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Minimax approach to sequential Bernoulli trials

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Abstract

In two Bernoulli sequences each with the probability p_i of success for $i = 1, 2$, we choose either sequence based on the previous observations. We consider the problem to maximize the expectation of the number of success. In this paper we obtain a minimax procedure which minimizes the maximum regret for all possible values of parameters among the class of rather simple procedures. The numerical treatment is also given.

1. Introduction

Suppose that there are two Bernoulli sequences $X_{11}, X_{12}, \dots, X_{1n}, \dots; X_{21}, X_{22}, \dots, X_{2n}, \dots$ each with probability of success $P\{X_{i1} = 1\} = p_i$, $i = 1, 2$. And we consider to choose some procedure N times as follows. For $j = 1, \dots, N$, let Y_j be a random variable defined by

$$Y_j = \begin{cases} 1, & \text{if one takes the procedure 1 at the } j\text{-th trial,} \\ 0, & \text{if one takes the procedure 2 at the } j\text{-th trial.} \end{cases}$$

Then we want to maximize

$$T := \sum_{j=1}^{N} \{Y_j X_{1j} + (1 - Y_j) X_{2j}\}. \tag{1.1}$$

This is a case of sequential medical trial formulated by Armitage (1975). The same case is sometimes called the two-armed bandit problem. There has been a substantial amount of literature published since 1950's (see, *e.g.* Maurice (1959), Anscombe (1963), Chernoff (1967), Armitage (1985), Bather (1981, 1985), Bather and Coad (1992), Bather

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and Simons (1983)). But it seems that the definitive answer to the problem is yet to be established.

In this paper we shall obtain a minimax procedure which minimizes the maximum regret for all possible values of parameters among the class of rather simple procedures. We shall discuss two types of procedures, one the fixed sample procedure, where we fix n and observe $X_{11}, \ldots, X_{1n}; X_{2,n+1}, \ldots, X_{2,2n}$, and for the remaining $(N-2n)$ cases we observe all from one of the sequence depending on $\sum_{j=1}^{n} X_{1j} \lesseqgtr \sum_{j=n+1}^{2n} X_{2j}$. And we seek for such n depending N which minimizes the maximum regret for all values of p_1 and p_2. The second procedure is such that we make pairs of observations $(X_{1,2j-1}, X_{2,2j})$, $j = 1, 2, \ldots$ sequentially and when $\sum X_{1,2j-1} - \sum X_{2,2j}$ reaches either k or $-k$, we stop the paired observations and the rest are all taken from either of the sequences. We want to choose such k depending on N which minimizes the maximum regret. It is shown that the second procedure is superior to the first, because the minimax regret for the second procedure is $0.375N^{1/2}$, where N is large, while for the first it is $0.246N^{2/3}$. These approximations are actually quite accurate for N not necessarily large, *e.g.* $N = 100$. Lastly it will be shown that for any sequential procedure the minimax regret can not be smaller than the magnitude of order $N^{1/2}$, and one lower bound is given by $0.2649N^{1/2}$ (which is not sharp). The results of this paper are mostly analogous to that of Bather and Simons (1985), but the approach is not the same.

2. Minimax regret solution for a fixed sample procedure

Now, let us consider the fixed sample procedure. We are to take first n observations from the first population and then next n observations from the second, and compare $\sum_{j=1}^{n} X_{1j}$ and $\sum_{j=n+1}^{2n} X_{2j}$ and if the former is larger than the latter, we take the remaining from the first population, and if the latter is larger from the second, and if both happen to be equal, we choose between the two populations randomly with equal probability. Then the expected number of success is expressed as

$$E(T) = n(p_1+p_2)+(N-2n)\left\{p_1\left(P\left\{\sum_{j=1}^{n} X_{1j} > \sum_{j=n+1}^{2n} X_{2j}\right\} + \frac{1}{2}P\left\{\sum_{j=1}^{n} X_{1j} = \sum_{j=n+1}^{2n} X_{2j}\right\}\right)\right.$$
$$\left. + p_2\left(P\left\{\sum_{j=1}^{n} X_{1j} < \sum_{j=n+1}^{2n} X_{2j}\right\} + \frac{1}{2}P\left\{\sum_{j=1}^{n} X_{1j} = \sum_{j=n+1}^{2n} X_{2j}\right\}\right)\right\}.$$

For the sake of simplicity of notation, we write

$$\tilde{P}\left\{\sum_{j=1}^{n} X_{1j} \geq \sum_{j=n+1}^{2n} X_{2j}\right\} := P\left\{\sum_{j=1}^{n} X_{1j} > \sum_{j=n+1}^{2n} X_{2j}\right\} + \frac{1}{2}P\left\{\sum_{j=1}^{n} X_{1j} = \sum_{j=n+1}^{2n} X_{2j}\right\}$$

Now the maximum possible expected number of success is $N\max(p_1,p_2)$, hence the regret is defined as

$$R := N\max(p_1,p_2) - E(T).$$

It follows that for $p_1 > p_2$

$$R = \triangle \left\{ n + (N-2n)\tilde{P}\left\{ \sum_{j=1}^{n} X_{1j} \le \sum_{j=n+1}^{2n} X_{2j} \right\} \right\},$$

where $\triangle = p_1 - p_2$. We want to choose n so that $\sup_{p_1,p_2} R$ is minimized. Assuming that n is large, we can approximate the distribution of $\sum_{j=1}^{n} X_{1j} - \sum_{j=n+1}^{2n} X_{2j}$ by the normal distribution, and we get

$$\begin{aligned}
&\tilde{P}\left\{ \sum_{j=1}^{n} X_{1j} \le \sum_{j=n+1}^{2n} X_{2j} \right\} \\
&= P\left\{ \sum_{j=1}^{n} X_{1j} < \sum_{j=n+1}^{2n} X_{2j} \right\} + \frac{1}{2}P\left\{ \sum_{j=1}^{n} X_{1j} = \sum_{j=n+1}^{2n} X_{2j} \right\} \\
&\approx \Phi\left(n(p_2 - p_1)/\sqrt{np_1(1-p_1) + np_2(1-p_2)} \right) \\
&= 1 - \Phi\left(\sqrt{n}\triangle/\sqrt{p_1(1-p_1) + p_2(1-p_2)} \right), \qquad (2.1)
\end{aligned}$$

where Φ is the standard normal distribution function. Note that by the definition of $\tilde{P}$, as above, the continuity correction is not required here. Accuracy of the approximations will be checked below and it will be shown that the approximation does not affect the result. Since

$$\begin{aligned}
p_1(1-p_1) + p_2(1-p_2) &= p_1 + p_2 - (p_1^2 + p_2^2) \\
&= \frac{1}{2} - \frac{1}{2}(p_1 + p_2 - 1)^2 - \frac{1}{2}(p_1 - p_2)^2 \\
&\le \frac{1}{2}(1 - \triangle^2)
\end{aligned}$$

for given $\triangle$, the denominator of $\Phi(\cdot)$ in (2.1) is maximized when $p_1 + p_2 = 1$, hence we have

$$\begin{aligned}
&\sup_{p_1 > p_2} R \\
&\approx \triangle \left[n + (N-2n)\left\{ 1 - \Phi\left(\sqrt{\frac{2n}{1-\triangle}}\triangle \right) \right\} \right] =: R_\triangle \text{ (say)}. \qquad (2.2)
\end{aligned}$$

