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Nonstandard arguments and the stability of generic structures

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Abstract

Generic 構造の研究に超準的手法を導入する．本稿では特に，Wagner が行った generic 構造の安定性の強さについての研究 [4] に，弱冠の新しい結果を付け足す．Wagner は saturated な generic 構造が安定になる為の十分条件 DS と ω-安定になる為の十分条件 DW を定義した．本稿では DS を簡略化し，DS と DW の間の関係を調べる．

1 Preliminaries

Let L be a countable relational language. Let $\mathbb{K}$ be a nonempty class of finite L-structures closed under isomorphisms and substructures (we consider the emptyset as an L-structure). Suppose $A \leq B$ is a reflexive and transitive relation on elements $A \subseteq B$ of $\mathbb{K}$, which is invariant under isomorphisms. If $A \leq B$ holds, we say that A is closed in B. We also assume that $(\mathbb{K}, \leq)$ satisfies the following properties:

1. $\emptyset \leq A$,
2. $A \subseteq B \subseteq C, A \leq C \Longrightarrow A \leq B$,
3. $A \leq B \Longrightarrow A \cap C \leq B \cap C$.

Let $(\mathbb{K}, \leq)$ be as above. Let N be an L-structure whose any finite substructure belongs to $\mathbb{K}$. Note that for any $A \subseteq N$, there is a unique smallest closed superset of A in N. We call this set the closure of A.

Definition 1 *Let $A \subseteq B$. We say that B is a minimal extension of A if the following conditions are satisfied*

- $A \not\leq B$

- *$A \leq B'$ for any $A \subseteq B' \subset B$.*

Definition 2 *Let $\leq$ be a closed relation on $\mathbb{K}$. Then we say that $(\mathbb{K}, \leq)$ satisfies finite closure axiom if there is no infinite chain $(A_i)_{i<\omega}$ of elements of $\mathbb{K}$ such that A_{i+1} is a minimal extension of A_i for each $i < \omega$.*

We assume that $(\mathbb{K}, \leq)$ satisfies the finite closure axiom in this paper. We say that an L-structure N has finite closures if for any finite $A \subseteq N$, the closure of A is also finite. Put $\bar{\mathbb{K}} = \{N : L\text{-structure} \mid A \in \mathbb{K} \text{ for any } A \subset_{\text{fin}} N\}$.

Fact 3 *[2] Let $\leq$ be a closed relation on $\mathbb{K}$. Then the following are equivalent:*

1. *$\mathbb{K}$ satisfies finite closure axiom.*
2. *Every member of $\bar{\mathbb{K}}$ has finite closures.*
3. *Every ω-saturated member of $\bar{\mathbb{K}}$ has finite closures.*
4. *Some ω-saturated member of $\bar{\mathbb{K}}$ has finite closures.*

Definition 4 *Let M be an L-structure. We say that M is a $\mathbb{K}$-generic structure if the following conditions are satisfied:*

1. *M is countable.*
2. *$\forall A \subset_{fin} M, A \in \mathbb{K}$ (i.e. $M \in \bar{\mathbb{K}}$).*
3. *$A \leq M, A \leq B \in \mathbb{K} \Rightarrow \exists B' \leq M$ such that $B' \cong_A B$.*

Fact 5 *Suppose that $(\mathbb{K}, \leq)$ satisfies the finite closure axiom. Then a $\mathbb{K}$-generic structure is unique.*

Definition 6 *Let d be a function from $\{A : A \leq_{\text{fin}} M\}$ to $\mathbb{R}_{\geq 0}$. We say d is a dimension function for M if for all $A, B \leq_{\text{fin}} M$,*

1. *$A \subset B \Longrightarrow d(A) \leq d(B)$*
2. *(Monotonicity) $d(\overline{A \cup B}) + d(A \cap B) \leq d(A) + d(B)$*
3. *$A \cong B \Longrightarrow d(A) = d(B)$*

For arbitrary $A \subset_{\text{fin}} M$, we put $d(A) = d(\overline{A})$. We define $d(A/B)$ the relative dimension of A over B. For finite A, B, $d(A/B) = d(AB) - d(B)$. For finite A, arbitrary B, $d(A/B) = \inf\{d(A/B_0) : B_0 \subset_{\text{fin}} B\}$. It is easy to check that these two definitions has the same value in the case A and B are finite.

2 Nonstandard arguments

Let M be the $\mathbb{K}$-generic structure and d be a dimension function for M. We consider M to be a 3-sorted structure

$$(M \cup P \cup \mathbb{R}; F, \in, d \leq, \cdots)$$

where P, F, $\in$ are as above, d is the dimension function of M, $\leq$ is the closed relation on $P \times P$.

