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Independence in generic structures

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Abstract
Wagner [W] proved that in generic structures forking independence and independence defined by dimension function are essentially the same. He proved the result under the assumption that the closure of a finite set is also finite. Verbovskiy and Yoneda [VY] provided some notions for studying generic structures without this finiteness condition and eliminated the finiteness assumption from the result. Here we give a very short proof of the result.

1 Introduction
Let \( L = \{ R_i : i \in \omega \} \) and for each \( i \in \omega \) let \( \alpha_i > 0 \) be given. \( \delta \) is the function assigning to each finite \( L \)-structure the value \( |A| - \sum \alpha_i |R_i^A| \). Let \( K \) be the class of all finite \( L \)-structures \( A \) such that \( \delta(A_0) \geq 0 \) for every substructure \( A_0 \) of \( A \). \( K_0 \) is a subclass of \( K \) and \( M \) is a stable structure all of whose finite substructures belong to \( K_0 \). \( \mathcal{M} \) is a big model of \( T = Th(M) \). The following proposition is proved by Wagner [W] under the finite closure assumption. Later Verbovskiy and Yoneda [VY] eliminated the finiteness assumption from the result. Here we give a direct proof. We do not assume the finiteness condition.

**Proposition 1** Let \( B, C \) be closed sets in \( \mathcal{M} \). Suppose that \( A = B \cap C \) is algebraically closed. Suppose also that \( B \) and \( C \) are independent over \( A \). Then (1) \( B \) and \( C \) are free over \( A \) and (2) \( BC \) is closed.

In section 1, we recall some definitions and state basic lemmas on generic structures. In section 2, we prove the above proposition by a straightforward method. We assume that the reader has some knowledge of stability theory. In particular, the reader is supposed to know the notion Morley sequence.
2 Preliminaries

Definition 2 1. Let $A \subset B \in K$. We say that $A$ is closed in $B$ (in symbol $A \leq B$) if whenever $X \subset B - A$ then $\delta(X/A)(= \delta(XA) - \delta(A)) \geq 0$.

2. Let $A \subset N$, where $N \models T$.
   (a) We say that $A$ is closed in $N$ if whenever $B$ is a finite subset of $N$ then $A \cap B \leq B$.
   (b) The closure of $A$ (in $N$) is the minimum closed set containing $A$. (The closure always exists.) The closure of $A$ is written as $cl(A)$.

Lemma 3 For every $A$, $cl(A) \subset acl(A)$.

Proof. Let $N \prec \mathcal{M}$ be a small model with $N \succ A$ and choose the closure $C$ of $A$ in $N$. Then, by $N \prec \mathcal{M}$, $C$ is the closure of $A$ in $\mathcal{M}$. Suppose that there is $c \in C$ which is nonalgebraic over $A$. Then we can choose an element $d \in \mathcal{M} - N$ with $tp(c/A) = tp(d/A)$. Let $\sigma$ be an $A$-automorphism sending $c$ to $d$. Then we would have two different closures $C$ and $\sigma(C)$. A contradiction.

Lemma 4 Let $A \subset B_{0} \leq B_{1}$ and $A \subset C_{0} \leq C_{1}$. Suppose that $B_{1}$ and $C_{1}$ are free over $A$. If $B_{1}C_{1}$ is closed then $B_{0}C_{0}$ is also closed.

Proof. We assume $B_{1}C_{1}$ is closed. Let $X \subset \mathcal{M} - B_{0}C_{0}$ be a finite set and put $X_{B} = X \cap B_{1}$, $X_{C} = X \cap C_{1}$, and $\hat{X} = X - B_{1}C_{1}$. Then we have the following inequalities:

\[
\begin{align*}
\delta(X/B_{0}C_{0}) &= \delta(\hat{X}/B_{0}C_{0}X_{B}X_{C}) + \delta(X_{B}X_{C}/B_{0}C_{0}) \\
&\geq \delta(\hat{X}/B_{1}C_{1}) + \delta(X_{B}X_{C}/B_{0}C_{0}) \\
&\geq \delta(X_{B}X_{C}/B_{0}C_{0}) \\
&= \delta(X_{B}/X_{C}B_{0}C_{0}) + \delta(X_{B}/B_{0}C_{0}).
\end{align*}
\]

By the freeness and $B_{0} \leq B_{1}$, $\delta(X_{B}/X_{C}B_{0}C_{0}) = \delta(X_{B}/B_{0}) \geq 0$. Similarly, $\delta(X_{B}/B_{0}C_{0}) \geq 0$. So we have $\delta(X/B_{0}C_{0}) \geq 0$. 

3 Proof of the Proposition

Let $B' = acl(B)$ and $C' = acl(C)$. If we prove $B'C' = B' \otimes_A C' \leq \mathcal{M}$, then $BC = B \otimes_A C \leq \mathcal{M}$ follows from lemma. So we can assume that $B$ and $C$ are algebraically closed. By $B \perp_A C$, we can choose sequences $\{B_i : i \in \omega\}$ and $\{C_i : i \in \omega\}$ satisfying the following conditions:

1. $\{B_i : i \in \omega\}$ is a Morley sequence of $tp(B/A)$;
2. $\{C_i : i \in \omega\}$ is a Morley sequence of $tp(C/A)$;
3. $\{B_i : i \in \omega\}$ and $\{C_i : i \in \omega\}$ are independent over $A$, so the set $\{B_i : i \in \omega\} \cup \{C_i : i \in \omega\}$ is an independent set over $A$.
4. $tp(B_iC_j/A) = tp(BC/A)$, for any $i,j \in \omega$.