The same applies to the case when $p_1 < p_2$. Now, for fixed n, in order to calculate the maximum value of the right-hand side(RHS) of (2.2) with respect to $\triangle$, we have

$$\begin{aligned}\frac{dR_\triangle}{d\triangle} &= n + (N-2n)\left\{1-\Phi\left(\frac{\sqrt{2n}\triangle}{\sqrt{1-\triangle^2}}\right)\right\} \\ &\quad - (N-2n)\triangle\left\{\frac{d}{d\triangle}\left(\frac{\sqrt{2n}\triangle}{\sqrt{1-\triangle^2}}\right)\right\}\phi\left(\frac{\sqrt{2n}\triangle}{\sqrt{1-\triangle^2}}\right) \\ &= n + (N-2n)\left\{1-\Phi\left(\frac{\sqrt{2n}\triangle}{\sqrt{1-\triangle^2}}\right)\right\} \\ &\quad - (N-2n)\frac{1}{1-\triangle^2}\cdot\frac{\sqrt{2n}\triangle}{\sqrt{1-\triangle^2}}\phi\left(\frac{\sqrt{2n}\triangle}{\sqrt{1-\triangle^2}}\right),\end{aligned}$$

where $\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. If we transform as $\xi = \sqrt{2n}\triangle/\sqrt{1-\triangle^2}$ and denote $R_\triangle$ by $R(\xi)$ or $R_{\triangle,n}$, we obtain

$$\frac{dR(\xi)}{d\triangle} = n + (N-2n)\{1-\Phi(\xi)\} - (N-2n)\left(1+\frac{\xi^2}{2n}\right)\xi\phi(\xi),$$

since

$$\frac{1}{1-\triangle^2} = 1 + \frac{\triangle^2}{1-\triangle^2} = 1 + \frac{\xi^2}{2n}.$$

Hence, by putting $f := N/n$, we have

$$\frac{1}{N-2n}\frac{dR(\xi)}{d\triangle} = \frac{1}{f-2} + 1 - \Phi(\xi) - \left(1+\frac{\xi^2}{2n}\right)\xi\phi(\xi).$$

Now define

$$H(\xi) = 1 - \Phi(\xi) - \left(1+\frac{\xi^2}{2n}\right)\xi\phi(\xi).$$

Then we have

$$\begin{aligned}H'(\xi) &= -\phi(\xi) + \left(\frac{\xi^4}{2n} - \frac{3\xi^2}{2n} + \xi^2 - 1\right)\phi(\xi) \\ &= \left\{\frac{\xi^4}{2n} - \left(1+\frac{3}{2n}\right)\xi^2 - 2\right\}\phi(\xi).\end{aligned}$$

Since the equation

$$\frac{\xi^4}{2n} - \left(1+\frac{3}{2n}\right)\xi^2 - 2 = 0$$

has one positive and one negative roots in ξ^2, we have one positive root in ξ, which is denoted as ξ_0. Then, for $\xi > \xi_0$, $H'(\xi) > 0$, and for $0 \le \xi < \xi_0$, $H'(\xi) < 0$, hence $H(\xi)$ is minimized in the range of $\xi > 0$ at $\xi = \xi_0$, and it is easily shown that $H(0) > 0$ and $H(\infty) := \lim_{\xi\to\infty} H(\xi) = 0$. Therefore, if

$$H(\xi_0) < -1/(f-2),$$

we have two roots ξ_1 and ξ_2 with $\xi_1 < \xi_0 < \xi_2$ for the equation $dR_\triangle/d\triangle = 0$, and it is shown that $R_\triangle$ is locally maximized when $\xi = \xi_1$. And, if $H(\xi) > -1/(f-2)$, $dR_{\triangle,n}/d\triangle$ is always positive, hence $R_{\triangle,n}$ is maximized when $\triangle = 1$, *i.e.* $\xi = \infty$ and $\max_\triangle R_\triangle = n$. And, in the case $H(\xi) < -1/(f-2)$ we have

$$\sup_\triangle R_\triangle = \max\{R(\infty), R(\xi_1)\} = \max\{n, R(\xi_1)\}.$$

The condition $R(\xi_1) \gtreqless n$ is equivalent to

$$\triangle\{1 + (f-2)(1 - \Phi(\xi_1))\} \gtreqless 1,$$

Where $\triangle = (\xi_1/\sqrt{2n})/\sqrt{1 + (\xi_1^2/(2n))}$. Since

$$\triangle = \frac{\xi}{\sqrt{2n}} \Big/ \sqrt{1 + \frac{\xi^2}{2n}},$$

it follows that a sufficient condition for $R(\xi_1)$ to be the maximum is

$$1 + (f-2)(1 - \Phi(\xi_1)) \geq \sqrt{2n + \xi_1^2} \Big/ \xi_1. \tag{2.3}$$

Since ξ_1 satisfies $H(\xi_1) = -1/(f-2)$, the left-hand side(LHS) of (2.3) is equal to

$$(f-2)\left(1 + \frac{\xi_1^2}{2n}\right)\xi_1\phi(\xi_1),$$

hence

$$\frac{\xi_1^2}{2n}\sqrt{2n + \xi_1^2}\ \phi(\xi_1) \geq \frac{1}{f-2} = \left(1 + \frac{\xi_1^2}{2n}\right)\xi_1\phi(\xi_1) - \{1 - \Phi(\xi_1)\}. \tag{2.4}$$

Since

$$\frac{d}{d\xi}\xi^2\phi(\xi) = \xi(2 - \xi^2)\phi(\xi),$$

it follows that, for $\xi_1 < \sqrt{2}$, the LHS of (2.4) is increasing and the RHS of (2.4) is decreasing. Let ξ_1^* be a solution of ξ of the equation

$$\frac{\xi^2}{2n}\sqrt{2n + \xi^2}\ \phi(\xi) = \left(1 + \frac{\xi^2}{2n}\right)\xi\phi(\xi) - \{1 - \Phi(\xi)\}.$$

