A Theory of Superstructures

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1 Axiom

We shall consider a first order theory of a language \( \mathcal{L}_\in = \{ \in \} \) on the classical logic with equality "\=" , where the symbol \( \in \) is a membership relation.

We adopt the following abbreviations:

\[
\begin{align*}
\text{Set}(x) & \equiv \exists y \ y \in x, \\
\forall x \in y \varphi(x) & \equiv \forall x \ [x \in y \Rightarrow \varphi(x)], \\
\exists x \in y \varphi(x) & \equiv \exists x \ [x \in y \land \varphi(x)], \\
x \subseteq y & \equiv \forall z \in x \ z \in y, \\
x \notin y & \equiv \neg x \in y, \\
x \subsetneq y & \equiv x \subseteq y \land \exists u \in y \ u \notin x, \\
\text{Trans}(x) & \equiv \forall y \ x \subseteq y, \\
\forall x \subseteq y \varphi(x) & \equiv \forall x \ [x \subseteq y \Rightarrow \varphi(x)], \\
\text{Wo}_{\subseteq}(x) & \equiv \forall y \subseteq x \ [\text{Set}(y) \Rightarrow \exists u \in y \ \forall z \in y \ u \subseteq z] \land \forall z \in x \ \text{Set}(z), \\
\text{Mater}(x, y) & \equiv \exists y \subseteq x \ \exists u \in y \ z \in u, \\
\exists! x \varphi(x) & \equiv \exists x \varphi(x) \land \forall x_1 \forall x_2 [\varphi(x_1) \land \varphi(x_2) \Rightarrow x_1 = x_2], \\
\exists! x \in y \varphi(x) & \equiv \exists! x \ [x \in y \land \varphi(x)].
\end{align*}
\]

We call \( \text{Mater}(x, y) \) that \( x \) is a set of materials of \( y \). In \( \text{ZF} \) set theory, \( \text{Mater}(x, y) \) means that \( x \) is a subset of union of \( y \).

A formula \( \varphi \) of \( \mathcal{L}_\in \) is restricted or bounded if all quantifiers in \( \varphi \) are of either form

\( \forall x \in y \) or \( \exists x \in y \).

Here is an axiom system of a theory of Superstructures.

1. Extensionality of nonempty sets:

\[
\forall x \forall y [\text{Set}(x) \land x \subseteq y \land y \subseteq x \Rightarrow x = y].
\]
2. Pair: 
\[\forall x \forall y \exists u [x \in u \land y \in u].\]

3. Transitive superset: 
\[\forall x \exists u [x \subseteq u \land \text{Trans}(u)].\]

4. Power: 
\[\forall x \exists u \forall y \subseteq x y \in u.\]

5. Infinity: 
\[\exists u [\text{Set}(u) \land \text{Wo}(u) \land \forall y \in u \exists v \in u y \subsetneq v].\]

6. Strong foundation: 
\[\forall x [\text{Set}(x) \land \forall y \in x \exists u \in x \text{Mater}(u, y) \Rightarrow \exists u \in x \neg \text{Set}(u)].\]

7. Choice: 
\[\forall x [\forall y \in x \exists u \in y \exists! v \in x u \in v \Rightarrow \exists w \forall y \in x \exists u \in y u \in w].\]

8. Restricted separation: If \(\varphi(y, z)\) is a restricted formula, then 
\[\forall p \forall x \exists u \forall y [y \in u \Leftrightarrow y \in x \land \varphi(y, p)].\]

9. \(\in\)-induction schema: 
\[\forall x [\forall y \in x \psi(y) \Rightarrow \psi(x)] \Rightarrow \forall x \psi(x).\]

We denote 1–9 by SS and 1–8 by SS\(_0\).

2 Universe

In this section, we consider the universe of SS\(_0\), and comulative hierarchy of SS.

By Infinity and Restricted separation, there is an \(a\) such that \(\neg \text{Set}(a)\), and by Power, there is \(b\) such that 
\[\forall x \subseteq a x \in b, \text{ or } \forall x [\neg \text{Set}(x) \Rightarrow x \in b].\]
By Restricted separation and Extensionality, there is a unique \(-\) such that

\[ \forall x \ [x \in - \iff \neg \text{Set}(x)]. \]

By Pair and Restricted separation, there is an unordered pair \(c\) for every \(a\) and \(b\) such that

\[ \forall x \ [x \in c \iff \[c = a \lor c = b\]]. \]

We denote such \(c\) by \(\{a, b\}\) and \(\{a, a\}\) by \(\{a\}\). We define an ordered pair \((a, b)\) by \(\{\{a\}, \{a, b\}\}\).

Let \(\varphi(x)\) be a restricted formula and suppose \(\exists x \in a \ \varphi(x)\). Then, by Restricted separation and Extensionality of nonempty sets, there is a unique \(b\) such that

\[ \forall x \ [x \in b \iff x \in a \land \varphi(x)]. \]

We denote such \(b\) by \(\{x \in a | \varphi(x)\}\).

By Power, there is a \(b\) for each \(a\)

\[ \forall x \subseteq a \ x \in b. \]

We denote \(\{x \in b | x \subseteq a\}\) by \(\mathcal{P}(a)\). Note that \(-\subseteq\mathcal{P}(a)\) for every \(a\).

