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Commutativity of localized self-homotopy groups of symplectic groups

Daisuke Kishimoto, Akira Kono, and Tomoaki Nagao

May 13, 2011

Abstract

The self-homotopy group of a topological group $G$ is the set of homotopy classes of self-maps of $G$ equipped with the group structure inherited from $G$. We determine the set of primes $p$ such that the $p$-localization of the self-homotopy group of $\text{Sp}(n)$ is commutative. As a consequence, we see that this group detects the homotopy commutativity of $p$-localized $\text{Sp}(n)$ by its commutativity almost all cases.

1 Introduction

For a group-like space $G$, the pointed homotopy set $[X, G]$ has a natural group structure inherited from $G$. We will always assume $[X, G]$ as a group with this group structure. This group has been studied for a long time, and there are many applications especially to the H-structure of $G$. See [1] and [9], for example. Put $X = G$. Then the group $[G, G]$ is called the self-homotopy group of $G$ and denoted by $\mathcal{H}(G)$. The self homotopy group $\mathcal{H}(G)$ has also been studied extensively, especially, in connection with the H-structure of $G$, see [2], [12] and [11]. In particular, it is shown in [12] the following.

**Theorem 1.1** (Kono and Ōshima [12]). Let $G$ be a compact, connected Lie group. Then $\mathcal{H}(G)$ is commutative if and only if $G$ is isomorphic with $T^n$ ($n \geq 0$), $T^n \times \text{Sp}(1)$ ($0 \leq n \leq 2$) or $\text{SO}(3)$, where $T^n$ denotes the $n$-dimensional torus.

Then we can say that for a connected Lie group $G$, $\mathcal{H}(G)$ reflects the homotopy commutativity of $G$ to its commutativity effectively, since we have Hubbuck’s torus theorem [8].

Localize at the prime $p$ in the sense of [7]. Then it is an interesting problem to consider for a fixed $G$, how the H-structure of $G(p)$ changes when we vary $p$. Kaji and the first named author obtained a result for a Lie group $G$ when $p$ is relatively large [9], [10]. Let us turn to the self homotopy group $\mathcal{H}(G)$. Let $X$ be a finite complex, and let $G$ be a path-connected group-like space. Then the group $[X, G]$ is known to be nilpotent, and then we can consider its localization $[X, G]_p$ at the prime $p$ in the sense of [7]. Moreover, there is a natural isomorphism of groups:

$$[X, G]_p \cong [X_p, G_p]$$
See [7]. Then if $G$ is a connected Lie group, it is also an interesting problem to consider how the group structure of $\mathcal{H}(G)_{(p)}$ changes if we vary $p$ as is considered for $G_{(p)}$. Recently, Hamanaka and the second named author obtained:

**Theorem 1.2** (Hamanaka and Kono [5]). $\mathcal{H}(SU(n))_{(p)}$ is commutative if and only if $p > 2n$ except for $n = 2$ and $(p, n) = (5, 3), (7, 4), (11, 6), (13, 7)$.

As is shown in [13], $SU(n)_{(p)}$ is homotopy commutative if and only if $p > 2n$. Then we can say that $\mathcal{H}(SU(n))_{(p)}$ detects the homotopy commutativity of $SU(n)_{(p)}$ very well.

The aim of this paper is to consider the above problem for $G = \text{Sp}(n)$, and we will prove:

**Theorem 1.3.** The group $\mathcal{H}(\text{Sp}(n))_{(p)}$ is commutative if and only if $p > 4n$ except for $n = 1$ and $(p, n) = (3, 2), (5, 3), (7, 2), (11, 3), (19, 5), (23, 6)$.

Since $\text{Sp}(n)_{(p)}$ is homotopy commutative if and only if $p > 4n$ except for $(p, n) = (3, 2)$ by [13], we get:

**Corollary 1.1.** $\text{Sp}(n)_{(p)}$ is homotopy commutative if and only if $\mathcal{H}(\text{Sp}(n))_{(p)}$ is commutative except for $n = 1$ and $(p, n) = (5, 3), (7, 2), (11, 3), (19, 5), (23, 6)$.

**Remark 1.1.** Let $p$ be an odd prime. As is well known [4], there is a homotopy equivalence $B\text{Sp}(n)_{(p)} \cong B\text{Spin}(2n+1)_{(p)}$, and then, in particular, we have $\mathcal{H}(\text{Sp}(n))_{(p)} \cong \mathcal{H}(\text{Spin}(2n+1))_{(p)}$. Thus the above results for $\mathcal{H}(\text{Sp}(n))_{(p)}$ implies those for $\mathcal{H}(\text{Spin}(2n+1))_{(p)}$. We also have a similar result for $\mathcal{H}(\text{Spin}(2n))_{(p)}$ when $p$ is an odd prime [6].

## 2 Calculating commutators in the group $[X, \text{Sp}(n)]$

Throughout this section, all spaces will be localized at the prime $p$.