Such sequences can be found by using an easy compactness argument.

(1) Freeness: By way of a contradiction, we assume there are tuples $\emptyset \neq \overline{b} \in B - A$, $\emptyset \neq \overline{c} \in C - A$ and $\overline{a} \in A$ with $R_i(\overline{b}, \overline{c}, \overline{a})$. By condition 4, we can find $\overline{b}_i \in B$ and $\overline{c}_i \in C$ such that for any $i,j \in \omega$, $tp(\overline{b}_i\overline{c}_j\overline{a}) = tp(\overline{b}\overline{c}\overline{a})$. So $R(\overline{b}_i, \overline{c}_j, \overline{a})$ holds for any $(i,j) \in \omega^2$. We fix $n \in \omega$. Then we have the following inequality:

$$\delta(\bigcup_{i<n}\overline{b}_i\overline{c}_i\overline{a} \leq n|\overline{b}\overline{c}\overline{a}| - \alpha_i n^2 .$$

This right value is negative for a sufficiently large $n$. A contradiction.

(2) Suppose that $BC$ is not closed and choose finite tuples $\overline{d} \in acl(BC) - BC$, $\overline{b} \in B$ and $\overline{c} \in C$ with $\varepsilon := \delta(\overline{d}/\overline{b}\overline{c}) < 0$.

By condition 4 above, for all $i,j \in \omega$, we can choose $\overline{b}_i \in B_i$, $\overline{c}_i \in C_i$ and $\overline{d}_{ij}$ such that $tp(\overline{b}\overline{c}\overline{d}_BC) = tp(\overline{b}_i\overline{c}_i\overline{d}_{ij}B_iC_j)$.

Claim A $(\bigcup_{(i,j) \in \omega^2} \overline{d}_{ij}) \cap (\bigcup_{i \in \omega} B_iC_i) = \emptyset$

Suppose otherwise and choose $i,j,m$ and $e \in \overline{d}_{ij} \cap (B_mC_m)$. By symmetry, we may assume $e \in B_m$. So we have $e \in acl(B_iC_j) \cap B_m$. By choice of $\overline{d}$ (and $\overline{d}_{ij}$), $m \neq i$. So, from $B_iC_j \perp_A B_m$, we have $e \in acl(A) = A$. So we must have $\overline{d}_{ij} \cap A \neq \emptyset$, a contradiction.

Claim B $\overline{d}_{ij}$'s are disjoint.
By way of a contradiction, we assume $e \in \bar{d}_{ij} \cap \bar{d}_{i'j'}$ for some pair $(i, j) \neq (i', j')$. First assume $\{i, j\} \cap \{i', j'\} = \emptyset$. Then, by the independence of $B_iC_j$ and $B_{i'}C_{j'}$ over $A$, we have $e \in A$, so we have $\bar{d}_{ij} \cap A \neq \emptyset$, a contradiction. Then, since other cases are similar, we can assume $i = i'$ and $j \neq j'$. In this case, we have $e \in aclB_i = B_i$. Again, this is a contradiction.

So, as in (1), we have

$$\delta(\bigcup_{(i,j)\in \mathbb{N}^2} \bar{d}_{(i,j)} \cup \bigcup_{i<\mathbb{N}} \bar{b}_i \bar{c}_i) \leq \delta(\bigcup_{(i,j)\in \mathbb{N}^2} \bar{d}_{(i,j)}/\bigcup_{i<\mathbb{N}} \bar{b}_i \bar{c}_i) + \delta(\bigcup_{i<\mathbb{N}} \bar{b}_i \bar{c}_i) \leq n^2 \varepsilon + n \delta(\bar{b}_0 \bar{c}_0).$$

For a sufficiently large $n$, we get a contradiction.

**Remark 5**

1. In our proof of Proposition 1, we did not use the "genericity" of the structure $M$. If we assume the "genericity", the converse of Proposition 1 is true by the following argument. Suppose that $BC = B \otimes_A C \leq \mathcal{M}$. Let $\{C_i : i < \alpha\}$ be a sufficiently long Morley sequence of $tp(C/A)$. Then, by stability, there is $i$ such that $B$ and $C_i$ are independent over $A$. By proposition $BC_i = B \otimes_A C_i \leq \mathcal{M}$. Then we have $BC \cong_A BC_i$ and that they are closed. So they have the same type over $A$, hence $BC = B \otimes_A C \leq \mathcal{M}$. (For details see [W] or [VY].)

2. The assumption that $A$ is algebraically closed is necessary in general. But Ikeda [I] showed that the algebraicity assumption can be eliminated if $(L = \{R(*, *)\}$ and) $K_0$ is closed under subgraphs.

**References.**

