On the binary digits of algebraic numbers *

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Abstract

Borel conjectured that all algebraic irrational numbers are normal in base 2. However, very few is known on this problem. We derive new, improved lower bounds of the number of digit changes in the binary expansions of algebraic irrational numbers.

1 Introduction

Borel [3] proved that almost all positive number $\xi$ are normal in every integral base $\alpha \geq 2$. Namely, every string of $l$ consecutive base-$\alpha$ digits occurs with average frequency tending to $1/\alpha^l$ in the $\alpha$-ary expansion of such $\xi$. It is widely believed that all algebraic irrational numbers are normal in each integral base. However, very few is known on this problem, which was first formulated by Borel [4]. For instance it is still unknown whether the word 11 occurs infinitely often in the binary expansion of $\sqrt{2}$.

In this paper we study the binary expansions of algebraic irrational numbers. In what follows, let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{Z}^+$ the set of positive integers. Denote the integral and fractional parts of a real number $\xi$ by $\lfloor \xi \rfloor$ and $\{ \xi \}$, respectively. Moreover, let $\lceil \xi \rceil$ be the minimal integer not less than $\xi$. Then the binary expansion of a positive number $\xi$ is written by

$$\xi = \sum_{n=-\infty}^{\infty} s(\xi, n)2^n,$$

where

$$s(\xi, n) = \lfloor 2^{-n} \xi \rfloor - 2\lfloor 2^{-n-1} \xi \rfloor \in \{0, 1\}. \quad (1.1)$$

There are several ways to measure the complexity of the binary expansions of real numbers. First we introduce the block complexity. Let $\beta(\xi, N)$ be the total number of distinct blocks of $N$ digits in the binary expansion of $\xi$, that is,

$$\beta(\xi, N) = \text{Card}\{s(\xi, i+1), \ldots, s(\xi, i+N)) \in \{0, 1\}^N \mid i \in \mathbb{Z}\},$$

where Card denotes the cardinality. If $\xi$ is normal in base 2, then $\beta(\xi, N) = 2^N$ for any $N \in \mathbb{Z}^+$. Suppose that $\xi$ is an algebraic irrational number. Bugeaud and Evertse [7] showed for any positive $\delta$ with $\delta < 1/11$ that

$$\limsup_{N \to \infty} \frac{\beta(\xi, N)}{N(\log N)^\delta} = \infty.$$

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Secondly we estimate the number of nonzero digits in the binary expansion of \( \xi \). For each integer \( N \), let
\[
\lambda(\xi, N) = \text{Card}\{n \in \mathbb{Z} \mid n \geq -N, s(\xi, n) \neq 0\}.
\]
Bailey, Borwein, Crandall and Pomerance [1] verified for any algebraic irrational \( \xi \) with degree \( D \geq 2 \) that there exists a positive computable constant \( C_0(\xi) \) depending only on \( \xi \) satisfying
\[
\lambda(\xi, N) \geq C_0(\xi) N^{1/D}
\]
for all sufficiently large integers \( N \). Rivoal [17] improved the constant \( C_0(\xi) \) for certain classes of algebraic irrational \( \xi \). For example, let \( \xi^* = 0.558\ldots \) be the unique positive zero of the polynomial \( 8X^3 - 2X^2 + 4X - 3 \) and \( \varepsilon \) an arbitrary positive real number with \( \varepsilon < 1 \). Theorem 7.1 in [1] implies for any sufficiently large \( N \in \mathbb{N} \) that
\[
\lambda(\xi^*, N) \geq (1 - \varepsilon)16^{-1/3}N^{1/3}.
\]
On the other hand, using Corollary 2 in [17], we obtain
\[
\lambda(\xi^*, N) \geq (1 - \varepsilon)N^{1/3}
\]
for all sufficiently large \( N \in \mathbb{N} \).