Put $G_n = \text{Sp}(n)$ and $X_n = G_\infty / G_n$. Let $q_k \in H^{4k}(BG_n; \mathbb{Z}_{(p)})$ be the $k$-th universal symplectic Pontrjagin class. Then the cohomology of $G_n$ is given by

$$H^*(G_n; \mathbb{Z}_{(p)}) = \Lambda(x_3, x_7, \ldots, x_{4n-1}), \quad x_{4k-1} = \sigma(q_k),$$

where $\sigma$ is the cohomology suspension. We also have

$$H^*(X_n; \mathbb{Z}_{(p)}) = \Lambda(y_{4n+3}, y_{4n+7}, \ldots), \quad \pi^*(x_i) = y_i$$

for the projection $\pi : G_\infty \to X_n$. Put $b_{4k+2} = \sigma(y_{4k+3}) \in H^*(\Omega X_n; \mathbb{Z}_{(p)})$ for $k \geq n$. We write a map $X \to K(\mathbb{Z}_{(p)}, k)$ corresponding to the cohomology class $x \in H^k(X; \mathbb{Z}_{(p)})$ by $x$, ambiguously. Then, in particular, since $b_{4k+2}$ is a loop map, the map $b_{4k+2} : [X, \Omega X_n] \to H^{4k+2}(X; \mathbb{Z}_{(p)})$ is a homomorphism.

Now we recall from [15] how to determine the (non)triviality of commutators in the group $[X, G_n]$. Apply the functor $[X, -]$ to the fibre sequence

$$\Omega G_\infty \xrightarrow{\Omega \pi} \Omega X_n \xrightarrow{\delta} G_n \to G_\infty$$
Proof. Let $\xi \in [X, \Omega X_n]$ such that $\delta_\ast(\lambda) = [\alpha, \beta]$ and $\Phi(\lambda)$ is not in the image of $\Phi \circ (\Omega \pi)_\ast$, then $[\alpha, \beta]$ is not trivial.

2. Suppose that $\Phi$ is injective. Then $[\alpha, \beta]$ is not trivial if and only if there exists the above $\lambda$.

In order to use Proposition 2.1, we need to describe $\lambda^\ast(b_{4m+2})$ explicitly, where $\lambda$ is as in Proposition 2.1. In [15], it is shown that we can choose $\lambda$ as:

**Lemma 2.1.** For $\alpha, \beta \in [X, G_n]$, there exists $\lambda \in [X, \Omega X_n]$ such that $\delta_\ast(\lambda) = [\alpha, \beta]$ and for $k \geq n$,

$$\lambda^\ast(b_{4k+2}) = \sum_{i+j=k+1, 1 \leq i, j \leq n} \alpha^\ast(x_{4i-1})\beta^\ast(x_{4j-1}).$$

We next describe $(\Omega \pi)_\ast(\xi)$ through the map $b_{4k+2} : [X, \Omega X_n] \to H^{4k+2}(X; \mathbb{Z}(p))$ for $\xi \in \widetilde{KSp}^{-2}(X)_{(p)}$ to use Proposition 2.1. Let $c' : G_n \to U(2n)$ denote the complexification map. We also denote the complexification $\widetilde{KSp}^\ast(X)_{(p)} \to \widetilde{K}^\ast(X)_{(p)}$ by $c'$. Let $\text{ch}_k$ denote the $2k$-dimensional part of the Chern character.

**Lemma 2.2.** For $\xi \in \widetilde{KSp}^{-2}(X)_{(p)}$, we have

$$(b_{4k+2} \circ (\Omega \pi))_\ast(\xi) = (-1)^{k+1}(2k+1)!\text{ch}_{2k+1}(c'(\xi)).$$

**Proof.** Let $c_k$ be the $k$-th universal Chern class. Then we have $c'(c_{2k}) = (-1)^k q_k$, and thus

$$(b_{4k+2} \circ (\Omega \pi))_\ast(\xi) = c^2(q_{k+1})(\xi) = (-1)^{k+1}c'(c^2(q_{2k+2}))(\xi) = (-1)^{k+1}(2k+1)!\text{ch}_{2k+1}(c'(\xi)).$$

\[\square\]

### 3 Proof of Theorem 1.3 for $p$ odd

Throughout this section, we localize all spaces at the odd prime $p$ unless otherwise is specified.

For a given positive integer $n$, let $m$ be an arbitrary integer satisfying $m < n \leq 2m$. Let $\epsilon_{4k-1}$ be a generator of $\pi_{4k-1}(G_n) \cong \mathbb{Z}_{(p)}$ for $k \leq n$. Then we have

$$(\epsilon_{4k-1})^\ast(x_{4k-1}) = \begin{cases} (2k-1)!u_{4k-1} & \text{if } k \text{ is odd} \\ 2(2k-1)!u_{4k-1} & \text{if } k \text{ is even} \end{cases} \quad (3.1)$$

