A note on Schmitt–Vogel lemma

INTRODUCTION

Let $S$ be a polynomial ring over an infinite field $k$, and $I$ a squarefree monomial ideal of $S$. The arithmetical rank of $I$ is defined by

$$\text{ara } I := \min \{ r : \text{there exist } a_1, \ldots, a_r \in I \text{ such that } (a_1, \ldots, a_r) = \sqrt{I} \}.$$  

For ideals $J \subset I \subset S$, $J$ is said to be a reduction of $I$ if there exists some $s \in \mathbb{N}$ such that

$$I^{s+1} = JJ^s.$$  

Note that when this is the case, $\sqrt{J} = \sqrt{I}$ holds. The analytic spread of $I$ is defined by

$$l(I) := \min \{ \mu(J) : J \text{ is a reduction of } I \},$$

where $\mu(J)$ denotes the minimal number of generators of $J$. The existence of the minimal reduction shows $\text{ara } I \leq l(I)$. On the other hand, it is known by Lyubeznik [3] that $\text{pd}_S S/I \leq \text{ara } I$, where $\text{pd}_S S/I$ denotes the projective dimension of $S/I$. Therefore we have the following inequalities:

$$\text{pd}_S S/I \leq \text{ara } I \leq l(I).$$

In the study of the arithmetical rank, Schmitt–Vogel lemma [5, Lemma, pp. 249] is an important and useful tool, because it gives a sufficient condition for ideals $J \subset I$ to hold $\sqrt{J} = \sqrt{I}$. In this report, we give a sufficient condition for an ideal $J$ with $J \subset I$ to be a reduction of $I$ by refining Schmitt–Vogel lemma. As an application of our theorem, we prove $l(I) = \text{pd}_S S/I$ for the ideal

$$I = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q}),$$

where $x_{11}, \ldots, x_{qi_q}$ are variables in $S$ pairwise distinct. Schmitt and Vogel [5] proved $\text{ara } I = \text{pd}_S S/I$ for this ideal $I$ using their lemma.
1. Main Theorem

In this section, we consider an arbitrary commutative ring $R$ with unitary. Our main result of this report is the following:

**Theorem 1.1.** Let $R$ be a commutative ring with unitary. Let $P_0, P_1, \ldots, P_r \subset R$ be finite subsets, and we set

$$P = \bigcup_{\ell=0}^{r} P_{\ell},$$

$$g_{\ell} = \sum_{a \in P_{\ell}} a, \quad \ell = 0, 1, \ldots, r.$$

Assume that

(C1) $\# P_0 = 1$.

(C2) For all $\ell > 0$ and $a, a'' \in P_{\ell}$ ($a \neq a''$), there exist some $\ell'$ ($0 \leq \ell' < \ell$), $a' \in P_{\ell'}$, and $b \in (P)$ such that $aa'' = a'b$.

Then we have $(g_0, g_1, \ldots, g_r)$ is a reduction of $(P)$.

On the other hand, Schmitt–Vogel lemma is the following:

**Proposition 1.2** (Schmitt–Vogel [5, Lemma, pp. 249]). Let $R$ be a commutative ring with unitary. Let $P_0, P_1, \ldots, P_r \subset R$ be finite subsets, and we set

$$P = \bigcup_{\ell=0}^{r} P_{\ell},$$

$$g_{\ell} = \sum_{a \in P_{\ell}} a, \quad \ell = 0, 1, \ldots, r.$$

Assume that

(C1) $\# P_0 = 1$.

(C2)' For all $\ell > 0$ and $a, a'' \in P_{\ell}$ ($a \neq a''$), there exist some $\ell'$ ($0 \leq \ell' < \ell$) and $a' \in P_{\ell'}$ such that $aa'' \in (a')$.

Then we have $\sqrt{(g_0, g_1, \ldots, g_r)} = \sqrt{(P)}$.