We define the nonstandard model M^* of M by a sufficiently saturated extension of this structure

$$(M \cup P \cup \mathbb{R}, F, \in, d \leq, \cdots) \prec (M^* \cup P^* \cup \mathbb{R}^*, F^*, \in^*, d^*, \leq^* \cdots)$$

Definition 7 *A set $A \in F^*$ is said to be a hyperfinite set. For $A \subseteq M$, $A^* \in F^*$ is said to be a hyperfinite extension of A if*

- *$M^* \models a \in^* A^*$ for each $a \in A$, and*
- *$M^* \models A^* \subseteq^* A$.*

write $A \subset_{hf} A^*$, $A^* \supset_{hf} A$

By saturation, a hyperfinite extension of A always exists.

Lemma 8 *For any subseteq A of M, there exists a hyperfinite extension of A.*

Proof: It is enough to prove that the following set of formulas is satisfiable:

$$\Gamma(X) = \{a \in^* X | a \in A\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$

But for any finite subseteq A_0 of A, A_0 realizes the following set of formulas:

$$\{a \in^* X | a \in A_0\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$

So, by compactness, $\Gamma(X)$ is satisfiable.

Let x, y be two nonstandard (or standarad) real numbers. We write $x \approx y$ if $|x - y| < 1/n$ for each $n \in \omega$.

Lemma 9 *For $r \in \mathbb{R}$, $\bar{a} \in M$ and $A \subset M$, the following are equivalent.*

1. *$d(\bar{a}/A) = r$;*

2. $d^*(\bar{a}/A^*) \approx r$ *for any* $A^* \supset_{hf} A$;

3. $d^*(\bar{a}/A^*) \approx r$, *for some* $A^* \supset_{hf} A$.

Proof: $(1 \to 2)$: By monotonicity of d, there are $A_n \subset_{\text{fin}} A$ $(n = 1, 2, ...)$ such that $\forall X \in F$

$$A_n \subset X \subset A \to r \leq d(\bar{a}/X) \leq r + 1/n.$$

These statements hold also in M^*. So if A^* is a hyperfinite extension of A, then we have

$$r \leq d^*(\bar{a}/A^*) \leq r + 1/n \quad (n = 1, 2, ...)$$

So we have $d^*(\bar{a}/A^*) \approx r$.

$(2 \to 3)$: trivial.

$(3 \to 1)$: We assume 3 and choose a witness A^*. Then $(d^*(\bar{a}/A^*) \approx r)$. Suppose 1 is not the case. Then there is $s \neq r$ such that $d(\bar{a}/A) = s$. By $1 \Rightarrow 2$, we have $d^*(\bar{a}/A^*) \approx s$. A contradiction.

Note that $M \models \forall A \in P \exists! \overline{A}\ (A \subseteq \overline{A} \leq M \wedge \forall X A \subseteq X \leq M \to \overline{A} \subseteq X)$. This formula holds also in M^*. For $X \in P^*$, we write $\overline{X}$ as the "closure" of X in M^*. In this paper, $M \models X \in F^* \to \overline{X} \in F^*$ because $\mathbb{K}$ satisfies the finite closure condition.

3 Main result

Definition 10 *([4])*

1. *Let* $A, B \subset_{fin} M$ *and* $C \subset M$. *Then we say* A *and* B *are d-independent over* C *and write* $A \perp^d_C B$ *if the following conditions are satisfied:*
   - $d(A/BC) = d(A/C)$, *and*
   - $\overline{AC} \cap \overline{BC} = \overline{C}$.

2. *For arbitrary* $A, B, C \subset M$, *we say* A *and* B *are d-independent over* C *if for each* $A_0 \subset_{fin} A, B_0 \subset_{fin} B$, $A_0 \perp^d_C B_0$

Note that for closed sets A, B, A and B are d-independent over $A \cap B$ if and only if for each $A_0 \subset_{\text{fin}} A, B_0 \subset_{\text{fin}} B$, $d(A_0/B_0(A \cap B)) = d(A_0/A \cap B)$.

Definition 11 *Let A and B be closed subsets of M. Then we say A and B are d^*-independent over $A \cap B$ if the following conditions are satisfied: there exist a hyperfinite extension A^* of A and a hyperfinite extension B^* of B such that*

- *A^* and B^* are both closed*
- *$d(A^*/B^*) = d(A^*/A^* \cap B^*)$*

Wagner's definition of DS (a sufficient condition for saturated M to be stable) is as follows:
For any closed A, B, if $\forall n \in \omega, \forall A_0 \subset_{\text{fin}} A, \forall B_0 \subset_{\text{fin}} B$, $A_0 \subset \exists A' \leq_{\text{fin}} A, B_0 \subset \exists B' \leq_{\text{fin}} B$ such that

$$d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n,$$

then A and B are free over $A \cap B$ and AB is closed.

On the other hands, Wagner's definition of DW (a sufficient condition for saturated M to be ω-stable) is as follows:

- for any closed A, B, if $A \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{A \cap B} B$, then A and B are free over $A \cap B$ and AB is closed and
- for any $\bar{a}$ and X, there exists finite $X_0 \subseteq X$ such that $d(\bar{a}/X_0) = d(\bar{a}/X)$.

