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Lagrangian Simulation of Vortex Sheet Dynamics

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1 Introduction

Vorticity is the curl of velocity, $\omega = \nabla \times u$. Before discussing vortex dynamics, we note that vortices are formed by the motion of a solid body in a fluid. This was appreciated by scientists in the 19th century; for example consider the comment by Felix Klein [11], referring to the stirring of a spoon in a cup of coffee, *"At the end of his famous dissertation on the motion of vortices, Helmholtz describes an easy method to produce vortices, that anyone can try out anyday in their cup of coffee. ... Of course one may demand that the motion which has been described here only qualitatively be formulated quantitatively, that is, derived from the differential equations of hydrodynamics. ... I will leave this task to other mathematicians ... (August 20, 1909)."* Indeed these matters continue to interest mathematicians and other scientists to the present day.

The topics to be discussed in this report are: incompressible fluid flow, Lagrangian formulation, Kelvin-Helmholtz instability, vortex sheet model, point vortex method, regularized vortex sheet motion, chaotic dynamics in vortex sheet flow, and vortex ring simulations. For background we cite Chorin-Marsden [5] for basic fluid dynamics and Majda-Bertozzi [17] for mathematical theory.

2 Incompressible fluid flow

With velocity $u(x,t)$ and pressure $p(x,t)$, the incompressible Navier-Stokes equations are

$$u_t + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \quad \nabla \cdot u = 0, \tag{1}$$

where ν is the fluid viscosity. We're especially interested in the zero-viscosity limit, $\nu \to 0$, and taking the curl in this limit gives the vorticity equation,

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u. \tag{2}$$

The velocity can be recovered from the vorticity by convolution,

$$u(x,t) = \int K(x - \tilde{x}) \times \omega(\tilde{x}, t) d\tilde{x}, \tag{3}$$

where

$$K(x) = -\frac{x}{4\pi |x|^3} \tag{4}$$

is the Biot-Savart kernel and the integral is over all space. The flow map is denoted by $x(\alpha, t)$, where α is a Lagrangian parameter labelling the fluid particles. The equation for the flow map is a version of Equation (3),

$$\frac{\partial x}{\partial t}(\alpha, t) = \int K(x(\alpha,t) - x(\tilde{\alpha}, t)) \times \nabla_\alpha x(\tilde{\alpha}, t) \cdot \omega_0(\tilde{\alpha})\, d\tilde{\alpha}, \tag{5}$$

where $\omega_0(\alpha)$ is the initial vorticity. This is a Lagrangian formulation of vortex dynamics, in contrast to the Eulerian formulation in Equation (2).

3 Kelvin-Helmholtz instability

Figure 1 shows an experimental visualization of Kelvin-Helmholtz instability in a shear layer. The flow is from left to right and the upper stream is moving faster than the lower stream. The lower stream has been seeded with flourescent particles which are illuminated by a laser sheet. The interface rolls up into a sequence of spiral vortices, a generic pattern in shear flows. There is much interest in understanding the growth, structure, and further dynamics of the vortices. Motivated by our interest in the zero-viscosity limit, we shall consider the vortex sheet model in which the transition zone between the two streams has zero thickness.

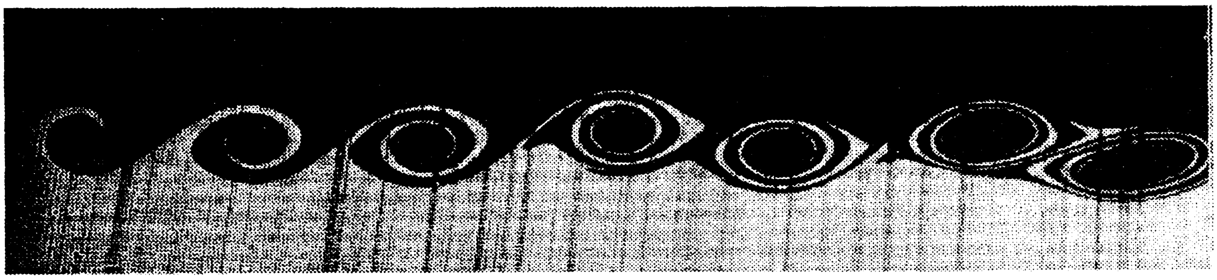

Figure 1: Experimental visualization of Kelvin-Helmholtz instability in a shear layer, photograph by F.A. Roberts, P.E. Dimotakis and A. Roshko, in M. Van Dyke (1982) *An Album of Fluid Motion*, Parabolic Press.

