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(Model theoretic techniques for constructing infinite structures)

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Nonstandard arguments and the stability of generic structures

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Abstract

Generic 構造の研究に超準的手法を導入する。本稿では特に、Wagner が行った generic 構造の安定性の強さについての研究 [4] に、弱冠の新しい結果を付け足す。Wagner は saturated な generic 構造が安定になる為の十分条件 DS と ω-安定になる為の十分条件 DW を定義した。本稿では DS を簡略化し、DS と DW の間の関係を調べる。

1 Preliminaries

Let $L$ be a countable relational language. Let $\mathbb{K}$ be a nonempty class of finite $L$-structures closed under isomorphisms and substructures (we consider the emptyset as an $L$-structure). Suppose $A \subseteq B$ is a reflexive and transitive relation on elements $A \subseteq B$ of $\mathbb{K}$, which is invariant under isomorphisms. If $A \subseteq B$ holds, we say that $A$ is closed in $B$. We also assume that $(\mathbb{K}, \subseteq)$ satisfies the following properties:

1. $\emptyset \subseteq A$,
2. $A \subseteq B \subseteq C, A \subseteq C \implies A \subseteq B$,
3. $A \subseteq B \implies A \cap C \subseteq B \cap C$.

Let $(\mathbb{K}, \subseteq)$ be as above. Let $N$ be an $L$-structure whose any finite substructure belongs to $\mathbb{K}$. Note that for any $A \subseteq N$, there is a unique smallest closed superset of $A$ in $N$. We call this set the closure of $A$.

Definition 1 Let $A \subseteq B$. We say that $B$ is a minimal extension of $A$ if the following conditions are satisfied

- $A \not\subseteq B$
• $A \leq B'$ for any $A \subseteq B' \subset B$.

**Definition 2** Let $\leq$ be a closed relation on $\mathbb{K}$. Then we say that $(\mathbb{K}, \leq)$ satisfies finite closure axiom if there is no infinite chain $(A_i)_{i<\omega}$ of elements of $\mathbb{K}$ such that $A_{i+1}$ is a minimal extension of $A_i$ for each $i < \omega$.

We assume that $(\mathbb{K}, \leq)$ satisfies the finite closure axiom in this paper. We say that an $L$-structure $N$ has finite closures if for any finite $A \subseteq N$, the closure of $A$ is also finite. Put $\overline{\mathbb{K}} = \{N : L$-structure $|A \in \mathbb{K}$ for any $A \subset \text{fin } N\}$.

**Fact 3** [2] Let $\leq$ be a closed relation on $\mathbb{K}$. Then the following are equivalent:

1. $\mathbb{K}$ satisfies finite closure axiom.
2. Every member of $\overline{\mathbb{K}}$ has finite closures.
3. Every $\omega$-saturated member of $\overline{\mathbb{K}}$ has finite closures.
4. Some $\omega$-saturated member of $\overline{\mathbb{K}}$ has finite closures.

**Definition 4** Let $M$ be an $L$-structure. We say that $M$ is a $\mathbb{K}$-generic structure if the following conditions are satisfied:

1. $M$ is countable.
2. $\forall A \subset \text{fin } M, A \in \mathbb{K}$ (i.e. $M \in \overline{\mathbb{K}}$).
3. $A \leq M, A \leq B \in \mathbb{K} \Rightarrow \exists B' \leq M$ such that $B' \cong A B$.

**Fact 5** Suppose that $(\mathbb{K}, \leq)$ satisfies the finite closure axiom. Then a $\mathbb{K}$-generic structure is unique.