3
where \( u_l \) is a generator of \( H^1(S^l; \mathbb{Z}_{(p)}) \). Define a map \( \theta : S^{4m-1} \times S^{4m+3} \to G_n \) by the composition

\[
S^{4m-1} \times S^{4m+3} \xrightarrow{\epsilon_{4m-1} \times \epsilon_{4m+3}} G_n \times G_n \xrightarrow{\mu} G_n,
\]

where \( \mu \) is the multiplication of \( G_n \). Then by (3.1), we have for \( k < l \):

\[
\theta^*(x_{4k-1}x_{4l-1}) = \begin{cases} 
2(2m-1)!(2m+1)!u_{4m-1} \otimes u_{4m+1} & (k, l) = (m, m+1) \\
0 & \text{otherwise}
\end{cases} \quad (3.2)
\]

Let \( j : G_n \to G_{2m} \) be the inclusion, and let \( \psi^2 : BG_n \to BG_n \) be the unstable Adams operation of degree 2 [18]. We consider the commutator \([j \circ \Omega \psi^2, j]\) in \([G_n, G_{2m}]\) by pulling back to \( S^{4m-1} \times S^{4m+3} \) through \( \theta \). By Lemma 2.1, there exists \( \lambda \in [G_n, \Omega X_{2m}] \) such that \( \delta_i(\lambda) = [j \circ \Omega \psi^2, j] \) and

\[
\lambda^*(b_{8m+2}) = \sum_{i+j=n+1 \leq i, j \leq n} (\Omega \psi^2)^*(x_{4i-1})x_{4j-1}.
\]

By definition of \( \psi^2 \), we have \((\Omega \psi^2)^*(x_{4k-1}) = 2^{2k}x_{4k-1} \). Then we get

\[
\lambda^*(b_{8m+2}) = \sum_{i+j=n+1 \leq i, j \leq n} 2^{2i}x_{4i-1}x_{4j-1}
\]

and thus by (3.2),

\[
\theta^* \circ \lambda^*(b_{8m+2}) = 2^{2m}(-3)(2m-1)!(2m+1)!u_{4m-1} \otimes u_{4m+1}.
\]

On the other hand, we have \( KSp^{-2}(S^{4m-1} \times S^{4m+3}) \cong \mathbb{Z}_{(p)} \) and its generator \( \xi \) can be chosen to satisfy

\[
\text{ch}_{4m+1}(c'(\xi)) = (4m+1)!u_{4m-1} \otimes u_{4m+3}.
\]

If we see that \( \theta^* \circ \lambda^*(b_{8m+2}) \) is not in the \( \mathbb{Z}_{(p)} \)-module generated by \((4m+1)!u_{4m-1} \otimes u_{4m+3} \), by Proposition 2.1, we can conclude that \( \theta^*((j \circ \Omega \psi^2, j)) = j \circ [\Omega \psi^2, 1_{G_n}] \circ \theta \) is non-trivial which implies \( \mathcal{H}(G_n) \) is not commutative. Put \( m \) as in the following table. Then we can easily see that \( m \) satisfies \( m < n \leq 2m \) and \( \frac{(4m+1)!}{(2m-1)!(2m+1)!} = (4m+1)(\frac{4m}{2m-1}) = 0 \) (p) by Lucas’ formula, and thus \( \mathcal{H}(G_n) \) is not commutative in these cases.

<table>
<thead>
<tr>
<th>( p &lt; n )</th>
<th>( m \equiv 0 ) (p), ( 0 &lt; n - m \leq p )</th>
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<tr>
<td>( p = n )</td>
<td>( m = p - 1 )</td>
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<tr>
<td>( n &lt; p &lt; n + 3 ) (( p \geq 13 ))</td>
<td>( m = p - 3 )</td>
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<tr>
<td>( n + 3 \leq p &lt; 2n )</td>
<td>( m = \frac{p - 3}{2} )</td>
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<tr>
<td>( 2n &lt; p &lt; 4n - 1 ) (( p \equiv -1 ) (4))</td>
<td>( m = \frac{p + 1}{4} )</td>
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<tr>
<td>( 2n &lt; p &lt; 4n - 1 ) (( p \equiv 1 ) (4), ( p &gt; 5 ))</td>
<td>( m = \frac{p + 3}{4} )</td>
</tr>
<tr>
<td>( (p, n) = (5, 2) )</td>
<td>( m = 1 )</td>
</tr>
<tr>
<td>( (p, n) = (7, 6) )</td>
<td>( m = 5 )</td>
</tr>
<tr>
<td>( (p, n) = (11, 9), (11, 10) )</td>
<td>( m = 8 )</td>
</tr>
</tbody>
</table>
Recall from [13] that $G_n$ is homotopy commutative if $p > 4n$ or $(p, n) = (3, 2)$ which implies $\mathcal{H}(G_n)$ is commutative for $p > 4n$ or $(p, n) = (3, 2)$ . Then the remaining cases are:

