A Theory of Superstructures (Model theoretic techniques for constructing infinite structures)

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A Theory of Superstructures

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1 Axiom

We shall consider a first order theory of a language $L_{\in} = \{\in\}$ on the classical logic with equality "="", where the symbol $\in$ is a membership relation.

We adopt the following abbreviations:

- $\text{Set}(x) \equiv \exists y \ y \in x$,
- $\forall x \ y \varphi(x) \equiv \forall x \ [x \in y \Rightarrow \varphi(x)]$,
- $\exists x \ y \varphi(x) \equiv \exists x \ [x \in y \land \varphi(x)]$,
- $x \subseteq y \equiv \forall z \ [x \in z \Rightarrow z \in y]$,
- $x \notin y \equiv \neg x \in y$,
- $x \subsetneq y \equiv x \subseteq y \land \exists u \in y u \notin x$,
- $\text{Trans}(x) \equiv \forall y \ [x \subseteq y]$,
- $\forall x \subseteq y \varphi(x) \equiv \forall x \ [x \subseteq y \Rightarrow \varphi(x)]$,
- $\text{Wo}_{\subseteq}(x) \equiv \forall y \subseteq x \ [\text{Set}(y) \Rightarrow \exists u \ y \in y \subseteq z \land \forall z \in x \ \text{Set}(z)]$,
- $\text{Mater}(x, y) \equiv \forall z \in x \exists u \in y z \in u$,
- $\exists! x \varphi(x) \equiv \exists x \varphi(x) \land \forall x_1 \forall x_2 [\varphi(x_1) \land \varphi(x_2) \Rightarrow x_1 = x_2]$,
- $\exists! x \in y \varphi(x) \equiv \exists! x \ [x \in y \land \varphi(x)]$.

We call $\text{Mater}(x, y)$ that $x$ is a set of materials of $y$. In ZF set theory, $\text{Mater}(x, y)$ means that $x$ is a subset of union of $y$.

A formula $\varphi$ of $L_{\in}$ is restricted or bounded if all quantifiers in $\varphi$ are of either form $\forall x \in y$ or $\exists x \in y$.

Here is an axiom system of a theory of Superstructures.

1. Extensionality of nonempty sets:

$$\forall x \forall y \ [\text{Set}(x) \land x \subseteq y \land y \subseteq x \Rightarrow x = y].$$
2. Pair: 
\[ \forall x \forall y \exists u \ [x \in u \land y \in u] . \]

3. Transitive superset: 
\[ \forall x \exists u \ [x \subseteq u \land \text{Trans}(u)] . \]

4. Power: 
\[ \forall x \exists u \forall y \subseteq x \ y \in u . \]

5. Infinity: 
\[ \exists u \ [\text{Set}(u) \land \text{Wo}_\subseteq(u) \land \forall y \in u \ \exists v \in u \ y \subsetneq v] . \]

6. Strong foundation: 
\[ \forall x \ [\text{Set}(x) \land \forall y \in x \exists u \in x \ \text{Mater}(u, y) \Rightarrow \exists u \in x \neg \text{Set}(u)] . \]

7. Choice: 
\[ \forall x \ [\forall y \in x \exists u \in y \exists! v \in x u \in v \Rightarrow \exists w \forall y \in x \exists! u \in y u \in w] . \]

8. Restricted separation: If \( \varphi(y, z) \) is a restricted formula, then 
\[ \forall p \ \forall x \exists u \forall y \ [y \in u \iff y \in x \land \varphi(y, p)] . \]

9. \( \in \)-induction schema: 
\[ \forall x \ [\forall y \in x \varphi(y) \Rightarrow \psi(x)] \Rightarrow \forall x \psi(x) . \]

We denote 1–9 by \( \text{SS} \) and 1–8 by \( \text{SS}_0 \).

## 2 Universe

In this section, we consider the universe of \( \text{SS}_0 \), and comulative hierarchy of \( \text{SS} \).

By Infinity and Restricted separation, there is an \( a \) such that \( \neg \text{Set}(a) \), and by Power, there is \( b \) such that
\[ \forall x \subseteq a \ x \in b, \ \text{or} \ \forall x \ [\neg \text{Set}(x) \Rightarrow x \in b] . \]
By Restricted separation and Extensionality, there is a unique $\rightarrow$ such that
\[\forall x [x \in \rightarrow \iff \neg \text{Set}(x)].\]

By Pair and Restricted separation, there is an unordered pair $c$ for every $a$ and $b$ such that
\[\forall x [x \in c \iff [c = a \lor c = b]].\]
We denote such $c$ by $\{a, b\}$ and $\{a, a\}$ by $\{a\}$. We define an ordered pair $(a, b)$ by $\}\{a\}, \{a, b\}\}$.

Let $\varphi(x)$ be a restricted formula and suppose $\exists x \in a \varphi(x)$. Then, by Restricted separation and Extensionality of nonempty sets, there is a unique $b$ such that
\[\forall x [x \in b \iff x \in a \land \varphi(x)].\]
We denote such $b$ by $\{x \in a \mid \varphi(x)\}$.

By Power, there is a $b$ for each $a$
\[\forall x \subseteq a \ x \in b.\]
We denote $\{x \in b \mid x \subseteq a\}$ by $\mathcal{P}(a)$. Note that $\rightarrow \subseteq \mathcal{P}(a)$ for every $a$.