Then, $R(\xi_1) > n$ for $\xi_1 < \xi_1^*$, $R(\xi_1) < n$ for $\xi_1 > \xi_1^*$ and $R(\xi_1^*) = n$. On the other hand, the relation between n and ξ_1 is

$$\triangle + \triangle(f-2)\{1 - \Phi(\xi_1)\} = 1$$
$$\left(\frac{1}{1 - \triangle^2}\right)\xi_1\phi(\xi_1) - \{1 - \Phi(\xi_1)\} = \frac{1}{f-2},$$

where $\Delta = \xi_1/\sqrt{2n+\xi_1^2}$. Then we have

$$\frac{1}{f-2} + 1 - \Phi(\xi_1) = \frac{1}{1-\Delta^2}\,\xi_1\phi(\xi_1) = \frac{1}{\Delta(f-2)},$$

which implies

$$\frac{1}{f-2} = \frac{\Delta}{1-\Delta^2}\,\xi_1\phi(\xi_1) = \frac{1}{1-\Delta^2}\,\xi_1\phi(\xi_1) - \{1-\Phi(\xi_1)\}.$$

Since

$$\frac{1}{1+\Delta}\,\xi_1\phi(\xi_1) = 1 - \Phi(\xi_1),$$

it follows that

$$\Delta = \frac{\xi_1\phi(\xi_1)}{1-\Phi(\xi_1)} - 1.$$

Hence we obtain Δ from given ξ_1, get f from Δ, and get n and N from f. This means that the relation between n and N is determined through ξ_1. The correspondences $n = n(\xi_1^*)$ and $f = f(\xi_1^*)$ to ξ_1^* is given by Table 1. Since $\partial R/\partial n < 0$ in such a region of n and f, in a similar way to the above we have

$$\frac{\partial}{\partial n}\sup_{\Delta} R_{\Delta} = \left.\frac{\partial R}{\partial n}\right|_{\Delta=\Delta^*(n)} < 0.$$

If for given N we choose ξ_1^* such that $N = n^* f^*$ with the correspondences of $n^* = n(\xi_1^*)$ and $f^* = f(\xi_1^*)$, then n^* and f^* give a minimax solution of R. Since n must be an integer, we shall obtain the integer close to n^*. If there exists an integer n_0^* near to n^* such that

$$n_0^* \le \sup_{\Delta} R_{\Delta,n_0^*} < n_0^* + 1,$$

then it is minimax. Indeed,

$$\begin{aligned} &\sup_{\Delta} R_{\Delta,n} \ge R_{1,n} = n > n_0^* && \text{for } n > n_0^*, \\ &\sup_{\Delta} R_{\Delta,n} > \sup_{\Delta} R_{\Delta,n_0^*} && \text{for } n < n_0^*. \end{aligned}$$

Summarizing the above, we have the rule as follows. *For fixed N, find the value n^* corresponding to N from Table 1 (use interpolation if necessary). Take the integer n_0^* closest to n^*. If the condition*

$$n_0^* \le \sup_{\Delta} R_{\Delta,n_0^*} < n_0^* + 1$$

is satisfied, n_0^* is the minimax solution for fixed N. If the condition is not satisfied, try neighboring integers until the above is satisfied. For all practical cases, n_0^* can be considered to be the minimax solution.

ξ	f	Δ	n	N	$f/N^{1/3}$
.7519	∞	0	∞	∞	4.106
.752	11116	.00040	1.79×10^6	1.99×10^{10}	4.105
.754	1049	.00422	1.60×10^4	1.68×10^7	4.100
.756	550	.00805	4.41×10^3	2.43×10^6	4.094
.758	373	.01188	2.04×10^3	7.59×10^5	4.088
.760	282	.01572	1.17×10^3	3.30×10^5	4.083
.762	227	.01957	758	1.72×10^5	4.077
.764	189	.02342	532	1.007×10^5	4.072
.766	163	.02728	394	6.41×10^4	4.066
.768	143	.03115	304	4.33×10^4	4.061
.770	127	.03502	241	3.06×10^4	4.055
.772	114	.03890	197	2.25×10^4	4.050
.774	104	.04279	163	1.70×10^4	4.044
.776	95.3	.04668	138	1.31×10^4	4.039
.778	88.0	.05058	118	1.04×10^4	4.033
.780	81.7	.05449	102	8.35×10^3	4.028
.785	69.3	.06429	74.3	5.15×10^3	4.015
.790	60.2	.07412	56.5	3.40×10^3	4.002
.795	53.1	.08400	44.5	2.36×10^3	3.989
.800	47.5	.09392	36.0	1.71×10^3	3.976
.805	43.0	.1039	29.7	1.28×10^3	3.963
.810	39.2	.1139	25.0	989	3.951
.815	36.1	.1239	21.3	768	3.938
.820	33.4	.1340	18.4	613	3.926
.825	31.0	.1441	16.0	497	3.914
.830	29.0	.1543	14.1	409	3.902
.835	27.2	.1645	12.5	340	3.890
.840	25.6	.1748	11.2	286	3.878
.850	22.8	.1954	9.10	208	3.855
.860	20.6	.2162	7.54	155	3.832
.870	18.7	.2372	6.35	119	3.810
.880	17.2	.2583	5.42	92.9	3.788

Table 1 The relation between n and N through given ξ

Now consider the case when Δ is small, if $\xi_1 = 0.7519$, then $\Delta = 0$, hence $1/(f-2) = 0$, *i.e.* $f \to \infty$, *i.e.* $n \to \infty$ if $\Delta \to 0$. Then we have for small Δ

$$f = 2 + \frac{1-\Delta^2}{\Delta\xi_1\phi(\xi_1)} \approx 4.4228\Delta^{-1},$$

and also

$$n = \frac{(1-\Delta^2)\xi_1^2}{2\Delta^2} \approx 0.2827\Delta^{-2},$$

hence

$$N = nf \approx 1.2502\triangle^{-3},$$

i.e.

$$N^{1/3} \approx 1.0773\triangle^{-1}.$$

For large N

$$f \approx 4.105N^{1/3}.$$

For example, we consider the case $N = 1000$. It is seen from Table 1 that $N = 1280$, 981 correspond to $\xi = 0.805$, 0.81, respectively. Since, for $1280^{1/3} = 10.86$, $981^{1/3} = 9.94$, $f/N^{1/3} = 3.963$, 3.951, respectively, by the method of interpolation for $1000^{1/3} = 10.00$ we have $f/N^{1/3} = 3.952$. Hence $f = 3.952 \times 10 = 39.52$, $n = 10000/39.52 = 25.30$. Now, taking $n = 25$, we see that $\sup_\triangle R_{25,\triangle}$ has the value 25.41 at $\triangle = 0.1137$. Hence $n = 25$ gives the minimax solution of R.