**Definition 6** Let $d$ be a function from $\{A : A \leq \text{fin } M\}$ to $\mathbb{R}_{\geq 0}$. We say $d$ is a dimension function for $M$ if for all $A, B \leq \text{fin } M$,

1. $A \subset B \Rightarrow d(A) \leq d(B)$
2. (Monotonicity) $d(A \cup B) + d(A \cap B) \leq d(A) + d(B)$
3. $A \cong B \Rightarrow d(A) = d(B)$

For arbitrary $A \subset \text{fin } M$, we put $d(A) = d(\overline{A})$. We define $d(A/B)$ the relative dimension of $A$ over $B$. For finite $A, B$, $d(A/B) = d(AB) - d(B)$. For finite $A$, arbitrary $B$, $d(A/B) = \inf\{d(A/B_0) : B_0 \subset \text{fin } B\}$. It is easy to check that these two definitions has the same value in the case $A$ and $B$ are finite.
2 Nonstandard arguments

Let $M$ be the $\mathbb{K}$-generic structure and $d$ be a dimension function for $M$. We consider $M$ to be a 3-sorted structure

$$(M \cup P \cup \mathbb{R}; F, \in, d \leq, \cdots)$$

where $P, F, \in$ are as above, $d$ is the dimension function of $M$, $\leq$ is the closed relation on $P \times P$.

We define the nonstandard model $M^*$ of $M$ by a sufficiently saturated extension of this structure

$$(M \cup P \cup \mathbb{R}, F, \in, d \leq, \cdots) \prec (M^* \cup P^* \cup \mathbb{R}^*, F^*, \in^*, d^*, \leq^* \cdots)$$

**Definition 7** A set $A \in F^*$ is said to be a hyperfinite set. For $A \subseteq M$, $A^* \in F^*$ is said to be a hyperfinite extension of $A$ if

- $M^* \models a \in^* A^*$ for each $a \in A$, and
- $M^* \models A^* \subseteq^* A$.

write $A \subset hf A^*$, $A^* \supset hf A$

By saturation, a hyperfinite extension of $A$ always exists.

**Lemma 8** For any subseteq $A$ of $M$, there exists a hyperfinite extension of $A$.

**Proof:** It is enough to prove that the following set of formulas is satisfiable:

$$\Gamma(X) = \{a \in^* X | a \in A\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$

But for any finite subseteq $A_0$ of $A$, $A_0$ realizes the following set of formulas:

$$\{a \in^* X | a \in A_0\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$

So, by compactness, $\Gamma(X)$ is satisfiable.

Let $x, y$ be two nonstandard (or standard) real numbers. We write $x \approx y$ if $|x - y| < 1/n$ for each $n \in \omega$.

**Lemma 9** For $r \in \mathbb{R}$, $\bar{a} \in M$ and $A \subset M$, the following are equivalent.

1. $d(\bar{a}/A) = r$;
2. \( d^*(\overline{a}/A^*) \approx r \) for any \( A^* \supset hf A \);

3. \( d^*(\overline{a}/A^*) \approx r \), for some \( A^* \supset hf A \).

Proof: (1 \( \rightarrow \) 2): By monotonicity of \( d \), there are \( A_n \subset_{\text{fin}} A \) (\( n = 1, 2, \ldots \)) such that \( \forall X \in F \)

\[
A_n \subset X \subset A \rightarrow r \leq d(\overline{a}/X) \leq r + 1/n.
\]

These statements hold also in \( M^* \). So if \( A^* \) is a hyperfinite extension of \( A \), then we have

\[
r \leq d^*(\overline{a}/A^*) \leq r + 1/n \quad (n = 1, 2, \ldots)
\]

So we have \( d^*(\overline{a}/A^*) \approx r \).

(2 \( \rightarrow \) 3): trivial.

(3 \( \rightarrow \) 1): We assume 3 and choose a witness \( A^* \). Then \( d^*(\overline{a}/A^*) \approx r \).

Suppose 1 is not the case. Then there is \( s \neq r \) such that \( d(\overline{a}/A) = s \). By 1 \( \Rightarrow \) 2, we have \( d^*(\overline{a}/A^*) \approx s \). A contradiction.

Note that \( M \models \forall A \in P \exists! \overline{A} (A \subseteq \overline{A} \leq M \wedge \forall X A \subseteq X \leq M \rightarrow \overline{A} \subseteq X) \).

