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CONNECTION OF WKB SOLUTIONS AT A HYPERBOLIC FIXED POINT

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1. Introduction

We consider the semiclassical Schrödinger equation

$$-h^2\Delta u + V(x)u = Eu, \tag{1}$$

where h is the semiclassical small parameter, $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$ and E is an energy parameter possibly depending on h.

An asymptotic solution of the form

$$u(x,h) = e^{\frac{i}{h}\psi(x)}b(x,h), \quad b(x,h) \sim \sum_{l=0}^{\infty}\left(\frac{h}{i}\right)^l b_l(x). \tag{2}$$

is called *WKB solution.* The functions ψ and b are called *phase function* and *symbol* (or *amplitude*) respectively.

Suppose we are given a WKB solution u of the form (2) locally near a non-singular point x^0, see §2. This means that $\psi(x)$ and each $b_k(x)$ satisfy there the eikonal equation and the transport equations respectively, see (6) and (7) below. Then it is well known that we can continue them along the Hamilton flow γ passing through (x^0,ξ^0), $\xi^0 = \frac{\partial\psi}{\partial x}(x^0)$, so long as γ is defined and the associated Lagrangian manifold $\Lambda = \{(x,\xi) \in \mathbb{R}^{2d};\ \xi = \frac{\partial\psi}{\partial x}(x)\}$, which carries the curve, projects diffeomorphically to the x-space.

A connection problem arises when γ converges to a *fixed point* or when Λ presents a *caustics.*

In this text, we assume that γ converges to a hyperbolic fixed point that we assume the origin $(x,\xi) = (0,0)$ of the phase space. In our Schrödinger setting, this means that a wave reaches a local non-degenerate maximum of the potential at $x = 0$. The aim of this text is to describe the reflected

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wave at an arbitrary point near $(0,0)$, under the condition that (2) holds microlocally on the incoming stable manifold associated to the fixed point.

More precisely, there exist incoming and outgoing stable manifolds Λ_-, Λ_+ respectively associated to the hyperbolic fixed point, see §3.1. It is proved in [BFRZ] that if a distribution solution to (1) is microlocally 0 on Λ_-, then it is microlocally 0 in a full neighborhood of the fixed point, and in particular on Λ_+. Here a h-dependent distribution $u(x,h) \in \mathcal{S}'(\mathbb{R}^d)$ is said to be microlocally 0 in an open set in the phase space if its Bargmann transform

$$[Tu](x,\xi;h) := \int_{\mathbb{R}^d} e^{\frac{i}{h}(x-y)\cdot\xi - \frac{(x-y)^2}{2h}} u(y,h)\, dy$$

is $\mathcal{O}(h^\infty)$ there. For more details see [BFRZ], page 72, §2.2. Microlocal terminology in the C^∞ category.

Our problem is formulated as follows:

Problem: *Assume that* $u(x,h) - e^{\frac{i}{h}\psi(x)}b(x,h)$ *is microlocally* 0 *on* Λ_-. *Find the asymptotic form of* u *on* Λ_+.

The rest of the paper is organized as follows. First of all, we recall the standard construction of WKB solutions at a non-singular point. For more details, see for example [Ma-Fe]. In the second part, we expose some geometric properties about a hyperbolic fixed point, and we write the WKB solution u as superpositions of time-dependent WKB solutions near Λ_- due to the idea of [He-Sj]. See (21) below. In the third part, we review the theory of *expandible solution* introduced also in [He-Sj]. Finally we calculate the large time asymptotic expansion of the phase and the symbol, to obtain the main results Theorem 5.1, Proposition 5.5 on the outgoing stable manifold.

2. WKB solution at a non-singular point

Consider a partial differential equation on $\mathbb{R}^d$:

$$P(x,hD;h)u = 0, \tag{3}$$

where

$$P(x,hD;h) = \sum_{k\geq 0}(-ih)^k p_k(x,hD), \quad p_k(x,hD) = \sum_{|\alpha|\leq m} a_{k,\alpha}(x)(hD)^\alpha.$$

Here $D = (-i\frac{\partial}{\partial x_1}, \cdots, -i\frac{\partial}{\partial x_d})$, the coefficients $a_{k,\alpha}$ are smooth, and h is a small positive parameter. We have in mind the Schrödinger equation (1), with E depending on h: $E = E_0 + \frac{h}{i}E_1 + \left(\frac{h}{i}\right)^2 E_2 + \cdots$.

We look for a WKB solution of the form (2).

