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Vanishing S-curvature of Randers spaces

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Abstract
We give a necessary and sufficient condition on a Randers space for the existence
of a measure for which Shen’s S-curvature vanishes everywhere. Moreover, if it
exists, such a measure coincides with the Busemann-Hausdorff measure up to a
constant multiplication.

Keywords: Randers spaces, S-curvature, Ricci curvature
Mathematics Subject Classification (2000): 53C60

1 Introduction
This short article is concerned with a characterization of Randers spaces admitting mea-

sures with vanishing S-curvature. A Randers space (due to Randers [Ra]) is a special
kind of Finsler manifold (M, F) whose Finsler structure F : TM → [0, ∞) is written
as F(v) = α(v) + β(v), where α is a norm induced from a Riemannian metric on M and
β is a one-form on M. Randers spaces are important in applications and reasonable for
concrete calculations. See [AIM] and [BCS, Chapter 11] for more on Randers spaces.

We equip a Finsler manifold (M, F) with an arbitrary smooth measure m. Then
the S-curvature S(v) ∈ R of v ∈ TM introduced by Shen (see [Sh, §7.3]) measures the
difference between m and the volume measure of the Riemannian structure induced from
the tangent vector field of the geodesic η with ℎ(0) = v (see §2.2 for the precise definition).
The author’s recent work [Oh], [OS] on the weighted Ricci curvature (in connection with
optimal transport theory) shed new light on the importance of this quantity.

A natural and important question arising from the theory of weighted Ricci curvature is:
when does (M, F) admit a measure m with S ≡ 0? If such a measure exists, then we
can choose it as a good reference measure. Our main result provides a complete answer
to this question for Randers spaces.

Theorem 1.1 A Randers space (M, F) admits a measure m with S ≡ 0 if and only if
β is a Killing form of constant length. Moreover, then m coincides with the Busemann-
Hausdorff measure up to a constant multiplication.

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1
It has been observed by Shen [Sh, Example 7.3.1] that a Randers space with the Busemann-Hausdorff measure satisfies $S \equiv 0$ if $\beta$ is a Killing form of constant length. Our theorem asserts that his condition on $\beta$ is also necessary for the existence of $m$ with $S \equiv 0$, and then it immediately follows that $m$ must be a constant multiplication of the Busemann-Hausdorff measure.

On the one hand, Shen’s result (the “if” part of Theorem 1.1) ensures that there is a rich class of non-Riemannian Randers spaces satisfying $S \equiv 0$. On the other hand, the “only if” part says that the class of general Randers spaces is much wider and many Randers spaces have no measures with $S \equiv 0$. This means that there are no canonical (reference) measures on such Finsler manifolds (in respect of the weighted Ricci curvature). Therefore, for a general Finsler manifold, it is natural to start with an arbitrary measure, as was discussed in [Oh] and [OS].

2 Preliminaries for Finsler geometry

We first review the basics of Finsler geometry. Standard references are [BCS] and [Sh]. We will follow the notations in [BCS] with a little change (e.g., we use $v^i$ instead of $y^i$).

2.1 Finsler structures

Let $M$ be a connected $n$-dimensional $C^\infty$-manifold with $n \geq 2$, and $\pi : TM \to M$ be the natural projection. Given a local coordinate $(x^i)_{i=1}^n : U \to \mathbb{R}^n$ of an open set $U \subset M$, we will always denote by $(x^i; v^i)_{i=1}^n$ the local coordinate of $\pi^{-1}(U)$ given by $v = \sum_i v^i(\partial/\partial x^i)|_{\pi(v)}$.

A $C^\infty$-Finsler structure is a function $F : TM \to [0, \infty)$ satisfying the following conditions:

(I) $F$ is $C^\infty$ on $TM \setminus \{0\}$;

(II) $F(cv) = cF(v)$ for all $v \in TM$ and $c \geq 0$;

(III) The $n \times n$ matrix

$$g_{ij}(v) := \frac{1}{2} \frac{\partial(F^2)}{\partial v^i \partial v^j}(v)$$

is positive-definite for all $v \in TM \setminus \{0\}$.