This formula holds also in \( M^* \). For \( X \in P^* \), we write \( \overline{X} \) as the "closure" of \( X \) in \( M^* \). In this paper, \( M \models X \in F^* \rightarrow \overline{X} \in F^* \) because \( \mathbb{K} \) satisfies the finite closure condition.

3 Main result

Definition 10 ([4])

1. Let \( A, B \subset_{\text{fin}} M \) and \( C \subset M \). Then we say \( A \) and \( B \) are \( d \)-independent over \( C \) and write \( A \downarrow^d_C B \) if the following conditions are satisfied:

- \( d(A/BC) = d(A/C) \), and
- \( \overline{AC} \cap \overline{BC} = \overline{C} \).

2. For arbitrary \( A, B, C \subset M \), we say \( A \) and \( B \) are \( d \)-independent over \( C \) if for each \( A_0 \subset_{\text{fin}} A, B_0 \subset_{\text{fin}} B, A_0 \downarrow^d_C B_0 \)

Note that for closed sets \( A, B, A \) and \( B \) are \( d \)-independent over \( A \cap B \) if and only if for each \( A_0 \subset_{\text{fin}} A, B_0 \subset_{\text{fin}} B, d(A_0/B_0(A \cap B)) = d(A_0/A \cap B) \).
Definition 11 Let $A$ and $B$ be closed subsets of $M$. Then we say $A$ and $B$ are $d^*$-independent over $A \cap B$ if the following conditions are satisfied: there exist a hyperfinite extension $A^*$ of $A$ and a hyperfinite extension $B^*$ of $B$ such that

- $A^*$ and $B^*$ are both closed
- $d(A^*/B^*) = d(A^*/A^* \cap B^*)$

Wagner's definition of DS (a sufficient condition for saturated $M$ to be stable) is as follows:
For any closed $A, B$, if $\forall n \in \omega, \forall A_0 \subseteq_{\text{fin}} A, \forall B_0 \subseteq_{\text{fin}} B$, $A_0 \subset \exists A' \leq_{\text{fin}} A, B_0 \subset \exists B' \leq_{\text{fin}} B$ such that
\[
d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n,
\]
then $A$ and $B$ are free over $A \cap B$ and $AB$ is closed.

On the other hands, Wagner's definition of DW (a sufficient condition for saturated $M$ to be $\omega$-stable) is as follows:

- for any closed $A, B$, if $A \downarrow_{A \cap B}^d B$, then $A$ and $B$ are free over $A \cap B$ and $AB$ is closed and
- for any $\overline{a}$ and $X$, there exists finite $X_0 \subseteq X$ such that $d(\overline{a}/X_0) = d(\overline{a}/X)$.

Theorem 12 For arbitrary closed $A, B$, the following are equivalent:

1. $\forall n \in \omega, \forall A_0 \subseteq_{\text{fin}} A, \forall B_0 \subseteq_{\text{fin}} B$, $A_0 \subset \exists A' \leq_{\text{fin}} A, B_0 \subset \exists B' \leq_{\text{fin}} B$ such that $d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n$

2. $A \downarrow_{A \cap B}^d B$

3. $A \downarrow_{A \cap B} d$

Proof: (1 $\rightarrow$ 2): Assume 1. Then by saturatedness, There exist a closed hyperfinite extension $A^*$ of $A$ and a closed hyperfinite extension $B^*$ of $B$ such that for all $n \in \omega$,
\[
d^*(A^*) + d^*(B^*) \leq d^*(A^*B^*) + d^*(A^* \cap B^*) + 1/n.
\]
The other direction
\[
d^*(A^*) + d^*(B^*) \geq d^*(A^*B^*) + d^*(A^* \cap B^*)
\]
is clear by monotonicity.
So we have

\[ d^*(A^*) + d^*(B^*) \approx d^*(A^*B^*) + d^*(A^* \cap B^*), \]
equivalently,

\[ d^*(A^*/B^*) \approx d^*(A^*/A^* \cap B^*). \]

