j^* \otimes [x_j, x_{i_3}, \dots, x_{i_{k+1}}, x_i, x_{i_2}]\\ - x_j^* \otimes [x_j, x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}].\end{aligned}$$

Hence, by **(A5)**, we obtain that the $\delta_{i_2,p}$-part is equal to zero modulo $V_n(k+1)$. Similarly, we see that the $\delta_{i_3,p}$-part and the $\delta_{i_4,p}$-part of the equation above are equal to zero modulo $V_n(k+1)$. Therefore we obtain $[(\mathbf{A5}), \tau_1(K_{pq})] \equiv 0$.

Case II. $i_2 = i_3 \neq i_4$.

If $i_2 = i_3 = m$ and $i_4 \neq m$, using **(A3)**, we have

$$\begin{aligned}&[(\mathbf{A5}), \tau_1(K_{pq})]\\ &\equiv \delta_{m,p}(\underline{x_i^* \otimes [x_i, x_m, x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}]}_{\textcircled{4}} - x_i^* \otimes [x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}, x_m, x_i]\\ &\quad + x_i^* \otimes [x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}, x_i, x_m] + x_i^* \otimes [x_i, x_m, x_m, x_i, x_{i_4}, \dots, x_{i_{k+1}}]\\ &\quad \underline{-x_i^* \otimes [x_i, x_m, x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}]}_{\textcircled{4}} - x_i^* \otimes [x_i, x_m, x_i, x_{i_4}, \dots, x_{i_{k+1}}, x_m]\\ &\quad - x_m^* \otimes [x_m, x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}, x_m] - x_m^* \otimes [x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}, x_m, x_m])\\ &\quad - \delta_{i_4,p}(x_i^* \otimes [x_i, x_m, x_m, x_i, x_{i_4}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_m, x_i, x_{i_4}, \dots, x_{i_{k+1}}, x_m]\\ &\qquad + x_{i_4}^* \otimes [x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}, x_m, x_{i_4}]).\end{aligned}$$

In the $\delta_{m,p}$-part, $\textcircled{4} = 0$. From **(A8)** and (3) of Lemma 5.1, the other terms are equal to

$$\begin{aligned}&\underline{-x_j^* \otimes [x_j, x_m, x_{i_4}, \dots, x_{i_{k+1}}, x_m, x_i] + x_j^* \otimes [x_j, x_m, x_{i_4}, \dots, x_{i_{k+1}}, x_i, x_m]}_{\textcircled{6}}\\ &\quad \underline{+x_j^* \otimes [x_j, x_m, x_m, x_i, x_{i_4}, \dots, x_{i_{k+1}}]}_{\textcircled{7}} \underline{-x_j^* \otimes [x_j, x_m, x_i, x_{i_4}, \dots, x_{i_{k+1}}, x_m]}_{\textcircled{5}}\\ &\quad - x_j^* \otimes [x_i, x_m, x_j, x_{i_4}, \dots, x_{i_{k+1}}, x_m] \underline{-x_j^* \otimes [x_j, x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}, x_m]}_{\textcircled{7}}\\ &\quad - x_j^* \otimes [x_m, x_i, x_j, x_{i_4}, \dots, x_{i_{k+1}}, x_m] \underline{+x_j^* \otimes [x_j, x_i, x_{i_4}, \dots, x_{i_{k+1}}, x_m, x_m]}_{\textcircled{5}}\end{aligned}$$

modulo $V_n(k+1)$. Then $\textcircled{5} \equiv 0$ by **(A5)**, and

$$\begin{aligned}\textcircled{6} &\equiv -x_j^* \otimes [x_j, x_m, x_i, x_m, x_{i_4}, \dots, x_{i_{k+1}}] + x_j^* \otimes [x_j, x_i, x_m, x_m, x_{i_4}, \dots, x_{i_{k+1}}]\\ &\equiv x_j^* \otimes [x_m, x_i, x_j, x_m, x_{i_4}, \dots, x_{i_{k+1}}]\end{aligned}$$

by **(A5)** and (1) of Lemma 5.1. Similarly,

$$\textcircled{7} \equiv x_j^* \otimes [x_i, x_m, x_j, x_{i_4}, \dots, x_{i_{k+1}}, x_m].$$

Hence, using (2) of Lemma 5.1, we see that the $\delta_{m,p}$-part $\equiv 0$. Similarly, we can show the $\delta_{i_4,p}$-part $\equiv 0$, and hence,

$$[(\mathbf{A5}), \tau_1(K_{pq})] \equiv 0.$$

By an argument similar to the above, we show $[(\mathbf{A5}), \tau_1(K_{pq})] \equiv 0$ for the other cases $i_2 = i_4 \neq i_3$, $i_3 = i_4 \neq i_2$ and $i_2 = i_3 = i_4$. Furthermore we obtain $[(\mathbf{A5}), \tau_1(K_{pqr})]$, $[(\mathbf{A4}), \tau_1(K_{pq})]$, $[(\mathbf{A4}), \tau_1(K_{pqr})]$, $[(\mathbf{A8}), \tau_1(K_{pq})]$ and $[(\mathbf{A8}), \tau_1(K_{pqr})] \in V_n(k+1)$. We leave it to the reader for exercises. This completes the proof of Theorem 5.1. □

5.2. **Contractions and trace maps.** The main purpose of this subsection is to detect the module $S^k H_{\mathbf{Q}}$ and $H_{\mathbf{Q}}^{[k-2,1^2]}$ in the cokernel $(\tau'_{k,N})_{\mathbf{Q}}$ using trace maps. For $k \geq 1$ and $1 \leq l \leq k+1$, let $\varphi_l^k : H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$ be the contraction map defined by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_l}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_{l-1}} \otimes x_{j_{l+1}} \otimes \cdots \otimes x_{j_{k+1}}.$$

For the natural embedding $\iota_n^{k+1} : \mathcal{L}_n(k+1) \to H^{\otimes(k+1)}$, we obtain a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\Phi_l^k = \varphi_l^k \circ (id_{H^*} \otimes \iota_n^{k+1}) : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We also call Φ_l^k a contraction map.

