<table>
<thead>
<tr>
<th>Title</th>
<th>A note on Schmitt-Vogel lemma (Algorithmic and Computational Theory in Algebra and Languages)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>KIMURA, Kyouko; TERAI, Naoki; YOSHIDA, Ken-ichi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2008), 1604: 125-130</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/139921">http://hdl.handle.net/2433/139921</a></td>
</tr>
<tr>
<td>Right</td>
<td>Type</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
A note on Schmitt–Vogel lemma

名古屋大学・大学院多元数理科学研究科
木村 杏子 (Kyouko KIMURA)
Graduate School of Mathematics
Nagoya University

佐賀大学・文化教育学部
寺井 直樹 (Naoki TERAI)
Department of Mathematics
Faculty of Culture and Education
Saga University

名古屋大学・大学院多元数理科学研究科
吉田 健一 (Ken-ichi YOSHIDA)
Graduate School of Mathematics
Nagoya University

INTRODUCTION

Let $S$ be a polynomial ring over an infinite field $k$, and $I$ a squarefree monomial ideal of $S$. The arithmetical rank of $I$ is defined by

$$\text{ara } I := \min \left\{ r : \text{there exist } a_1, \ldots, a_r \in I \text{ such that } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I} \right\}.$$ 

For ideals $J \subset I \subset S$, $J$ is said to be a reduction of $I$ if there exists some $s \in \mathbb{N}$ such that

$$I^{s+1} = JI^s.$$ 

Note that when this is the case, $\sqrt{J} = \sqrt{I}$ holds. The analytic spread of $I$ is defined by

$$l(I) := \min \{ \mu(J) : J \text{ is a reduction of } I \},$$

where $\mu(J)$ denotes the minimal number of generators of $J$. The existence of the minimal reduction shows $\text{ara } I \leq l(I)$. On the other hand, it is known by Lyubeznik [3] that $\text{pd}_S S/I \leq \text{ara } I$, where $\text{pd}_S S/I$ denotes the projective dimension of $S/I$. Therefore we have the following inequalities:

$$\text{pd}_S S/I \leq \text{ara } I \leq l(I).$$

In the study of the arithmetical rank, Schmitt–Vogel lemma [5, Lemma, pp. 249] is an important and useful tool, because it gives a sufficient condition for ideals $J \subset I$ to hold $\sqrt{J} = \sqrt{I}$. In this report, we give a sufficient condition for an ideal $J$ with $J \subset I$ to be a reduction of $I$ by refining Schmitt–Vogel lemma. As an application of our theorem, we prove $l(I) = \text{pd}_S S/I$ for the ideal

$$I = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q}),$$

where $x_{11}, \ldots, x_{qi_q}$ are variables in $S$ pairwise distinct. Schmitt and Vogel [5] proved $\text{ara } I = \text{pd}_S S/I$ for this ideal $I$ using their lemma.
1. Main Theorem

In this section, we consider an arbitrary commutative ring \( R \) with unitary. Our main result of this report is the following:

**Theorem 1.1.** Let \( R \) be a commutative ring with unitary. Let \( P_0, P_1, \ldots, P_r \subset R \) be finite subsets, and we set

\[
P = \bigcup_{\ell=0}^{r} P_{\ell},
\]

\[
g_{\ell} = \sum_{a \in P_{\ell}} a, \quad \ell = 0, 1, \ldots, r.
\]

Assume that

(C1) \( \#P_0 = 1 \).

(C2) For all \( \ell > 0 \) and \( a, a'' \in P_{\ell} \) (\( a \neq a'' \)), there exist some \( \ell' \) (\( 0 \leq \ell' < \ell \)), \( a' \in P_{\ell'} \), and \( b \in (P) \) such that \( aa'' = a'b \).

Then we have \( (g_0, g_1, \ldots, g_r) \) is a reduction of \( (P) \).