Now we consider the asymptotic behaviour of the number of digit changes in the binary expansions of real numbers \( \xi \). Let \( N \) be an integer. The number \( \gamma(\xi, N) \) of digit changes, introduced in [6], is defined by
\[
\gamma(\xi, N) = \text{Card}\{n \in \mathbb{Z} \mid n \geq -N, s(\xi, n) \neq s(\xi, 1 + n)\}.
\]
Note that \( \gamma(\xi, N) < \infty \) because \( s(\xi, n) = 0 \) for all sufficiently large \( n \in \mathbb{N} \). Suppose again that \( \xi \) is an algebraic irrational number of degree \( D \geq 2 \). In [6] Bugeaud proved, using Ridout’s theorem [16], that
\[
\lim_{N \to \infty} \frac{\gamma(\xi, N)}{\log N} = \infty.
\]
In the same paper he showed, using a quantitative version of Ridout’s theorem [12], that
\[
\gamma(\xi, N) \geq 3(\log N)^{6/5}(\log \log N)^{-1/4}
\]
for every sufficiently large \( N \in \mathbb{N} \). Moreover, improving the quantitative parametric subspace theorem from [9], Bugeaud and Evertse [7] verified the following: There exist an effectively computable absolute constant \( C_1 > 0 \) and an effectively computable constant \( C_2(\xi) > 0 \), depending only on \( \xi \), satisfying
\[
\gamma(\xi, N) \geq C_1 \frac{(\log N)^{3/2}}{(\log(6D))^{1/2}(\log \log N)^{1/2}}
\]
for any integer \( N \) with \( N \geq C_2(\xi) \).

Note that if \( \xi \) is normal, then the word 10 occurs in the binary expansion of \( \xi \) with frequency tending to \( 1/4 \). Thus, it is widely believed that the function \( \gamma(\xi, N) \) should grow linearly in \( N \). The main purpose of this paper is to improve lower bounds of the function \( \gamma(\xi, N) \) for certain classes of algebraic irrational numbers. Now we state the main results.
THEOREM 1.1. Let $\xi > 0$ be an algebraic irrational number with minimal polynomial $A_D X^D + A_{D-1} X^{D-1} + \cdots + A_0 \in \mathbb{Z}[X]$, where $A_D > 0$. Assume that there exists an odd prime number $p$ which divides all coefficients $A_D, A_{D-1}, \ldots, A_1$, but not the constant term $A_0$. Let $\varepsilon$ be an arbitrary positive number with $\varepsilon < 1$ and $r$ the minimal positive integer such that $p$ divides $(2^r - 1)$. Then there exists an effectively computable positive constant $C(\xi, \varepsilon)$ depending only on $\xi$ and $\varepsilon$ such that

$$\gamma(\xi, N) \geq (1 - \varepsilon)p^{1/D} r^{1/D} A_D^{-1/D} N^{1/D}$$

for any integer $N$ with $N \geq C(\xi, \varepsilon)$.

For instance, let $A$ and $D$ be positive integers such that $A^{-1/D}$ is an irrational number of degree $D$. Assume that there is an odd prime $p$ which divides $A$. Let $\varepsilon$ be any positive number with $\varepsilon < 1$ and $r$ defined as in Theorem 1.1. Then, since the minimal polynomial of $A^{-1/D}$ is $A X^D - 1$, by Theorem 1.1 we obtain

$$\gamma(A^{-1/D}, N) \geq (1 - \varepsilon)p^{1/D} r^{1/D} A^{-1/D} N^{1/D}$$

for every integer $N$ with $N \geq C(A^{-1/D}, \varepsilon)$. In the case where $A = 3$ and $D = 2$, we get $p = 3$ and $r = 2$. Hence

$$\gamma \left( \frac{1}{\sqrt{3}}, N \right) \geq \frac{1 - \varepsilon}{\sqrt{2}} \sqrt{N}$$

for each integer $N$ with $N \geq C(1/\sqrt{3}, \varepsilon)$.

### 2 Signed binary representations

In this section we study signed binary expansions of a nonzero integer $n$ of the form

$$n = \sum_{i=0}^{l-1} a_i 2^i,$$

where, for $0 \leq i \leq l - 1$, $a_i \in \{-1, 0, 1\}$ and $a_{l-1} \neq 0$. The sequence of signed bits is usually written with most-significant digits $a_{l-1}$ first. In a sequence of signed bits, the $\overline{1}$ will denote $-1$. Thus, 1000$\overline{1}$ is a signed bit representation for 15. We are interested in finding a representation of $n$ having the minimal Hamming weight, or the number of nonzero digits. Note that the minimality of a Hamming weight does not determine a unique representation in general, as the example $19 = 2^4 + 2 + 1 = 2^4 + 2^2 - 1$ shows. Reitwiesner [15] proved for each integer $n$ that there exists a unique signed expansion (2.1) satisfying