The action of P on u of the form (2) is given by

$$\big[P(x,hD;h)u\big](x,h) = e^{\frac{i}{h}\psi(x)}\sum_{j\geq 0}\left(\frac{h}{i}\right)^j \big[R_j(x,\nabla_x)b\big](x,h),$$

where $R_j(x,\nabla_x)$ is a jth order real differential operator. In particular,

$$R_0 = p_0(x,\nabla_x\psi)$$

and

$$R_1 = (\nabla_\xi p_0)(x,\nabla_x\psi)\cdot\nabla_x + \frac{1}{2}\mathrm{Tr}\left(\nabla_\xi^2 p_0(x,\nabla_x\psi)\nabla_x^2\psi\right) + p_1(x,\nabla_x\psi).$$

Here $\nabla_w^2 = \left(\frac{\partial^2}{\partial w_j\partial w_k}\right)_{1\leq j,k\leq d}$, $w = x$ or ξ.

If the symbol b has the development of the form (2), then we are led to $\sum_{j+l=k} R_j b_l = 0$ for all $k\in\mathbb{N} = \{0,1,2,\dots\}$, i.e.

(4) $$p_0(x,\nabla_x\psi) = 0,$$

(5) $$\big[R_1(x,\nabla_x)b_k\big](x,h) = -\sum_{j=1}^{k}\big[R_{j+1}(x,\nabla_x)b_{k-j}\big](x,h), \qquad k\geq 0.$$

Here the right hand side of (5) is 0 when $k=0$.

The equation (4) is called *eikonal equation* or *Hamilton-Jacobi equation* and (5) is called *transport equation.* In the Schrödinger case, these equations reduce to

(6) $$|\nabla_x\psi|^2 + V(x) = E_0,$$

(7) $$2\nabla_x\psi\cdot\nabla_x b_k + (\Delta\psi - E_1)\,b_k = -\Delta b_{k-1} + \sum_{l=1}^{k} E_{l+1}b_{k-l}.$$

Here the right hand side of (7) is 0 when $k=0$.

Remark 2.1. *The differential operator $P(x,hD;h)$ can be generalized to h-pseudo-differential operator $P = \mathrm{Op}_h(p)$:*

$$[\mathrm{Op}_h(p)u](x,h) := \frac{1}{(2\pi h)^d}\int\!\!\int e^{\frac{i}{h}(x-y)\cdot\xi}p(x,\xi)u(y)\,dy\,d\xi,$$

where $p = p(x,\xi)$ is a symbol belonging to a symbol class $S_{2d}(\langle(x,\xi)\rangle^m)$, i.e. $p(x,\xi)\in C^\infty(\mathbb{R}^{2d};\mathbb{R})$ and for any multi-indices α,β,

$$\left|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)\right| \leq C_{\alpha\beta}\langle(x,\xi)\rangle^m, \qquad \langle(x,\xi)\rangle = (1+|x|^2+|\xi|^2)^{1/2}.$$

Furthermore, p can depend on h:

$$p(x,\xi;h)\sim\sum_{j=0}^{\infty}h^j p_j(x,\xi)$$

with $p_j(x,\xi) \in S_{2d}(\langle(x,\xi)\rangle^m)$*, in the sense that*

$$\left|\partial_x^\alpha \partial_\xi^\beta \left(p(x,\xi;h) - \sum_{j=0}^{N} p_j(x,\xi)h^j \right)\right| \le C_{N\alpha\beta} h^{N+1} \langle(x,\xi)\rangle^m.$$

2.1. **Eikonal equation.** First we solve the eikonal equation (4). Set $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} = \mathbb{R}^d$.

Proposition 2.1. *Suppose* $p_0(x,\xi)$ *is smooth in a neighborhood of a point* $(a,b) \in \mathbb{R}^{2d}$ *and that* $p_0(a,b) = 0$, $\dfrac{\partial p_0}{\partial \xi_1}(a,b) \neq 0$. *Then for any* $\psi_0(x')$ *smooth near* a' *satisfying* $\nabla_{x'}\psi_0(a') = b'$*, there exists unique solution to the Cauchy problem*

$$\begin{cases} p_0(x, \nabla_x \psi) &= 0, \\ \psi\big|_{x_1 = a_1} &= \psi_0(x'). \end{cases} \tag{8}$$

Proof. Let Λ_0 be the $(n-1)$-dimensional manifold

$$\Lambda_0 = \{(x,\xi) \in p_0^{-1}(0);\ x_1 = a_1, \xi' = \nabla_{x'}\psi_0(x')\},$$

and Λ its evolution by the Hamilton flow

$$\Lambda = \bigcup_{|t|<\epsilon} \exp tH_{p_0}(\Lambda_0).$$

Here $H_{p_0} = \nabla_\xi p_0 \cdot \nabla_x - \nabla_x p_0 \cdot \nabla_\xi$ is the Hamilton vector field. Notice that H_{p_0} is transversal to Λ_0 by the assumption $\frac{\partial p_0}{\partial \xi_1}(a,b) \neq 0$.

