Construction of a limiting Carleman weight

Hideki TAKUWA

1   Background and motivation

Let n be an integer greater than or equal to 2 and $x = (x^1, \cdots, x^n) \in \Omega$, where Ω is a bounded domain in $\mathbf{R}^n$. For $A(x) = (A_j(x))_{1 \leq j \leq n} \in C^2(\bar{\Omega}, \mathbf{R}^n)$, and $q \in L^\infty(\Omega, \mathbf{C})$, the linear operator of second order is defined as

$$(1.\ 1) \qquad \mathcal{L}(x, D_x) = \sum_{j=1}^n (D_{x^j} + A_j(x))^2 + q(x),$$

where $D_{x^j} = -i\,\partial_{x^j}$ $(i = \sqrt{-1})$. We assume that 0 is not an eigenvalue of the operator $\mathcal{L}(x, D_x) : H^2(\Omega) \cap H_0^1 \to L^2(\Omega)$. For $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ to the boundary value problem

$$(1.\ 2) \qquad \begin{cases} \mathcal{L}(x, D_x)u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega. \end{cases}$$

For $A \in C^2(\bar{\Omega}, \mathbf{R}^n)$ and $q \in L^\infty(\Omega)$, the operator $\mathcal{N}_{A,q} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is defined by $\mathcal{N}_{A,q}(f) = (\frac{\partial}{\partial\nu} + iA \cdot \nu)u|_{\partial\Omega}$ for $f \in H^{1/2}(\partial\Omega)$, where $\nu = \nu(x)$ is the unit outer normal vector on $\partial\Omega$. The map $\mathcal{N}_{A,q}$ is called Dirichlet-to-Neumann map.

The inverse problem is that the potential dA and q are determined from the map Dirichlet-to-Neumann map $\mathcal{N}_{A,q}$. Since Calderón first proposed and gave a method to solve this problem, this problem is often called Calderón problem.

For the operator $\mathcal{L}(x, D_x)$, Green formula gives

$$(1.\ 3) \qquad \begin{aligned} & \big(\mathcal{L}(x, D_x)u, v\big)_{L^2(\Omega)} - \big(u, \mathcal{L}(x, D_x)v\big)_{L^2(\Omega)} \\ & = \Big(u, \big(\frac{\partial}{\partial\nu} + i\nu \cdot A\big)v\Big)_{L^2(\partial\Omega)} - \Big(\big(\frac{\partial}{\partial\nu} + i\nu \cdot A\big)u, v\Big)_{L^2(\partial\Omega)} \end{aligned}$$

for $u, v \in H^1(\Omega)$ with $\Delta u, \Delta v \in L^2(\Omega)$, where $(f, g)_{L^2(\Omega)} = \int_\Omega f(x)\overline{g(x)}dx$ and $(f, g)_{L^2(\partial\Omega)} = \int_{\partial\Omega} f(x)\overline{g(x)}dS(x)$. Calderón proposed that the special solution is useful to our inverse problem. After the pioneering work by Calderón, the many improvements were done in this topics. The brief history can be found in [5] and its references.

We study the construction of the special solution for the equation in (1.2). Let $h > 0$. The special solution $u = u(x; h)$ has the form

$$(1.\ 4) \qquad u(x; h) = e^{\frac{1}{h}\Phi(x)}\big(a_0(x) + hr(x; h)\big).$$

Roughly speaking, Calderón and Bukhgeim-Uhlmann used the linear phase function $\Phi_0(x) = (a + ib) \cdot x$ for $a, b \in \mathbf{R}^n$ with $|a| = |b|$ and $a \cdot b = 0$. Since the equality

$$\Delta\Big(e^{\frac{1}{h}(a+ib)\cdot x}\Big) = \frac{1}{h^2}\big\{(|a|^2 - |b|^2) + 2ia \cdot b\big\}e^{\frac{1}{h}(a+ib)\cdot x}$$

The importance for the linear complex valued phase function $\Phi_0(x)$ is explained.