1. $p = 4n - 1$
2. $(p, n) = (7, 5)$
3. $(p, n) = (5, 4)$
4. $(p, n) = (5, 3)$

### 3.1 Case 1

In this case we have a homotopy equivalence [14] $G_n \simeq \prod_{k=1}^n S^{4k-1}$. Assume $n \geq 14$. We define $\alpha \in \mathcal{H}(G_n)$ by the composite

$$G_n \xrightarrow{\rho} S^3 \times S^7 \times S^{11} \times S^{15} \times S^{4n-37} \xrightarrow{q} S^{4n-1} \xrightarrow{\epsilon_{4n-1}} G_n,$$

where $\rho$ is the projection and $q$ is the pinch map onto the top cell. We also define $\beta \in \mathcal{H}(G_n)$ by

$$G_n \xrightarrow{\rho'} S^{4n-1} \xrightarrow{\epsilon_{4n-1}} G_n,$$

where $\rho'$ is the projection. Then we have

$$[\alpha, \beta] = \gamma \circ (\epsilon_{4n-1} \times \epsilon_{4n-1}) \circ ((q \circ \rho) \times \rho') \circ \Delta,$$

where $\gamma : G_n \times G_n \to G_n$ and $\Delta : G_n \to G_n \times G_n$ denote the commutator map of $G_n$ and the diagonal map, respectively. Now one can easily see $((q \circ \rho) \times \rho') \circ \Delta$ induces an injection $[S^{4n-1} \times S^{4n-1}, G_n] \to \mathcal{H}(G_n)$. On the other hand, we have $\gamma \circ (\epsilon_{4n-1} \times \epsilon_{4n-1}) = (\epsilon_{4n-1}, \epsilon_{4n-1}) \circ q'$, where $q' : S^{4n-1} \times S^{4n-1} \to S^{8n-2}$ is the pinch map onto the top cell and $(-, -)$ means a Samelson product. Then since $q'$ induces an injection $\pi_{8n-2}(G_n) \to [S^{4n-1} \times S^{4n-1}, G_n]$ and the Samelson product $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle \in \pi_{8n-2}(G_n)$ is non-trivial by [3], we obtain that the commutator $[\alpha, \beta]$ is non-trivial. Thus $\mathcal{H}(G_n)$ is not commutative.

We next assume $8 \leq n \leq 13$. By looking at the homotopy groups of spheres [16], the above Samelson product $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle$ factors as $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle = i \circ \alpha_1(3)$, where $i : S^3 \to G_n$ is the inclusion and $\alpha_1(2k - 1)$ is a generator of $\pi_{2k+2p-4}(S^{2k-1}) \cong \mathbb{Z}/p$. Put $X = S^3 \times S^7 \times S^{11} \times S^{4n-13} \times S^{4n-9} \times S^{4n-5}$. We define $\alpha, \beta \in \mathcal{H}(G_n)$ by

$$G_n \xrightarrow{\rho} X \xrightarrow{q} S^{3p-3} \xrightarrow{\alpha_1(p)} S^p \xrightarrow{\epsilon_p} G_n$$

and

$$G_n \xrightarrow{\rho'} S^p \xrightarrow{\epsilon_p} G_n,$$

respectively, where $\rho$ and $\rho'$ are the projections and $q$ is the pinch map onto the top cell. Then we get

$$[\alpha, \beta] = i \circ \alpha_1(3) \circ \alpha_1(2p) \circ q' \circ ((q \circ \rho) \times \rho') \circ \Delta.$$
where \( q' : S^{3p-3} \times S^p \rightarrow S^{4p-3} \) is the pinch map. As is seen above, the maps \( i \) and \( q' \circ ((q \circ \rho) \times \rho') \circ \Delta \) induce injections \( \pi_{4p-3}(S^3) \rightarrow \pi_{4p-3}(G_n) \) and \( \pi_{4p-3}(G_n) \rightarrow \mathcal{H}(G_n) \), respectively. Since \( \alpha_1(3) \circ \alpha_1(2p) \neq 0 \) as in [16], we obtain that the commutator \([\alpha, \beta]\) is non-trivial. Thus \( \mathcal{H}(G_n) \) is not commutative.

For \( n \leq 7 \), the case 1 occurs only when \( n = 1, 2, 3, 5, 6 \). We only prove the case \( n = 6 \) since the remaining cases are quite similarly proved. Note that for \( n = 6 \) in the case 1, we have \( p = 23 \). One can easily see that the dimension of cells of \( G_6/\sqrt[k]{S^{4k-1}} \) is in the set \( I = \{0\} \cup \bigcup_{k=2}^{6} \{4(n_1 + \cdots + n_k) - k \mid 1 \leq n_1 < \cdots < n_k \leq 6\} \). On the other hand, Since \( G_6 \cong \prod_{k=1}^{6} S^{4k-1} \), we see that the homotopy groups of \( G_6 \) in dimension \( k \in I \) for all \( k \in I \) are trivial by looking at the homotopy groups of spheres [16]. Then the inclusion \( \sqrt[k]{S^{4k-1}} \rightarrow G_6 \) induces an injection \( \mathcal{H}(G_6) \rightarrow \bigoplus_{k=1}^{6} \pi_{4k-1}(G_6) \), and so \( \mathcal{H}(G_6) \) is commutative.