We see that, for ϵ small, Λ is a Lagrangian manifold, the projection of which to the x-space is diffeomorphic and that $\Lambda \subset p_0^{-1}(0)$(conservation of energy). These facts mean that Λ is represented by a generating function $\psi(x)$,

$$\Lambda = \{(x,\xi) \in \mathbb{R}^{2d};\ \xi = \nabla_x \psi(x)\}$$

and that this $\psi(x)$ is the solution to (8). □

Remark 2.2. *In the Schrödinger case, the domain* $\{x \in \mathbb{R}^d;\ V(x) < E_0\}$ *is called classically allowed region. If a point* $a \in \mathbb{R}^d$ *belongs to the classically allowed region, there exist real* b*'s such that* $|b|^2 + V(a) = E_0$ *and* $\nabla_\xi p_0(a,b) = 2b \neq 0$.

Definition 2.1. *For a Lagrangian manifold* Λ*, a point* $(x,\xi) \in \Lambda$ *will be called non-singular (with respect to the projection on* $\mathbb{R}_x^d$*) if it has a neighborhood admitting a diffeomorphic projection on* $\mathbb{R}_x^d$*. It is singular in the opposite case.*

2.2. **Transport equation.** Next, we study the transport equations.

Let us parametrize Λ_0 by $y' \in \mathbb{R}^{d-1}$:

$$\Lambda_0 = \left\{(x(y'), \xi(y')) \in p_0^{-1}(0);\ x_1 = x_1^0,\ x' = y',\ \xi' = \nabla_{x'}\psi_0(y')\right\}.$$

Put

$$\left(x(t,y'), \xi(t,y')\right) = \exp tH_{p_0}\left(x(y'), \xi(y')\right), \quad J(t,y') = \det \frac{\partial x(t,y')}{\partial(t,y')}.$$

Notice that the fact that (a,b) is a non-singular point of Λ means $J(t,y') \neq 0$ for small t.

Proposition 2.2. *On the curve $x = x(t,y')$, the first order differential operator R_1 can rewritten as*

$$\tag{9} \begin{aligned} &(\nabla_\xi p_0)(x, \nabla_x\psi) \cdot \nabla_x b_k + \tfrac{1}{2}\mathrm{Tr}\left(\nabla_\xi^2 p_0(x,\nabla_x\psi)\nabla_x^2\psi\right) b_k \\ &= \frac{1}{\sqrt{|J(t,y')|}}\frac{d}{dt}\left(\sqrt{|J(t,y')|}b_k\right) - \frac{1}{2}\mathrm{Tr}(\nabla_x\nabla_\xi p_0)b_k. \end{aligned}$$

Hence in particular

$$b_0(x(t,y')) = \sqrt{\frac{J(0,y')}{J(t,y')}}b_0(x(y'))\exp\left(\frac{1}{2}\int_0^t \mathrm{Tr}(\nabla_x\nabla_\xi p)d\tau\right).$$

Proof. Differentiating by (t,y') the canonical equation

$$\frac{d}{dt}x(t,y') = \nabla_\xi p_0\left(x(t,y'), \nabla_x\psi(x(t,y'))\right),$$

one obtains

$$\frac{d}{dt}\frac{\partial x(t,y')}{\partial(t,y')} = \nabla_x\nabla_\xi p_0 \cdot \frac{\partial x(t,y')}{\partial(t,y')} + \nabla_\xi^2 p_0 \nabla_x^2\psi \cdot \frac{\partial x(t,y')}{\partial(t,y')},$$

and taking the determinant one gets

$$\frac{d}{dt}J(t,y') = \mathrm{Tr}(\nabla_x\nabla_\xi p_0 + \nabla_\xi^2 p_0\nabla_x^2\psi)J(t,y'),$$

that is

$$\mathrm{Tr}(\nabla_x\nabla_\xi p_0 + \nabla_\xi^2 p_0\nabla_x^2\psi) = \frac{d}{dt}(\log|J|).$$