The positive-definite matrix $(g_{ij}(v))$ defines a Riemannian structure $g_v$ of $T_xM$ through

$$g_v\left(\sum_i a^i \frac{\partial}{\partial x^i}, \sum_j b^j \frac{\partial}{\partial x^j}\right) := \sum_{i,j} g_{ij}(v) a^i b^j. \quad (2.1)$$

Note that $g_v(v, v) = F(v)^2$. This inner product $g_v$ is regarded as the best approximation of $F|_{T_xM}$ in the direction $v$. Indeed, the unit sphere of $g_v$ is tangent to that of $F|_{T_xM}$ at $v/F(v)$ up to the second order. If $(M, F)$ is Riemannian, then $g_v$ always coincides with the original Riemannian metric. As usual, $(g^ij)$ will stand for the inverse matrix of $(g_{ij})$. 

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We define the Cartan tensor
\[ A_{ijk}(v) := \frac{F(v)}{2} \frac{\partial g_{ij}}{\partial v^k}(v) \]
for \( v \in TM \setminus \{0\} \), and remark that \( A_{ijk} \equiv 0 \) holds if and only if \((M, F)\) is Riemannian.
We also define the formal Christoffel symbol
\[ \gamma^i_{jk}(v) := \frac{1}{2} \sum_l g^i_{lj} \left( \frac{\partial g_{kl}}{\partial x^j}(v) + \frac{\partial g_{kl}}{\partial x^j}(v) - \frac{\partial g_{jk}}{\partial x^l}(v) \right) \]
for \( v \in TM \setminus \{0\} \). Then the geodesic equation is written as \( \ddot{\eta} + G(\dot{\eta}) = 0 \) with the geodesic spray coefficients
\[ G^i(v) := \sum_{j,k} \gamma^i_{jk}(v)v^jv^k \]
for \( v \in TM \) \((G^i(0) := 0 \text{ by convention})\). Using these, we further define the nonlinear connection
\[ N^i_{\eta}(v) := \sum_k \left\{ \gamma^i_{jk}(v)v^k - \frac{1}{F(v)} A^i_{jk}(v)G^k(v) \right\} \]
for \( v \in TM \) \((N^i_{\eta}(0) := 0 \text{ by convention})\), where \( A^i_{jk}(v) := \sum_l g^i_{jl}A_{ljk}(v) \). Note that (see [BCS, Exercise 2.3.3])
\[ N^i_{\eta}(v) = \frac{1}{2} \frac{\partial G^i}{\partial v^j}(v). \]

### 2.2 S-curvature and weighted Ricci curvature

We choose an arbitrary positive \( C^\infty \)-measure \( m \) on a Finsler manifold \((M, F)\). Fix a unit vector \( v \in F^{-1}(1) \) and let \( \eta : (-\varepsilon, \varepsilon) \to M \) be the geodesic with \( \eta(0) = v \). Along \( \eta \), the tangent vector field \( \dot{\eta} \) defines the Riemannian metric \( g_{\dot{\eta}} \) via (2.1). Denoting the volume form of \( g_{\dot{\eta}} \) by \( \text{vol}_{\dot{\eta}} \), we decompose \( m \) into \( m(dx) = e^{-\Psi(\dot{\eta})}\text{vol}_{\dot{\eta}}(dx) \) along \( \eta \). Then we define the \( \textsf{s} \)-curvature of \( v \) by
\[ S(v) := \frac{d((\Psi \circ \dot{\eta})''(t))}{dt}(0). \]
We extend this definition to all \( w = cv \) with \( c \geq 0 \) by \( S(w) := cS(v) \). Clearly \( S \equiv 0 \) holds on Riemannian manifolds with the volume measure.

The \textit{weighted Ricci curvature} is defined in a similar manner as follows:

(i) \( \text{Ric}_n(v) := \text{Ric}(v) + (\Psi \circ \eta)''(0) \) if \( S(v) = 0 \), \( \text{Ric}_n(v) := -\infty \) otherwise;
(ii) \( \text{Ric}_N(v) := \text{Ric}(v) + (\Psi \circ \eta)''(0) - S(v)^2/(N - n) \) for \( N \in (n, \infty) \);
(iii) \( \text{Ric}_\infty(v) := \text{Ric}(v) + (\Psi \circ \eta)''(0) \).

Here \( \text{Ric}(v) \) is the usual (unweighted) Ricci curvature of \( v \). The author [Oh] shows that bounding \( \text{Ric}_N \) from below by \( K \in \mathbb{R} \) is equivalent to the curvature-dimension condition \( \text{CD}(K, N) \), and then there are many analytic and geometric applications. Observe that the bound \( \text{Ric}_n \geq K > -\infty \) makes sense only when the \( \textsf{s} \)-curvature vanishes everywhere.
Therefore the class of such special triples \((M, F, m)\) deserves a particular interest. We remark that, if there are two measures \(m_1, m_2\) on \((M, F)\) satisfying \(S \equiv 0\), then \(m_1 = c \cdot m_2\) holds for some positive constant \(c\).

