A note on independence in generic structures

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Abstract

We show that if $K$ is closed under quasi-substructures then $\text{tp}(B/C)$ does not fork over $B \cap C$ if and only if $B$ and $C$ are free over $B \cap C$ and $BC$ is closed for any closed $B, C \in K$.

Our notations and definitions are standard (see [1], [5] for reference).

Let $L = \{R_0, R_1, \ldots\}$ be a countable relational language, where each $R_i$ is symmetric and irreflexive, i.e., if $\models R_i(\bar{a})$ then the elements of $\bar{a}$ are without repetition and $\models R_i(\sigma(\bar{a}))$ for any permutation $\sigma$. Thus, for any $L$-structure $A$ and $R_i$ with arity $n$, $R^A_i$ can be thought of as a set of $n$-element subsets of $A$.

For a finite $L$-structure $A$, a predimension of $A$ is defined by $\delta(A) = |A| - \sum \alpha_i |R^A_i|$, where $0 < \alpha_i \leq 1$. Let $\delta(B/A)$ denote $\delta(BA) - \delta(A)$.

For $A \subset B$, $A$ is closed in $B$ (write $A \leq B$), if $\delta(X/A \cap X) \geq 0$ for any finite $X \subset B$. The closure $A$ in $B$ is defined by $\text{cl}_B(A) = \cap \{C : A \leq C \leq B\}$.

Let $K^*$ be the class of all finite $L$-structures $A$ with $\delta(B) \geq 0$ for any $B \subset A$. Fix a subclass $K$ of $K^*$ that is closed under substructures. A countable $L$-structure $M$ is $K$-generic, if (i) any finite $A \subset M$ belongs to $K$; (ii) for any $A \leq B \in K$ with $A \leq M$ there is $B' \cong_A B$ with $B' \leq M$.

Let $\mathcal{M}$ be a big model of a $K$-generic structure. Note that if $K$-generic structure $M$ is saturated then $\mathcal{M}$ also satisfies (i) and (ii). We abbreviate $\text{cl}_\mathcal{M}(\star)$ to $\text{cl}(\star)$. $K$ has finite closures, if there is no chain $A_0 \subset A_1 \subset \cdots$ of $A_i \in K$ with $\delta(A_{i+1}/A_i) < 0$ for each $i \in \omega$. Note that $K$ has finite closures if and only if $\text{cl}(A)$ is finite for any finite $A \subset \mathcal{M}$.

For $A, B, C$ with $B \cap C \subset A$, $B$ and $C$ are free over $A$ (write $B \perp_A C$), if $R^{ABC} = R^{AB} \cup R^{AC}$ for every $R \in L$. Note that $B \perp_A C$ if and only if $\delta(\overline{b}/\overline{c}\overline{a}) = \delta(\overline{b}/\overline{a})$ for any $\overline{b} \in B - A, \overline{c} \in C - A$ and $\overline{a} \in A$.

Assumption $L$ is a countable relational language. $K$ is a class of finite $L$-structure $A$ with $\delta(B) \geq 0$ for any $B \subset A$. Moreover $K$ is closed under substructures and has finite closures. $\mathcal{M}$ is a big model of a saturated $K$-generic structure.

Definition Let $A$ and $B$ be $L$-structures. Then $A$ is said to be a quasi-substructure of $B$, if the universe of $A$ is contained in that of $B$, and $R^A$ is contained in $R^B$ for every $R \in L$. If $L$ is a language of a graph, then the notion of quasi-substructures coincides with that of subgraphs.

For $A, B \subset \mathcal{M}$, we denote $B^{\text{Aut}(\mathcal{M}/A)} = \{\sigma(b) : b \in B, \sigma \in \text{Aut}(\mathcal{M}/A)\}$.

Lemma 1 Let $B, C \leq \mathcal{M}$ with $A = B \cap C$. Then $B^{\text{Aut}(\mathcal{M}/A)} = B$ implies $B \perp_A C$ and $BC \leq \mathcal{M}$.

Proof Since $K$ is closed under quasi-substructures, there is $B' \cong_A B$ with $B' \perp_A C$ and $B'C \in K$. By genericity, we can assume that $(B' \leq)B'C \leq \mathcal{M}$. So we have $tp(B/A) = tp(B'/A)$. By $B^{\text{Aut}(\mathcal{M}/A)} = B$, we have $B = B'$ as a set. Hence $B \perp_A C$ and $BC \leq \mathcal{M}$.

For $A \leq B$, $B$ is said to be minimal over $A$, if $C = A$ or $C = B$ for any $A \subset C \leq B$.