On the other hand, Schmitt–Vogel lemma is the following:

**Proposition 1.2** (Schmitt–Vogel [5, Lemma, pp. 249]). Let \( R \) be a commutative ring with unitary. Let \( P_0, P_1, \ldots, P_r \subset R \) be finite subsets, and we set

\[
P = \bigcup_{\ell=0}^{r} P_{\ell},
\]

\[
g_{\ell} = \sum_{a \in P_{\ell}} a, \quad \ell = 0, 1, \ldots, r.
\]

Assume that

(C1) \( \#P_0 = 1 \).

(C2)' For all \( \ell > 0 \) and \( a, a'' \in P_{\ell} \) (\( a \neq a'' \)), there exist some \( \ell' \) (\( 0 \leq \ell' < \ell \)) and \( a' \in P_{\ell'} \) such that \( aa'' \in (a') \).

Then we have \( \sqrt{(g_0, g_1, \ldots, g_r)} = \sqrt{(P)} \).

Second condition of Theorem 1.1 is stronger than that of Schmitt–Vogel lemma, but Theorem 1.1 has a stronger conclusion than Schmitt–Vogel lemma.

**Remark 1.3.** Schmitt–Vogel lemma allows us to add some exponent \( e(a) \) for each \( a \in P_{\ell} \) in the sum \( g_{\ell} \), i.e., we may put

\[
g_{\ell} = \sum_{a \in P_{\ell}} a^{e(a)}.
\]

In particular, we can take \( g_{\ell} \) as homogeneous if \( R \) is graded. But a similar statement does not hold for our theorem.

Instead of proving Theorem 1.1, we will give a detailed explanation of an example in Section 3, which illustrates the outline of the proof of the theorem. See also [2].
2. An Application

In this section, we apply Theorem 1.1 to some ideals and calculate the analytic spread of them.

Consider the ideal

$$(2.1) \quad I = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q}),$$

where $x_{11}, \ldots, x_{qi_q}$ are variables in $S$ pairwise distinct.

**Lemma 2.1.** For the above ideal $I$,

$$pd_{S} S/I = \sum_{s=1}^{q} i_s - q + 1.$$  \hspace{1cm} \Box

**Proof.** For an integer $q \geq 1$, we set $I_q = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q})$, $r_q = \sum_{s=1}^{q} i_s - q + 1$. We prove the lemma by induction on $q$. The case $q = 1$ is clear. Suppose that $q \geq 2$. If we put $P = (x_{q1}, \ldots, x_{qi_q})$, then $I_q = I_{q-1} \cap P$ and $r_q = r_{q-1} + \text{height } P - 1 = r_{q-1} + pd_{S} S/P - 1$. Consider Mayer–Vietoris sequence

$$0 \rightarrow S/I_q \rightarrow S/I_{q-1} \oplus S/P \rightarrow S/(I_{q-1} + P) \rightarrow 0.$$

Since $pd_{S} S/I = \max\{i : Tor^S_i(k, S/I) \neq 0\}$, the long exact sequence

$$\cdots \rightarrow 0 = Tor^S_{r_q+1}(k, S/I_{q-1}) \oplus Tor^S_{r_q+1}(k, S/P) \rightarrow Tor^S_{r_q+1}(k, S/(I_{q-1} + P))$$

implies $r_q = pd_{S} S/I_q$. \hspace{1cm} \Box

Schmitt–Vogel [5] proved $\text{ara } I = pd_{S} S/I$ (see also Schenzel–Vogel [4]). They proved it by applying Schmitt–Vogel lemma to

$$P_\ell = \{x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q} : \ell_1 + \cdots + \ell_q = \ell + q\}, \quad \ell = 0, 1, \ldots, r,$$

where $r = \sum_{s=1}^{q} i_s - q$. These $P_0, P_1, \ldots, P_r$ also satisfy the assumption of Theorem 1.1. Thus $J = (g_0, g_1, \ldots, g_r)$ is a reduction of $I$. Since

$$r + 1 = pd_{S} S/I = \text{ara } I \leq l(I) \leq r + 1,$$

we have $l(I) = pd_{S} S/I$. Therefore we have the following corollary:

**Corollary 2.2.** Let $I = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q})$. Then we have

$$l(I) = pd_{S} S/I.$$  \hspace{1cm} \Box

In particular, $(g_0, g_1, \ldots, g_r)$ is a minimal reduction of $I$.