Theorem 12 *For arbitrary closed A, B, the following are equivalent:*

1. *$\forall n \in \omega$, $\forall A_0 \subset_{\text{fin}} A$, $\forall B_0 \subset_{\text{fin}} B$, $A_0 \subset \exists A' \leq_{\text{fin}} A$, $B_0 \subset \exists B' \leq_{\text{fin}} B$ such that $d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n$*
2. *$A \mathop{\smile\!\!\!\!\!\!\mid}^{d^*}_{A \cap B} B$*
3. *$A \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{A \cap B} B$*

Proof: (1 $\to$ 2): Assume 1. Then by saturatedness, There exist a closed hyperfinite extension A^* of A and a closed hyperfinite extension B^* of B such that for all $n \in \omega$,

$$d^*(A^*) + d^*(B^*) \leq d^*(A^*B^*) + d^*(A^* \cap B^*) + 1/n.$$

The other direction

$$d^*(A^*) + d^*(B^*) \geq d^*(A^*B^*) + d^*(A^* \cap B^*)$$

is clear by monotonicity.

So we have

$$d^*(A^*) + d^*(B^*) \approx d^*(A^*B^*) + d^*(A^* \cap B^*),$$

equivalently,

$$d^*(A^*/B^*) \approx d^*(A^*/A^* \cap B^*).$$

$(2 \to 1)$: Fix any $n \in \omega$, $A_0 \subset_{\text{fin}} A$, and $B_0 \subset_{\text{fin}} B$. Let $A^* \supset_{\text{hf}} A$ and $B^* \supset_{\text{hf}} B$ be a wittness of d^*-independent. By the finite closure condition, we can take A^* and B^* to be both closed. Then A^* and B^* satisfy the following formula:

- $A_0 \subset \exists A^* \leq_{\text{fin}} A$, $B_0 \subset \exists B^* \leq_{\text{fin}} B$, and
- $d(A^*) + d(B^*) \leq d(A^*B^*) + d(A^* \cap B^*) + 1/n$.

Because M is an elementary substructure of M^*, we can take expected sets.

$(2 \to 3)$: Let A^* and B^* be witness of d^*-independence. Take any $A' \subset_{\text{fin}} A$ and any $B' \subset_{\text{fin}} B$. Then $d(A^*/B^*) \approx d(A^*/A^* \cap B^*)$. By transpositon, $d(B^*/A^*) \approx d(B^*/A^* \cap B^*)$. By monotonicity of d, $d(B^*/A'A^* \cap B^*) \approx d(B^*/A^* \cap B^*)$. By transposition, $d(A'/B^*) \approx d(A'/A^* \cap B^*)$. By Monotonicity, $d(A'/B'A^* \cap B^*) \approx d(A'/A^* \cap B^*)$. By Lemma 9, $d(A'/B'A \cap B) = d(A'/A \cap B)$.

$(3 \to 2)$: Take a closed hyperfinite extension A^* of A and a closed hyperfinite extension B^* of B. By compactness, it is enough to prove that for any $A_0 \subset_{\text{fin}} A$, the following set of formulas are satisfiable:

1. $X \in F$
2. $X \subseteq A$
3. $A_0 \subseteq X$
4. X is closed
5. $d(X/B^*) \approx d(X/X \cap B^*)$

We show $A_0^* = \overline{A_0(A^* \cap B^*)}$ is a realization of the above set of formulas. 1, 2, 3, and 4 are clear.

5. First,

$$\begin{aligned} d(A_0^*/B^*) &= d(A_0^*B^*) - d(B^*) \\ &= d(A_0B^*) - d(B^*) \\ &= d(A_0/B^*) \\ &\approx d(A_0/B). \end{aligned}$$

Second,

$$\begin{aligned} d(A_0^*/A_0^* \cap B^*) &= d(A_0^*) - d(A_0^* \cap B^*) \\ &= d(A_0(A^* \cap B^*)) - d(A_0^* \cap B^*) \\ &\leq d(A_0(A^* \cap B^*)) - d(A^* \cap B^*) \\ &= d(A_0/A^* \cap B^*) \\ &\approx d(A_0/A \cap B) \end{aligned}$$

Finally, by the d-independence of A and B, $d(A_0/B) = d(A_0/A \cap B)$.
Hence, $d(A_0^*/A_0^* \cap B^*) \lesssim d(A_0^*/B^*)$. The other direction is clear.

Consequence
DS is equivalent to the first condition of DW. In particular, DW is a stronger condition than DS.

Fact 13 *[3] Let T be stable. Then the following are equivalent:*

1. *T is superstable.*
2. *For any $B \subset \mathcal{M}$ and $p \in S(B)$, there is finite $A \subseteq B$ such that p does not fork over A.*

So, we have the following corollary.

Corollary 14 *Suppose DS and that for any closed set A, B, $A \perp^d_{A \cap B} B$ if and only if $A \perp_{A \cap B} B$. Then $T = \mathrm{Th}(M)$ is ω-stable or merely stable.*

This corollary is a partial solution of Baldwin's problem[1].

References

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