We rewrite \(S(v)\) according to [Sh, §7.3] for ease of later calculation. Recall that \(\eta\) is the geodesic with \(\dot{\eta}(0) = v\). Fix a local coordinate \((x^i)_{i=1}^n\) containing \(\eta\) and represent \(m\) along \(\eta\) as

\[
m(dx) = \sigma(\eta) \, dx^1 dx^2 \cdots dx^n = \frac{\sigma(\eta)}{\sqrt{\det(g_\eta)}} \theta(\eta)(dx).
\]

We have by definition

\[
S(v) = \frac{d}{dt} \bigg|_{t=0} \log \left( \frac{\sqrt{\det(g_{\dot{\eta}(t)})}}{\sigma(\eta(t))} \right) = \frac{1}{2 \det(g_v)} \frac{d}{dt} \bigg|_{t=0} \left[ \det(g_{\dot{\eta}(t)}) \right] - \sum_i \frac{v^i}{\sigma(x)} \frac{\partial \sigma}{\partial x^i}(x).
\]

Since \(\eta\) solves the geodesic equation \(\ddot{\eta} + G(\dot{\eta}) = 0\), the first term is equal to

\[
\frac{1}{2} \sum_{i,j,k} \left\{ g^{ij}(v) \frac{\partial g_{ij}}{\partial x^k}(v) v^k + g^{ij}(v) \frac{\partial g_{ij}}{\partial v^k}(v) \dot{\eta}^k(0) \right\} = \sum_i \left( \gamma^i_{\, jk}(v) v^k - \frac{1}{F(v)} A^i_{\, jk}(v) G^k(v) \right) = \sum_i N^i_i(v).
\]

Thus we obtain

\[
S(v) = \sum_i \left\{ N^i_i(v) - \frac{v^i}{\sigma(x)} \frac{\partial \sigma}{\partial x^i}(x) \right\}.
\]

Observe that \(S(cv) = cS(v)\) indeed holds for \(c \geq 0\) in this form.

### 2.3 Busemann-Hausdorff measure and Berwald spaces

Different from the Riemannian case, there are several constructive measures on a Finsler manifold, each of them is canonical in some sense and coincides with the volume measure for Riemannian manifolds. Among them, here we treat only the Busemann-Hausdorff measure which is actually the Hausdorff measure associated with the suitable distance structure if \(F\) is symmetric in the sense that \(F(v) = F(-v)\) holds for all \(v \in TM\).

Roughly speaking, the Busemann-Hausdorff measure is the measure such that the volume of the unit ball of each tangent space equals the volume of the unit ball in \(\mathbb{R}^n\). Precisely, using a basis \(w_1, w_2, \ldots, w_n \in T_x M\) and its dual basis \(\theta^1, \theta^2, \ldots, \theta^n \in T^*_x M\), the Busemann-Hausdorff measure \(m_{BH}(dx) = \sigma_{BH}(x) \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n\) is defined as

\[
\frac{\omega_n}{\sigma_{BH}(x)} = \text{vol}_{\mathbb{R}^n} \left( \left\{ (c^i) \in \mathbb{R}^n \mid F\left( \sum_i c^i w_i \right) < 1 \right\} \right),
\]

where \(\text{vol}_{\mathbb{R}^n}\) is the Lebesgue measure and \(\omega_n\) is the volume of the unit ball in \(\mathbb{R}^n\).

Let \((M, F)\) be a Berwald space (see [BCS, Chapter 10] for the precise definition). Then it is well known that \(S \equiv 0\) holds for the Busemann-Hausdorff measure (see [Sh, Proposition 7.3.1]). In fact, along any geodesic \(\eta : [0, l] \rightarrow M\), the parallel transport \(T_{0,t} : T_{\eta(0)} M \rightarrow T_{\eta(t)} M\) with respect to \(g_\eta\) preserves \(F\). Therefore choosing parallel vector fields along \(\eta\) as a basis yields that \(\sigma_{BH}\) is constant on \(\eta\), which yields \(S \equiv 0\).
3 Proof of Theorem 1.1

