Nonstandard arguments and the stability of generic structures

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Abstract


1 Preliminaries

Let L be a countable relational language. Let K be a nonempty class of finite L-structures closed under isomorphisms and substructures (we consider the emptyset as an L-structure). Suppose A ≤ B is a reflexive and transitive relation on elements A ⊆ B of K, which is invariant under isomorphisms. If A ≤ B holds, we say that A is closed in B. We also assume that (K, ≤) satisfies the following properties:

1. \( \emptyset \leq A \),
2. \( A \subseteq B \subseteq C, A \leq C \Rightarrow A \leq B \),
3. \( A \leq B \Rightarrow A \cap C \leq B \cap C \).

Let (K, ≤) be as above. Let N be an L-structure whose any finite substructure belongs to K. Note that for any \( A \subseteq N \), there is a unique smallest closed superset of A in N. We call this set the closure of A.

Definition 1 Let \( A \subseteq B \). We say that B is a minimal extension of A if the following conditions are satisfied

- \( A \not\leq B \)
• $A \leq B'$ for any $A \subseteq B' \subset B$.

**Definition 2** Let $\leq$ be a closed relation on $\mathbb{K}$. Then we say that $(\mathbb{K}, \leq)$ satisfies finite closure axiom if there is no infinite chain $(A_i)_{i<\omega}$ of elements of $\mathbb{K}$ such that $A_{i+1}$ is a minimal extension of $A_i$ for each $i < \omega$.

We assume that $(\mathbb{K}, \leq)$ satisfies the finite closure axiom in this paper. We say that an $L$-structure $N$ has finite closures if for any finite $A \subseteq N$, the closure of $A$ is also finite. Put $\overline{\mathbb{K}} = \{ N : \text{L-structure} \mid A \in \mathbb{K} \text{ for any } A \subset \text{fin} \ N \}$.

**Fact 3** [2] Let $\leq$ be a closed relation on $\mathbb{K}$. Then the following are equivalent:

1. $\mathbb{K}$ satisfies finite closure axiom.
2. Every member of $\overline{\mathbb{K}}$ has finite closures.
3. Every $\omega$-saturated member of $\overline{\mathbb{K}}$ has finite closures.
4. Some $\omega$-saturated member of $\overline{\mathbb{K}}$ has finite closures.

**Definition 4** Let $M$ be an $L$-structure. We say that $M$ is a $\mathbb{K}$-generic structure if the following conditions are satisfied:

1. $M$ is countable.
2. $\forall A \subset \text{fin} M, A \in \mathbb{K}$ (i.e. $M \in \overline{\mathbb{K}}$).
3. $A \leq M, A \leq B \in \mathbb{K} \Rightarrow \exists B' \leq M$ such that $B' \cong A B$.

**Fact 5** Suppose that $(\mathbb{K}, \leq)$ satisfies the finite closure axiom. Then a $\mathbb{K}$-generic structure is unique.

**Definition 6** Let $d$ be a function from $\{ A : A \leq \text{fin} M \}$ to $\mathbb{R}_{\geq 0}$. We say $d$ is a dimension function for $M$ if for all $A, B \leq \text{fin} M$,

1. $A \subset B \implies d(A) \leq d(B)$
2. (Monotonicity) $d(A \cup B) + d(A \cap B) \leq d(A) + d(B)$
3. $A \cong B \implies d(A) = d(B)$

For arbitrary $A \subset \text{fin} M$, we put $d(A) = d(\overline{A})$. We define $d(A/B)$ the relative dimension of $A$ over $B$. For finite $A, B$, $d(A/B) = d(AB) - d(B)$. For finite $A$, arbitrary $B$, $d(A/B) = \inf\{d(A/B_0) : B_0 \subset \text{fin} B\}$. It is easy to check that these two definitions has the same value in the case $A$ and $B$ are finite.
2 Nonstandard arguments

Let $M$ be the $\mathbb{K}$-generic structure and $d$ be a dimension function for $M$. We consider $M$ to be a 3-sorted structure

$$(M \cup P \cup \mathbb{R}; F, \in, d \leq, \cdots)$$

where $P$, $F$, $\in$ are as above, $d$ is the dimension function of $M$, $\leq$ is the closed relation on $P \times P$.

We define the nonstandard model $M^*$ of $M$ by a sufficiently saturated extension of this structure

$$(M \cup P \cup \mathbb{R}, F, \in, d \leq, \cdots) \prec (M^* \cup P^* \cup \mathbb{R}^*, F^*, \in^*, d^* \leq^*, \cdots)$$

**Definition 7** A set $A \in F^*$ is said to be a hyperfinite set. For $A \subseteq M$, $A^* \in F^*$ is said to be a hyperfinite extension of $A$ if

- $M^* \models a \in^* A^*$ for each $a \in A$, and
- $M^* \models A^* \subseteq^* A$.

write $A \subset_{hf} A^*$, $A^* \supset_{hf} A$

By saturation, a hyperfinite extension of $A$ always exists.