$$a_i a_{i+1} = 0$$

for any $i \geq 0$. We will call this representation the signed separated binary expansion of $n$, say the SSB expansion of $n$ for short. We write the Hamming weight of the SSB expansion of $n$ by

$$\nu(n) = \sum_{i=0}^{l-1} |a_i|.$$
For instance, if \( l \geq 2 \), then \( \nu(2^l - 1) = 2 \). For convenience, let \( \nu(0) := 0 \). It is known for every integer \( n \) that \( \nu(n) \) is the minimal Hamming weight among the signed binary expansions of \( n \) (for instance, see [5]). In particular, since

\[
n = 1 + \cdots + 1 \quad \text{or} \quad n = -1 - \cdots - 1, \tag{2.2}
\]

we get

\[
\nu(n) \leq |n|. \tag{2.2}
\]

SSB expansions have applications in the optimal design of arithmetical hardware [2, 15], in coding theory [13], and in cryptography [14]. For detailed information concerning the SSB expansions of integers, see [5, 8, 10, 11].

We now show that the function \( \nu \) satisfies the convexity relations which are analogues of Theorem 4.2 in [1].

**LEMMA 2.1.** Let \( m \) and \( n \) be integers. Then

\[
\nu(m + n) \leq \nu(m) + \nu(n) \tag{2.3}
\]

and that

\[
\nu(mn) \leq \nu(m)\nu(n). \tag{2.4}
\]

**Proof.** It is easy to check (2.3) and (2.4) in the case of \( mn = 0 \). Thus, we may assume that \( mn \neq 0 \). Let

\[
\Lambda := \{ \pm 2^l \mid l \in \mathbb{N} \}.
\]

Then there exist \( \lambda_1, \ldots, \lambda_{\nu(m)}, \lambda'_1, \ldots, \lambda'_{\nu(n)} \in \Lambda \) such that

\[
m = \sum_{k=1}^{\nu(m)} \lambda_k, \quad n = \sum_{h=1}^{\nu(n)} \lambda'_h. \tag{2.5}
\]

We have

\[
m + n = \sum_{k=1}^{\nu(m)} \lambda_k + \sum_{h=1}^{\nu(n)} \lambda'_h
\]

and

\[
mn = \sum_{k=1}^{\nu(m)} \sum_{h=1}^{\nu(n)} \lambda_k \lambda'_h.
\]

Observe that \( \lambda_k \lambda'_h \in \Lambda \) for any \( k \) and \( h \). Hence, using the minimality of the Hamming weights of SSB expansions, we obtain (2.3) and (2.4). \( \square \)
Note that the SSB expansion (2.5) of an integer $m$ satisfies the following:

$$|\lambda_i| \neq |\lambda_j| \text{ for } 1 \leq i < j \leq \nu(m).$$  \hfill (2.6)

In fact, if $|\lambda_i| = |\lambda_j|$ for some $1 \leq i < j \leq \nu(m)$, then

$$m = \begin{cases} 
2\lambda_i + \sum_{k=1, k \neq i,j}^{\nu(m)} \lambda_k & \text{(if } \lambda_i = \lambda_k), \\
\sum_{k=1, k \neq i,j}^{\nu(m)} \lambda_k & \text{(if } \lambda_i = -\lambda_k).
\end{cases}$$

The equality above contradicts the minimality of the Hamming weights of SSB expansions. Combining (2.2) and (2.3), we obtain, for all integers $m$ and $n$,

$$|\nu(m + n) - \nu(m)| \leq |n|. \hfill (2.7)$$

In fact, we get

$$\nu(m + n) - \nu(m) \leq \nu(n) \leq |n|$$

and

$$\nu(m) - \nu(m + n) \leq \nu(-n) \leq |n|.$$  

The SSB expansions of real numbers are also introduced in [8]. Let $V$ be a nonempty finite word on the alphabet $\{0, 1, \tilde{T}\}$. We write the right-infinite word $VVV \ldots$ by $V^\omega$. Let

$$K := \{x = x_0x_1x_2 \ldots \in \{0, 1, \tilde{T}\}^\mathbb{N} \mid x_i x_{i+1} = 0 \text{ for any } i \geq 1\}.$$  