For n not large, we shall obtain the range of N for which specified value of n gives the minimax solution. Suppose that n and N are given and denote $R_\triangle$ as $R_{n,N}(\triangle)$ and consider it as a function of $\triangle$. As is shown above, when $f := N/n$ is not too small, $R_{n,N}(\triangle)$ has one local maximum at $\triangle = \triangle^*_{n,N}$ in the range $0 \leq \triangle < 1$. Now, for given n, let $\mathfrak{N}(n)$ be the set of values of N such that $R_{n,N}(\triangle^*_{n,N}) \leq n + 1$. Since, for $N_1 < N_2$

$$R_{n,N_1}(\triangle^*_{n,N_1}) < R_{n,N_2}(\triangle^*_{n,N_1}) \leq R_{n,N_2}(\triangle^*_{n,N_2}),$$

it follows that $N \in \mathfrak{N}(n)$ if and only if there exist $N^*(n)$ such that $1 \leq N \leq N^*(n)$. And also it has been shown that $R_{n,N}(\triangle^*_{n,N})$ is decreasing in n, hence $N^*(n)$ is increasing in n. Then, for N satisfying $N^*(n-1) < N < N^*(n)$, n gives the minimax solution, because

$$R_{n-1,N}(\triangle^*_{n,N}) > n, \qquad R_{n,N}(\triangle^*_{n,N}) \leq n + 1,$$

hence

$$\sup_\triangle R_{n,N}(\triangle) = \max\{n, R_{n,N}(\triangle^*_{n,N})\} \leq n + 1,$$

and for $n' > n$, $\sup_\triangle R_{n',N}(\triangle) \geq n' \geq n + 1$, and for $n' < n$, $\sup_\triangle R_{n',N}(\triangle) \geq R_{n',N}(\triangle^*_{n',N}) \geq R_{n,N}(\triangle_{n,N})$. For small n we must use the exact formula for the probability instead of normal approximation. Thus, for $n = 1$

$$\tilde{P}\{X_{11} > X_{12}\} = p_1 q_2 + \{(p_1 p_2 + q_1 q_2)/2\},$$

and when $p_1 = p_2$, we have

$$R_{1,N} = (p_1 - p_2)[1 + (N-2)\{p_2 q_1 + \{(p_1 p_2 + q_1 q_2)/2\}\}].$$

Putting $\triangle := p_1 - p_2$, $p_2 q_1 + \{(p_1 p_2 + q_1 q_2)/2\}$ is shown to be equal to $(1-\triangle)/2$, hence

$$R_{1,N}(\triangle) = \frac{\triangle}{2}\{2 + (N-2)(1-\triangle)\} = \frac{\triangle}{2}\{N - (N-2)\triangle\}$$

which is maximized when $\Delta = N/\{2(N-2)\}$ for $N \geq 4$, and we have

$$\sup_{\Delta} R_{1,N}(\Delta) = \frac{N^2}{8(N-2)}$$

and the range of N in which $\sup_{\Delta} R_{1,N}(\Delta) \leq 2$ is obtained from the inequality $N^2 - 16N - 32 \leq 0$, that is, $N \leq 8 + \sqrt{96} \fallingdotseq 18.80$. For $N = 3$, $\sup_{\Delta} R_{1,N}(\Delta) = R_{1,N}(1) = 9/8$. And when $N = 1,\ 2$, $R_{0,N}(\Delta) = N/2 \leq 1$. Hence $n = 1$ gives the minimax solution for $4 \leq N \leq 18$. In a similar way to the above, we get $N^*(2) = 29$, $N^*(3) = 49$, which implies that the minimax solution is given as

$$n = \begin{cases} 1 & \text{for } 3 \leq N \leq 18, \\ 2 & \text{for } 19 \leq N \leq 29, \\ 3 & \text{for } 30 \leq N \leq 49. \end{cases}$$

3. Minimax regret solution for the sequential procedure

Now we consider the second type procedure. We continue to observe the pair (X_{1j}, X_{2j}), $j = 1, 2, \ldots, N$ as long as $|\sum_j X_{1j} - \sum_j X_{2j}| < k$, and stop when $\sum_{j=1}^{n} X_{1j} - \sum_{j=1}^{n} X_{2j} = k$ or $= -k$ and take for the remaining $N - 2n$ cases from the first population in the former case and for the second population for the latter case. There is some probability that the paired observation does not stop until $2k \geq N$. Such a probability can be evaluated, but if N is large as compared with k, the probability can be ignored.

Denote $S_j := X_{1j} - X_{2j}$ $(j = 1, \ldots, N)$, and define random variables n^+ and n^- as $n^+ = n$ and $n^- = 0$ when $\sum_{j=1}^{n} S_j = k$ is first satisfied, and $n^- = n$ and $n^+ = 0$ when $\sum_{j=1}^{n} S_j = -k$ is first done. Put $\bar{n} := n^+ + n^-$. Let $p_1 > p_2$. Since $R = \bar{n}\Delta$ for $n^+ > 0$ and $R = \{\bar{n} + (N - 2\bar{n})\}\Delta$ for $n^- > 0$, where $\Delta = p_1 - p_2$, it follows that

$$E(R) = [NP\{n^- > 0\} + E(n^+ - n^-)]\Delta.$$

In order to calculate the probability $P\{n^- > 0\}$, assume that we start from S_0 not necessarily equal to 0 and stop as soon as $|\sum_{j=0}^{n} S_j| = k$, and denote

$$\pi(j) := P\{n^- > 0 | S_0 = j\} \qquad (j = 0, \pm 1, \ldots, \pm k).$$

Then we have the recurrence equation (ignoring the case when the procedure does not stop before $2k > N$)

$$\pi(j) = \pi_+ \pi(j+1) + \pi_0 \pi(j) + \pi_- \pi(j-1) \qquad (j = 0, \pm 1, \ldots, \pm k)$$

where $\pi(k+1) = \pi(-k-1) = 0$, $\pi_+ = P\{S_j > 0\} = p_1 q_2$, $\pi_- = P\{S_j < 0\} = q_1 p_2$ and $\pi_0 = P\{S_j = 0\} = 1 - \pi_+ - \pi_- = p_1 p_2 + q_1 q_2$. Hence $\pi(j)$ can be written as $\pi(j) = a + b\gamma^j$ $(j = 0, \pm 1, \ldots, \pm k)$, where

$$\gamma = \frac{\pi_-}{\pi_+} = \frac{p_1 q_2}{p_2 q_1}.$$

Since $\pi(k) = 0$ and $\pi(-k) = 1$, it follows that

$$a = -\frac{\gamma^{2k}}{1-\gamma^k}, \qquad b = \frac{\gamma^k}{1-\gamma^{2k}}, \qquad P\{n^- > 0\} = \pi(0) = \frac{\gamma^k}{1+\gamma^k}.$$

