<table>
<thead>
<tr>
<th>Title</th>
<th>A Theory of Superstructures (Model theoretic techniques for constructing infinite structures)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>MURAKAMI, Masahiko</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1602: 17-21</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/139891">http://hdl.handle.net/2433/139891</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A Theory of Superstructures

法政大学 村上 雅彦 (Masahiko MURAKAMI)
Hosei University

1 Axiom

We shall consider a first order theory of a language $L_{\in} = \{\in\}$ on the classical logic with equality "\=". Axiom system of a theory of Superstructures.

1. Extensionality of nonempty sets:

$$\forall x \forall y [\text{Set}(x) \land x \subseteq y \land y \subseteq x \Rightarrow x = y].$$
2. Pair:
\[ \forall x \forall y \exists u [x \in u \land y \in u]. \]

3. Transitive superset:
\[ \forall x \exists u [x \subseteq u \land \text{Trans}(u)]. \]

4. Power:
\[ \forall x \exists u \forall y \subseteq x \; y \in u. \]

5. Infinity:
\[ \exists u [\text{Set}(u) \land \text{Wo}_\subseteq(u) \land \forall y \in u \exists v \in y \; y \in v]. \]

6. Strong foundation:
\[ \forall x [\text{Set}(x) \land \forall y \in x \exists u \in x \; \text{Mater}(u, y) \Rightarrow \exists u \in x \neg \text{Set}(u)]. \]

7. Choice:
\[ \forall x [\forall y \in x \exists u \in y \exists! v \in u \; u \in v \Rightarrow \exists w \forall y \in x \exists! u \in y \; u \in w]. \]

8. Restricted separation: If \( \varphi(y, z) \) is a restricted formula, then
\[ \forall p \forall x \exists u \forall y [y \in u \Leftrightarrow y \in x \land \varphi(y, p)]. \]

9. \( \in \)-induction schema:
\[ \forall x [\forall y \in x \psi(y) \Rightarrow \psi(x)] \Rightarrow \forall x \psi(x). \]

We denote 1–9 by SS and 1–8 by SS\(_0\).

2 Universe

In this section, we consider the universe of SS\(_0\), and cumulative hierarchy of SS.

By Infinity and Restricted separation, there is an \( a \) such that \( \neg \text{Set}(a) \), and by Power, there is \( b \) such that
\[ \forall x \subseteq a \; x \in b, \text{ or } \forall x [\neg \text{Set}(x) \Rightarrow x \in b]. \]
By Restricted separation and Extensionality, there is a unique $\varphi$ such that
\[
\forall x \left[ x \in \varphi \iff \neg \text{Set}(x) \right].
\]

By Pair and Restricted separation, there is an unordered pair $c$ for every $a$ and $b$ such that
\[
\forall x \left[ x \in c \iff c = a \lor c = b \right].
\]
We denote such $c$ by $\{a, b\}$ and $\{a, a\}$ by $\{a\}$. We define an ordered pair $(a, b)$ by $\{(a), (a, b)\}$.

Let $\varphi(x)$ be a restricted formula and suppose $\exists x \in a \varphi(x)$. Then, by Restricted separation and Extensionality of nonempty sets, there is a unique $b$ such that
\[
\forall x \left[ x \in b \iff x \in a \land \varphi(x) \right].
\]
We denote such $b$ by $\{x \in a \mid \varphi(x)\}$. By Power, there is a $b$ for each $a$
\[
\forall x \subseteq a \exists b.
\]
We denote $\{x \in b \mid x \subseteq a\}$ by $\mathcal{P}(a)$. Note that $\varphi \subseteq \mathcal{P}(a)$ for every $a$.

By Transitive superset, for every $x$, there is $t$ such that $\text{Trans}(t) \and x \subseteq t$, define a transitive closure of $x$ by:
\[
\text{TC}(x) = \begin{cases} x & \text{if } x \in \varphi \\ \{y \in t \mid \forall z \in \mathcal{P}(t) \left[ \text{Trans}(z) \land x \subseteq z \implies y \in z \right] \} & \text{if } x \notin \varphi \end{cases}.
\]
When $a \varphi \subseteq \varphi$, we denote the union $\{x \in \text{TC}(a) \mid \exists y \in a \ x \in y\}$ by $\cup a$. When $\{a, b\} \varphi \subseteq \varphi$, we denote $\cup \{a, b\}$ by $a \cup b$.

As in ZF, we define maps, injections, surjections, bijections.