Then $K$ is a nonempty compact subset of $\{0, 1, \tilde{T}\}$ endowed with the weak topology, so two points are close if they agree on a sufficiently large beginning block. Dajani, Kraaikamp, and Liardet [8] introduced the continuous map $f$ from $K$ onto the interval $[-2/3, 2/3]$ given by

$$f(x) = \sum_{i=1}^{-1} x_i 2^i = : 0.x_0x_1x_2 \ldots.$$  

We will call this representation the SSB expansion of $f(x)$. Note that $f$ is not injective, namely, the SSB expansion of a given real number $\eta \in [-2/3, 2/3]$ is not unique. In fact, we have

$$\frac{1}{3} = 0.(01)^\omega = 0.1(0\tilde{T})^\omega.$$  

Let $\xi$ be any real number. We take a sufficiently large positive integer $R$ such that

$$\eta := 2^{-R}\xi \in \left[-\frac{2}{3}, \frac{2}{3}\right].$$  

Using the SSB expansion of $\eta = \sum_{i=-\infty}^{1} x_i 2^i$, we can define the SSB expansion of $\xi$ as follows:

$$\xi = 2^R \sum_{i=-\infty}^{-1} x_i 2^i =: \sum_{i=-\infty}^{R-1} y_i 2^i = y_{R-1} \ldots y_0 y_{-1} y_{-2} \ldots,$$  

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where \( y_i = x_{i-R} \) for \( i \leq R - 1 \). Dajani, Kraaikamp, and Liardet [8] identified \( K \) up to a countable set, with the interval \([-2/3, 2/3]\). They studied the dynamical properties of \( K \), equipped with its Borel \( \sigma \)-algebra \( B_K \) both under the shift \( \sigma \) and the odometer \( \tau \). We now give the SSB expansions of rational numbers.

**Lemma 2.2.** The SSB expansion of a rational number \( \xi \) is ultimately periodic. Moreover, let \( r \) be the period of the ordinary binary expansion of \( \xi \). Then \( r \) is also the period of the SSB expansion of \( \xi \).

**Proof.** Without loss of generality, we may assume that \( \xi > 0 \). In fact, let \( \sum_{i=-\infty}^R a_i 2^i \) be the SSB expansion of \( \xi \). Then the SSB expansion of \( -\xi \) is \( \sum_{i=-\infty}^R (-a_i) 2^i \). Moreover, we may assume that \( \xi < 1/2 \). Since \( \xi \) is a rational number, its ordinary binary expansion is ultimately periodic. Namely, we have

\[
\xi = 0.0UV^\omega,
\]

where \( U \) and \( V \) are words on the alphabet \( \{0,1\} \) and the length of \( V \) is \( r \). Put

\[
\xi_k := 0.0UV \ldots V_k.
\]

(2.8)

It is easy to obtain the SSB expansion of \( \xi_k \) from its ordinary binary expansion (2.8): apply the following rule repeatedly, working from right to left (least-significant first):

replace any sequence \( 01 \ldots 1 \) by \( 10 \ldots 0 \).

Hence, the SSB expansion of \( \xi_k \) satisfies the following:

\[
\xi_k = 0.U'V_1V_2 \ldots V_3V_3,
\]

where \( U', V_i \) are words on the alphabet \( \{0,1,\bar{T}\} \) and the length of \( V_i \) is \( r \) for \( i = 1,2,3 \). Therefore, we deduce that the SSB expansion of \( \xi \) is

\[
\xi = 0.U'V_1V_2^\omega,
\]

which implies Lemma 2.2. \( \square \)

**Lemma 2.3.** Let \( b \) be an integer and \( p \) an odd prime number. Assume that \( p \) does not divide \( b \). Let \( r \) be the minimal positive integer such that \( p \) divides \( 2^r - 1 \). Then, for each \( N \in \mathbb{N} \),

\[
\nu \left( \left\lfloor \frac{2^N b}{p} \right\rfloor \right) \geq \frac{N}{r} - 4.
\]

**Proof.** Put \( \xi := b/p \). We show that the SSB expansion of \( \xi \) is written by

\[
\xi =: \sum_{i=-\infty}^R a_i 2^i = U.V_1V_2^\omega,
\]

(2.9)
where \( U, V_j \) are words on the alphabet \( \{0, 1, \hat{T}\} \) and the length of \( V_j \) is \( r \) for \( j = 1, 2 \). For the proof of (2.9), we may assume that \( \xi > 0 \).