Note that we have a minimal reduction of $I$ explicitly.

**Remark 2.3.** In general, $l(I) \neq pd_{S} S/I$ for a squarefree monomial ideal $I$. For example, if $\mu(I) - \text{height}(I) = 1$ and $S/I$ is Cohen–Macaulay, then $\text{height}(I) = pd_{S} S/I = \text{ara } I < l(I) = \mu(I)$; see [1].
3. An Example

In this section, we give one example to illustrate the outline of the proof of Theorem 1.1.

Let us consider the ideal

\[ I = (x_1, x_2, x_3) \cap (y_1, y_2, y_3). \]

This is a special form of the ideal (2.1). The minimal graded resolution of \( S/I \) is

\[
0 \rightarrow S(-6) \rightarrow S(-5)^6 \rightarrow S(-4)^{15} \rightarrow S(-3)^{18} \rightarrow S(-2)^9 \rightarrow S \rightarrow S/I \rightarrow 0
\]

and \( \text{pd}_S S/I(=3+3-2+1)=5 \). Then

\[
\begin{align*}
P_0 &= \{x_1y_1\}, \\
P_1 &= \{x_1y_2, x_2y_1\}, \\
P_2 &= \{x_1y_3, x_2y_2, x_3y_1\}, \\
P_3 &= \{x_2y_3, x_3y_2\}, \\
P_4 &= \{x_3y_3\}.
\end{align*}
\]

Let us see conditions of Theorem 1.1. Since \( \#P_0 = 1 \), (C1) is satisfied. For the assumption (C2), we have the following equations:

\[
\begin{align*}
(3.1) & \quad P_1: \quad x_1y_2 \cdot x_2y_1 = x_1y_1 \cdot x_2y_2 \in (P_0)(P_2), \\
(3.2) & \quad P_2: \quad \left\{ \begin{array}{l}
x_1y_3 \cdot x_2y_2 = x_1y_2 \cdot x_2y_3 \in (P_1)(P_3), \\
x_2y_2 \cdot x_3y_1 = x_2y_1 \cdot x_3y_2 \in (P_1)(P_3), \
\end{array} \right.
\end{align*}
\]

\[
(3.3) \quad P_3: \quad x_2y_3 \cdot x_3y_2 = x_2y_2 \cdot x_3y_3 \in (P_2)(P_4).
\]

Thus (C2) is also satisfied.

Now we shall see \( J = (g_0, g_1, g_2, g_3, g_4) \) is a reduction of \( I \), where

\[
\begin{align*}
g_0 &= x_1y_1, \\
g_1 &= x_1y_2 + x_2y_1, \\
g_2 &= x_1y_3 + x_2y_2 + x_3y_1, \\
g_3 &= x_2y_3 + x_3y_2, \\
g_4 &= x_3y_3.
\end{align*}
\]

We put

\[
I_\ell = \left( \bigcup_{j=0}^{\ell} P_j \right), \quad \ell = 0, 1, 2, 3, 4.
\]

Note that \( I_4 = I \). It is enough to show

\[
I_\ell^{2^\ell} \subset JJ^{2^{\ell-1}}, \quad \ell = 0, 1, 2, 3, 4
\]

in order to see that \( J \) is a reduction of \( I \). We show this by induction on \( \ell \). In fact, we show

\[
I_\ell^{2^\ell} \subset I_{\ell-1}^{2^{\ell-1}} I^{2^\ell-2^{\ell-1}} + JJ^{2^{\ell-1}}, \quad \ell = 0, 1, 2, 3, 4.
\]
Step 1: The case $\ell = 0$. In this case, $I_0 = (P_0) = (x_1y_1) = (g_0) \subset J$.

Step 2: The case $\ell = 1$. We want to show $I_1^2 \subset I_0 I + JI$. To see this, it is enough to show that $a_1a_2 \in I_0 I + JI$ for all $a_1, a_2 \in P_1$ (we do not assume $a_1 \neq a_2$). When $a_1 \neq a_2$, (3.1) shows $a_1a_2 \in I_0 I$. When $a_1 = a_2 = a$, we use $g_1$. For example, 

$$(x_1y_2)^2 = (g_1 - x_2y_1)x_1y_2 = g_1x_1y_2 - x_2y_1 \cdot x_1y_2 = g_1x_1y_2 - x_1y_1 \cdot x_2y_2 \in JI + I_0 I.$$ 

Therefore $I_1^2 \subset I_0 I + JI$ holds.