4 Vortex sheet model

In the vortex sheet model of a shear layer, the tangential fluid velocity has a discontinuous jump across the interface between the streams. A schematic picture in 2D is shown in Figure 2. The sheet is defined by a curve in 2D flow and a surface in 3D flow. In either case, the vorticity is a delta-function supported on the interface.

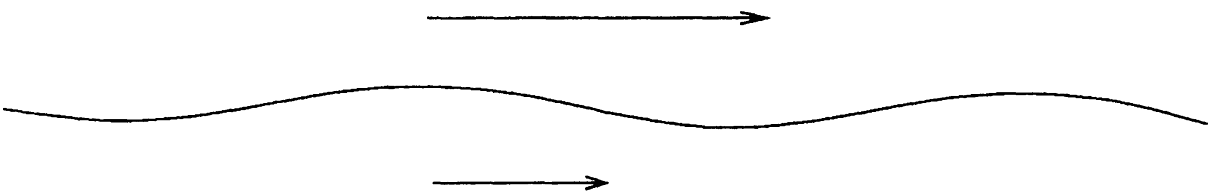

Figure 2: Schematic picture of a vortex sheet in 2D flow.

If the sheet is initially flat, then it remains so forever. According to linear stability theory, there exist exponential perturbations that grow arbitrarily fast and this implies that the initial value problem is ill-posed in the sense of Hadamard. To discuss the nonlinear dynamics, we consider the vortex sheet as a complex curve, $z(\alpha, t)$, where α is a parameter along the sheet. The Birkhoff-Rott equation governs the nonlinear evolution of the sheet and is given by

$$\frac{\partial \bar{z}}{\partial t}(\alpha, t) = \mathrm{pv} \int K(z(\alpha, t) - z(\tilde{\alpha}, t)) d\tilde{\alpha}, \tag{6}$$

where pv denotes the principal value and

$$K(z) = \frac{1}{2\pi i z} \tag{7}$$

is the Cauchy kernel [1]. Equation (6) is simply the flow map Equation (5) for the case of a vortex sheet in 2D flow. Concerning the mathematical theory, Sulem et al. [23] proved local existence of analytic solutions and Moore [18] showed using formal asymptotics that a curvature singularity forms at a finite time. Several groups have proven the existence of global weak solutions when the vorticity has one sign [17]. There has been further study of the Moore singularity [2] and analytic solutions [24, 25]. However the aim of this report is to describe numerical simulations.

5 Point vortex method

Figure 3 depicts the approximation of a continuous vortex sheet by discrete point vortices. Letting $z_j(t), j = 1 : N$ denote the points and discretizing the Birkhoff-Rott equation (6), we obtain ODEs for the motion of the points,

$$\frac{d\bar{z}_j}{dt} = \sum_{\substack{k=1 \\ k \neq j}}^{N} K(z_j - z_k)\, w_k, \tag{8}$$

where w_k are suitable quadrature weights. Early computations using small values of N gave apparently smooth roll-up, but Birkhoff [1] used larger values of N and obtained irregular motion. There was a confused state of affairs until the work of Moore [18] on singularity formation. It was then shown numerically that the point vortex method converges up to the critical time [12], but it is necessary to regularize the problem to proceed further in time into the roll-up regime.

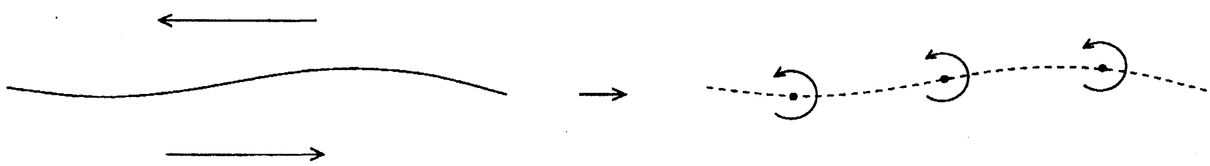

Figure 3: Point vortex approximation of a vortex sheet.