For any $x_i^* \otimes [x_{i_1}, \ldots, x_{i_{k+1}}] \in H^* \otimes_{\mathbf{Z}} H^{\otimes k}$, and $1 \leq m \leq k+1$, let

$$\Phi_{l,m}^k(x_i^* \otimes [x_{i_1}, \ldots, x_{i_{k+1}}])$$

denote the element obtained by the contraction of x_i^* with the only element x_{j_m}. For example,

$$\begin{aligned}
&\Phi_{1,2}^3(x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}]) \\
&\quad = \Phi_{1,2}^3(x_i^* \otimes (x_{i_1} \otimes x_{i_2} \otimes x_{i_3} - x_{i_2} \otimes x_{i_1} \otimes x_{i_3} - x_{i_3} \otimes x_{i_1} \otimes x_{i_2} \\
&\qquad\qquad + x_{i_3} \otimes x_{i_2} \otimes x_{i_1})) \\
&\quad = -\delta_{ii_2} x_{i_1} \otimes x_{i_3}, \\
&\Phi_{1,3}^3(x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_3}]) = -\delta_{ii_3} x_{i_1} \otimes x_{i_2} + \delta_{ii_3} x_{i_2} \otimes x_{i_1}.
\end{aligned}$$

Then we have

$$\Phi_l^k(x_i^* \otimes [x_{i_1}, \ldots, x_{i_{k+1}}]) = \sum_{m=1}^{k+1} \Phi_{l,m}^k(x_i^* \otimes [x_{i_1}, \ldots, x_{i_{k+1}}]).$$

Here we make sure that $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k)$ is written as a quotient module of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k)$, which is used to define the trace maps later. For each $k \geq 5$, if we set $Q_n(k) := (\Gamma_n(k) \cap K)\Gamma_n(k+1) \big/ \Gamma_n(k+1)$, we have an exact sequence

$$0 \to Q_n(k) \to \mathcal{L}_n(k) \to \mathcal{L}_n^N(k) \to 0$$

of $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian groups. This induces an exact sequence

$$0 \to H^* \otimes_{\mathbf{Z}} Q_n(k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k) \to 0.$$

Therefore we can regard $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k)$ as a quotient module of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k)$ by $H^* \otimes_{\mathbf{Z}} Q_n(k)$. Since the basic commutators of types

$$[x_{i_1}, \ldots, x_{i_k}] \quad \text{and} \quad [x_{i_1}, \ldots, x_{i_{k-2}}, [x_{i_{k-1}}, x_{i_k}]]$$

form a basis of the free abelian group $\mathcal{L}_n^N(k)$ by Theorem 4.1, those of type

$$[c_1, c_2] \quad \text{for} \quad \mathrm{wt}(c_1), \mathrm{wt}(c_2) \geq 3$$

and

$$[c_1, c_2, c_3] \quad \text{for} \quad \mathrm{wt}(c_1), \mathrm{wt}(c_2), \mathrm{wt}(c_3) \geq 2$$

form a basis of $Q_n(k)$.

5.2.1. **The Morita trace**. Here we recall the Morita trace map. Let

$$\mathrm{Tr}_{[k]} : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H$$

be the composition map of the contraction Φ_1^k and the natural projection $f_{[k]} : H^{\otimes k} \to S^k H$ defined by

$$f_{[k]}(x_{i_1} \otimes \cdots \otimes x_{i_k}) = x_{i_1} \cdots x_{i_k}.$$

The Morita trace was introduced with remarkable pioneer works by Shigeyuki Morita who showed that $\mathrm{Tr}_{[k]}$ is surjective and vanishes on the image of the Johnson homomorphism τ_k for $n \geq 3$ and $k \geq 2$. This shows that $S^k H_{\mathbf{Q}}$ appears in the irreducible decomposition of $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$ and $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ as a $\mathrm{GL}(n, \mathbf{Z})$-module. We call $S^k H_{\mathbf{Q}}$ the Morita obstruction.

Let $c = [c_1, c_2] \in \Gamma_n(k+1)$ be a basic commutator of weight $k+1$ such that $\mathrm{wt}(c_1), \mathrm{wt}(c_2) \geq 2$. Then for any $1 \leq i \leq n$,

$$\Phi_1^k(x_i^* \otimes c) = \Phi_1^k(x_i^* \otimes c_1) \otimes c_2 - \Phi_1^k(x_i^* \otimes c_2) \otimes c_1 \in H^{\otimes k}.$$

Hence $\mathrm{Tr}_{[k]}(x_i^* \otimes c) = 0$. This shows that $\mathrm{Tr}_{[k]}$ factors through $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k)$. Therefore we see that the Morita obstruction $S^k H_{\mathbf{Q}}$ also appears in $\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})$.