Observe that

\[
\xi = \lfloor \xi \rfloor + \frac{u}{2^r - 1}
\]

for some integer \( u \) with \( 0 \leq u < 2^r - 1 \). Thus, the ordinary binary expansion of \( \xi \) satisfies

\[
\xi = U'.W^w,
\]

where \( U' \) and \( W \) are words on the alphabet \( \{0, 1\} \) and the length of \( W \) is \( r \). Hence, in the same way as the proof of Lemma 2.2, we obtain (2.9). For each \( N \in \mathbb{N} \), let

\[
\xi_N := \sum_{i=-N}^{R} a_i 2^i.
\]

Then \( 2^N \xi_N \) is an integer whose SSB expansion is written by

\[
2^N \xi_N = U V_1 V_2 \ldots V_2' V',
\]

where \( V' \) is the prefix of \( V_2 \) of length \( r \lfloor N/r \rfloor \). By the assumptions of Lemma 2.3, it is clear that the expansion (2.9) is not finite, namely, at least one letter of the word \( V_2 \) is not zero. Hence, we obtain

\[
\nu(2^N \xi_N) \geq \left\lfloor \frac{N}{r} \right\rfloor - 1. \tag{2.10}
\]

Observe that

\[
|2^N \xi - 2^N \xi_N| = \left| 2^N \sum_{i=-\infty}^{-N-1} a_i 2^i \right| = \left| \sum_{i=-\infty}^{-1} a_i 2^i \right|.
\]

The second statement of Lemma 1 in [8] implies that

\[
|2^N \xi - 2^N \xi_N| \leq \frac{2}{3}.
\]

and so

\[
|\lfloor 2^N \xi \rfloor - 2^N \xi_N| \leq 2.
\]

Therefore, combining (2.7) and (2.10), we obtain

\[
\nu(\lfloor 2^N \xi \rfloor) \geq \nu(2^N \xi_N) - 2 \geq \left\lfloor \frac{N}{r} \right\rfloor - 3,
\]

which implies Lemma 2.3.
3 Proof of Theorem 1.1

We study the relations between the number $\gamma(\xi, N)$ of digit changes and the value $\nu([2^N\xi^h])$ for $h \in \mathbb{Z}^+$ and $N \in \mathbb{N}$. Let $\eta_1$ and $\eta_2$ be any real numbers. Then it is easily seen that

\begin{equation}
||\eta_1 + \eta_2| - (|\eta_1| + |\eta_2|)| \leq 1 \tag{3.1}
\end{equation}

and that

\begin{equation}
||\eta_1 - \eta_2| - (|\eta_1| - |\eta_2|)| \leq 1. \tag{3.2}
\end{equation}

**Lemma 3.1.** Let $\xi$ be a positive real number. Then, for any $h \in \mathbb{Z}^+$ and $N \in \mathbb{N}$,

\begin{equation}
\nu([2^N\xi^h]) \leq (\gamma(\xi, N) + 1)^h + 2^{h+1}\max\{1, \xi^h\}. \tag{3.3}
\end{equation}

**Proof.** Let $\tau := \gamma(\xi, N)$. We first consider the case of $h = 1$. Using the definition of $\tau$ and

\[ 1 \ldots 1 0 \ldots 0 = 2^{h+i} - 2^k, \]

we obtain

\begin{equation}
\nu([2^N\xi]) \leq 2\left\lfloor \frac{T}{2} \right\rfloor \leq \tau + 1 \tag{3.4}
\end{equation}

because $\nu([2^N\xi])$ is the minimal Hamming weight among the signed binary expansions of $[2^N\xi]$.