By Infinity, fixing $\alpha$ such that
\[
\alpha \notin \varphi \land \text{Wo}_\varphi(\alpha) \land \forall y \in \alpha \exists v \in \alpha \ y \subseteq v,
\]
there is a unique $\subseteq$-least element $0_\alpha$ in $\alpha$: $\forall x \in \alpha \ 0_\alpha \subseteq x$. For every $x \in a$, there is unique $x'$ such that
\[
\forall y \in \alpha \ [x' \subseteq y \iff x \varphi y].
\]
We denote such $x'$ by $x +_\alpha 1$. We can define a minimal unbounded well-ordered set $\mathbb{N}_\alpha$ with order relation $\subseteq$ by
\[
\mathbb{N}_\alpha = \left\{ x \in \alpha \mid \forall y \in \alpha \left[ 0 \subseteq y \land \forall z \in \alpha \left[ z \subseteq y \implies z +_\alpha 1 \subseteq y \right] \Rightarrow x \varphi y \right\}.
\]
Then we have Restricted induction principle:

$$\varphi(0) \land \forall n \in \mathbb{N}_\alpha [\varphi(n) \Rightarrow \varphi(n +_{\alpha} 1)] \Rightarrow \forall n \in \mathbb{N}_\alpha \varphi(n),$$

where $\varphi(n)$ is restricted. Then we have that $\mathbb{N}_\alpha$ is unique up to isomorphism, so we denote a structure of natural numbers by $\langle \mathbb{N}, \leq, +1, 0 \rangle$.

Since $u \in y$ implies $\text{Mater}(y, u)$, we have, by Strong foundation, foundation principle:

$$\forall x [\text{Set}(x) \Rightarrow \exists y \in x \forall u \in x u \notin y].$$

We shall show dual foundation principle:

$$\forall x [\text{Set}(x) \Rightarrow \exists y \in x \forall u \in x y \notin u].$$

Suppose, on contrary, there is $x$ such that $\text{Set}(x)$ and $\forall y \in x \exists u \in x y \in u$. Since $\text{Mater}(\text{TC}(x), \text{TC}(x))$, we have, by Strong foundation, there is $a \in \mathbb{W}$ such that $a \in \{\text{TC}(x)\}$, which is contradiction.

Let $\mathbb{N}$ be a structure of natural numbers. we define the predicate “$x$ has rank $n$” by

$$\rho(n, x) \equiv \bar{\rho}(n, \text{TC}(x) \cup \{x\}, x),$$

$$\bar{\rho}(n, t, x) \equiv \exists f : t \rightarrow \mathbb{N} \left[ \forall y \in t f(y) = \cup \{k \in \mathbb{N} \mid k = 0 \lor \exists z \in y k = f(z) + 1 \} \right. \left. \wedge n = f(x) \right].$$

Then every $x$ has a unique rank.

In SS, applying the following $\psi(x)$ to $\in$-induction schema, we have obtained cumulative hierarchy $W_n$. We cannot prove that there is $W_n$ for every $n \in \mathbb{N}$.

$$\psi(x) \equiv \forall n \in \mathbb{N} [\rho(n, x) \Rightarrow \exists W_n \forall y [y \in W_n \Leftrightarrow \exists k \leq n \rho(k, y)]]$$

### 3 Models

We construct models for SS in ZFC. We say a model $W$ is ZF-standard if the membership relation $\in$ of $W$ is that of ZFC.

Given a set $X$, we define the iterated power set $V_n(X)$ over $X$ recursively by

$$V_0(X) = X, \text{ and } V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The superstructure $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set $X$ is said to be a base set if $\emptyset \notin X$ and each element of $X$ is disjoint from $V(X)$. 
If $X$ is a base set then $V(X)$ is a \textit{ZF}-standard model for \textit{SS}. In $V(X)$, we see $X \cup \{\emptyset\} = -$ and $\mathbb{P}(a) = \mathbb{P}(a) \cup -$.

Let $X$ and $Y$ are infinite base sets, and let $j: V(X) \rightarrow V(Y)$ be a nontrivial bounded elementary embedding — $\langle V(X), V(Y), j \rangle$ is a nonstandard universe. Then the transitive closure $W$ of ran $j$ within $V(Y)$ is a model for \textit{SS}. In $W$, we see $j(\mathbb{N})$ is a structure of natural numbers if $\mathbb{N}$ is a structure of natural numbers in $V(X)$, and there is no $W_\nu$ for nonstandard $\nu \in j(\mathbb{N}) \setminus j^* \mathbb{N}$.