5.2.2. **Trace map for** $H^{[k-2,1^2]}$. Next we detect $H_{\mathbf{Q}}^{[k-2,1^2]}$ in the cokernel $(\tau'_{k,N})_{\mathbf{Q}}$. Let $\mu : H^{\otimes k} \to \Lambda^3 H \otimes_{\mathbf{Z}} S^{k-3} H$ be a homomorphism defined by

$$x_{i_1} \otimes \cdots \otimes x_{i_k} \mapsto (x_{i_1} \wedge x_{i_2} \wedge x_{i_3}) \otimes x_{i_4} \cdots x_{i_k}.$$

Since $H^{[k-2,1^2]}$ is considered as a quotient module of $\Lambda^3 H \otimes_{\mathbf{Z}} S^{k-3} H$ (see [6]), we have a natural projection $\nu : \Lambda^3 H \otimes_{\mathbf{Z}} S^{k-3} H \to H^{[k-2,1^2]}$. Let

$$\mathrm{Tr}_{[k-2,1^2]} : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{[k-2,1^2]}$$

be the composition of Φ_4^k and $f_{[k-2,1^2]} := \nu \circ \mu$. The map $\mathrm{Tr}_{[k-2,1^2]}$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism. We call it the trace map for $H^{[k-2,1^2]}$. In the following, we show

Theorem 5.2. *For $n \geq 3$ and $k \geq 3$,*

1. $\mathrm{Tr}_{[k-2,1^2]}^{\mathbf{Q}}$ *is surjective,*
2. $\mathrm{Tr}_{[k-2,1^2]} \circ \tau'_k \equiv 0$.

To show part (2) of the theorem above, it suffices to show that $\mathrm{Tr}_{[k-2,1^2]}$ vanishes on **(A1)**, ..., **(A8)** in Theorem 5.1.

Lemma 5.2. *For $k \geq 5$,*

1. $\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [A, B]) = 0$ *for* $\mathrm{wt}(A)$, $\mathrm{wt}(B) \geq 3$,
2. $\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [A, B, C]) = 0$ *for* $\mathrm{wt}(A)$, $\mathrm{wt}(B)$, $\mathrm{wt}(C) \geq 2$.

Proof. For part (1), we may assume $\mathrm{wt}(A) \geq \mathrm{wt}(B)$. If $\mathrm{wt}(B) = 4$, we have

$$\Phi_4^k(x_i^* \otimes [A, B]) = \Phi_4^k(x_i^* \otimes A) \otimes B - \Phi_4^k(x_i^* \otimes B) \otimes A.$$

If $\mathrm{wt}(A) \geq 4$ and $\mathrm{wt}(B) = 3$,

$$\Phi_4^k(x_i^* \otimes [A, B]) = \Phi_4^k(x_i^* \otimes A) \otimes B - B \otimes \Phi_1^{k-3}(x_i^* \otimes A).$$

If $\mathrm{wt}(A) = \mathrm{wt}(B) = 3$,

$$\Phi_4^k(x_i^* \otimes [A, B]) = A \otimes \Phi_1^{k-3}(x_i^* \otimes B) - B \otimes \Phi_1^{k-3}(x_i^* \otimes A).$$

Hence, we obtain $\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [A,B]) = 0$ for any case. Similarly, we see part (2). This completes the proof of Lemma 5.2. □

From this lemma, we verify that $\mathrm{Tr}_{[k-2,1^2]}$ vanishes on **(A1)** and **(A2)**.

Lemma 5.3. *For $k \geq 3$ and $4 \leq m \leq k+1$,*

$$f_{[k-2,1^2]} \circ \Phi_{4,m}^k(x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}]) = 0.$$

Proof. Since the element $[x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}]$ in $H^{\otimes k}$ is written as a sum of elements of types

$$A \otimes [x_{i_1}, \dots, x_{i_{m-1}}] \otimes x_{i_m} \otimes B \quad \text{or} \quad A \otimes x_{i_m} \otimes [x_{i_1}, \dots, x_{i_{m-1}}] \otimes B,$$

$\Phi_{4,m}^k(x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}])$ is a sum of elements of types

$$\delta_{ii_4}[x_{i_1}, x_{i_2}, x_{i_3}] \otimes B$$

or

$$\delta_{ii_m} A \otimes [x_{i_1}, \dots, x_{i_{m-1}}] \otimes B \quad \text{for} \quad A \in H^{\otimes 3}.$$

Considering the image of $f_{[k-2,1^2]}$, we obtain the required result. This completes the proof of Lemma 5.3. □

Corollary 5.1. *For $k \geq 3$,*

$$\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}]) = 0$$

if $i \neq i_1, i_2, i_3$. That is, $\mathrm{Tr}_{[k-2,1^2]}$ vanishes on **(A3)**.

Proof. Since

$$\begin{aligned} &\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}]) \\ &\qquad = \sum_{m=1}^{k+1} f_{[k-2,1^2]} \circ \Phi_{4,m}^k(x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}]), \end{aligned}$$

we immediately obtain the required result from Lemma 5.3. □

Lemma 5.4. *For $k \geq 3$, and $i \neq i_2, i_3$,*

$$\begin{aligned} &\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}]) \\ &\quad = -\sum_{2 \leq l_3 < l_2 < l_1 \leq k+1} (x_{i_{l_1}} \wedge x_{i_{l_2}} \wedge x_{i_{l_3}}) \otimes x_{i_2} \cdots \check{x_{i_{l_3}}} \cdots \check{x_{i_{l_2}}} \cdots \check{x_{i_{l_1}}} \cdots x_{i_{k+1}}. \end{aligned}$$

Here $\check{y}$ means removing y in the product.

Proof. From Lemma 5.3 and $i \neq i_2, i_3$, we see

$$\begin{aligned} &\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}]) \\ &\qquad = f_{[k-2,1^2]} \circ \Phi_{4,1}^k(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, \dots, x_{i_{k+1}}]). \end{aligned}$$

On the other hand, in general, if we write $[x_{j_1}, x_{j_2}, x_{j_3}, \dots, x_{j_{k+1}}] \in H^{\otimes k+1}$ as a sum of elements $x_{j'_1} \otimes \cdots \otimes x_{j'_{k+1}}$, the sum of the elements such that $j'_4 = j_1$ is given by

$$-\sum_{2 \leq l_3 < l_2 < l_1 \leq k+1} x_{j_{l_1}} \otimes x_{j_{l_2}} \otimes x_{j_{l_3}} \otimes x_{j_1} \otimes \cdots \check{x_{j_{l_3}}} \cdots \check{x_{j_{l_2}}} \cdots \check{x_{j_{l_1}}} \cdots \otimes x_{j_{k+1}}.$$

Hence we obtain the required result. This completes the proof of Lemma 5.4. □

This lemma induces

Corollary 5.2. *For $k \geq 3$, we have:*

(1) *For $i \neq i_2, i_3, \quad j \neq i_3, i_4$ and $i \neq j$,*

$$\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_2}]) = 0.$$

(2) *For $i \neq i_2, i_3, i_4$,*

$$\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [x_i, x_{i_2}, x_{i_3}, \ldots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_2}]) = 0.$$

Hence we verify that $\mathrm{Tr}_{[k-2,1^2]}$ vanishes on **(A4)** and **(A5)**.