For $\Phi(x) \in C^\infty(\Omega, \mathbf{C})$

$$(1.\ 5)$$

$$e^{-\frac{1}{h}\Phi(x)}\mathcal{L}_{A,q}(x, D_x)\Big(e^{-\frac{1}{h}\Phi(x)}\big(a_0(x) + hr(x; h)\big)\Big)$$

$$= \Big\{\sum_{j=1}^{n}\Big(D_{x^j} + \frac{1}{ih}\frac{\partial}{\partial x^j}\Phi(x) + A_j(x)\Big)^2 + q(x)\Big\}\big(a_0(x) + hr(x; h)\big)$$

$$= \Big\{-\frac{1}{h^2}\sum_{j=1}^{n}\Big(\frac{\partial\Phi}{\partial x^j}\Big)^2 + \frac{1}{h}L_1(x, D_x) + \mathcal{L}_{A,q}(x, D_x)\Big\}\big(a_0(x) + hr(x; h)\big)$$

where $L_1(x, D_x)$ is the linear operator of first order with complex coefficients.

Kenig-Sjöstrand-Uhlmann [3] showed the strategy to construct the special solution (1.4). this idea had already used in the previous results (for example, see [1]). The strategy consists of three steps. The first step is to solve the eikonal equation $p_2(\nabla\Phi(x)) = \sum_{j=1}^{n}(\partial_{x^j}\Phi(x))^2 = 0$ and determine the complex phase function $\Phi(x) = \varphi(x) + i\psi(x)$. The second step is to determine the amplitude function $a_0(x)$ by solving the transport equation $L_1(x, D_x)a_0(x) = 0$. Since the phase function $\Phi(x)$ is complex valued, the transport equation of principal type should be satisfied a condition for bicharacteristics. The final step is to determine the lower order term $r(x; h)$. The combination between Carleman estimate, that is, a weighted L^2 estimate, and Riesz representation theorem, gives the existence of the function $r(x; h)$.

In Carleman estimate, the real part $\varphi(x)$ of the phase function $\Phi(x)$ is the weight function for the estimate.

Green formula (1.3) and the special solution (1.4) show the information of some transformations of the potentials dA and q. The support theorem for them implies the uniqueness results. The original works as [1] used the linear phase functions $\Phi_0(x) = (a + ib) \cdot x$, so the support theorem for Fourier transform gives the uniqueness.

Kenig-Sjöstrand-Uhlmann [3] proposed the new phase function $\operatorname{Re} \Phi(x) = \varphi(x) = \log|x|$. They call this function a limiting Carleman weight. This name came from the third step as above. To show Carleman estimate, Poisson bracket vanishes identically on the zero set of the symbol. The limiting Carleman weight will be defined in the next section.

2 Construction of a limiting Carleman weight of radial type

The solvability of the eikonal equation is discussed in this section.

Let Ω be a domain in $\mathbb{R}^n$ with $\Omega^c = \mathbb{R}^n \backslash \Omega \neq \emptyset$, $x = (x^1, \cdot, x^n) \in \Omega$ and $x_0 \in \Omega^c$. Let $G = (g_{jk})_{1 \leq j,k \leq n}$ be a real constant matrix with $g_{jk} = g_{kj} \in \mathbb{R}$ and $\det G \neq 0$. Its inverse matrix is denoted by $G^{-1} = (g^{jk})$. The matrix G define the pseudo-Riemann metric g on $\mathbb{R}^n$ and the pseudo-distance function $d(x, x_0)$ between $x \in \Omega$ and $x_0 \in \Omega^c$ on $(\mathbb{R}^n, g)$ by

$$g = \sum_{j,k=1}^{n} g_{jk} dx^j dx^k$$

(2. 1)

$$\{d(x, x_0)\}^2 = \|x - x_0\|_G^2 = \sum_{j,k=1}^{n} g_{jk}(x^j - x_0^j)(x^k - x_0^k)$$

respectively. We shall construct a solution $\Phi(x) = \varphi(x) + i\psi(x) \in C^2(\Omega, \mathbb{C})$ to the eikonal equation

$$0 = p_2(\nabla \Phi(x)) = \sum_{j,k=1}^{n} g^{jk} \frac{\partial \Phi(x)}{\partial x^j} \frac{\partial \Phi(x)}{\partial x^k}$$