### 3.2 Case 2

In this case, we have \( G_5 \cong B_1 \times B_2 \times S^{11} \), where \( B_k \) is an \( S^{4k-1} \)-bundle over \( S^{4k+11} \) for \( k = 1, 2 \), see [14]. We first calculate \( K^*(G_5)(7) \). Note that \( K^*(B_k) \) for \( k = 1, 2 \) and \( K^*(S^{11})(7) \) are free \( \mathbb{Z}(7) \)-module, we have

\[
K^*(G_5)(7) \cong K^*(B_1)(7) \otimes K^*(B_2)(7) \otimes K^*(S^{11})(7).
\]

Let \( A_k \) be the \((4k + 11)\)-skeleton of \( B_k \) for \( k = 1, 2 \). Then we have \( A_2 \cong \Sigma^4 A_1 \).

Let \( u' \) be the composite of the inclusions \( \Sigma A_1 \rightarrow \Sigma G_5 \rightarrow B G_5 \rightarrow B U(\infty) \). Since \( A_1 \) is a retract of \( \Sigma CP^7 \), we get \( \text{ch}(u') = \Sigma t_3 + \frac{1}{11} \Sigma t_{15} \) where \( t_3, t_{15} \) are generators of \( H^*(A_1; \mathbb{Z}(7)) \) with \( |t_k| = k \) and \( \Sigma \) stands for the suspension isomorphism. Let \( v' \) be the composite of the pinch map \( \Sigma A_1 \rightarrow S^{16} \) and a generator of \( \pi_{16}(B U(\infty)) \cong \mathbb{Z}(7) \). Then we see \( \text{ch}(v') = \Sigma t_{15} \) by choosing a suitable generator of \( \pi_{16}(B U(\infty)) \). Consider the exact sequence

\[
0 \rightarrow \tilde{K}^{-1}(S^{15})(7) \rightarrow \tilde{K}^{-1}(A_1)(7) \rightarrow \tilde{K}^{-1}(S^3)(7) \rightarrow 0
\]

induced from the cofibre sequence \( S^3 \rightarrow A_1 \rightarrow S^{15} \). Then we get \( \tilde{K}^{-1}(A_1)(7) \) is generated by \( u' \) and \( v' \). Since the inclusion \( A_k \rightarrow B_k \) induces an isomorphism \( \tilde{K}^{-1}(B_k)(7) \rightarrow \tilde{K}^{-1}(A_k)(7) \), we get

\[
K^*(G_5)(7) = \Lambda(u_1, u_2, v_1, v_2, w), \quad |u_k| = |v_k| = |w| = -1
\]

such that for \( k = 1, 2 \),

\[
\text{ch}(u_k) = \Sigma x_{4k-1} + \frac{1}{7} \Sigma x_{4k+11}, \quad \text{ch}(v_k) = \Sigma x_{4k+11}, \quad \text{ch}(w) = \Sigma x_{11}.
\]

Since \( q \circ c' = 2 \) for the quaternionization \( q : K^*(G_5)(7) \rightarrow KSp^*(G_5)(7) \), we obtain:

**Lemma 3.1.** \( KSp^{-2}(G_5)(7) \) is a free \( \mathbb{Z}(7) \)-module with a basis \( \{a_1, \ldots, a_{10}\} \) such that

\[
\text{ch}_{15}(c'(a_k)) = \begin{cases} 
\frac{1}{7} x_{11} x_{19} & k = 1 \\
x_{11} x_{19} & k = 2 \\
0 & k \neq 1, 2.
\end{cases}
\]
Let \( \alpha \) be the composite of the projection \( G_5 \to B_2 \) and the inclusion \( B_2 \to G_5 \). We consider the commutator \([1_{G_5}, \alpha]\). By Lemma 2.1, there exists \( \lambda \in [G_5, \Omega X_3] \) such that \( \delta_* (\lambda) = [1_{G_5}, \alpha] \) and \( \lambda^* (b_{30}) = x_{11} x_{19} \). On the other hand, it follows from Lemma 2.2 and Lemma 3.1 that the image of the map \( b_{30} \circ (\Omega \pi)_* : KSp^{-2} (G_5 \wedge 7) \to H^{30} (G_5; \mathbb{Z}_{(7)}) \) is generated by \( 7x_{11} x_{19} \). Then by Proposition 2.1, we conclude that \([1_{G_5}, \alpha]\) is non-trivial which implies \( \mathcal{H}(G_5) \) is not commutative.