Now for $\nu(j) := E(\bar{n}|S_0 = j)$, we have

$$\begin{aligned} &\nu(j) = \pi_+\nu(j+1) + \pi_0\nu(j) + \pi_-\nu(j-1) + 1 \qquad (j = 0, \pm 1, \ldots, \pm(k-1)),\\ &\nu(k) = \nu(-k) = 0, \end{aligned}$$

and then the solution is given by the form of

$$\nu(j) = a' + b'j + c'\gamma^j \qquad (j = 0, \pm 1, \ldots, \pm k).$$

From the above equation we have

$$\begin{aligned} &-\pi_+\nu(j+1) + (1-\pi_0)\nu(j) - \pi_-\nu(j-1) = 1\\ &-\pi_+\{\nu(j+1) - \nu(j)\} + \pi_-\{\nu(j) - \nu(j-1)\} = 1, \end{aligned}$$

since $\pi_0 = 1 - \pi_+ - \pi_-$. Substituting $\nu(j) = a' + b'j + c'\gamma^j$ $(j = 0, \pm 1, \ldots, \pm k)$, we obtain

$$-b'(\pi_+ - \pi_-) = 1, \qquad b' = -1/(\pi_+ - \pi_-).$$

Since $\nu(k) = a' + b'k + c'\gamma^k = 0$ and $\nu(-k) = a' - b'k + c'\gamma^k = 0$, we have

$$\nu(0) = E(\bar{n}|S_0 = 0) = a' + c' = \frac{1}{\pi_+ - \pi_-}\left(\frac{1-\gamma^k}{1+\gamma^k}\right).$$

Now we calculate

$$E(n^+) = \sum_n nP\{n^+ = n\}, \qquad E(n^-) = \sum_n nP\{n^- = n\}.$$

For a path of $S_1, \ldots, S_n$ with $n^+ = n$, there exists a symmetric path to them with respect to x-axis such that $n^- = n$, and the ratio of probabilities of such paths in all the cases is $1 : \gamma^k$. Then we have

$$P\{n^- = n\} = \gamma^k P\{n^- = n\},$$

which implies $E(n^-) = \gamma^k E(n^+)$.
Since

$$E(n^+ + n^-) = E(\bar{n}) = \frac{1}{\pi_+ - \pi_-}\left(\frac{1-\gamma^k}{1+\gamma^k}\right),$$

we have

$$E(n^+ - n^-) = \frac{1}{\pi_+ - \pi_-}\left(\frac{1-\gamma^k}{1+\gamma^k}\right)^2.$$

Noting that $\pi_+ - \pi_- = p_1(1-p_2) - p_2(1-p_1) = p_1 - p_2 = \triangle$, we obtain

$$E(R) = N\triangle\left(\frac{\gamma^k}{1+\gamma^k}\right) + k\left(\frac{1-\gamma^k}{1+\gamma^k}\right)^2,$$

where

$$\gamma = \frac{\pi_-}{\pi_+} = \frac{p_2(1-p_1)}{p_1(1-p_2)} = \frac{(p_1+\triangle)(1-p_1)}{p_1(1-p_1+\triangle)} = \frac{1-\{\triangle/(1-p_1)\}}{1+(\triangle/p_1)},$$

hence, for $\triangle \le p_1 \le 1$

$$0 \le \gamma \le \left(\frac{1-\triangle}{1+\triangle}\right)^2 =: \gamma_\triangle.$$

For given $\triangle$, we have

$$\sup_{|p_1-p_2|=\triangle} E(R) = \max\left\{N\triangle\left(\frac{\gamma_\triangle^k}{1+\gamma_\triangle^k}\right) + k\left(\frac{1-\gamma_\triangle^k}{1+\gamma_\triangle^k}\right)^2,\ k\right\}.$$

So, for given N and k we shall obtain

$$\sup_\triangle\left\{N\triangle\frac{\gamma_\triangle^k}{1+\gamma_\triangle^k} + k\left(\frac{1-\gamma_\triangle^k}{1+\gamma_\triangle^k}\right)^2\right\}. \tag{3.1}$$

Putting $z_\triangle := (1-\gamma_\triangle)/(1+\gamma_\triangle)$, we can rewrite $\{\cdots\}$ in (3.1) by

$$\bar{R} := \frac{N\triangle}{2}(1-z_\triangle) + kz_\triangle^2.$$

Then we have

$$\frac{d\bar{R}}{dz_\triangle} = -\frac{N\triangle}{2} + 2kz_\triangle + \frac{N}{2}(1-z_\triangle)\frac{d\triangle}{dz_\triangle}.$$

Since

$$\begin{aligned}
\frac{dz_\triangle}{d\triangle} = \frac{d\gamma_\triangle}{d\triangle}\cdot\frac{dz_\triangle}{d\gamma_\triangle} &= \left\{\frac{d}{d\triangle}\left(\frac{1-\triangle}{1+\triangle}\right)^2\right\}\left\{\frac{d}{d\gamma_\triangle}\left(\frac{1-\gamma_\triangle^k}{1+\gamma_\triangle^k}\right)^2\right\} \\
&= \frac{2(1-\triangle)}{(1+\triangle)^2}\cdot\frac{2k\gamma_\triangle^{k-1}(1-\gamma_\triangle^k)}{(1+\gamma_\triangle^k)^2} \\
&= \frac{4(1-\triangle)}{(1+\triangle)^2}\cdot\frac{k}{\gamma_\triangle}z_\triangle(1-z_\triangle) \\
&= \frac{4k}{1-\triangle}z_\triangle(1-z_\triangle),
\end{aligned}$$

it follows that

$$\begin{aligned}
\frac{d\bar{R}}{dz_\triangle} &= -\frac{N\triangle}{2} + 2kz_\triangle + \frac{N(1-\triangle)}{4kz_\triangle} \\
&= \frac{N}{2}\left(\frac{1-\triangle}{2kz_\triangle} - \triangle\right) + 2kz_\triangle.
\end{aligned}$$

Now, since $z_\triangle$ is monotone increasing function of $\triangle$ and $z_0 = 0$, $z_1 = 1$, it follows that

$$\frac{d\bar{R}}{dz} = \begin{cases} \infty & \text{for } z = 0, \\ -\frac{N}{2} + 2k & \text{for } z = 1. \end{cases}$$

If $N > 4k$, there is at least a solution $z_\triangle = z_\triangle^*$ of $d\bar{R}/dz_\triangle = 0$, hence such $\triangle^*$ also exists that $z_{\triangle^*} = z_\triangle^*$. The solution of the equation

$$\frac{d^2\bar{R}}{dz_\triangle^2} = \frac{N}{2}\left\{-\frac{1-\triangle}{2kz_\triangle^2} - \left(1 + \frac{1}{2kz_\triangle}\right)\frac{d\triangle}{dz_\triangle}\right\} + 2k = 0$$

is not necessarily unique, but we consider one of the solutions corresponding to the maximum. Then, since

$$\frac{N}{2} - \frac{2k}{1-\triangle^2}\left(\frac{N\triangle}{2} - 2kz_\triangle\right)(1 + z_\triangle) = 0,$$

it follows that

$$\frac{N}{2}\left\{1 - \frac{2k\triangle(1+z_\triangle)}{1-\triangle^2}\right\} + \frac{2kz_\triangle(1+z_\triangle)}{1-\triangle^2} = 0,$$

i.e.