Lemma 5.5. *For $k \geq 3$ and $i \neq i_1, i_2$,*

$$\begin{aligned}\mathrm{Tr}_{[k-2,1^2]}&(x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}]) \\ &= -\sum_{j=4}^{k+1} 2(x_{i_j} \wedge x_{i_1} \wedge x_{i_2}) \otimes x_{i_4} \cdots \check{x_{i_j}} \cdots x_{i_{k+1}}.\end{aligned}$$

Proof. From Lemma 5.3 and $i \neq i_1, i_2$, we see

$$\begin{aligned}\mathrm{Tr}_{[k-2,1^2]}&(x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}]) \\ &= f_{[k-2,1^2]} \circ \Phi^k_{4,3}(x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}]).\end{aligned}$$

In general, an element $[x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_{k+1}}] \in H^{\otimes k+1}$ is written as a sum of elements of types

$$A \otimes x_{j_3} \otimes [x_{j_1}, x_{j_2}] \otimes B \quad \text{or} \quad A \otimes [x_{j_1}, x_{j_2}] \otimes x_{j_3} \otimes B.$$

Hence $\Phi^k_{4,3}(x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}])$ is a written as a sum of

$$A \otimes [x_{i_1}, x_{i_2}] \otimes B$$

for $\mathrm{wt}(A) = 3$, or

$$x_{i_j} \otimes [x_{i_1}, x_{i_2}] \otimes B$$

for $4 \leq j \leq k+1$. Then $f_{[k-2,1^2]}(A \otimes [x_{i_1}, x_{i_2}] \otimes B) = 0$ for $\mathrm{wt}(A) = 3$.

On the other hand, in $[x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_{k+1}}] \in H^{\otimes k+1}$, the sum of the elements of type $x_{j_l} \otimes [x_{j_1}, x_{j_2}] \otimes B$ is given by

$$-\sum_{l=4}^{k+1} x_{j_l} \otimes [x_{j_1}, x_{j_2}] \otimes x_{j_4} \otimes \cdots \check{x_{j_l}} \cdots \otimes x_{j_{k+1}}.$$

From this, we obtain Lemma 5.5. □

Lemma 5.4 induces

Corollary 5.3. *For $k \geq 3$ and any $\gamma \in \mathfrak{S}_{k-2}$,*

$$\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{j_1}, \ldots, x_{j_{k-2}}] - x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{j_{\gamma(1)}}, \ldots, x_{j_{\gamma(k-2)}}]) = 0.$$

That is, $\mathrm{Tr}_{[k-2,1^2]}$ *vanishes on* **(A6)** *and* **(A7)**.

Furthermore, by an argument similar to that in Lemmas 5.4 and 5.5, we obtain

Lemma 5.6. *For $i, j \neq i_2$ and $i \neq j$,*

$$\begin{aligned}\mathrm{Tr}_{[k-2,1^2]}(x_i^* \otimes [x_i, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}] \\ - x_j^* \otimes [x_i, x_{i_2}, x_j, x_{i_4}, \ldots, x_{i_{k+1}}]) = 0.\end{aligned}$$

Proof. We leave the calculations to the reader for exercises. □

Therefore $\mathrm{Tr}_{[k-2,1^2]}$ vanishes on **(A8)**. Finally, we consider the surjectivity of $\mathrm{Tr}^{\mathbf{Q}}_{[k-2,1^2]}$. Since $n \geq 3$, we can choose distinct $1 \leq i, j, l \leq n$. Then from Lemma 5.5,

$$\begin{aligned}\mathrm{Tr}^{\mathbf{Q}}_{[k-2,1^2]}(x_i^* \otimes [x_j, x_l, x_i, x_i, \ldots, x_i]) &= -2(k-2)(x_i \wedge x_j \wedge x_l) \otimes x_i \cdots x_i, \\ &\neq 0\end{aligned}$$

in $H_{\mathbf{Q}}^{[k-2,1^2]}$. Since $H_{\mathbf{Q}}^{[k-2,1^2]}$ is irreducible, we see that $\mathrm{Tr}^{\mathbf{Q}}_{[k-2,1^2]}$ is surjective. This completes the proof of Theorem 5.2. As a corollary, we obtain

Corollary 5.4. *For $n \geq 3$ and $k \geq 3$,*

(1) $H_{\mathbf{Q}}^{[k-2,1^2]} \subset \mathrm{Coker}(\tau'_{k,\mathbf{Q}})$,

(2) $H_{\mathbf{Q}}^{[k-2,1^2]} \subset \mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})$.

Proof. Part (1) is clear. Part (2) follows from the fact that $\mathrm{Tr}_{[k-2,1^2]}$ factors through $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k)$ since $\mathrm{Tr}_{[k-2,1^2]}$ vanishes on **(A1)** and **(A2)**. □

5.3. **An upper bound on** $\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})$**.** In this subsection, we show that $\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})$ is a direct sum of $S^k H_{\mathbf{Q}}$ and $H_{\mathbf{Q}}^{[k-2,1^2]}$ as a $\mathrm{GL}(n, \mathbf{Z})$-module for $n \geq k+2$. To show this, it suffices to show that $\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})$ is generated by

$$\binom{n+k-1}{k} + \frac{(k-2)(k-1)}{2}\binom{n+k-3}{k}$$

elements for $n \geq k+2$ since we have already shown that $\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}}) \supset S^k H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[k-2,1^2]}$.