(2. 2)

$$= \sum_{j,k=1}^{n} g^{jk} \left(\frac{\partial \varphi(x)}{\partial x^j} \frac{\partial \varphi(x)}{\partial x^k} - \frac{\partial \psi(x)}{\partial x^j} \frac{\partial \psi(x)}{\partial x^k} \right)$$

$$+ 2i \sum_{j,k=1}^{n} g^{jk} \frac{\partial \varphi(x)}{\partial x^j} \frac{\partial \psi(x)}{\partial x^k},$$

where $\nabla = (\partial_x^1, \cdot \partial x^n)$. The eikonal equation (2.2) appears if we construct a special solution (1.4) to the linear equation of second order

$$P(x, D_x)u = \left\{ \sum_{j,k=1}^{n} g^{jk}(D_{x^j} - A_j(x))(D_{x^k} - A_k) + q(x) \right\} u(x; h) = 0.$$

The eikonal equation (2.2) is obtained as we get it as the top term in (1.5) for the flat Laplacian $\Delta = \sum_{j=1}^{n} \partial_{x^j}^2$. The solvability of the eikonal equation $p_2(\nabla\Phi(x)) = 0$ is equivalent to find two functions φ and ψ that satisfy

$$\begin{aligned}
(2.\ 3) \qquad & \sum_{j,k=1}^{n} g^{jk}\left(\frac{\partial\,\varphi(x)}{\partial\,x^j}\frac{\partial\,\varphi(x)}{\partial\,x^k} - \frac{\partial\,\psi(x)}{\partial\,x^j}\frac{\partial\,\psi(x)}{\partial\,x^k}\right) = 0, \\
& \sum_{j,k=1}^{n} g^{jk}\frac{\partial\,\varphi(x)}{\partial\,x^j}\frac{\partial\,\psi(x)}{\partial\,x^k} = 0.
\end{aligned}$$

Let the real part $\varphi(x) = \operatorname{Re}\Phi(x)$ be fixed. The symbols $a(x,\xi)$ and $b(x,\xi)$ are defined by

$$\begin{aligned}
(2.\ 4) \qquad & a(x,\xi) = \sum_{j,k=1}^{n} g^{jk}\left(\frac{\partial\,\varphi(x)}{\partial\,x^j}\frac{\partial\,\varphi(x)}{\partial\,x^k} - \xi_j\xi_k\right), \\
& b(x,\xi) = \sum_{j,k=1}^{n} g^{jk}\frac{\partial\,\varphi(x)}{\partial\,x^j}\xi_k.
\end{aligned}$$

Imaginary part $\psi(x) = \operatorname{Im}\Psi(x)$ is the solution to the system of nonlinear partial differential equations of first order

$$(2.\ 5) \qquad a(x, \nabla\psi(x)) = b(x, \nabla\psi(x)) = 0 \qquad \text{in } \Omega.$$

A class of phase functions is defined to guarantee the solvability of the system of equations (2.5):

Definition 2.1 *The function $\varphi(x) \in C^2(\Omega, \mathbf{R})$ is called a limiting Carleman weight for $P(D_x) = \sum_{j,k=1}^{n} g^{jk} D_{x^j} D_{x^k}$ on Ω if and only if*

$$(2.\ 6) \qquad \{a, b\}(x, \xi) = \sum_{j=1}^{n}\left(\frac{\partial\,a}{\partial\,\xi_j}\frac{\partial\,b}{\partial\,x^j} - \frac{\partial\,a}{\partial\,x^j}\frac{\partial\,b}{\partial\,\xi_j}\right)(x, \xi) = 0,$$

on $J = \{(x,\xi) \in T^*\Omega \mid a(x,\xi) = b(x,\xi) = 0\}$.

If the set J is a submanifold in $T^*\Omega$, this condition means that the manifold J is involutive. This condition is well known for the solvability of the system of first order equations.