6 Regularized vortex sheet motion

The vortex blob method for vortex sheets was introduced by Chorin and Bernard [4]. The term vortex-blob means that the vorticity is smeared out in space as opposed to being localized as for a point vortex. This amounts to replacing the singular Cauchy kernel by a regularized approximation, and the following form is popular,

$$K_\delta(z) = \frac{1}{2\pi i z} \cdot \frac{|z|^2}{|z|^2+\delta^2}, \tag{9}$$

where δ is a smoothing parameter [13]. Figure 4 shows the solution obtained with $\delta = 0.25$ and $N = 400$, and periodic boundary conditions. The results resemble the experiment in Figure 1. Recent computations have demonstrated the onset of chaos as the smoothing parameter δ is made smaller.

7 Chaos in vortex sheet flow

Figure 5 shows the computation of a vortex sheet in free space [15]. The flow is axisymmetric and a cross-section is shown, and the sheet in this case represents a vortex ring. These computations use adaptive point insertion to maintain resolution. The sheet rolls up smoothly at early times, but an irregular wake appears at late times due to chaotic dynamics. Rom-Kedar, Leonard and Wiggins [20] analyzed a relevant point vortex model and the underlying mechanism is similar to the strained elliptic vortex discovered by Kida [10]. The results indicate that vortex cores undergo a self-sustained oscillation leading to resonances with the underlying rotational motion in the core.

8 Vortex sheet dynamics in 3D

The stability of vortex rings is a topic of considerable interest [8]. We imagine that the ring is initially a circular disk vortex sheet, shown schematically in Figure 6 (left). The surface is defined by $x(\Gamma, \theta, t)$, where Γ and θ are Lagrangian parameters; Γ is the circulation across the vortex filaments and θ is a parameter along the filaments. The evolution equation is

$$\frac{\partial x}{\partial t} = \mathrm{pv} \iint K(x, \tilde{x}) \times \frac{\partial \tilde{x}}{\partial \theta} d\tilde{\Gamma} d\tilde{\theta}, \tag{10}$$

where $x = x(\Gamma, \theta, t), \tilde{x} = x(\tilde{\Gamma}, \tilde{\theta}, t)$, and $K(x, y)$ is the Biot-Savart kernel from Equation (4), as derived by Caflisch [3] and Kaneda [9]. This is another analog of the flow map Equation (5). In the computation, the sheet is