In general, $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k+1)$ is generated by

$$\mathfrak{G} := \{x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \,|\, 1 \leq i, i_j \leq n\}.$$

Hence $\mathrm{Coker}(\tau'_{k,N})$ is also generated by these elements.

Lemma 5.7. *For $n \geq 3$ and $k \geq 1$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] = 0 \in \mathrm{Coker}(\tau'_{k,N})$$

if $i_l \neq i$ for $1 \leq l \leq k+1$.

Proof. We show the lemma by induction on k. For $k = 1$, we have $\tau'_{1,N}(K_{ii_1i_2}) = x_i^* \otimes [x_{i_1}, x_{i_2}]$. Assume $k \geq 2$. By the inductive hypothesis, there exists a certain $\sigma \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1,N}(\sigma) = x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_k}].$$

On the other hand, we have $\tau'_{1,N}(K_{ii_{k+1}}) = x_i^* \otimes [x_i, x_{i_{k+1}}]$. Then

$$\tau'_{k,N}([K_{ii_{k+1}}, \sigma]) = x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}].$$

This completes the proof of Lemma 5.7. □

Let $\mathfrak{F}$ be a set consisting of elements $x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}]$ of $\mathfrak{G}$ such that $i_l = i$ for some $1 \leq l \leq n$, and $i_m \neq i$ for $m \neq l$.

Lemma 5.8. *For $n \geq k+1$, $\mathrm{Coker}(\tau'_{k,N})$ is generated by $\mathfrak{F}$.*

Proof. Take any $x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] \in \mathfrak{G}$ such that $i_{l_1} = i_{l_2} = i$ for distinct l_1, l_2. Since $n \geq k+1$, there exists a certain $j \in \{1, 2, \dots, n\}$ such that $j \neq i, i_l$ for $1 \leq l \leq k+1$. Set

$$\sigma := \begin{cases} K_{iji_{k+1}}, & i \neq i_{k+1}, \\ K_{ij}^{-1}, & i = i_{k+1}. \end{cases}$$

Then

$$\tau'_{1,N}(\sigma) = x_i^* \otimes [x_j, x_{i_{k+1}}].$$

On the other hand, from Lemma 5.7, there exists a certain $\sigma' \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1,N}(\sigma') = x_j^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_k}].$$

Then we obtain

$$\tau'_{k,N}([\sigma, \sigma']) = x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] - \sum_{l=1}^{k} \delta_{ii_l} x_j^* \otimes [x_{i_1}, \dots, x_{i_{l-1}}, [x_j, x_{i_{k+1}}], x_{i_{l+1}}, \dots, x_k].$$

Observing the Jacobi identity

$$[Z, [X, Y]] = [[Z, X], Y] - [[Z, Y], X]$$

in the graded Lie algebra $\mathrm{gr}(\mathcal{A}'_n)$, we see that the right-hand side of the equation above is equal to

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] + \delta_{ii_1} x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_2}, \dots, x_{i_k}] - \sum_{l=2}^{k} \delta_{ii_l} \Big(x_j^* \otimes [x_{i_1}, \dots, x_{i_{l-1}}, x_j, x_{i_{k+1}}, x_{i_{l+1}}, \dots, x_k] - x_j^* \otimes [x_{i_1}, \dots, x_{i_{l-1}}, x_{i_{k+1}}, x_j, x_{i_{l+1}}, \dots, x_k] \Big).$$

This completes the proof of Lemma 5.8. □

Lemma 5.9. *For $n \geq 3$ and $k \geq 2$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_k}, x_i] = 0 \in \mathrm{Coker}(\tau'_{k,N})$$

if $i_l \neq i$ for $1 \leq l \leq k$.

Proof. We show the lemma by induction on k. For $k = 2$, we have

$$\tau'_{2,N}([K_{ii_1}, K_{ii_2}]) = x_i^* \otimes [x_i, x_{i_2}, x_{i_1}] - x_i^* \otimes [x_i, x_{i_1}, x_{i_2}] = x_i^* \otimes [x_{i_1}, x_{i_2}, x_i].$$

Assume $k \geq 3$. By the inductive hypothesis, there exists a certain $\sigma \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1,N}(\sigma) = x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_i].$$

On the other hand, we have $\tau'_{1,N}(K_{ii_k}) = x_i^* \otimes [i, i_k]$. Then, by the Jacobi identity,

$$\begin{aligned} \tau'_{k,N}([K_{ii_k}, \sigma]) &= x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, x_i, x_{i_k}] - x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k-1}}, [x_i, x_{i_k}]] \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_k}, x_i]. \end{aligned}$$

This completes the proof of Lemma 5.9. □

Lemma 5.10. *For $k \geq 5$, $n \geq k+2$ and $4 \leq l \leq k-1$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{l-1}}, x_i, x_{i_{l+1}}, \dots, x_{i_{k+1}}] = 0 \in \mathrm{Coker}(\tau'_{k,N})$$

if $i_m \neq i$ for $m \neq l$.