For the flat Laplacian Δ, that is, $(G = Id)$, Poisson bracket is calculated as

$$(2.\ 7) \qquad \{a, b\}(x, \xi) = -8 \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial x^j \partial x^k}(x) \left(\frac{\partial \varphi}{\partial x^j}(x) \frac{\partial \varphi}{\partial x^k}(x) - \xi_j \xi_k \right)$$

Poisson bracket for the linear phase $\varphi_1(x) = a \cdot x$ vanishes identically for all part of $T^*\Omega$. Another example is the radial function $\varphi_1(x) = \log|x - x_0|$. where $|x - x_0|$ is the usual distance in $\mathbf{R}^n$. This function was given by Kenig-Sjöstrand-Uhlmann [3]. When the spatial dimension n is equal to 2, there are plenty of limiting Carleman weight. For $n = 2$, set $z = x^1 + ix^2$ and $f(z) = \varphi(x^1, x^2) + i\psi(x^1, x^2)$ for $\varphi, \psi \in C^\infty(D, \mathbb{R})$, where D is a domain in $\mathbf{R}^2$. If the function $f(z)$ is holomorphic, Cauchy-Riemann equations

$$\frac{\partial \varphi}{\partial x^1}(x^1, x^2) = \frac{\partial \psi}{\partial x^2}(x^1, x^2), \qquad \frac{\partial \varphi}{\partial x^2}(x^1, x^2) = -\frac{\partial \psi}{\partial x^1}(x^1, x^2).$$

The properties in (2.3) for $n = 2$ and $G = Id$ are easily obtained from Cauchy-Riemann equations. Uhlmann-Wang [6] used polynomial functions $\varphi(x_1, x_2) = \mathrm{Re}\,(x_1 + ix_2)^k$ and applied this weight function to an inverse problem.

Kenig-Sjöstrand-Uhlmann [3] proposed the limiting Carleman weight φ_1 and its counter part ψ_1 as

$$(2.\ 8) \qquad \begin{aligned} \varphi_1(x) &= \log|x - x_0|, \\ \psi_1(x) &= \mathrm{dist}_{S^{n-1}}\left(\frac{x - x_0}{|x - x_0|}, \omega \right) = \theta, \quad \omega \in S^{n-1} \end{aligned}$$

where $s^{n-1} = \{\omega \in \mathbf{R}^n|\ |\omega| = (\sum_{j=1}^{n} \omega^j)^{1/2} = 1\}$ and $\mathrm{dist}_{S^{n-1}}(p, q)$ is the distance for $p, q \in S^{n-1}$. The exact expression of ψ is calculate from the property

$$\cos\theta = \left\langle \frac{x - x_0}{|x - x_0|}, \omega \right\rangle = \sum_{j=1}^{n} \frac{1}{|x - x_0|}(x^j - x_0^j)\omega^j$$

Since there are plenty of holomorphic functions, the limiting Carleman weights should be much. One of the motivations of this article is to explain the meaning of the limiting Carleman weight of the radial type as $\varphi_1(x) = \log|x - x_0|$. In fact, the limiting Carleman weight of radial type can be constructed even for non-elliptic case.

Theorem 2.1 *Let* $f = f(t) \in C^2(\mathbf{R}\backslash\{0\}, \mathbf{R})$ *with* $f'(t) \neq 0$ *for* $t \neq 0$. *Set the function* $\varphi(x)$ *of radial type associated with* G *by*

(2. 9) $\qquad \varphi(x) = f(t)|_{t=||x-x_0||_G^2} = f(||x - x_0||_G^2).$

Then (A) *and* (B) *are equivalent:*

(A) *The function* $\varphi(x)$ *is a limiting Carleman weight for* $P(D_x)$.

(2. 10)
(B) $\quad f(t) = \begin{cases} C_1 \log t + C_2 & t > 0 \\ C_3 \log(-t) + C_4 & t < 0, \end{cases} \qquad C_1, C_2, C_3, C_4 \in \mathbf{R}.$

The sketch of proof is given in Section 3.

Once the limiting Carleman weight of radial type φ is constructed, any spherical functions $\psi = \psi(x)$ are orthogonal to the function φ with respect to G.

Lemma 2.1 *Let* M *be a real symmetric matrix with* $\det M \neq 0$ *For the function of radial type* $\varphi(x) = f(||x - x_0||_G^2)$ *with respect to* G, *the spherical function* $\psi(x) = g(s^1, \cdots, s^n)|_{s=\frac{x-x_0}{||x-x_0||_M}}$ *for* M *is orthogonal to* $\varphi(x)$ *in the following sense*

(2. 11) $\qquad \langle \varphi, \psi \rangle_G = \sum_{j,k=1}^{n} g^{jk} \frac{\partial \varphi}{\partial x^j}(x) \frac{\partial \psi}{\partial x^k}(x) = b(x, \nabla \psi) = 0,$

for $x \in \{x \in \Omega | \, ||x - x_0||_M \neq 0\}$.