Second condition of Theorem 1.1 is stronger than that of Schmitt–Vogel lemma, but Theorem 1.1 has a stronger conclusion than Schmitt–Vogel lemma.

**Remark 1.3.** Schmitt–Vogel lemma allows us to add some exponent $e(a)$ for each $a \in P_{\ell}$ in the sum $g_{\ell}$, i.e., we may put

$$g_{\ell} = \sum_{a \in P_{\ell}} a^{e(a)}.$$

In particular, we can take $g_{\ell}$ as homogeneous if $R$ is graded. But a similar statement does not hold for our theorem.

Instead of proving Theorem 1.1, we will give a detailed explanation of an example in Section 3, which illustrates the outline of the proof of the theorem. See also [2].
2. An application

In this section, we apply Theorem 1.1 to some ideals and calculate the analytic spread of them.

Consider the ideal

\[(2.1) \quad I = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q}),\]

where $x_{11}, \ldots, x_{qi_q}$ are variables in $S$ pairwise distinct.

**Lemma 2.1.** For the above ideal $I$,

\[\text{pd}_S S/I = \sum_{s=1}^{q} i_s - q + 1.\]

**Proof.** For an integer $q \geq 1$, we set

\[I_q = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q}),\]

\[r_q = \sum_{s=1}^{q} i_s - q + 1.\]

We prove the lemma by induction on $q$. The case $q = 1$ is clear. Suppose that $q \geq 2$. If we put $P = (x_{q1}, \ldots, x_{qi_q})$, then $I_q = I_{q-1} \cap P$ and $r_q = r_{q-1} + \text{height} P - 1 = r_{q-1} + \text{pd}_S S/P - 1$. Consider Mayer–Vietoris sequence

\[0 \to S/I_q \to S/I_{q-1} \oplus S/P \to S/(I_{q-1} + P) \to 0.\]

Since $\text{pd}_S S/I = \max\{i : \text{Tor}^S_i(k, S/I) \neq 0\}$, the long exact sequence

\[\cdots \to 0 \to \text{Tor}^S_{r_q+1}(k, S/I_{q-1}) \oplus \text{Tor}^S_{r_q+1}(k, S/P) \to \text{Tor}^S_{r_q+1}(k, S/(I_{q-1} + P)) \to \text{Tor}^S_{r_q}(k, S/I_q) \to \text{Tor}^S_{r_q}(k, S/I_{q-1}) \oplus \text{Tor}^S_{r_q}(k, S/P) = 0 \to \cdots\]

implies $r_q = \text{pd}_S S/I_q$. \(\Box\)

Schmitt–Vogel [5] proved $\text{ara} I = \text{pd}_S S/I$ (see also Schenzel–Vogel [4]). They proved it by applying Schmitt–Vogel lemma to

\[P_\ell = \{x_{1\ell_1}x_{2\ell_2}\cdots x_{q\ell_q} : \ell_1 + \cdots + \ell_q = \ell + q\}, \quad \ell = 0, 1, \ldots, r,\]

where $r = \sum_{s=1}^{q} i_s - q$. These $P_0, P_1, \ldots, P_r$ also satisfy the assumption of Theorem 1.1. Thus $J = (g_0, g_1, \ldots, g_r)$ is a reduction of $I$. Since

\[r + 1 = \text{pd}_S S/I = \text{ara} I \leq \ell(I) \leq r + 1,\]

we have $\ell(I) = \text{pd}_S S/I$. Therefore we have the following corollary:

**Corollary 2.2.** Let $I = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q})$. Then we have

\[\ell(I) = \text{pd}_S S/I.\]

In particular, $(g_0, g_1, \ldots, g_r)$ is a minimal reduction of $I$.

Note that we have a minimal reduction of $I$ explicitly.