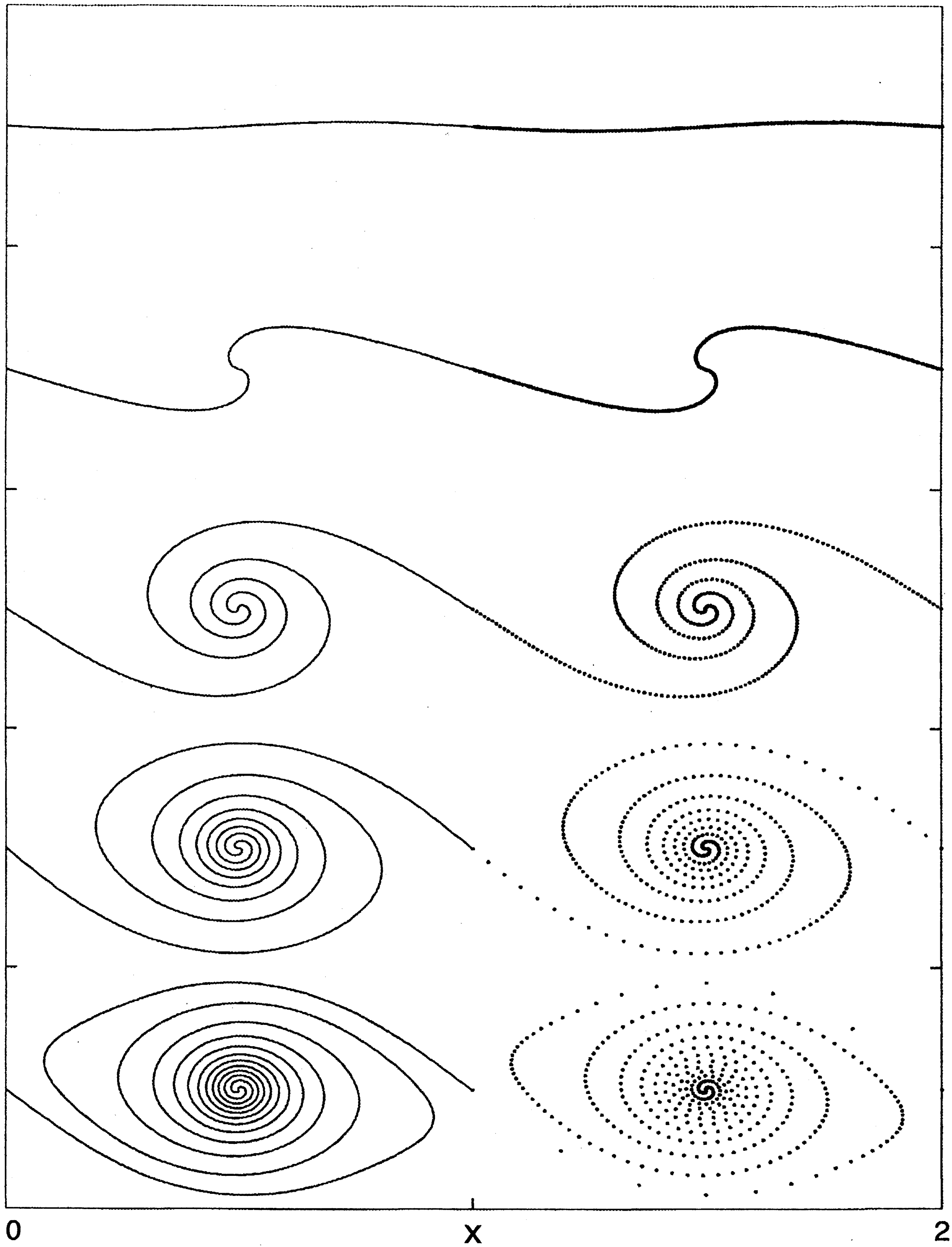

Figure 4: Time sequence of regularized vortex sheet roll-up with smoothing parameter $\delta = 0.25$ and $N = 400$ points [13]. Two wavelengths are plotted, the points are on the right and an interpolating curve is on the left.

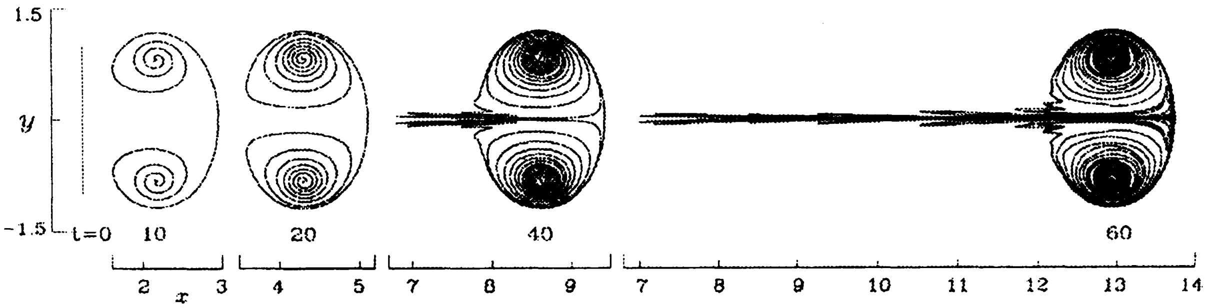

Figure 5: Time sequence of axisymmetric vortex sheet roll-up into a vortex ring (cross-section). The wake is due to chaotic dynamics in the flow [15].

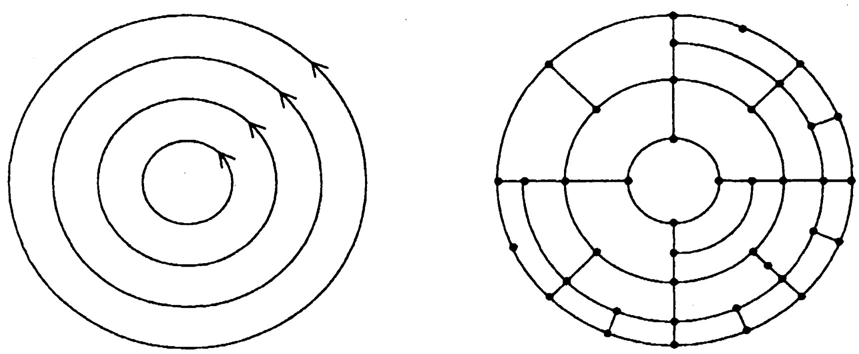

Figure 6: Schematic picture of a circular disk vortex sheet in 3D flow. left: vortex filaments shown as circles, right: discretization by panels [7].