$$N = \frac{8k^2 z_\triangle(1+z_\triangle)}{2k\triangle(1+z_\triangle) - (1-\triangle^2)},$$

which implies that for given k the relation with $\triangle$ maximizing $\bar{R}$ is provided. We also have

$$\begin{aligned} \bar{R}^* = \max \bar{R} &= \frac{N\triangle}{2}(1 - z_\triangle) + kz_\triangle^2 \\ &= \frac{4k^2\triangle z_\triangle(1-z_\triangle^2)}{2k\triangle(1+z_\triangle) - (1-\triangle^2)} + kz_\triangle^2. \end{aligned}$$

If k is comparatively large with a sufficiently large N, letting $\eta = k\triangle$ we have

$$\gamma = \left\{\frac{1-(\eta/k)}{1+(\eta/k)}\right\}^2, \qquad z = \frac{\{1+(\eta/k)\}^{2k} - \{1-(\eta/k)\}^{2k}}{\{1+(\eta/k)\}^{2k} + \{1-(\eta/k)\}^{2k}},$$

hence

$$\begin{aligned} z &\approx \frac{e^{2\eta} - e^{-2\eta}}{e^{2\eta} + e^{-2\eta}} = \frac{1 - e^{-4\eta}}{1 + e^{-4\eta}} =: z_\eta, \\ \frac{N}{k^2} &\approx \frac{8z_\eta(1+z_\eta)}{2\eta(1+z_\eta) - 1}, \\ \frac{\bar{R}^*}{k} &\approx \frac{4\eta z_\eta(1-z_\eta^2)}{2\eta(1+z_\eta) - 1} + z_\eta^2. \end{aligned}$$

Since each RHS of the above depends on only η, putting

$$H(\eta) := \frac{4z_\eta(1+z_\eta)}{2\eta(1+z_\eta)-1}$$

we have

$$k = \left\{\frac{N}{2H(\eta)}\right\}^{1/2},$$

and

$$\bar{R}^* \approx \sqrt{\frac{N}{2}}\left\{\eta(1-z_\eta)H(\eta)^{1/2} + z_\eta^2 H(\eta)^{-1/2}\right\}. \tag{3.2}$$

Letting η^* be η minimizing the RHS of (3.2), we obtain the value of k minimizing $\bar{R}^*$, *i.e.* the minimax solution is given by the form of

$$k^* = \{2H(\eta^*)\}^{-1/2}N^{1/2}.$$

By a numerical calculation, we have $\eta^* \fallingdotseq 0.552$, and then $H(\eta^*) \fallingdotseq 5.8424$, $\bar{R}^* = 0.53033\sqrt{N/2} = 0.3750N^{1/2}$, and $k^* = 0.2925N^{1/2}$.

For example, when $N = 1000$, we have $k^* \fallingdotseq 9.25$. If we take $k = 9$ as the nearest to the value, then $R(\triangle)$ must be the maximum value at near point to $\triangle = \eta^*/k \fallingdotseq 0.0613$, hence we shall calculate $R(\triangle)$ in the neighborhood of $\triangle \fallingdotseq 0.061$. Let

$$R_k := N\triangle\left(\frac{\gamma^k}{1+\gamma^k}\right) + k\left(\frac{1-\gamma^k}{1+\gamma^k}\right)^2.$$

When $k = 9$, the values of R_k in a neighborhood of $\triangle \fallingdotseq 0.061$ are as follows.

$\triangle$	$(1-\gamma^k)/(1+\gamma^k)$	R_k
0.058	0.77992	11.8568
0.059	0.78690	11.8594
0.060	0.79368	11.8589
0.061	0.80027	11.8556

Hence, for $k = 9$, R_9 has the maximum value 11.8594 at $\triangle = 0.059$. On the other hand, when $\triangle = 0.059$, we have, for $k = 10$, $R_{10} = 11.9313$, hence $\sup_\triangle R_{10} > \sup_\triangle R_9$. In a similar way to the above, for $k = 8$, $R_8 = 12.094$ when $\triangle = 0.059$, hence $k = 9$ gives the minimax solution. Then $R_9 = 11.8594$ is seen to be extremely close to the value $0.3750\sqrt{N} \fallingdotseq 11.8585$. Letting $N = 100$, we have $k^* = 2.925$. Taking $k = 3$ as the closest integer to the value of k^*, we see that R_3 has the maximum value 3.7314 at near point to $\triangle = 0.19$. If, for the value of $\triangle$, $k = 2$ or 4, then R_k has the values 4.194 or 4.435, respectively, hence $k = 3$ gives the minimax solution, and then the value of R_3 is seen to be very close to $0.375\sqrt{N} = 3.75$. As is seen in the above, the approximation is very accurate for not so large N.

For the case of small k, the exact computation is not difficult and we have

$$k = \begin{cases} 1 & \text{for } 3 \leq N \leq 25, \\ 2 & \text{for } 26 \leq N \leq 73. \end{cases}$$

On the other hand, if $0.2925N^{1/2} \leq 1.5$ or 2.5, then $N \leq 26$ or 73, hence the approximation is sufficiently accurate for even such regions of N. And such a procedure is seen to be more efficient than a fixed sample one.

Therefore we have the rule as follows. *For fixed N, the minimax solution for k can be obtained to be the nearest integer to $k^* = 0.2925N^{1/2}$.*

4. The lower bound for the order of minimax risk

In this section, we shall show that the order of minimax risk is not smaller than $N^{1/2}$, under any procedure. Now we carry out two sequential procedures i $(i = 1, 2)$ N times, and denote their results by $X_{11}, \ldots, X_{1N}$ and $X_{21}, \ldots, X_{2N}$, respectively. Since, for each $j = 2, \ldots, N$, Y_j depends on only $X_{11}, \ldots, X_{1,\,j-1}$ and $X_{21}, \ldots, X_{2,\,j-1}$, it follows from (1.1) that

$$\begin{aligned} E(T) &= E[\sum_{j=1}^{N} Y_j X_{1j} + \sum_{j=1}^{N} (1 - Y_j) X_{2j}] \\ &= Np_2 + (p_1 - p_2) \sum_{j=1}^{N} E(Y_j). \end{aligned}$$

Hence, the regret is given by

$$\begin{aligned} &\Delta\{N - \sum_{j=1}^{N} E(Y_j | p_1, p_2)\} \quad \text{for } p_1 > p_2, \\ &\Delta \sum_{j=1}^{N} E(Y_j | p_1, p_2) \quad \text{for } p_1 < p_2. \end{aligned}$$

In the minimax solution, when $p_1 = p_2$, we can deduce $\sum_{j=1}^{N} E(Y_j | p_1, p_2) = N/2$ from the symmetry of the problem. And now, comparing the case when $p_1 = (1 + \Delta)/2$ and $p_2 = (1 - \Delta)/2$ and the case when $p_1 = p_2 = 1/2$, we obtain the maximum value of

$$\sum_{j=1}^{N} E(Y_j | p_1 = (1 + \Delta)/2,\ p_2 = (1 - \Delta)/2)$$

under the condition $\sum_{j=1}^{N} E(Y_j | p_1 = p_2 = 1/2) = N/2$. For each j, Y_j depends on only $X_{11}, \ldots, X_{1,\,j-1};\ X_{21}, \ldots, X_{2,\,j-1}$, but we relax the condition and assume that for each j,