Step 3: The case $\ell = 2$. We want to show $I_2^3 \subset I_1^2 I^2 + JI^3$. To see this, we only check $a_1a_2a_3a_4 \in I_1^2 I^2 + JI^3$ for all $a_1, a_2, a_3, a_4 \in P_0 \cup P_1 \cup P_2$. There are two cases:

(i) $a_1, a_2, a_3, a_4 \in P_2$;

(ii) for some $i$, $a_i \in P_0 \cup P_1$.

In case (i), there are two cases dividing large. The first one is that $a_1 \neq a_2$ and $a_3 \neq a_4$ by renumbering $a_1, a_2, a_3, a_4$. In this case, it is easy to check $a_1a_2a_3a_4 \in I_1^2 I^2$ because of (3.2). For example,

$$(x_1y_3)^2x_2y_2 \cdot x_3y_1 = (x_1y_3 \cdot x_2y_2)(x_1y_3 \cdot x_3y_1) = (x_1y_2 \cdot x_1y_1)(x_2y_3 \cdot x_3y_3) \in I_1^2 I^2.$$ 

The second one is that there are no such a renumbering on $a_1, a_2, a_3, a_4$. In this case, we use $g_2$ as in the case $\ell = 1$. For example,

$$(x_1y_3)^3x_2y_2 = (x_1y_3)^2(g_2 - x_2y_2 - x_3y_1)x_2y_2$$

$$= g_2(x_1y_3)^2x_2y_2 - (x_1y_3)^2(x_2y_2)^2 - (x_1y_3)^2x_3y_1 \cdot x_2y_2$$

$$= g_2(x_1y_3)^2x_2y_2 - (x_1y_3 \cdot x_2y_2)^2 - (x_1y_3 \cdot x_3y_1)(x_1y_3 \cdot x_2y_2)$$

$$\in JI^3 + I_1^2 I^2.$$ 

In case (ii), if there are two indices $i$ (say, $i_1, i_2$) such that $a_i \in P_0 \cup P_1$, then $a_{i_1}a_{i_2} \in I_1^2$ and $a_1a_2a_3a_4 \in I_1^2 I^2$ hold. Next, we consider the case that there is only one $i$ such that $a_i \in P_0 \cup P_1$. We may assume $a_1 \in P_0 \cup P_1$ and $a_2, a_3, a_4 \in P_2$. Then we need to make only one pair of distinct elements from $a_2, a_3, a_4$. It is weaker requirement than that of case (i). In fact, to make one pair of distinct elements, we only need two of $a_2, a_3, a_4$. For example,

$$(x_1y_3)^2 = x_1y_3(g_2 - x_2y_2 - x_3y_1)$$

$$= g_2x_1y_3 - x_1y_3 \cdot x_2y_2 - x_1y_3 \cdot x_3y_1$$

$$\in JI + I_1 I.$$ 

Step 4: The case $\ell = 3$. We want to show $I_3^5 \subset I_2^4 I^4 + JI^7$. In this case, the same argument as in Step 3 is also usable. We omit here.

Step 5: The case $\ell = 4$. It is clear that $I_4^{16} \subset I_3^8 I^8 + JI^{15}$ since $\#P_4 = 1$. Therefore we obtain that $J$ is a reduction of $I$.

Remark 3.1. The reduction number $r_J(I)$ is defined by

$$r_J(I) := \min\{s : I^{s+1} = JI^s \}.$$
Above argument gives an upper bound of $r_J(I)$. But this is very big in general. In fact, in the above argument, we only see $I^{2^4} = JI^{2^4-1}$, that is, $r_J(I) \leq 2^4 - 1 = 15$. But $r_J(I) = 3$, i.e., $I^4 = JI^3$ holds.

REFERENCES