Lemma 2 Let $B, C \leq \mathcal{M}$ with $A = B \cap C$. If $tp(B/A)$ is algebraic then $B \perp_A C$ and $BC \leq \mathcal{M}$.

Proof We can assume that $B$ is minimal over $A$. Since $tp(B/A)$ is algebraic, we can take a maximal set $(B =)B_1, \ldots, B_n$ of conjugates of $B$ over $A$ satisfying $B_i \neq B_j$ as a set for each $i, j$ with $1 \leq i < j \leq n$. By minimality, we have $B_i \cap B_j = A$.

Claim: $\perp\{B_i\}_i$ and $B_1\ldots B_n \leq \mathcal{M}$.

Proof: Since $K$ is closed under quasi-substructures, for each $i$ there is a copy $B'_i$ of $B_i$ over $A$ with $\perp\{B'_i\}_i$ and $(A \leq)B'_1\ldots B'_n \in K$. We can assume that
B'_1...B'_n \leq \mathcal{M}$. By maximality of $n$, we have $B_1...B_n = B'_1...B'_n$ as a set. Hence $\downarrow \{B_i\}_i$ and $B_1...B_n \leq \mathcal{M}$.

We devide into two cases.

Case: There is $i$ with $B_i \subset C$. By claim, $BB_i \leq \mathcal{M}$. By induction hypothesis, we have $B \perp B_i C$ and $BB_i C = BC \leq \mathcal{M}$. Again, by claim, $B \perp B_i C$.

Case: For any $i$, $B_i \not\subset C$. By minimality, we have $B_i \cap C = A$. Let $B^* = B_1...B_n$. Then $B^* \aut(\mathcal{M}/A) = B^*$. By lemma, $B \perp B_i C$, and hence $B \perp {}_A C$.

The following fact is due to Wagner [5]. Recently, Tsuboi [4] gave a short proof of this fact.

**Fact 3**  Let $B, C \leq \mathcal{M}$ with $A = B \cap C$ algebraically closed. Then $B \downarrow_A C$ iff $B \perp {}_A C$ and $BC \leq \mathcal{M}$.

The following theorem is a generalization of results obtained in [1] and [3].

**Theorem**  Let $K$ be closed under quasi-substructures. Let $B, C \leq \mathcal{M}$ with $A = B \cap C$. Then $B \downarrow_A C$ if and only if $B \perp {}_A C$ and $BC \leq \mathcal{M}$.

**Proof**  $(\Rightarrow)$ Suppose that $B \downarrow_A C$. First we show $B \perp {}_A C$. Let $B' = \acl(A) \cap B$ and $C' = \acl(A) \cap C$. By lemma 2, $B \cup \acl(A), C \cup \acl(A) \leq \mathcal{M}$. So, by fact 3, $B \perp \acl(A)$. By lemma 2, $B \perp_B \acl(C)$. So $B \perp_B C$. Again, by lemma 2, $B' \perp_A C$. Hence $B \perp_A C$. Next we show $BC \leq \mathcal{M}$. By lemma 2, $BC \cup \acl(A) \leq \mathcal{M}$. So it is enough to show that $BC \cap X \leq X$ for any finite $X \leq BC \cup \acl(A)$. For $D \subset \mathcal{M}$ let $X_D$ denote $X \cap D$. Take any $\bar{e} \in X - X_B X_C$. By lemma 2, we have $B' C \leq \mathcal{M}$, and so $X_B X_C \leq \mathcal{M}$. By lemma 2 and fact 3, we have $B \perp_B \acl(A)$ and $B \perp \acl(A) C$, and so $X_B \perp_X B' \bar{e} X_C'$ and $X_B \perp_{X_B', X_C'} X_C$. Therefore

\[
\delta(\bar{e}/X_B X_C) = \delta(\bar{e}/X_B X_C) + \delta(X_B/\bar{e} X_B X_C) - \delta(X_B/X_B X_C) \\
\geq \delta(X_B/\bar{e} X_B X_C) - \delta(X_B/X_B X_C) \\
= \delta(X_B/\bar{e} X_B X_C') - \delta(X_B/X_B X_C') \\
= \delta(X_B/X_B X_C') - \delta(X_B/X_B X_C') = 0.
\]

Hence $X_B X_C \leq X$.

$(\Leftarrow)$ Suppose that $B \perp {}_A C$ and $BC \leq \mathcal{M}$. Take $B'$ with $B' \downarrow_A C$ and $\tp(B'/A) =$
tp(\(B/A\)). By the only-if part of this theorem, we have \(B' \perp_A C\) and \(B'C \leq \mathcal{M}\). Thus we have \(tp(B'/C) = tp(B/C)\) and hence \(B \downarrow_A C\).

Reference


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