**Remark 2.3.** In general, $\ell(I) \neq \text{pd}_S S/I$ for a squarefree monomial ideal $I$. For example, if $\mu(I) - \text{height}(I) = 1$ and $S/I$ is Cohen–Macaulay, then $\text{height}(I) = \text{pd}_S S/I = \text{ara} I < \ell(I) = \mu(I)$; see [1].
3. An example

In this section, we give one example to illustrate the outline of the proof of Theorem 1.1.

Let us consider the ideal

\[ I = (x_1, x_2, x_3) \cap (y_1, y_2, y_3). \]

This is a special form of the ideal (2.1). The minimal graded resolution of \( S/I \) is

\[ 0 \to S(-6) \to S(-5)^6 \to S(-4)^{15} \to S(-3)^{18} \to S(-2)^9 \to S \to S/I \to 0 \]

and \( \text{pd}_S S/I(=3+3-2+1)=5. \) Then

\[ P_0 = \{x_1y_1\}, \]
\[ P_1 = \{x_1y_2, x_2y_1\}, \]
\[ P_2 = \{x_1y_3, x_2y_2, x_3y_1\}, \]
\[ P_3 = \{x_2y_3, x_3y_2\}, \]
\[ P_4 = \{x_3y_3\}. \]

Let us see conditions of Theorem 1.1. Since \#P_0 = 1, (C1) is satisfied. For the assumption (C2), we have the following equations:

(3.1) \[ P_1: \quad x_1y_2 \cdot x_2y_1 = x_1y_1 \cdot x_2y_2 \in (P_0)(P_2), \]
\[ x_1y_3 \cdot x_2y_2 = x_1y_2 \cdot x_2y_3 \in (P_1)(P_3), \]

(3.2) \[ P_2: \quad \begin{align*}
    x_1y_3 \cdot x_3y_1 &= x_1y_1 \cdot x_3y_3 \in (P_0)(P_4), \\
    x_2y_2 \cdot x_3y_1 &= x_2y_1 \cdot x_3y_2 \in (P_1)(P_3), \\
\end{align*} \]

(3.3) \[ P_3: \quad x_2y_3 \cdot x_3y_2 = x_2y_2 \cdot x_3y_3 \in (P_2)(P_4). \]

Thus (C2) is also satisfied.

Now we shall see \( J = (g_0, g_1, g_2, g_3, g_4) \) is a reduction of \( I \), where

\[ g_0 = x_1y_1, \]
\[ g_1 = x_1y_2 + x_2y_1, \]
\[ g_2 = x_1y_3 + x_2y_2 + x_3y_1, \]
\[ g_3 = x_2y_3 + x_3y_2, \]
\[ g_4 = x_3y_3. \]

We put

\[ I_\ell = \left( \bigcup_{j=0}^{\ell} P_j \right), \quad \ell = 0, 1, 2, 3, 4. \]

Note that \( I_4 = I \). It is enough to show

\[ I_\ell^{2\ell} \subset JI^{2\ell-1}, \quad \ell = 0, 1, 2, 3, 4 \]

in order to see that \( J \) is a reduction of \( I \). We show this by induction on \( \ell \). In fact, we show

\[ I_\ell^{2\ell} \subset I_{\ell-1}^{2^{\ell-1}}I^{2^{\ell-2}} + JI^{2^{\ell-1}}, \quad \ell = 0, 1, 2, 3, 4. \]
Step 1: The case $\ell = 0$. In this case, $I_0 = (P_0) = (x_1y_1) = (g_0) \subset J$.

Step 2: The case $\ell = 1$. We want to show $I_1^2 \subset I_0 I + JI$. To see this, it is enough to show that $a_1a_2 \in I_0 I + JI$ for all $a_1, a_2 \in P_1$ (we do not assume $a_1 \neq a_2$). When $a_1 \neq a_2$, (3.1) shows $a_1a_2 \in I_0 I$. When $a_1 = a_2 = a$, we use $g_1$. For example,

$$(x_1y_2)^2 = (g_1 - x_2y_1)x_1y_2 = g_1x_1y_2 - x_2y_1 \cdot x_1y_2 = g_1x_1y_2 - x_1y_1 \cdot x_2y_2 \in JI + I_0I.$$

Therefore $I_1^2 \subset I_0 I + JI$ holds.