### 3.3 Case 3

In this case, we have a homotopy equivalence \( G_4 \cong B_1 \times B_2 \) where \( B_k \) is an \( S^{4k-1} \)-bundle over \( S^{4k+7} \) for \( k = 1, 2 \) [14]. As in the previous case, we have

\[
K^* (G_4)_3 = \Lambda(u_1, u_2, v_1, v_2), \quad |u_k| = |v_k| = -1
\]

such that for \( k = 1, 2 \),

\[
\text{ch}(u_k) = \Sigma x_{4k-1} + \frac{1}{5!} \Sigma x_{4k+7}, \quad \text{ch}(v_k) = \Sigma x_{4k+7},
\]

and thus we obtain:

**Lemma 3.2.** \( \widetilde{KSp}^{-2} (G_4)_5 \) is a free \( \mathbb{Z}_{(5)} \)-module with a basis \( \{a_1, \ldots, a_6\} \) such that

\[
\text{ch}_{11}(c'(a_k)) = \begin{cases} 
\frac{1}{5} x_7 x_{15} & k = 1 \\
x_7 x_{15} & k = 2 \\
0 & k \neq 1, 2.
\end{cases}
\]

Let \( \psi^2 : BG_4 \to BG_4 \) be the unstable Adams operation of degree 2 as above. We consider \([\Omega \psi^2, 1_{G_4}]\). By Lemma 2.1, there exists \( \lambda \in [G_4, \Omega X_4] \) such that \( \delta_* (\lambda) = [\Omega \psi^2, 1_{G_4}] \) and

\[
\lambda^* (b_{22}) = 2^4 x_7 x_{15} + 2^8 x_{15} x_7 = 2^4 \cdot 3 \cdot 5 x_7 x_{15}.
\]

Then by Lemma 2.2 and Lemma 3.2, we see that \( \lambda^* (b_{22}) \) is not in the image of \( b_{22} \circ (\Omega \pi)_* \). Then by Proposition 2.1, we obtain \([\Omega \psi^2, 1_{G_4}]\) is not trivial, and thus \( \mathcal{H}(G_4) \) is not commutative.

### 3.4 Case 4

This case is very special. We first show:

**Lemma 3.3.** The map \( (b_{14} \times b_{18})_* : [G_3, \Omega X_3] \to H^{14} (G_3; \mathbb{Z}_{(5)}) \oplus H^{18} (G_3; \mathbb{Z}_{(5)}) \) is injective.

**Proof.** Note that the 23-skeleton of \( X_3 \) is \( A = S^{15} \cup e^{19} \cup e^{23} \). Then since \( G_3 \) is of dimension 21, the inclusion \( A \to X_3 \) induces an isomorphism of groups \( [G_4, \Omega A] \cong [G_4, \Omega X_3] \). Since for \( k \leq 23 \), \( \pi_k (A) \) is in the stable range. Then one can easily see that

\[
\pi_k (A) \cong \begin{cases} 
\mathbb{Z}_{(5)} & k = 15, 19, 23 \\
0 & k \neq 15, 19, 23 \text{ and } k \leq 23.
\end{cases}
\]

Thus we can easily deduce that \([G_3, \Omega X_3]\) is a free \( \mathbb{Z}_{(5)} \)-module. On the other hand, the rationalization of the map \((b_{14} \times b_{18})_*\) is injective. Then the proof is completed. \( \square \)
As in the case 2, we have
\[ K^*(G_3)_{(5)} = \Lambda(u, v, w), \quad |u| = |v| = |w| = -1 \]
such that
\[ \text{ch}(u) = \Sigma x_3 + \frac{1}{5!} \Sigma x_{11}, \quad \text{ch}(v) = \Sigma x_{11}, \quad \text{ch}(w) = \Sigma x_7. \]
Then we get \( K\hat{S}_{p}^{-2}(G_3)_{(5)} \) is a free \( \mathbb{Z}_{(5)} \)-module with a basis \( \{a_1, a_2, a_3\} \) such that
\[ \text{ch}(e'(a_1)) = x_3 x_{11}, \quad \text{ch}(e'(a_2)) = \frac{1}{5} x_7 x_{11}, \quad \text{ch}(e'(a_3)) = x_7 x_{11}. \]
Thus we obtain:

**Lemma 3.4.** The image of \( (b_{14} \times b_{18})_* \circ (\Omega \pi)_* : K\hat{S}_{p}^{-2}(G_3)_{(5)} \to H^{14}(G_3; \mathbb{Z}_{(5)}) \oplus H^{18}(G_3; \mathbb{Z}_{(5)}) \)
is generated by \( 5x_3 x_{11} \) and \( x_7 x_{11} \).

Let \( \alpha, \beta \in \mathcal{H}(G_1) \). Then for a degree reason, we have \( \alpha^*(x_{4k-1}) = \alpha_{4k-1} x_{4k-1} \) and \( \beta^*(x_{4k-1}) = \beta_{4k-1} x_{4k-1} \), where \( \alpha_i, \beta_i \in \mathbb{Z}_{(5)} \). Moreover, since \( P\chi x_3 = x_11 \), we have \( \alpha_3 \equiv \alpha_{11}, \beta_3 \equiv \beta_{11} \) (5).