To satisfy the distance condition $0 = a(x, \nabla \psi) = ||\nabla \psi||_G^2 - ||\nabla \varphi||_G^2$, the matrix M is taken as $M = G$.

Lemma 2.2 *Let* $\varphi(x) = \frac{1}{2} \log ||x - x_0||_G^2$. *For* $\omega \in \mathbf{R}^n$ *with* $||\omega||_G^2 = 1$, *set the family of functions* $h = h(s, \omega)$ *that are obtained by*

(2. 12) $\qquad \frac{\partial h}{\partial s^j}(s) = \frac{\omega^j}{\sqrt{1 - \langle \omega, s \rangle^2}}.$

Set the matrix $M = G$ *in Lemma 2.2. Then the function*

(2. 13) $\qquad \psi(x) = h(s^1, \cdots, s^n)|_{s=\frac{x-x_0}{||x-x_0||_G}} = h\left(\frac{x - x_0}{||x - x_0||_G}\right),$

satisfies

(2. 14) $\qquad a(x, \nabla \psi(x)) = 0 \quad x \in \Omega,$

It follows from Theorem 2.1, Lemma 2.1 and 2.1 that the special solutions $\Phi(x; x_0, \omega) = \varphi(x) + i\psi(x)$ to the eikonal equation (2.2) is obtained as

(2. 15) $\qquad \varphi(x) = \frac{1}{2} \log ||x - x_0||_G^2, \qquad \psi(x) = h(s^1, \cdots, s^n)|_{s=\frac{x-x_0}{||x-x_0||_G}},$

where the function h is defined in Lemma 2.2.

3 Idea of proof for Theorem 2.1

In this section the proof for Theorem 2.1 is given. The proof of the others are given in [4].

Sketch of proof for Theorem 2.1: For the distance function $d(x, x_0) = ||x - x_0||_G$, we have

$$\frac{\partial}{\partial x^l}||x - x_0||_G^2 = \frac{\partial}{\partial x^l}\sum_{j,k=1}^{n} g_{jk}(x^j - x_0^j)(x^k - x_0^k) = 2\sum_{j=1}^{n} g_{jl}(x^l - x_0^l).$$

Since we have obtained

$$(3.\ 1) \qquad \frac{\partial \varphi}{\partial x^l}(x) = \frac{df}{dt}(t)\Big|_{t=||x-x_0||_G^2} \times 2\sum_{\alpha=1}^{n} g_{l\alpha}(x^\alpha - x_0^\alpha),$$

and

$$(3.\ 2)$$
$$\frac{\partial^2 \varphi}{\partial x_j\, \partial x_k}(x) = 2\frac{df}{dt}(t)\Big|_{t=||x-x_0||_G^2} g_{jk}$$
$$+ 4\frac{d^2 f}{dt^2}(t)\Big|_{t=||x-x_0||_G^2} \Big(\sum_{p=1}^{n} g_{jp}(x^p - x_0^p)\Big)\Big(\sum_{q=1}^{n} g_{kq}(x^q - x_0^q)\Big),$$

the symbols $a = a(x, \xi)$ and $b = b(x, \xi)$ are expressed for the limiting Carleman weight of radial type as

$$(3.\ 3)$$
$$a(x, \xi) = \sum_{j,k=1}^{n} g^{jk}\Big(\frac{\partial \varphi(x)}{\partial x^j}\frac{\partial \varphi(x)}{\partial x^k} - \xi_j\xi_k\Big)$$
$$= 4\sum_{p,q=1}^{n} g_{pq}(x^p - x_0^p)(x^q - x_0^q) - \sum_{j,k=1}^{n} g^{jk}\xi_j\xi_k$$
$$= 4(f')^2||x - x_0||_G^2 - ||\xi||_{G^{-1}}^2,$$

and

$$(3.\ 4)$$
$$b(x, \xi) = \sum_{j,k=1}^{n} g^{jk}\frac{\partial \varphi}{\partial x^j}\xi_k = 2f'(||x - x_0||_G^2)\sum_{j=1}^{n}(x^j - x_0^j)\xi_j$$
$$= 2f'\langle x - x_0, \xi\rangle.$$