Step 3: The case $\ell = 2$. We want to show $I_2^4 \subset I_1^2I^2 + JI^3$. To see this, we only check $a_1a_2a_3a_4 \in I_1^2I^2 + JI^3$ for all $a_1, a_2, a_3, a_4 \in P_0 \cup P_1 \cup P_2$. There are two cases:

(i) $a_1, a_2, a_3, a_4 \in P_2$;

(ii) for some $i$, $a_i \in P_0 \cup P_1$.

In case (i), there are two cases dividing large. The first one is that $a_1 \neq a_2$ and $a_3 \neq a_4$ by renumbering $a_1, a_2, a_3, a_4$. In this case, it is easy to check $a_1a_2a_3a_4 \in I_1^2I^2$ because of (3.2). For example,

$$(x_1y_3)^2x_2y_2 \cdot x_3y_1 = (x_1y_3 \cdot x_2y_2)(x_1y_3 \cdot x_3y_1) = (x_1y_2 \cdot x_1y_1)(x_2y_3 \cdot x_3y_3) \in I_1^2I^2.$$

The second one is that there are no such a renumbering on $a_1, a_2, a_3, a_4$. In this case, we use $g_2$ as in the case $\ell = 1$. For example,

$$(x_1y_3)^3x_2y_2 = (x_1y_3)^2(g_2 - x_2y_2 - x_3y_1)x_2y_2$$

$$= g_2(x_1y_3)^2x_2y_2 - (x_1y_3)^2(x_2y_2)^2 - (x_1y_3)^2x_3y_1 \cdot x_2y_2$$

$$= g_2(x_1y_3)^2x_2y_2 - (x_1y_3 \cdot x_2y_2)^2 - (x_1y_3 \cdot x_3y_1)(x_1y_3 \cdot x_2y_2)$$

$$\in JI^3 + I_1^2I^2.$$

In case (ii), if there are two indices $i$ (say, $i_1, i_2$) such that $a_i \in P_0 \cup P_1$, then $a_{i_1}a_{i_2} \in I_1^2$ and $a_1a_2a_3a_4 \in I_1^2I^2$ hold. Next, we consider the case that there is only one $i$ such that $a_i \in P_0 \cup P_1$. We may assume $a_1 \in P_0 \cup P_1$ and $a_2, a_3, a_4 \in P_2$. Then we need to make only one pair of distinct elements from $a_2, a_3, a_4$. It is weaker requirement than that of case (i). In fact, to make one pair of distinct elements, we only need two of $a_2, a_3, a_4$. For example,

$$(x_1y_3)^2 = x_1y_3(g_2 - x_2y_2 - x_3y_1)$$

$$= g_2x_1y_3 - x_1y_3 \cdot x_2y_2 - x_1y_3 \cdot x_3y_1$$

$$\in JI + I_1I.$$

Step 4: The case $\ell = 3$. We want to show $I_3^8 \subset I_2^4I^4 + JI^7$. In this case, the same argument as in Step 3 is also usable. We omit here.

Step 5: The case $\ell = 4$. It is clear that $I_4^{16} \subset I_3^8I^8 + JI^{15}$ since $\# P_4 = 1$. Therefore we obtain that $J$ is a reduction of $I$.

Remark 3.1. The reduction number $r_J(I)$ is defined by

$$r_J(I) := \min\{s : I^{s+1} = JI^s\}.$$
Above argument gives an upper bound of $r_J(I)$. But this is very big in general. In fact, in the above argument, we only see $I^{2^4} = JI^{2^4-1}$, that is, $r_J(I) \leq 2^4 - 1 = 15$. But $r_J(I) = 3$, i.e., $I^4 = JI^3$ holds.

REFERENCES