Let \((M, F)\) be a Randers space, i.e., \(F(v) = \alpha(v) + \beta(v)\) such that \(\alpha\) is a norm induced from a Riemannian metric and that \(\beta\) is a one-form. In a local coordinate \((x^i)_{i=1}^n\), we can write

\[
\alpha(v) = \sqrt{\sum_{i,j} a_{ij}(x)v^i v^j}, \quad \beta(v) = \sum_i b_i(x)v^i
\]

for \(v \in T_x M\). The length of \(\beta\) at \(x\) is defined by \(\|\beta\|(x) := \sqrt{\sum_{i,j} a_{ij}(x)b_i(x)b_j(x)}\), which is necessarily less than 1 in order to guarantee \(F > 0\) on \(TM \setminus \{0\}\).

We denote the Christoffel symbol of \((a_{ij})\) by \(\gamma^i_{jk}\). We also define

\[
b^i(x) := \sum_j a^{ij}(x)b_j(x), \quad b_{ij}(x) := \frac{\partial b_i}{\partial x^j}(x) - \sum_k b_k(x)\gamma^k_{ij}(x).
\]

Note that \(b_{ij}\) is the coefficient of the covariant derivative \(\tilde{\nabla}\) of \(\beta\) with respect to \(\alpha\), namely \(\tilde{\nabla}_{\partial/\partial x^i}\beta = \sum_i b_{ij}dx^i\). We find by calculation that

\[
\frac{\partial \|\beta\|^2}{\partial x^i}(x) = 2 \sum_j b_{j|i}(x)b^j(x).
\]

We say that \(\beta\) is a Killing form if \(b_{ij} + b_{ji} \equiv 0\) holds on \(M\). The geodesic spray coefficients of \(F\) are given by (see [BCS, (11.3.11)]

\[
G^i(v) = \sum_{j,k} \gamma^i_{jk}(v)v^j v^k
\]

\[
= \sum_{j,k} \left[ \tilde{\gamma}^i_{jk}(x)v^j v^k + b_{j|k}(x)\left( a^{ij}(x)v^k - a^{ik}(x)v^j \right) \alpha(v) \right.
\]

\[
+ \left. b_{j|k}(x)v^i \left\{ v^j v^k + \left( b^k(x)v^i - b^i(x)v^k \right) \alpha(v) \right\} \right].
\]

\[
=: \sum_{j,k} \tilde{\gamma}^i_{jk}(x)v^j v^k + X^i(v) + Y^i(v).
\]

If \(S \equiv 0\) on \(T_x M\), then we deduce from (2.2) that \(\sum_i N^i_i(v)\) is linear in \(v \in T_x M\). We shall see that only this infinitesimal constraint is enough to imply the condition on \(\beta\) stated in Theorem 1.1. To see this, we calculate \(2N^i_i = \partial G^i/\partial v^i\) using (3.2). As the first term \(\sum_{j,k} \tilde{\gamma}^i_{jk}(x)v^j v^k\) comes from a Riemannian structure, it suffices to consider only the linearity of \(\sum_i \{\partial X^i/\partial v^i(v) + \partial Y^i/\partial v^i(v)\}\). For the sake of simplicity, we will omit evaluations at \(x\) and \(v\) in the following calculations.

We first obtain

\[
\sum_i \frac{\partial X^i}{\partial v^i} = \sum_{i,j} \left( b_{j|i} - b_{i|j} \right) a^{ij} \alpha + \sum_{i,j,k,l} b_{j|k}(a^{ij}v^k - a^{ik}v^j) \frac{a_{kl}v^l}{\alpha}
\]

\[
= \sum_{i,j} b_{j|i}(a^{ij} - a^{ji}) \alpha + \sum_{j,k} b_{j|k}(v^k v^j - v^j v^k) \alpha^{-1} = 0.
\]
As Euler’s theorem [BCS, Theorem 1.2.1] ensures
\[ \sum_i \frac{\partial}{\partial v^i} \left( \frac{v^i}{F} \right) = \frac{1}{F^2} \sum_i \left( F - v^i \frac{\partial F}{\partial v^i} \right) = \frac{n - 1}{F}, \]
we next observe
\[ \sum_i \frac{\partial Y^i}{\partial v^i} = \sum_{i,j} \frac{v^i}{F} \left\{ (b_{ij} + b_{ji}) v^j + \left( b_{ij} - b_{ji} \right) b^j \alpha + \sum_{k,l} b_{jk} (b^k v^j - b^j v^k) \frac{a_{il} v^l}{\alpha} \right\} \]
\[ + \frac{n - 1}{F} \sum_{j,k} b_{jk} \left\{ v^j v^k + \left( b^k v^j - b^j v^k \right) \alpha \right\} \]
\[ = \frac{n + 1}{2} \sum_{i,j} \left( b_{ij} + b_{ji} \right) \frac{v^i v^j}{F} + \left( n + 1 \right) \sum_{i,j} \left( b_{ij} - b_{ji} \right) b^j \alpha v^i \frac{F}{F}. \]