**Lemma 8** For any subset $A$ of $M$, there exists a hyperfinite extension of $A$.

**Proof:** It is enough to prove that the following set of formulas is satisfiable:

$$\Gamma(X) = \{a \in^* X | a \in A\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$ 

But for any finite subset $A_0$ of $A$, $A_0$ realizes the following set of formulas:

$$\{a \in^* X | a \in A_0\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$ 

So, by compactness, $\Gamma(X)$ is satisfiable.

Let $x, y$ be two nonstandard (or standard) real numbers. We write $x \approx y$ if $|x - y| < 1/n$ for each $n \in \omega$.

**Lemma 9** For $r \in \mathbb{R}$, $\bar{a} \in M$ and $A \subseteq M$, the following are equivalent.

1. $d(\bar{a}/A) = r$;
2. \( d^* (\overline{a}/A^*) \approx r \) for any \( A^* \supset h_f A \);

3. \( d^* (\overline{a}/A^*) \approx r \), for some \( A^* \supset h_f A \).

**Proof:** (1 \( \Rightarrow \) 2): By monotonicity of \( d \), there are \( A_n \subset \text{fin} A \) (\( n = 1, 2, \ldots \)) such that \( \forall X \in F \)

\[
A_n \subset X \subset A \rightarrow r \leq d(\overline{a}/X) \leq r + 1/n.
\]

These statements hold also in \( M^* \). So if \( A^* \) is a hyperfinite extension of \( A \), then we have

\[
r \leq d^* (\overline{a}/A^*) \leq r + 1/n \ (n = 1, 2, \ldots)
\]

So we have \( d^* (\overline{a}/A^*) \approx r \).

(2 \( \Rightarrow \) 3): trivial.

(3 \( \Rightarrow \) 1): We assume 3 and choose a witness \( A^* \). Then \( d^* (\overline{a}/A^*) \approx r \).

Suppose 1 is not the case. Then there is \( s \neq r \) such that \( d(\overline{a}/A) = s \). By 1 \( \Rightarrow \) 2, we have \( d^* (\overline{a}/A^*) \approx s \). A contradiction.

Note that \( M \models \forall A \in P \exists! \overline{A} (A \subseteq \overline{A} \leq M \wedge \forall X A \subseteq X \leq M \rightarrow \overline{A} \subseteq X) \).

This formula holds also in \( M^* \). For \( X \in P^* \), we write \( \overline{X} \) as the "closure" of \( X \) in \( M^* \). In this paper, \( M \models X \in F^* \rightarrow \overline{X} \in F^* \) because \( K \) satisfies the finite closure condition.

## 3 Main result

**Definition 10 ([4])**

1. Let \( A, B \subset \text{fin} M \) and \( C \subset M \). Then we say \( A \) and \( B \) are \( d \)-independent over \( C \) and write \( A \downarrow^d_C B \) if the following conditions are satisfied:

   \[ \bullet \ d(A/BC) = d(A/C) \], and

   \[ \bullet \ \overline{AC} \cap \overline{BC} = \overline{C} \]

2. For arbitrary \( A, B, C \subset M \), we say \( A \) and \( B \) are \( d \)-independent over \( C \) if for each \( A_0 \subset \text{fin} A, B_0 \subset \text{fin} B, A_0 \downarrow^d_C B_0 \)

Note that for closed sets \( A, B \), \( A \) and \( B \) are \( d \)-independent over \( A \cap B \) if and only if for each \( A_0 \subset \text{fin} A, B_0 \subset \text{fin} B, d(A_0/B_0(A \cap B)) = d(A_0/A \cap B) \).
Definition 11 Let $A$ and $B$ be closed subsets of $M$. Then we say $A$ and $B$ are $d^*$-independent over $A \cap B$ if the following conditions are satisfied: there exist a hyperfinite extension $A^*$ of $A$ and a hyperfinite extension $B^*$ of $B$ such that

- $A^*$ and $B^*$ are both closed
- $d(A^*/B^*) = d(A^*/A^* \cap B^*)$

Wagner's definition of DS (a sufficient condition for saturated $M$ to be stable) is as follows:
For any closed $A, B$, if $\forall n \in \omega, \forall A_0 \subset_{\text{fin}} A, \forall B_0 \subset_{\text{fin}} B, A_0 \subset \exists A' \leq_{\text{fin}} A, B_0 \subset \exists B' \leq_{\text{fin}} B$ such that

$$d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n,$$

then $A$ and $B$ are free over $A \cap B$ and $AB$ is closed.

On the other hands, Wagner's definition of DW (a sufficient condition for saturated $M$ to be $\omega$-stable) is as follows:

- for any closed $A, B$, if $A \downarrow_{A \cap B}^d B$, then $A$ and $B$ are free over $A \cap B$ and $AB$ is closed and
- for any $\overline{a}$ and $X$, there exists finite $X_0 \subseteq X$ such that $d(\overline{a}/X_0) = d(\overline{a}/X)$.