represented by a set of particles, $x_i(t), i = 1 : N$, which evolve by the system

$$\frac{dx_i}{dt} = \sum_{i=1}^{N} K_\delta(x_i, x_j) \times w_j, \tag{11}$$

where w_j are suitable weights. We use the following regularized Biot-Savart kernel in 3D,

$$K_\delta(x, y) = -\frac{x - y}{4\pi(|x - y|^2 + \delta^2)^{3/2}}. \tag{12}$$

There are two practical issues. First, we use a multipole treecode to reduce the cost of evaluating the right side of Equation (11) from $O(N^2)$ to $O(N \log N)$ [6, 22, 16]. Second, we use a panel method to discretize the sheet, shown schematically in Figure 6 (right) [7]. Figure 7 presents a simulation of azimuthal instability of a vortex ring using this method. The simulation started with $N = 7785$ and finished with $N = 1,117,875$. The results exhibit core axial flow as seen experimentally by Naitoh et al. [19].

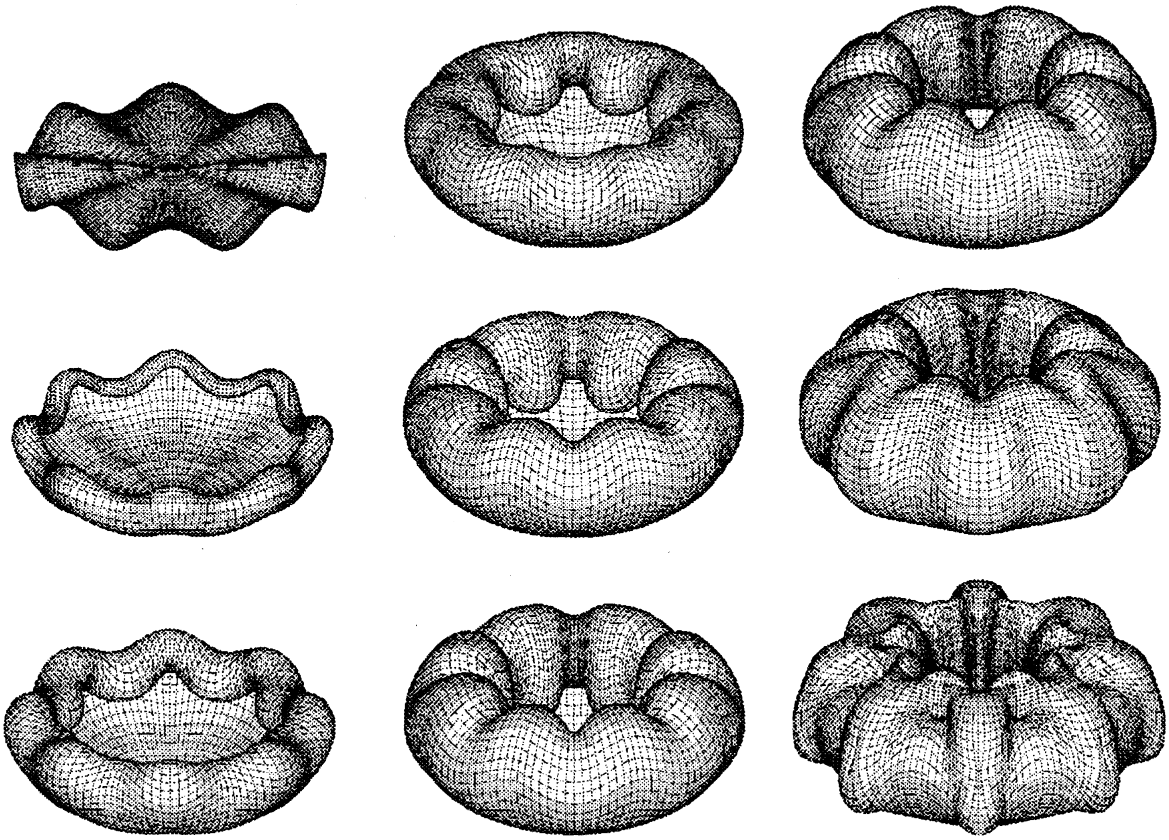

Figure 7: Time sequence of azimuthal instability of a vortex ring in 3D flow. The panels representing the sheet surface are shown as a wire mesh [7].

9 Conclusion

The mathematical and numerical study of vortex sheets is ongoing. Some opens problems include the further understanding of weak solutions and chaotic dynamics, and the numerical resolution of highly twisted vortex sheet surfaces in 3D flow. Vortex sheet separation from a sharp edge in 3D flow also requires further study.

Acknowledgements

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