Y_j depends on $X_{11}, \ldots, X_{1N};\ X_{21}, \ldots, X_{2N}$. Then, in order to maximize $\sum_{j=1}^N E(Y_j|p_1 = (1+\triangle)/2,\ p_2 = (1-\triangle)/2)$ under the condition $\sum_{j=1}^N E(Y_j|p_1 = p_2 = 1/2) = N/2$, for each j we take $Y_j = 1$ when

$$p_1^{\sum_{j=1}^N X_{1j}}(1-p_1)^{N-\sum_{j=1}^N X_{1j}} p_2^{\sum_{j=1}^N X_{2j}}(1-p_2)^{N-\sum_{j=1}^N X_{2j}} \geq c\left(\frac{1}{2}\right)^{2N}.$$

This means that for each j, $Y_j = 1$ if $\sum_{j=1}^N X_{2j} > \sum_{j=1}^N X_{1j}$, $Y_j = 0$ if $\sum_{j=1}^N X_{2j} < \sum_{j=1}^N X_{1j}$, and the value of Y_j is chosen at random if $\sum_{j=1}^N X_{2j} = \sum_{j=1}^N X_{1j}$. Then we have for a large N

$$P\{\sum_{j=1}^N X_{1j} > \sum_{j=1}^N X_{2j}\} \approx 1 - \Phi\left(\frac{\sqrt{2N}\triangle}{\sqrt{1-\triangle^2}}\right),$$

hence

$$\sup_{0\leq\triangle<1} R \geq \sup_{0\leq\triangle<1} N\triangle\left\{1-\Phi\left(\frac{\sqrt{2N}\triangle}{\sqrt{1-\triangle^2}}\right)\right\}, \tag{4.1}$$

so we consider $\triangle$ maximizing the RHS of (4.1). Since the RHS of (4.1) is given by

$$\begin{aligned}
&\sup_{0\leq\triangle<1} N\triangle\{1-\Phi(\sqrt{2N}\triangle)\} \\
&= \sqrt{\frac{N}{2}} \sup_{0\leq\triangle<1} \sqrt{2N}\triangle\{1-\Phi(\sqrt{2N}\triangle)\} \\
&= \sqrt{\frac{N}{2}} \sup_{\xi} \xi\{1-\Phi(\xi)\} \\
&\fallingdotseq 0.170\sqrt{\frac{N}{2}} \fallingdotseq 0.12N^{1/2},
\end{aligned}$$

it follows that for a large N

$$\sup_{0\leq\triangle<1} R > 0.12N^{1/2}. \tag{4.2}$$

The inequality (4.2) is not sharp, and the more accurate evaluation will be possible, but (4.2) is enough to show the order of minimax regret to be $N^{1/2}$.

Next we shall discuss the above more accurately. If we get the lower bound of

$$R^* := R(p_1, p_2) + R(p_2, p_1),$$

then the minimax value of R is not smaller than $R^*/2$. Since

$$R^* = \triangle[N - \sum_{j=1}^N \{E(Y_j|p_1, p_2) - E(Y_j|p_2, p_1)\}],$$

for each $j = 1, \ldots, N$, we shall obtain the lower bound of

$$E(Y_j|p_1, p_2) - E(Y_j|p_2, p_1).$$

Now, for each j, Y_j depends on only $Y_1, \ldots, Y_{j-1}$; $Y_1X_{11}, \ldots, Y_{j-1}X_{1,\,j-1}$; $(1-Y_1)X_{21}, \ldots, (1-Y_{j-1})X_{2,\,j-1}$, which is denoted by Z_{j-1}. For each j we express $Y_j = y_j(z_{j-1})$ as a function of z_{j-1}, and denote the probability function of Z_{j-1} by $p(z_{j-1}|p_1, p_2)$. Then we have for each j

$$\begin{aligned}&E(Y_j|p_1, p_2) - E(Y_j|p_2, p_1)\\&= \sum_{z_{j-1}} y_j(z_{j-1})\{p(z_{j-1}|p_1, p_2) - p(z_{j-1}|p_2, p_1)\}.\end{aligned} \tag{4.3}$$

In order maximize (4.3) we obtain for each j

$$p_j(z_{j-1}) = \begin{cases}1 & \text{for } p(z_{j-1}|p_1, p_2) > p(z_{j-1}|p_2, p_1),\\ 0 & \text{for } p(z_{j-1}|p_1, p_2) < p(z_{j-1}|p_2, p_1).\end{cases}$$

Now, let $P\{Y_1 = 1\} = P\{Y_1 = 0\} = 1/2$. For each $j \geq 2$, Y_j is determined so that it depends on only $Y_1, \ldots, Y_{j-1}$; $Y_1X_{11}, \ldots, Y_{j-1}X_{1,\,j-1}$; $(1-Y_1)X_{21}, \ldots, (1-Y_{j-1})X_{2,\,j-1}$. Since the conditional probability functions are given by

$$\begin{aligned}p(y_1x_{11}, (1-y_1)x_{21}|y_1) &= p_1^{y_1x_{11}}(1-p_1)^{y_1(1-x_{11})}p_2^{(1-y_1)x_{21}}(1-p_2)^{(1-y_1)(1-x_{21})}\\ p(y_2x_{12}, (1-y_2)x_{22}|z_1) &= p_1^{y_2(z_1)x_{12}}(1-p_1)^{y_2(z_1)(1-x_{12})}\\ &\quad\cdot p_2^{(1-y_2(z_1))x_{22}}(1-p_2)^{(1-y_2(z_1))(1-x_{22})},\end{aligned}$$

we consequently have

$$\begin{aligned}&p(y_1x_{11}, \ldots, y_{j-1}x_{1,j-1}, (1-y_1)x_{21}, \ldots, (1-y_{j-1})x_{2,j-1}|p_1, p_2)\\ &= \frac{1}{2}p_1^{\sum_{k=1}^{j-1} y_kx_{1k}}(1-p_1)^{\sum_{k=1}^{j-1} y_k - \sum_{k=1}^{j-1} y_kx_{1k}}p_2^{\sum_{k=1}^{j-1}(1-y_k)x_{2k}}(1-p_2)^{\sum_{k=1}^{j-1}(1-y_k)-\sum_{k=1}^{j-1}(1-y_k)x_{2k}}.\end{aligned}$$

Putting $w_{j-1} := \sum_{k=1}^{j-1} y_k$, $u_{j-1} := \sum_{k=1}^{j-1} y_kx_{1k}$ and $v_{j-1} := \sum_{k=1}^{j-1}(1-y_k)x_{2k}$, we have

$$\begin{aligned}&p(y_1x_{11}, \ldots, y_{j-1}x_{1,j-1}, (1-y_1)x_{21}, \ldots, (1-y_{j-1})x_{2,j-1}|p_1, p_2)\\ &= \frac{1}{2}p_1^{u_{j-1}}(1-p_1)^{w_{j-1}-u_{j-1}}p_2^{v_{j-1}}(1-p_2)^{j-1-w_{j-1}-v_{j-1}}.\end{aligned}$$