Proof. Since $n \geq k+2$, there exists a certain $j \in \{1, 2, \dots, n\}$ such that $j \neq i, i_m$ for $1 \leq m \leq k+1$ and $m \neq l$. From Lemma 5.7, there exist $\sigma \in \mathcal{A}'_n(k-l+1)$ and $\tau \in \mathcal{A}'_n(l-1)$ such that

$$\tau'_{k-l+1,N}(\sigma) = x_i^* \otimes [x_j, x_{i_{l+1}}, \dots, x_{i_{k+1}}],$$
$$\tau'_{l-1,N}(\tau) = x_j^* \otimes [x_{i_1}, \dots, x_{i_{l-1}}, x_i].$$

Then we have

$$\tau'_{k,N}([\sigma,\tau]) = x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{l-1}}, x_i, x_{i_{l+1}}, \dots, x_{i_{k+1}}].$$

This completes the proof of Lemma 5.10. □

Lemma 5.11. *For $k \geq 2$,*

$$x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}] = x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_2}, \dots, x_{i_k}] \in \mathrm{Coker}(\tau'_{k,N})$$

if $i, j \neq i_2, \dots, i_{k+1}$ and $i \neq j$.

Proof. From Lemma 5.7, there exists a certain $\sigma \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1,N}(\sigma) = x_j^* \otimes [x_i, x_{i_2}, \dots, x_{i_k}].$$

Then,

$$\tau'_{k,N}([K_{iji_{k+1}}, \sigma]) = x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_2}, \dots, x_{i_k}].$$

This completes the proof of Lemma 5.11. □

Lemma 5.12. *For $n \geq k+2$,*

$$x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}] = x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}] \in \mathrm{Coker}(\tau'_{k,N})$$

if $i \neq i_2, \dots, i_{k+1}$.

Proof. Since $n \geq k+2$, there exists a certain $j \in \{1, 2, \dots, n\}$ such that $j \neq i, i_l$ for $3 \leq l \leq k+1$. From Lemma 5.11, there exists a certain $\sigma \in \mathcal{A}'_n(k-1)$ such that

$$\tau'_{k-1,N}(\sigma) = x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}] - x_j^* \otimes [x_j, x_{i_{k+1}}, x_{i_3}, \dots, x_{i_k}].$$

Then,

$$\begin{aligned}\tau'_{k,N}([\sigma, K_{ii_2}]) = {} & x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}] - x_i^* \otimes [x_i, x_{i_3}, \dots, x_{i_{k+1}}, x_{i_2}] \\ & - \delta_{ji_2} x_i^* \otimes [x_j, x_{i_{k+1}}, x_{i_3}, \dots, x_{i_k}, x_i].\end{aligned}$$

Hence from Lemma 5.9, we obtain the required result. This completes the proof of Lemma 5.12. □

Next, we consider the case where $k = 3$.

Lemma 5.13. *For $n \geq 4$, if $i \neq i_1, i_2, i_4$, then*

(1) $x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] = x_i^* \otimes [x_i, x_{i_4}, x_{i_2}, x_{i_1}] - x_i^* \otimes [x_i, x_{i_4}, x_{i_1}, x_{i_2}]$,
(2) $x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}] = x_i^* \otimes [x_{i_2}, x_{i_4}, x_i, x_{i_1}]$

in $\mathrm{Coker}(\tau'_{3,N})$.

Proof. From Lemma 5.9, there exists a certain $\sigma \in \mathcal{A}'_n(2)$ such that

$$\tau'_2(\sigma) = x_i^* \otimes [x_{i_1}, x_{i_2}, x_i].$$

Then, we obtain

$$\begin{aligned}\tau'_{3,N}([K_{ii_4},\sigma]) &= x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4}] - x_i^*\otimes[x_{i_1},x_{i_2},[x_i,x_{i_4}]]\\ &= x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4}] + x_i^*\otimes[x_i,x_{i_4},[x_{i_1},x_{i_2}]]\\ &= x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4}] - x_i^*\otimes[x_i,x_{i_4},x_{i_2},x_{i_1}] + x_i^*\otimes[x_i,x_{i_4},x_{i_1},x_{i_2}].\end{aligned}$$

Hence we have part (1). For part (2), from part (1), we have

$$\begin{aligned}x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4}] &= x_i^*\otimes[x_i,x_{i_4},x_{i_2},x_{i_1}] - x_i^*\otimes[x_i,x_{i_4},x_{i_1},x_{i_2}],\\ x_i^*\otimes[x_{i_2},x_{i_4},x_i,x_{i_1}] &= x_i^*\otimes[x_i,x_{i_1},x_{i_4},x_{i_2}] - x_i^*\otimes[x_i,x_{i_1},x_{i_2},x_{i_4}]\end{aligned}$$

in $\mathrm{Coker}(\tau'_{3,N})$. Then from Lemma 5.12, we obtain the required result. This completes the proof of Lemma 5.13. □

Lemma 5.14. *For $k \geq 5$ and $n \geq k+2$,*

$$x_i^*\otimes[x_{i_1},x_{i_2},\dots,x_{i_{k-1}},x_i,x_{i_{k+1}}] = 0 \in \mathrm{Coker}(\tau'_{k,N})$$

if $i_l \neq i$ for $l \neq k$.

Proof. Since $n \geq k+2$, there exists some $j \in \{1,\dots,n\}$ such that $j \neq i_l, i$ for $1 \leq m \leq k+1$ and $m \neq k$. From Lemmas 5.13 and 5.7, there exist some $\sigma \in \mathcal{A}'_n(3)$ and $\tau \in \mathcal{A}_n(k-3)$ such that

$$\begin{aligned}\tau'_{3,N}(\sigma) &= x_i^*\otimes[x_j,x_{i_{k-1}},x_i,x_{i_{k+1}}] - x_i^*\otimes[x_{i_{k-1}},x_{i_{k+1}},x_i,x_j],\\ \tau'_{k-3,N}(\tau) &= x_j^*\otimes[x_{i_1},\dots,x_{i_{k-2}}],\end{aligned}$$

respectively. Then,

$$\tau'_{k,N}([\sigma,\tau]) = x_i^*\otimes[x_{i_1},x_{i_2},\dots,x_{i_{k-1}},x_i,x_{i_{k+1}}].$$