Next suppose that $h \geq 2$. Put

\begin{align*}
\xi_1 &= \sum_{n=-N}^{N} s(\xi, n)2^n, \\
\xi_2 &= \sum_{n=-\infty}^{-N-1} s(\xi, n)2^n.
\end{align*}

Note that $2^N\xi_1 \in \mathbb{Z}$. We have

\[ 2^N\xi^h = 2^N(\xi_1 + \xi_2)^h = 2^N\xi_1^h + 2^N \sum_{i=1}^{h} \binom{h}{i} \xi_1^{h-i}\xi_2^i, \]

and so, by (3.1),

\[ ||[2^N\xi^h] - [2^N\xi_1^h]| \leq 1 + 2^N \sum_{i=1}^{h} \binom{h}{i} \xi_1^{h-i}\xi_2^i. \]

Hence, by (2.7), we get

\begin{equation}
\nu([2^N\xi^h]) \leq \nu([2^N\xi_1^h]) + 1 + 2^N \sum_{i=1}^{h} \binom{h}{i} \xi_1^{h-i}\xi_2^i. \tag{3.4}
\end{equation}
In what follows we estimate upper bounds of the right-hand side of (3.4). By (2.4) and (3.3),

\[ \nu \left( 2^N \xi_1^h \right) \leq \nu \left( 2^N \xi_1 \right)^h = \nu \left( \left[ 2^N \xi_1 \right]^h \right) \leq (\tau + 1)^h. \]

By the inequality above and (2.6), there exist \( a, b \in \mathbb{N} \) with \( a + b \leq (\tau + 1)^h \) and \( l_1, \ldots, l_a, k_1, \ldots, k_b \in \mathbb{N} \) satisfying the following:

\[ l_1 < \cdots < l_a, \quad k_1 < \cdots < k_b; \quad (3.5) \]

\[ 2^h \xi_1^h = \sum_{i=1}^a 2^{l_i} - \sum_{j=1}^b 2^{k_j}. \]

Let

\[ \theta_1 = \sum_{1 \leq i \leq a} 2^{l_i - (h-1)N} - \sum_{1 \leq j \leq b} 2^{k_j - (h-1)N}, \]

\[ \theta_2 = \sum_{1 \leq i \leq a} 2^{l_i - (h-1)N} - \sum_{1 \leq j \leq b} 2^{k_j - (h-1)N}. \]

Then \( \theta_1 \in \mathbb{Z} \) and

\[ \theta_1 + \theta_2 = 2^N \xi_1^h. \quad (3.6) \]

By (3.5) we have

\[ \sum_{1 \leq i \leq a} 2^{l_i - (h-1)N} < \sum_{i=1}^\infty 2^{-i} = 1 \]

and

\[ \sum_{1 \leq j \leq b} 2^{k_j - (h-1)N} < 1. \]

Thus

\[ |\theta_2| < 1. \quad (3.7) \]

Combining (3.6) and (3.7), we obtain

\[ ||2^N \xi_1^h| - \theta_1| \leq 1. \]

Hence, by (2.7)

\[ \nu \left( \left| 2^N \xi_1^h \right| \right) \leq \nu(\theta_1) + 1 \leq a + b + 1 \leq (\tau + 1)^h + 1. \quad (3.8) \]

Moreover, since \( \xi_1 \leq \xi \) and since \( \xi_2 \leq 2^{-N} \),

\[ 2^N \sum_{i=1}^h \binom{h}{i} \xi_1^{h-i} \xi_2^i \leq \sum_{i=0}^h \binom{h}{i} \max\{1, \xi^h\} = 2^h \max\{1, \xi^h\}. \quad (3.9) \]
Combining (3.4), (3.8), and (3.9), we conclude that

\[
\nu \left( \left\lfloor 2^N \xi \right\rfloor \right) \leq (\tau + 1)^h + 2^h \max\{1, \xi^h\} + 2
\leq (\tau + 1)^h + 2^{1+h} \max\{1, \xi^h\}.
\]

\[\square\]

Now we verify Theorem 1.1. Let \( A'_i = A_i / p \) for \( i = 1, 2, \ldots, D \). Then we have

\[
\sum_{h=1}^{D} A'_h 2^N \xi^h = -\frac{2^N A_0}{p}
\]

for each \( N \in \mathbb{N} \). Then by Lemma 2.3 we get

\[
\nu \left( \left\lfloor \frac{2^N A_0}{p} \right\rfloor \right) \geq \frac{N}{r} - 4. \tag{3.10}
\]

On the other hand, by (3.1) and (3.2)

\[
\left\lfloor \sum_{h=1}^{D} A'_h 2^N \xi^h \right\rfloor - \sum_{h=1}^{D} A'_h \left\lfloor 2^N \xi^h \right\rfloor \leq \sum_{h=1}^{D} |A'_h|.
\]