Let us consider the commutator \([\alpha, \beta]\). By Lemma 2.1, there exists \( \lambda \in [G_3, \Omega X_3] \) such that \( \delta_*(\lambda) = [\alpha, \beta] \) and
\[ \lambda^*(b_{14}) = (\alpha_3 \beta_{11} - \alpha_{11} \beta_3) x_3 x_{11}, \quad \lambda^*(b_{18}) = (\alpha_7 \beta_{11} - \alpha_{11} \beta_7) x_7 x_{11}. \]
Since \( \alpha_3 \beta_{11} - \alpha_{11} \beta_3 \equiv 0 \) (5), we obtain that \( (b_{14} \times b_{18})_* (\lambda) \) is in the image of \( (b_{14} \times b_{18})_* \circ (\Omega \pi)_* \) by Lemma 3.4. Thus by Proposition 2.1, \( \mathcal{H}(G_3) \) is commutative.

# Proof of Theorem 1.3 for \( p = 2 \)

Throughout this section, spaces will be localized at the prime 2. We only consider \( \mathcal{H}(G_n) \) for \( n \geq 2 \) since \( \mathcal{H}(G_1) \) is obviously commutative.

For \( m \geq 2 \), put \( N = 2^{m-2}. \) Let \( A = S^3 \cup e^7 \) be the 7-skeleton of \( G_\infty \), and let \( i : \Sigma A \to BG_\infty \) be the composite of inclusions \( \Sigma A \to \Sigma G_\infty \to BG_\infty \). We write generators of \( \tilde{H}^*(A; \mathbb{Z}_{(2)}) \) by \( t_3, t_7 \) where \( |t_k| = k \). Then by [17], we can deduce
\[ \text{ch}(e'(i)) = \Sigma u_3 - \frac{1}{6} \Sigma u_7. \] (4.1)

For a generator \( \beta_R \) of \( \widetilde{KO}(S^8)_{(2)} \), let \( \bar{\alpha} : \Sigma^{8N-8} A \to G_\infty \) be the adjoint of \( i \wedge \beta_R^{N-1} : \Sigma^{8N-7} A \to BG_\infty \). Then by (4.1), we get
\[ \bar{\alpha}^*(x_{8N-1}) = (4N - 1)! \Sigma^{8N-7} \text{ch}(e'(i)) = -(4N - 1)! \frac{1}{6} \Sigma^{8N-8} t_7. \]

Since the inclusion \( G_{4N} \to G_\infty \) is an \((16N + 2)\)-equivalence and \( \Sigma^{8N-8} A \) is of dimension \( 8N - 1 \), the map \( \bar{\alpha} : \Sigma^{8N-8} A \to G_\infty \) factors as the composite of the map \( \alpha : \Sigma^{8N-8} A \to G_{4N} \) and the inclusion \( G_{4N} \to G_\infty \). In particular, we have
\[ \alpha^*(x_{8N-1}) = -(4N - 1)! \frac{1}{6} \Sigma^{8N-8} t_7. \]
Let $\epsilon$ be a generator of $\pi_{8N+3}(G_{4N})$. Then we get
\[ \epsilon^s(x_{8N+3}) = (4N + 1)!w, \]
where $w$ denotes a generator of $H^{8N+3}(S^{8N+3}, \mathbb{Z}(2))$. Define a map $\theta : \Sigma^{8N-8}A \times S^{8N+3} \to G_{4N}$ by the composite
\[ \Sigma^{8N-8}A \times S^{8N+3} \xrightarrow{\alpha \times \iota} G_{4N} \times G_{4N} \xrightarrow{\mu} G_{4N}, \]
where $\mu$ is the multiplication of $G_{4N}$. Then by definition, we have:
\[ \theta^*(x_{4k-1}) = \begin{cases} 
-(4N-1)! \Sigma^{8N-8} t_7 \otimes 1 & k = 2N \\
(4N+1)! \otimes w & k = 2N + 1 \\
0 & k \neq 2N, 2N + 1 \end{cases} \tag{4.2} \]

Consider the commutator $[\Omega \psi^3, 1_{G_{4N}}]$ in $H(G_{4N})$ for the unstable Adams operation $\psi^3 : BG_{4N} \to BG_{4N}$ of degree 3. Then by Lemma 2.1, there exists $\lambda \in [G_{4N}, \Omega X_{4N}]$ such that
\[ \lambda^*(b_{16N+2}) = \sum_{i+j=4N+1, 1 \leq i, j \leq 4N} (\Omega \psi^3)^*(x_{4i-1})x_{4j-1} = \sum_{i+j=4N+1, 1 \leq i, j \leq 4N} 3x_{4i-1}x_{4j-1}. \]

Hence by (4.2), we get
\[ \theta^* \circ \lambda^*(b_{16N+2}) = 3^{4N-1} \cdot 4(4N - 1)! (4N + 1)! \Sigma^{8N-8} t_7 \otimes w, \tag{4.3} \]
here $\delta_\epsilon(\lambda \circ \theta)$ equals to the commutator $[(\Omega \psi^3) \circ \theta, \theta]$ in $[\Sigma^{8N-8}A \times S^{8N+3}, G_{4N}]$.