By Transitive superset, for every \(x\), there is \(t\) such that \(\text{Trans}(t) \land x \subseteq t\), define a transitive closure of \(x\) by:

\[
\text{TC}(x) = \begin{cases} 
  x & \text{if } x \in - \\
  \{y \in t | \forall z \in \mathcal{P}(t) [\text{Trans}(z) \land x \subseteq z \Rightarrow y \in z]\} & \text{if } x \notin -
\end{cases}
\]

When \(a \not\subseteq -\), we denote the union \(\{x \in \text{TC}(a) | \exists y \in a \ x \in y\}\) by \(\bigcup a\). When \(\{a, b\} \not\subseteq -\), we denote \(\bigcup \{a, b\}\) by \(a \cup b\).

As in \(\text{ZF}\), we define maps, injections, surjections, bijections.

By Infinity, fixing \(\alpha\) such that

\[ \alpha \notin - \land \text{Wo}(\alpha) \land \forall y \in \alpha \ \exists v \in \alpha \ y \subseteq v, \]

there is a unique \(\subseteq\)-least element \(0_\alpha\) in \(\alpha\): \(\forall x \in \alpha \ 0_\alpha \subseteq x\). For every \(x \in a\), there is unique \(x'\) such that

\[ \forall y \in \alpha \ [x' \subseteq y \iff x \not\subseteq y]. \]

We denote such \(x'\) by \(x +_\alpha 1\). We can define a minimal unbounded well-ordered set \(\mathbb{N}_\alpha\) with order relation \(\subseteq\) by

\[ \mathbb{N}_\alpha = \{x \in \alpha \mid \forall y \in \alpha \ [0 \subseteq y \land \forall z \in \alpha [z \subseteq y \Rightarrow z +_\alpha 1 \not\subseteq y] \Rightarrow x \not\subseteq y]\}. \]
Then we have Restricted induction principle:

\[ \varphi(0) \land \forall n \in \mathbb{N}_\alpha [\varphi(n) \Rightarrow \varphi(n + _\alpha 1)] \Rightarrow \forall n \in \mathbb{N}_\alpha \varphi(n), \]

where \( \varphi(n) \) is restricted. Then we have that \( \mathbb{N}_\alpha \) is unique up to isomorphism, so we denote a structure of natural numbers by \( \langle \mathbb{N}, \leq, +, 0 \rangle \).

Since \( u \in y \) implies \( \text{Mater}(y, u) \), we have, by Strong foundation, foundation principle:

\[ \forall x [\text{Set}(x) \Rightarrow \exists y \in x \forall u \in xu \not\in y]. \]

We shall show dual foundation principle:

\[ \forall x [\text{Set}(x) \Rightarrow \exists y \in x \forall u \in xy \not\in u]. \]

Suppose, on contrary, there is \( x \) such that \( \text{Set}(x) \) and \( \forall y \in x \exists u \in x y \in u \). Since \( \text{Mater}(\text{TC}(x), \text{TC}(x)) \), we have, by Strong foundation, there is \( a \in \mathbb{N} \) such that

\[ a \in \text{TC}(x), \text{TC}(x) \]

which is contradiction.

Let \( \mathbb{N} \) be a structure of natural numbers. We define the predicate "\( x \) has rank \( n \)" by

\[ \rho(n, x) \equiv \bar{\rho}(n, \text{TC}(x) \cup \{x\}, x), \]

\[ \bar{\rho}(n, t, x) \equiv \exists f : t \rightarrow \mathbb{N} [\forall y \in t f(y) = \bigcup \{k \in \mathbb{N} \mid k = 0 \lor \exists z \in y k = f(z) + 1\} \wedge n = f(x)]. \]

Then every \( x \) has a unique rank.

In SS, applying the following \( \psi(x) \) to \( \in \)-induction schema, we have obtained cumulative hierarchy \( W_n \). We cannot prove that there is \( W_n \) for every \( n \in \mathbb{N} \).

\[ \psi(x) \equiv \forall n \in \mathbb{N} [\rho(n, x) \Rightarrow \exists W_n \forall y [y \in W_n \iff \exists k \leq n \rho(k, y)]]. \]

3 Models

We construct models for SS in ZFC. We say a model \( W \) is ZF-standard if the membership relation \( \in \) of \( W \) is that of ZFC.

Given a set \( X \), we define the iterated power set \( V_n(X) \) over \( X \) recursively by

\[ V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)). \]

The superstructure \( V(X) \) is the union \( \bigcup_{n<\omega} V_n(X) \). The set \( X \) is said to be a base set if \( \emptyset \not\in X \) and each element of \( X \) is disjoint from \( V(X) \).
If $X$ is a base set then $V(X)$ is a ZF-standard model for SS. In $V(X)$, we see $X \cup \{\emptyset\} = \varnothing$ and $\mathcal{P}_\varnothing(a) = \mathcal{P}(a) \cup \varnothing$.

Let $X$ and $Y$ are infinite base sets, and let $j: V(X) \to V(Y)$ be a nontrivial bounded elementary embedding — $\langle V(X), V(Y), j \rangle$ is a nonstandard universe. Then the transitive closure $W$ of $\text{ran } j$ within $V(Y)$ is a model for SS. In $W$, we see $j(\mathbb{N})$ is a structure of natural numbers if $\mathbb{N}$ is a structure of natural numbers in $V(X)$, and there is no $W_\nu$ for nonstandard $\nu \in j(\mathbb{N}) \setminus j" \mathbb{N}$. 