This completes the proof of Lemma 5.14. □

Lemma 5.15. *For $k \geq 2$,*

$$\begin{aligned}&x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4},\dots,x_{i_{k+1}}]\\ &\quad = x_j^*\otimes[x_j,x_{i_4},\dots,x_{i_{k+1}},x_{i_2},x_{i_1}] - x_j^*\otimes[x_j,x_{i_4},\dots,x_{i_{k+1}},x_{i_1},x_{i_2}]\end{aligned}$$

$\in \mathrm{Coker}(\tau'_{k,N})$ *if $i,j \neq i_l$ for $l \neq 3$, and $i \neq j$.*

Proof. From Lemmas 5.9 and 5.7, there exist some $\sigma \in \mathcal{A}'_n(k-2)$ and $\tau \in \mathcal{A}_n(2)$ such that

$$\begin{aligned}\tau'_{2,N}(\sigma) &= x_i^*\otimes[x_j,x_{i_4},\dots,x_{i_{k+1}}],\\ \tau'_{k-2,N}(\tau) &= x_j^*\otimes[x_{i_1},x_{i_2},x_i],\end{aligned}$$

respectively. Then, by the Jacobi identity,

$$\begin{aligned}\tau_{k,N}([\sigma,\tau]) &= x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4},\dots,x_{i_{k+1}}]\\ &\qquad - x_j^*\otimes[x_{i_1},x_{i_2},[x_j,x_{i_4},\dots,x_{i_{k+1}}]]\\ &= x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4},\dots,x_{i_{k+1}}]\\ &\qquad + x_j^*\otimes[x_j,x_{i_4},\dots,x_{i_{k+1}},x_{i_1},x_{i_2}] - x_j^*\otimes[x_j,x_{i_4},\dots,x_{i_{k+1}},x_{i_2},x_{i_1}].\end{aligned}$$

This completes the proof of Lemma 5.15. □

Lemma 5.16. *For $k \geq 5$,*

$$x_i^*\otimes[x_{i_1},x_{i_2},x_i,x_{i_4},\dots,x_{i_{k+1}}] = x_j^*\otimes[x_{i_1},x_{i_2},x_j,x_{i_5},\dots,x_{i_{k+1}},x_{i_4}] \in \mathrm{Coker}(\tau'_{k,N})$$

if $i,j \neq i_l$ for $l \neq 3$, and $i \neq j$.

Proof. From Lemma 5.7, there exists some $\sigma \in \mathcal{A}_n(k-1)$ such that

$$\tau'_{k-1,N}(\sigma) = x_i^* \otimes [x_{i_1}, x_{i_2}, x_j, x_{i_5}, \ldots, x_{i_{k+1}}].$$

Then,

$$\begin{aligned}\tau'_{k,N}([\sigma, K_{ji_4i}]) &= x_i^* \otimes [x_{i_1}, x_{i_2}, [x_{i_4}, x_i], x_{i_5}, \ldots, x_{i_{k+1}}] \\ &\quad + x_j^* \otimes [x_{i_1}, x_{i_2}, x_j, x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}] \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_4}, x_i, x_{i_5}, \ldots, x_{i_{k+1}}] \\ &\quad - x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, x_{i_5}, \ldots, x_{i_{k+1}}] \\ &\quad + x_j^* \otimes [x_{i_1}, x_{i_2}, x_j, x_{i_5}, \ldots, x_{i_{k+1}}, x_{i_4}].\end{aligned}$$

Hence from Lemma 5.10, we obtain the required result. This completes the proof of Lemma 5.16. □

In the following, we consider the case where $n \geq k+2$. From Lemmas 5.9, 5.10 and 5.14, we see that $\mathrm{Coker}(\tau'_{k,N})$ is generated by elements $x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}]$ and $x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}]$ of $\mathfrak{F}$ such that $1 \leq i, i_l \leq n$ and $i \neq i_l$. Furthermore, if we set

$$s'(i, i_2, \ldots, i_{k+1}) := x_i^* \otimes [x_i, x_{i_2}, \ldots, x_{i_{k+1}}] \in \mathrm{Coker}(\tau'_{k,N})$$

for $i_l \neq i$, then from Lemmas 5.11 and 5.12, we see that $s'(i, i_2, \ldots, i_{k+1})$ does not depend on the choice of i such that $i \neq i_l$ for $2 \leq l \leq k+1$. Hence we can set

$$s(i_2, \ldots, i_{k+1}) := s'(i, i_2, \ldots, i_{k+1})$$

and have

$$s(i_2, \ldots, i_{k+1}) = s(i_3, \ldots, i_{k+1}, i_2) = \cdots = s(i_{k+1}, i_2, \ldots, i_k)$$

in $\mathrm{Coker}(\tau'_{k,N})$.

Next, set

$$u'(i_1, i_2, i, i_4, \ldots, i_{k+1}) := x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, \ldots, x_{i_{k+1}}] \in \mathrm{Coker}(\tau'_{k,N})$$

for $i_l \neq i$. From Lemma 5.15, we verify that

$$u'(i_1, i_2, i, i_4, \ldots, i_{k+1}) = s(i_4, \ldots, i_{k+1}, i_2, i_1) - s(i_4, \ldots, i_{k+1}, i_1, i_2)$$

and it also does not depend on the choice of i such that $i \neq i_l$ for $l \neq 3$. Hence we can set

$$u(i_1, i_2, i_4, \ldots, i_{k+1}) := u'(i_1, i_2, i, i_4, \ldots, i_{k+1}).$$

Here we consider some relations among the $u(i_1, i_2, i_4, \ldots, i_{k+1})$s. First, using

$$u(i_1, i_2, i_4, \ldots, i_{k+1}) = s(i_1, i_4, \ldots, i_{k+1}, i_2) - s(i_1, i_2, i_4, \ldots, i_{k+1}),$$

we obtain

$$u(j, j_1, j_2, \ldots, j_k) + u(j, j_2, \ldots, j_k, j_1) + \cdots + u(j, j_k, j_1, \ldots, j_{k-1}) = 0. \tag{8}$$