In order to apply Proposition 2.1, we next calculate the free part of $\tilde{KSp}^{-2}(\Sigma^{8N-8}A \times S^{8N+3})_{(2)}$. We know that the pinch map $q : \Sigma^{8N-8}A \times S^{8N+3} \to \Sigma^{16N-8}A$ induces an isomorphism between the free parts in $\tilde{KSp}^{-2}_{(2)}$. Then we calculate $\tilde{KSp}^{-2}(\Sigma^{16N-5}A)_{(2)}$. Consider the following commutative diagram of exact sequences induced from the cofibre sequence $S^{16N-2} \to \Sigma^{16N-5}A \to S^{16N+2}$.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{KSp}^{-2}(S^{16N+2})_{(2)} & \longrightarrow & \tilde{KSp}^{-2}(\Sigma^{16N-5}A)_{(2)} & \longrightarrow & \tilde{KSp}^{-2}(S^{16N-2})_{(2)} & \longrightarrow & 0 \\
& & c' = 1 & & c' = c' & & c' = 0 & & \\
0 & \longrightarrow & \tilde{K}^{-2}(S^{16N+2})_{(2)} & \longrightarrow & \tilde{K}^{-2}(\Sigma^{16N-5}A)_{(2)} & \longrightarrow & \tilde{K}^{-2}(S^{16N-2})_{(2)} & \longrightarrow & 0
\end{array}
\]

Put $u' = \beta_{c}^{8N-2} \wedge c'(i)$ and $v'$ to be the complexification of the composite of the pinch map $\Sigma^{16N-3}A \to S^{16N+4}$ and a generator of $\pi_{16N+4}(BSp(\infty))$, where $\beta_{c}$ is a generator of $\tilde{K}^{0}(S^{2})_{(2)}$. Then by (4.1), one sees that $\tilde{K}^{-2}(\Sigma^{16N-5}A)_{(2)}$ is generated by $u'$ and $v'$ such that
\[ \text{ch}(u') = \Sigma^{16N-5}t_3 - \frac{1}{6} \Sigma^{16N-5}t_7, \quad \text{ch}(v') = \Sigma^{16N-5}t_7.\]

Put $u = \lambda \wedge i$ and $v$ to be the composite of the pinch map $\Sigma^{16N-3}A \to S^{16N+4}$ and a generator of $\pi_{16N+4}(BSp(\infty))$, where $\lambda$ is a generator of $\tilde{K}O(S^{16N-4})_{(2)}$. Then by the above diagram, we obtain that $\tilde{KSp}^{-2}(\Sigma^{16N-5}A)_{(2)}$ is a free $\mathbb{Z}_{(2)}$-module generated by $u, v$ such that
\[ \text{ch}(c'(u)) = 2\Sigma^{16N-5}t_3 - \frac{1}{3} \Sigma^{16N-5}t_7, \quad \text{ch}(c'(v)) = \Sigma^{16N-5}t_7. \]

Summarizing, we get:
Lemma 4.1. The free part of $KSp^{-2} (\Sigma^{8N-8}A \times S^{8N+3})_2$ is generated by $\bar{u}$ and $\bar{v}$ such that
\[
\text{ch}(c'(\bar{u})) = 2\Sigma^{8N-8}t_3 \otimes w - \frac{1}{3}\Sigma^{8N-8}t_7 \otimes w, \quad \text{ch}(c'(\bar{v})) = \Sigma^{8N-8}t_7 \otimes w.
\]

For an integer $k$, we put $\nu_2(k) = m$ if $k = 2^m(2l - 1)$. Then in general, we have
\[
\nu_2(k!) = \left[ \frac{k}{2} \right] + \left[ \frac{k}{2^2} \right] + \left[ \frac{k}{2^3} \right] + \cdots,
\]
where $[x] = \max\{n \in \mathbb{Z} \mid n \leq x\}$.

Note that $H^{16N+2}(\Sigma^{8N-8}A \times S^{8N+3})$ is a free $\mathbb{Z}_2$-module. Then it follows from the above lemma, we obtain that the image of $b_{16N+2} \circ (\Omega \pi)_* : [\Sigma^{8N-8}A \times S^{8N+3}, \Omega X_{AN}] \to H^{16N+2}(\Sigma^{8N-8}A \times S^{8N+3}; \mathbb{Z}_2)$ is generated by $(8N + 1)!\Sigma^{8N-8}t_7 \otimes w$. It follows from (4.4) that $\nu_2((8N + 1)!) = 2^{m+1} - 1$ and $\nu_2(4(4N - 1)!(4N + 1)! = 2^{m+2} - m$. Then by Lemma 2.1, (4.3) and Lemma 4.1, we get that the commutator $[(\Omega \psi^3) \circ \theta, \theta]$ is non-trivial. If $N < n \leq 2N$, the map $\alpha$ and $\epsilon$ factors through the inclusion $j : G_n \to G_{4N}$, and so there exists a map $\hat{\theta} : \Sigma^{8N-8}A \times S^{8N+3} \to G_n$ such that $\theta = j \circ \hat{\theta}$. Then we obtain that $[(\Omega \psi^3) \circ \theta, \theta] = j \circ [(\Omega \psi^3, 1_{G_n}) \circ \hat{\theta}$ is non-trivial which implies that $\mathcal{H}(G_n)$ is not commutative.

References