By Transitive superset, for every $x$, there is $t$ such that $\text{Trans}(t) \land x \subseteq t$, define a transitive closure of $x$ by:
\[\text{TC}(x) = \begin{cases} x & \text{if } x \in \rightarrow \\ \{y \in t \mid \forall z \in \mathcal{P}(t) [\text{Trans}(z) \land x \subseteq z \Rightarrow y \in z] \} & \text{if } x \notin \rightarrow \end{cases}.\]

When $a \not\subset \rightarrow$, we denote the union $\{x \in \text{TC}(a) \mid \exists y \in a \ x \in y\}$ by $\bigcup a$. When $\{a, b\} \not\subset \rightarrow$, we denote $\bigcup \{a, b\}$ by $a \cup b$.

As in ZF, we define maps, injections, surjections, bijections.

By Infinity, fixing $\alpha$ such that
\[\alpha \notin \rightarrow \land \text{Wo}_\varsubsetneq(\alpha) \land \forall y \in \alpha \exists v \in \alpha \ y \subseteq v,\]
there is a unique $\subseteq$-least element $0_\alpha$ in $\alpha$: $\forall x \in \alpha \ 0_\alpha \subseteq x$. For every $x \in a$, there is unique $x'$ such that
\[\forall y \in \alpha [x' \subseteq y \iff x \subsetneq y].\]
We denote such $x'$ by $x +_\alpha 1$. We can define a minimal unbounded well-ordered set $\mathbb{N}_\alpha$ with order relation $\subseteq$ by
\[\mathbb{N}_\alpha = \{x \in \alpha \mid \forall y \in \alpha [0 \subsetneq y \land \forall z \in \alpha [z \subsetneq y \Rightarrow z +_\alpha 1 \subsetneq y] \Rightarrow x \subsetneq y]\}.\]
Then we have Restricted induction principle:

$$\varphi(0) \land \forall n \in \mathbb{N}_\alpha [\varphi(n) \Rightarrow \varphi(n + 1_{\alpha})] \Rightarrow \forall n \in \mathbb{N}_\alpha \varphi(n),$$

where $\varphi(n)$ is restricted. Then we have that $\mathbb{N}_\alpha$ is unique up to isomorphism, so we denote a structure of natural numbers by $(\mathbb{N}, \leq, +, 0)$.

Since $u \in y$ implies $\text{Mater}(y, u)$, we have, by Strong foundation, foundation principle:

$$\forall x [\text{Set}(x) \Rightarrow \exists y \in x \forall u \in x u \notin y].$$

We shall show dual foundation principle:

$$\forall x [\text{Set}(x) \Rightarrow \exists y \in x \forall u \in x y \notin u].$$

Suppose, on contrary, there is $x$ such that $\text{Set}(x)$ and $\forall y \in x \exists u \in y y \notin u$. Since $\text{Mater}(\text{TC}(x), \text{TC}(x))$, we have, by Strong foundation, there is $a \in -$ such that $a \in \{\text{TC}(x)\}$, which is contradiction.

Let $\mathbb{N}$ be a structure of natural numbers. we define the predicate “$x$ has rank $n$” by

$$\rho(n, x) \equiv \bar{\rho}(n, \text{TC}(x) \cup \{x\}, x),$$
$$\bar{\rho}(n, t, x) \equiv \exists f : t \rightarrow \mathbb{N} \left[ \forall y \in t f(y) = \bigcup \{k \in \mathbb{N} | k = 0 \lor \exists z \in y k = f(z) + 1\} \land n = f(x) \right].$$

Then every $x$ has a unique rank.

In SS, applying the following $\psi(x)$ to $\epsilon$-induction schema, we have obtained cumulative hierarchy $W_n$. We cannot prove that there is $W_n$ for every $n \in \mathbb{N}$.

$$\psi(x) \equiv \forall n \in \mathbb{N} [\rho(n, x) \Rightarrow \exists W_n \forall y [y \in W_n \Leftrightarrow \exists k \leq n \rho(k, y)]].$$

3 Models

We construct models for SS in ZFC. We say a model $W$ is $\text{ZF-standard}$ if the membership relation $\in$ of $W$ is that of ZFC.

Given a set $X$, we define the iterated power set $V_n(X)$ over $X$ recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The superstructure $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set $X$ is said to be a base set if $\emptyset \notin X$ and each element of $X$ is disjoint from $V(X)$.
If $X$ is a base set then $V(X)$ is a $\text{ZF}$-standard model for $\text{SS}$. In $V(X)$, we see $X \cup \{\emptyset\} = \emptyset$ and $\mathcal{P}_X(a) = \mathcal{P}(a) \cup \emptyset$.

Let $X$ and $Y$ are infinite base sets, and let $j : V(X) \to V(Y)$ be a nontrivial bounded elementary embedding — $\langle V(X), V(Y), j \rangle$ is a nonstandard universe. Then the transitive closure $W$ of $\text{ran } j$ within $V(Y)$ is a model for $\text{SS}$. In $W$, we see $j(\mathbb{N})$ is a structure of natural numbers if $\mathbb{N}$ is a structure of natural numbers in $V(X)$, and there is no $W_\nu$ for nonstandard $\nu \in j(\mathbb{N}) \setminus j"\mathbb{N}$. 