(2 \rightarrow 1): Fix any \( n \in \omega, A_0 \subset \text{fin} A, \) and \( B_0 \subset \text{fin} B. \) Let \( A^* \supset hf A \) and \( B^* \supset hf B \) be a witness of \( d^* \)-independent. By the finite closure condition, we can take \( A^* \) and \( B^* \) to be both closed. Then \( A^* \) and \( B^* \) satisfy the following formula:

- \( A_0 \subset \exists A^* \leq \text{fin} A, \) \( B_0 \subset \exists B^* \leq \text{fin} B, \) and
- \( d(A^*) + d(B^*) \leq d(A^*B^*) + d(A^* \cap B^*) + 1/n. \)

Because \( M \) is an elementary substructure of \( M^* \), we can take expected sets.

(2 \rightarrow 3): Let \( A^* \) and \( B^* \) be witness of \( d^*-\)independence. Take any \( A' \subset \text{fin} A \) and any \( B' \subset \text{fin} B. \) Then \( d(A^*/B^*) \approx d(A^*/A^* \cap B^*). \) By transposition, \( d(B^*/A^*) \approx d(B^*/A^* \cap B^*). \) By monotonicity of \( d, \) \( d(B^*/A'^*A^* \cap B^*) \approx d(B^*/A^* \cap B^*). \) By transposition, \( d(A'^*/B^*) \approx d(A'^*/A^* \cap B^*). \) By Monotonicity, \( d(A'^*/B'A^* \cap B^*) \approx d(A'^*/A^* \cap B^*). \) By Lemma 9, \( d(A'^*/B'A \cap B) = d(A'/A \cap B). \)

(3 \rightarrow 2): Take a closed hyperfinite extension \( A^* \) of \( A \) and a closed hyperfinite extension \( B^* \) of \( B. \) By compactness, it is enough to prove that for any \( A_0 \subset \text{fin} A, \) the following set of formulas are satisfiable:

1. \( X \in F \)
2. \( X \subseteq A \)
3. \( A_0 \subseteq X \)
4. \( X \) is closed
5. \( d(X/B^*) \approx d(X/X \cap B^*) \)

We show \( A_0^* = A_0(A^* \cap B^*) \) is a realization of the above set of formulas.
1, 2, 3, and 4 are clear.
5. First,

\[
\begin{align*}
d(A_0^*/B^*) &= d(A_0^*B^*) - d(B^*) \\
&= d(A_0B^*) - d(B^*) \\
&= d(A_0/B^*) \\
&\approx d(A_0/B).
\end{align*}
\]
Second,
\[
d(A_0^*/A_0^{*} \cap B^{*}) = d(A_0^*) - d(A_0^{*} \cap B^{*})
\]
\[
= d(A_0(A^* \cap B^{*})) - d(A_0^{*} \cap B^{*})
\]
\[
\leq d(A_0(A^* \cap B^{*})) - d(A^* \cap B^{*})
\]
\[
= d(A_0/A^* \cap B^{*})
\]
\[
\approx d(A_0/A \cap B)
\]

Finally, by the $d$-independence of $A$ and $B$, $d(A_0/B) = d(A_0/A \cap B)$. Hence, $d(A_0^*/A_0^{*} \cap B^{*}) \leq d(A_0^*/B^*)$. The other direction is clear.

Consequence

DS is equivalent to the first condition of DW. In particular, DW is a stronger condition than DS.

Fact 13 [3] Let $T$ be stable. Then the following are equivalent:

1. $T$ is superstable.

2. For any $B \subseteq \mathcal{M}$ and $p \in S(B)$, there is finite $A \subseteq B$ such that $p$ does not fork over $A$.

So, we have the following corollary.

Corollary 14 Suppose DS and that for any closed set $A, B$, $A \perp^d_{A \cap B} B$ if and only if $A \perp_{A \cap B} B$. Then $T = \text{Th}(M)$ is $\omega$-stable or merely stable.

This corollary is a partial solution of Baldwin’s problem[1].

References