Then we take for each j

$$Y_j = \begin{cases}1 & \text{for } p(y_1x_{11}, \ldots, y_{j-1}x_{1,j-1}, (1-y_1)x_{21}, \ldots, (1-y_{j-1})x_{2,j-1}|p_1, p_2)\\ & \qquad > p(y_1x_{11}, \ldots, y_{j-1}x_{1,j-1}, (1-y_1)x_{21}, \ldots, (1-y_{j-1})x_{2,j-1}|p_2, p_1),\\ 0 & \text{otherwise.}\end{cases}$$

In particular, we consider the case when

$$p_1 = (1+\triangle)/2 = p > p_2 = (1-\triangle)/2 = 1-p = q.$$

Since

$$\begin{aligned} &p(y_1x_{11},\dots,y_{j-1}x_{1,j-1},(1-y_1)x_{21},\dots,(1-y_{j-1})x_{2,j-1}|p_1,p_2) \\ &= \frac{1}{2}p^{u_{j-1}+(j-1-w_{j-1}-v_{j-1})}q^{w_{j-1}-u_{j-1}+v_{j-1}}, \end{aligned}$$

putting

$$Z_{j-1} = \sum_{k=1}^{j-1} Y_1X_{1k} + \sum_{k=1}^{j-1}(1-Y_2)(1-X_{2k})$$

we have

$$\begin{aligned} &p(y_1x_{11},\dots,y_{j-1}x_{1,j-1},(1-y_1)x_{21},\dots,(1-y_{j-1})x_{2,j-1}|p_1,p_2) \\ &= \frac{1}{2}p^{z_{j-1}}q^{j-1-z_{j-1}}. \end{aligned}$$

Hence we take

$$Y_j = \begin{cases} 1 & \text{for } Z_{j-1} > \frac{1}{2}(j-1), \\ 0 & \text{for } Z_{j-1} < \frac{1}{2}(j-1), \\ 1 & \text{with probability } 1/2 \text{ for } Z_{j-1} = \frac{1}{2}(j-1). \end{cases}$$

Since for each j

$$P\{Y_1X_{1j} + (1-Y_2)X_{2j} = 1|Y_j\} = p \quad (i=1,2),$$

it follows that Z_{j-1} is distributed as the binomial distribution $B(j-1,p)$. We also have for each j

$$\begin{aligned} E(Y_j) &= P\left\{Z_{j-1} > \frac{j-1}{2}\right\} + \frac{1}{2}\left\{Z_{j-1} = \frac{j-1}{2}\right\} \\ &=: \tilde{P}\left\{Z_{j-1} \geq \frac{j-1}{2}\right\}. \end{aligned}$$

Then we obtain

$$\sum_{j=1}^{N} E(Y_j) = \sum_{j=0}^{N-1} \tilde{P}\{Z_j \geq \frac{j}{2}\}, \tag{4.4}$$

where the RHS of (4.4) corresponding to the case $j=0$ is equal to $1/2$. In this case it is shown that there exists the best procedure independent of $\triangle$. And also

$$R = \triangle\{N - \sum_{j=1}^{N} E(Y_j|p_1,p_2)\} = \triangle\sum_{j=1}^{N-1}(1-\tilde{P}\{Z_j \geq 1/2\}).$$

Let the probability $\tilde{P}\{Z_j \geq 1/2\}$ approximate by the normal distribution, *i.e.*

$$\tilde{P}\{Z_j \geq 1/2\} = 1 - \Phi\left(\frac{\sqrt{j}\Delta}{\sqrt{1-\Delta^2}}\right) + \varepsilon_j.$$

Then

$$R \approx \Delta \sum_{j=1}^{N-1} \left\{1 - \Phi\left(\frac{\sqrt{j}\Delta}{\sqrt{1-\Delta^2}}\right)\right\} + \Delta \sum_{j=1}^{N-1} \varepsilon_j.$$

Putting $\xi = \sqrt{N}\Delta/\sqrt{1-\Delta^2}$, we have

$$R = \frac{\sqrt{N}\xi}{\sqrt{1+(\xi^2/N)}} \cdot \frac{1}{N} \sum_{j=1}^{N-1} \{1 - \Phi(\xi/\sqrt{N})\} + \Delta \sum_{j=1}^{N-1} \varepsilon_j.$$

Since, for a large N, the Riemann sum is approximated by the integral, we obtain

$$\frac{1}{N} \sum_{j=1}^{N-1} \{1 - \Phi(\xi/\sqrt{N})\} = \int_0^1 \{1 - \Phi(\sqrt{\eta}\xi)\} d\eta.$$

Transforming $\eta = \zeta^2/\xi^2$, we have

$$\begin{aligned}
&\int_0^1 \{1 - \Phi(\sqrt{\eta}\xi)\} d\eta \\
&= \frac{1}{\xi^2} \int_0^{\xi} 2\zeta \{1 - \Phi(\zeta)\} d\zeta \\
&= \frac{1}{\xi^2} [\xi^2 \{1 - \Phi(\xi)\} - \xi\phi(\xi) + \{\Phi(\xi) - \frac{1}{2}\}].
\end{aligned}$$

On the other hand, since $\varepsilon_j < C/j^2$, it follows that

$$RN^{-1/2} \approx \xi\{1 - \Phi(\xi)\} - \phi(\xi) + \frac{1}{\xi}\{\Phi(\xi) - \frac{1}{2}\}. \tag{4.5}$$

In order to obtain ξ maximizing the RHS of (4.5), differentiating the RHS with respect to ξ and letting it be zero, we obtain

$$1 - \Phi(\xi) - \frac{1}{\xi^2}\{\Phi(\xi) - \frac{1}{2}\} + \frac{1}{\xi}\phi(\xi) = 0,$$

which has the unique root in the range $\xi > 0$. The solution is given by $\xi \risingdotseq 1.247$, and then $RN^{-1/2} \risingdotseq 0.2649$. Hence the minimax value of R is not smaller than $0.2649N^{1/2}$ for a large N. For not so large N, the lower bound can be obtained from the exact calculation of the binomial probability.

Remark. Lai and Robbins (1985) obtain the asymptotic lower bound which gives in our formula

$$\liminf_{N\to\infty} \frac{1}{\log N} R(\triangle) \geq |\triangle|/I(p_1, p_2),$$

where $\triangle = p_1 - p_2$ and $I(p_1, p_2)$ is the Kullback–Leibler information number (see also Li and Zhang (1992)). But this formula is not of help in obtaining the bound for the minimax regret, since

$$\limsup_{|\triangle|\to 0} |\triangle|/I(p_1, p_2) = \infty.$$

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