From Lemma 5.16, we see

$$u(i_1, i_2, i_4, \ldots, i_{k+1}) = u(i_1, i_2, i_5, \ldots, i_{k+1}, i_4). \tag{9}$$

In general, for $k \geq 5$,

$$\begin{aligned} 0 &= x_i^* \otimes [x_{i_1}, x_{i_2}, [x_i, x_{i_4}], [x_{i_5}, x_{i_6}], x_{i_7}, \dots, x_{i_{k+1}}] \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, [x_{i_5}, x_{i_6}], x_{i_7}, \dots, x_{i_{k+1}}] \\ &\quad - x_i^* \otimes [x_{i_1}, x_{i_2}, x_{i_4}, x_i, [x_{i_5}, x_{i_6}], x_{i_7}, \dots, x_{i_{k+1}}] \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, [x_{i_5}, x_{i_6}], x_{i_7}, \dots, x_{i_{k+1}}] \\ &= x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}, \dots, x_{i_{k+1}}] \\ &\quad - x_i^* \otimes [x_{i_1}, x_{i_2}, x_i, x_{i_4}, x_{i_6}, x_{i_5}, x_{i_7}, \dots, x_{i_{k+1}}] \end{aligned}$$

in $\mathrm{Coker}(\tau'_{k,N})$. This shows

$$u(i_1, i_2, i_4, i_5, i_6, i_7, \dots, i_{k+1}) = u(i_1, i_2, i_4, i_6, i_5, i_7, \dots, i_{k+1}). \tag{10}$$

Observing the fact that for any l, the symmetric group $\mathfrak{S}_l$ of degree l is generated by a cyclic permutation of length l and a transposition, we verify that

$$u(i_1, i_2, j_1, j_2, \dots, j_{k-2}) = u(i_1, i_2, j_{\gamma(1)}, j_{\gamma(2)}, \dots, j_{\gamma(k-2)}) \tag{11}$$

for any $\gamma \in \mathfrak{S}_{k-2}$ by (9) and (10).

In order to reduce the generators more, we consider the rational case. By the same argument as above, we see that $\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})$ is also generated by the $s(i_2, \dots, i_{k+1})$s and $u(i_1, i_2, i_4 \dots, i_{k+1})$s as a $\mathbf{Q}$-vector space for $n \geq k+2$. Denote by W the subspace of $\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})$ generated by elements $u(i_1, i_2, i_3, \dots, i_k)$ for $i_1 > i_2 > i_3 \leq i_4 \leq \cdots \leq i_k$. Then we have

Lemma 5.17. *For $k \geq 5$, $n \geq k+2$, and any $1 \leq j_1, \dots, j_k \leq n$,*

$$u(j_1, j_2, j_3, \dots, j_k) \in W.$$

Proof. By (11), we may assume that $j_1 > j_2$ and $j_3 \leq \cdots \leq j_{k+1}$. Suppose $j_2 \leq j_3$. If $j_2 < j_3$, by (8), we obtain

$$\begin{aligned} u(j_1, j_2, j_3, \dots, j_k) &= -u(j_1, j_3, j_2, j_4, \dots, j_k) - u(j_1, j_4, j_2, j_3, j_5, \dots, j_k) \\ &\quad - \cdots - u(j_1, j_k, j_2, j_3, \dots, j_{k-1}) \in W. \end{aligned}$$

If $j_2 = j_3$, there exists some l such that $3 \leq l \leq k$ and

$$j_2 = j_3 = \cdots = j_l < j_{l+1} \leq \cdots \leq j_k.$$

Then, by (8), we see

$$\begin{aligned} (l-1)u(j_1, j_2, j_3, \dots, j_k) &= -u(j_1, j_{l+1}, j_2, \dots, j_l, j_{l+1}, \dots j_k) \\ &\quad - \cdots - u(j_1, j_k, j_2, \dots, j_{k-1}). \end{aligned}$$

Therefore, we obtain the required result. This completes the proof of Lemma 5.17. □

Now, if we set $V := \mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})/W$, we have

$$s(j_1, j_2, j_3, \dots, j_k) = s(j_2, j_1, j_3, \dots, j_k) \in V.$$

This shows

$$s(j_1, j_2, j_3, \dots, j_k) = s(j_{\gamma(1)}, j_{\gamma(2)}, j_{\gamma(3)}, \dots, j_{\gamma(k)}) \in V$$

for any $\gamma \in \mathfrak{S}_k$. In particular, V is generated by $s(j_1, j_2, j_3, \ldots, j_k)$ such that $1 \leq j_1 \leq \cdots \leq j_k \leq n$. Therefore we conclude that

Proposition 5.1. *For $k \geq 5$ and $n \geq k+2$, $\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}})$ is generated by*

$$\{s(i_1, i_2, i_3, \ldots, i_k) \,|\, 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

and

$$\{u(i_1, i_2, i_3, \ldots, i_k) \,|\, i_1 > i_2 > i_3 \leq i_4 \leq \cdots \leq i_k\}$$

as a $\mathbf{Q}$-vector space. In particular, the number of the generators above is

$$\binom{n+k-1}{k} + \frac{(k-2)(k-1)}{2}\binom{n+k-3}{k}.$$

Therefore we conclude that

Theorem 5.3. *For $n \geq k+2$,*

$$\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}}) = S^k H_{\mathbf{Q}} \oplus H_{\mathbf{Q}}^{[k-2,1^2]}.$$

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