By comparing the evaluations at \( v \) and \( -v \), the coefficients \( b_{ij} + b_{ji} \) in the first term must vanish for all \( i, j \), and hence \( \beta \) is a Killing form. For the second term, we find that \( \alpha/F \sum_j (b_{ij} - b_{ji}) b^j \) must be constant on each \( T_x M \). If \( \alpha/F \) is not constant on some \( T_x M \) (i.e., \( \|\beta\|(x) \neq 0 \)), then it holds that \( \sum_j (b_{ij} - b_{ji}) b^j = 0 \). Since \( \beta \) is a Killing form, we deduce from (3.1) that
\[ 0 = \sum_j (b_{ij} - b_{ji}) b^j = -2 \sum_j b_{ji} b^j = -\frac{\partial (\|\beta\|^2)}{\partial x^i}. \]
Therefore \( \beta \) has a constant length as required, for \( \|\beta\| \neq 0 \) is an open condition. If \( \alpha/F \) is constant on some \( T_x M \), then the above argument yields that \( \beta \equiv 0 \) on \( M \). This completes the proof of the “only if” part of Theorem 1.1.

For the “if” part, it is sufficient to show that the Busemann-Hausdorff measure satisfies \( S \equiv 0 \), that can be found in [Sh, Example 7.3.1]. We briefly repeat his discussion for completeness. We first observe from [Sh, (2.10)] that
\[ m_{BH}(dx) = \left( 1 - \|\beta\|(x)^2 \right)^{n+1/2} \sqrt{\det(a_{ij}(x))} \, dx^1 \cdots dx^n =: \sigma_{BH}(x) \, dx^1 \cdots dx^n. \]
Since \( \beta \) has a constant length, we have
\[ \sum_k \frac{v^k}{\sigma_{BH}(x)} \frac{\partial \sigma_{BH}}{\partial x^k}(x) = 1/2 \sum_{i,j,k} v^k a^{ij}(x) \frac{\partial a_{ij}}{\partial x^k}(x) = \sum_{i,j} \tilde{\gamma}^{ij}(x) v^j. \]

Therefore we conclude, by (2.2),
\[ S(v) = \frac{1}{2} \sum_{i,j,k} \frac{\partial}{\partial v^i} [\tilde{\gamma}^{jk}(x) v^j v^k] - \sum_k \frac{v^k}{\sigma_{BH}(x)} \frac{\partial \sigma_{BH}}{\partial x^k}(x) = 0. \]

We finally remark related known results and several consequences of Theorem 1.1.
Remark 3.1  (a) A Randers space is a Berwald space if and only if $\beta$ is parallel in the sense that $b_{ij} \equiv 0$ for all $i, j$ (see [BCS, Theorem 11.5.1]). Thanks to [Sh, Example 7.3.2], we know that a Killing form of constant length is not necessarily parallel.

(b) In [De], Deng gives a characterization of vanishing $S$-curvature for homogeneous Randers spaces endowed with the Busemann-Hausdorff measure.

(c) It is easy to construct a Randers space whose $\beta$ does not have a constant length. Hence many Finsler manifolds do not admit measures with $S \equiv 0$ (in other words, with $\text{Ric}_n \geq K > -\infty$).

(d) Another consequence of Theorem 1.1 is that only (constant multiplications of) the Busemann-Hausdorff measures can satisfy $S \equiv 0$ on Randers spaces. Then a natural question is the following:

**Question**  Is there a Finsler manifold $(M, F)$ on which some measure $m$ other than (a constant multiplication of) the Busemann-Hausdorff measure satisfies $S \equiv 0$? If yes, what kind of measure is $m$?

If such a measure exists, then it is more natural than the Busemann-Hausdorff measure in respect of the weighted Ricci curvature.

References