Since Poisson bracket between a and b for the general case are obtain as

$$
\{a,b\}(x,\xi) = -2\sum_{j,k=1}^{n}\frac{\partial^2 \varphi}{\partial x^j x^k}\left[\left(\sum_{p=1}^{n}g^{jp}\frac{\partial \varphi}{\partial x^p}\right)\left(\sum_{q=1}^{n}g^{kq}\frac{\partial \varphi}{\partial x^q}\right)\right.
$$
$$
\left.-\left(\sum_{p=1}^{n}g^{jp}\xi_p\right)\left(\sum_{q=1}^{n}g^{kq}\xi_q\right)\right],
$$

(3. 5)

we obtain Poisson bracket for function $\varphi(x) = f(\|x - x_0\|_G^2)$ of radial type as

$$
-\frac{1}{4}\{a,b\}(x,\xi) = 4(f')^3\|x - x_0\|_G^2 + f'\|\xi\|_{G^{-1}}^2
$$
$$
+ 8(f')^2 f''\|x - x_0\|_G^2 + 2f''\langle x - x_0, \xi\rangle^2
$$
$$
= 4(f')^3\|x - x_0\|_G^2 + f'\left(4(f')^2\|x - x_0\|_G^2 - a(x,\xi)\right)
$$
$$
+ 8(f')^2 f''\|x - x_0\|_G^4 + \frac{1}{2}f''\left(b(x,\xi)\right)^2
$$

(3. 6)

Since the limiting Carleman weight satisfies $\{a,b\}(x,\xi) = 0$ on $J = \{(x,\xi) \in T^*\Omega|\ a(x,\xi) = b(x,\xi) = 0\}$, we have

$$
0 = -\frac{1}{32}\{a,b\}(x,\xi)
$$
$$
= 8\|x - x_0\|_G^2\left(f'(\|x - x_0\|_G^2)\right)^2
$$
$$
\left(f'(\|x - x_0\|_G^2) + \|x - x_0\|_G^2 f''(\|x - x_0\|_G^2)\right)
$$

(3. 7)

for $(x,\xi) \in J$. Since $f' \neq 0$, the differential equation for f is obtained

$$
\frac{df}{dt}(t) + t\frac{d^2 f}{dt^2}(t) = 0 \qquad t > 0.
$$

(3. 8)

Since $\dfrac{d}{dt}\left(t\dfrac{dg}{dt}\right) = 0$ for $t \neq 0$, the solution to (3.8) is obtained as $f(t) = C_1 \log |t| + C_2$ for $C_1, C_2 \in \mathbf{R}$.

References

[1] A. Bukhgeim, G. Uhlmann, Recovering a Potential from Partial Cauchy Data Comm. PDE, 27(2002), 653-668.

[2] D. Ferreira, C. Kenig, J. Sjöstrand, G. Uhlmann, Determining the magnetic Schrödinger operator from partial Cauchy data Comm. Math. Phys. 271 (2007), no. 2, 467–488.

[3] C. Kenig, J. Sjöstrand, G. Uhlmann, The Calderón Problem with Partial Data. Ann. of Math. (2) 165 (2007), no. 2, 567–591.

[4] H. Takuwa, Construction of limiting Carleman weights of radial type for operators of second order, preprint.

[5] G. Uhlmann, Commentary on Alberto P. Calderón's paper: On an Inverse Boundary Value Problem to appear, Selecta, edited by A. Bellow, C.E. Kenig and P. Malliavin.

[6] G. Uhlmann and J.N. Wang, Reconstructing discontinuities using complex geometrical optics solutions, SIAM J. Appl. Math., to appear.

Faculty of Science and Engineering,
Doshisha University,
1-3, Tatara Miyakodani,
Kyotanabe City, 610-0394, JAPAN
e-mail: htakuwa@mail.doshisha.ac.jp