Theorem 12 For arbitrary closed $A, B$, the following are equivalent:

1. $\forall n \in \omega, \forall A_0 \subset_{\text{fin}} A, \forall B_0 \subset_{\text{fin}} B, A_0 \subset \exists A' \leq_{\text{fin}} A, B_0 \subset \exists B' \leq_{\text{fin}} B$ such that $d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n$

2. $A \downarrow_{A \cap B}^{d^*} B$

3. $A \downarrow_{A \cap B}^{d} B$

Proof: $(1 \rightarrow 2)$: Assume 1. Then by saturatedness, There exist a closed hyperfinite extension $A^*$ of $A$ and a closed hyperfinite extension $B^*$ of $B$ such that for all $n \in \omega,$

$$d^*(A^*) + d^*(B^*) \leq d^*(A^*B^*) + d^*(A^* \cap B^*) + 1/n.$$

The other direction

$$d^*(A^*) + d^*(B^*) \geq d^*(A^*B^*) + d^*(A^* \cap B^*)$$
is clear by monotonicity.

So we have

\[ d^*(A^*) + d^*(B^*) \approx d^*(A^*B^*) + d^*(A^* \cap B^*), \]

equivalently,

\[ d^*(A^*/B^*) \approx d^*(A^*/A^* \cap B^*). \]

\((2 \to 1)\): Fix any \( n \in \omega, A_0 \subset \text{fin} A, \) and \( B_0 \subset \text{fin} B. \) Let \( A^* \supset hf A \) and \( B^* \supset hf B \) be a witness of \( d^* \)-independent. By the finite closure condition, we can take \( A^* \) and \( B^* \) to be both closed. Then \( A^* \) and \( B^* \) satisfy the following formula:

- \( A_0 \subset \exists A^* \leq \text{fin} A, \) \( B \subset \exists B^* \leq \text{fin} B, \) and

- \( d(A^*) + d(B^*) \leq d(A^*B^*) + d(A^* \cap B^*) + 1/n. \)

Because \( M \) is an elementary substructure of \( M^* \), we can take expected sets.

\((2 \to 3)\): Let \( A^* \) and \( B^* \) be witness of \( d^* \)-independence. Take any \( A' \subset \text{fin} A \) and any \( B' \subset \text{fin} B. \) Then \( d(A'/B^*) \approx d(A^*/A^* \cap B^*). \) By transposition, \( d(B^*/A^*) \approx d(B^*/A^* \cap B^*). \) By monotonicity of \( d, \) \( d(B^*/A^* \cap B^*) \approx d(B^*/A^* \cap B^*). \) By transposition, \( d(A'/B^*) \approx d(A^*/A^* \cap B^*). \) By Monotonicity, \( d(A'/B^*/A^* \cap B^* \approx d(A'/A^* \cap B^*). \) By Lemma 9, \( d(A'/B'A \cap B) = d(A'/A \cap B). \)

\((3 \to 2)\): Take a closed hyperfinite extension \( A^* \) of \( A \) and a closed hyperfinite extension \( B^* \) of \( B. \) By compactness, it is enough to prove that for any \( A_0 \subset \text{fin} A, \) the following set of formulas are satisfiable:

1. \( X \in F \)
2. \( X \subseteq A \)
3. \( A_0 \subseteq X \)
4. \( X \) is closed
5. \( d(X/B^*) \approx d(X/X \cap B^*) \)

We show \( A_0^* = A_0(A^* \cap B^* \cap \text{fin} A) \) is a realization of the above set of formulas. 1, 2, 3, and 4 are clear.

5. First,

\[ d(A_0^*/B^*) = d(A_0^*B^*) - d(B^*) \]
\[ = d(A_0B^*) - d(B^*) \]
\[ = d(A_0/B^*) \]
\[ \approx d(A_0/B). \]
Second,

\[
d(A_0^*/A_0^* \cap B^*) = d(A_0^*) - d(A_0^* \cap B^*)
\]

\[
= d(A_0(A^* \cap B^*)) - d(A_0^* \cap B^*)
\]

\[
\leq d(A_0(A^* \cap B^*)) - d(A^* \cap B^*)
\]

\[
= d(A_0/A^* \cap B^*)
\]

\[
\approx d(A_0/A \cap B)
\]

Finally, by the \(d\)-independence of \(A\) and \(B\), \(d(A_0/B) = d(A_0/A \cap B)\).
Hence, \(d(A_0^*/A_0^* \cap B^*) \leq d(A_0^*/B^*)\). The other direction is clear.

Consequence
DS is equivalent to the first condition of DW. In particular, DW is a stronger condition than DS.

Fact 13 [3] Let \(T\) be stable. Then the following are equivalent:

1. \(T\) is superstable.
2. For any \(B \subset \mathcal{M}\) and \(p \in S(B)\), there is finite \(A \subseteq B\) such that \(p\) does not fork over \(A\).

So, we have the following corollary.

Corollary 14 Suppose DS and that for any closed set \(A, B\), \(A \vdash^d_{A \cap B} B\) if and only if \(A \vdash^d_{A \cap B} B\). Then \(T = \text{Th}(M)\) is \(\omega\)-stable or merely stable.

This corollary is a partial solution of Baldwin’s problem[1].

References