Hence, using (2.3), (2.7) and Lemma 3.1, we get

\[
\nu \left( \left\lfloor -\frac{2^N A_0}{p} \right\rfloor \right) = \nu \left( \left\lfloor \sum_{h=1}^{D} A'_h 2^N \xi^h \right\rfloor \right) \leq \nu \left( \sum_{h=1}^{D} A'_h \left\lfloor 2^N \xi^h \right\rfloor \right) + \sum_{h=1}^{D} |A'_h|
\leq \sum_{h=1}^{D} |A'_h| \left( \nu \left( \left\lfloor 2^N \xi^h \right\rfloor \right) + 1 \right)
\leq \sum_{h=1}^{D} |A'_h| \left( (\gamma(\xi, N) + 1)^h + 2^{h+1} \max\{1, \xi^h\} + 1 \right). \tag{3.11}
\]

Combining (3.10) and (3.11), we obtain, for every nonnegative integer \( N \),

\[
N \leq P(\gamma(\xi, N)), \tag{3.12}
\]

where \( P(X) \in \mathbb{R}[X] \) is a polynomial of degree \( D \) with leading coefficient \( rA'_D \). Thus, for any positive number \( R \), there is an effectively computable positive constant \( C'(\xi, R) \) depending only on \( \xi \) and \( R \) such that

\[
\gamma(\xi, N) \geq R
\]

for any integer \( N \) with \( N \geq C'(\xi, R) \). Let \( \varepsilon \) be an arbitrary positive number with \( \varepsilon < 1 \). Put

\[
\delta := -1 + (1 - \varepsilon)^{-D} > 0.
\]

By (3.12), there exists an effectively computable positive constant \( C(\xi, \varepsilon) \) depending only on \( \xi \) and \( \varepsilon \) such that, for every integer \( N \) with \( N \geq C(\xi, \varepsilon) \),

\[
N \leq (1 + \delta)rA'_D \gamma(\xi, N)^D,
\]
namely,

\[(1 - \varepsilon)p^{1/D}r^{-1/D}A_D^{-1/D}N^{1/D} \leq \gamma(\xi, N).\]

Therefore, we proved Theorem 1.1.

**REMARK 3.1.** The constant preceding \(N^{1/D}\) in Theorem 1.1 can be improved by considering the number \(\sigma\) of nonzero digits in the period of the SSB expansion of \(A_0/p\). In the same way as the proof of Lemma 2.3, we can show that

\[\nu \left( \left\lfloor -\frac{2^N A_0}{p} \right\rfloor \right) \geq \sigma \left( \frac{N}{r} - 1 \right) - 2 \geq \frac{\sigma}{r} N - 2\sigma - 2.\]

Let \(\varepsilon\) be an arbitrary positive number with \(\varepsilon < 1\). Then, by Lemma 3.1, there exists an effectively computable positive constant \(C''(\xi, \varepsilon)\) depending only on \(\xi\) and \(\varepsilon\) such that

\[\gamma(\xi, N) \geq (1 - \varepsilon) \left( \frac{\sigma p}{r A_D} \right)^{1/D} N^{1/D}\] (3.13)

for every integer \(N\) with \(N \geq C''(\xi, \varepsilon)\), which gives improvements of Theorem 1.1.

For instance, we consider the case of \(\xi = 1/\sqrt{5}\). Then we have \(A_2 = p = 5\) and \(r = 4\). Theorem 1.1 implies that

\[\gamma \left( \frac{1}{\sqrt{5}}, N \right) \geq \frac{1 - \varepsilon}{2} \sqrt{N}\]

for each integer \(N\) with \(N \geq C(1/\sqrt{5}, \varepsilon)\). Observe that the SSB expansion of \(A_0/p = -1/5\) is

\[\frac{1}{5} = 0.0\overline{01}\omega.\]

Hence, the number of nonzero digits in the SSB expansion of \(A_0/p\) is 2. Therefore, (3.13) implies that

\[\gamma \left( \frac{1}{\sqrt{5}}, N \right) \geq \frac{1 - \varepsilon}{\sqrt{2}} \sqrt{N}\]

for any integer \(N\) with \(N \geq C''(1/\sqrt{5}, \varepsilon)\).

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