The left hand side of (9) is equal to

$$\begin{aligned} &\frac{d}{dt}b_k + \frac{1}{2}\mathrm{Tr}(\nabla_x\nabla_\xi p_0 + \nabla_\xi^2 p_0\nabla_x^2\psi)b_k - \frac{1}{2}\mathrm{Tr}(\nabla_x\nabla_\xi p_0)b_k \\ &= \tfrac{d}{dt}b_k + \left(\tfrac{d}{dt}\log\sqrt{|J|}\right)b_k - \tfrac{1}{2}\mathrm{Tr}(\nabla_x\nabla_\xi p_0)b_k \\ &= \tfrac{1}{\sqrt{|J|}}\tfrac{d}{dt}(\sqrt{|J|}b_k) - \tfrac{1}{2}\mathrm{Tr}(\nabla_x\nabla_\xi p_0)b_k. \end{aligned}$$

□

Remark 2.3. *In the Schrödinger case,* $\nabla_x\nabla_\xi p_0 = 0$. *In case* $d=1$, *in particular,* $J(t) = \dot{x}(t) = 2\xi(t) = 2\sqrt{E-V(x(t))}$ *and hence* $b_0(x) = (E - V(x))^{-1/4}$.

Remark 2.4. *Let* (a,b) *be a point in* $p_0^{-1}(0) \subset \mathbb{R}^{2d}$. *In the one-dimensional case* $d=1$, *there exists only one Lagrangian manifold* Λ *which carries* (a,b). *This is just the Hamilton flow* $\{\exp tH_{p_0}(a,b)\}_{t\in\mathbb{R}}$. *The point* (a,b) *is singular if and only if* $\partial_\xi p_0(a,b) = 0$. *If, moreover,* $p_0 = \xi^2 + V(x)$, *this means* $\xi = 0$.

If $\partial_x p_0(a,0) = V'(a) \neq 0$ *then* $x=a$ *is a simple turning point.*

Otherwise, i.e. $\partial_x p_0(a,0) = V'(a) = 0$, $x=a$ *is a double or multiple turning point. In this case, the point* $(a,0)$ *is a fixed point of the Hamilton vector field.*

3. Hyperbolic fixed point

3.1. **Stable manifold.** We suppose that the function $p_0(x,\xi)$ defined in a neighborhood of the origin in $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$ behaves like

$$p_0(x,\xi) = |\xi|^2 - \sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}\big((x,\xi)^3\big) \quad \text{as} \quad (x,\xi) \to (0,0), \tag{10}$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ are constants.

Let us consider the canonical system of p_0:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = \begin{pmatrix} \nabla_\xi p_0(x(t),\xi(t)) \\ -\nabla_x p_0(x(t),\xi(t)) \end{pmatrix}. \tag{11}$$

The origin $(x,\xi) = (0,0)$ is a fixed point of the Hamilton vector field H_{p_0}. The linearization at the origin is

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = F_{p_0}\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}, \tag{12}$$

where F_{p_0} is the fundamental matrix

$$F_{p_0} := \left.\begin{pmatrix} \frac{\partial^2 p_0}{\partial x \partial \xi} & \frac{\partial^2 p_0}{\partial \xi^2} \\ -\frac{\partial^2 p_0}{\partial x^2} & -\frac{\partial^2 p_0}{\partial \xi \partial x} \end{pmatrix}\right|_{(x,\xi)=(0,0)} = \begin{pmatrix} 0 & 2\,\mathrm{Id} \\ \frac{1}{2}\mathrm{diag}\,(\lambda_j)^2 & 0 \end{pmatrix}.$$

This matrix has d positive eigenvalues $\{\lambda_j\}_{j=1}^d$ and d negative eigenvalues $\{-\lambda_j\}_{j=1}^d$. The eigenspaces $\Lambda_\pm^0$ corresponding to these positive and negative eigenvalues are respectively outgoing and incoming stable manifolds for the quadratic part q_0 of p_0, see Example 3.1 below :

$$\begin{aligned} \Lambda_\pm^0 = & \ \big\{(x,\xi) \in \mathbb{R}^{2d};\ \exp tH_{q_0}(x,\xi) \to (0,0) \quad \text{as} \quad t \to \mp\infty\big\} \\ = & \ \big\{(x,\xi) \in \mathbb{R}^{2d};\ \xi_j = \pm\tfrac{\lambda_j}{2} x_j,\ j = 1, \cdots, d\big\}. \end{aligned}$$

By the stable manifold theorem, we also have outgoing and incoming stable manifolds for p_0:

$$\Lambda_\pm = \left\{(x,\xi) \in \mathbb{R}^{2d};\ \exp tH_{p_0}(x,\xi) \to (0,0) \quad \text{as} \quad t \to \mp\infty\right\}.$$

These are Lagrangian manifolds and written in the form

$$\Lambda_\pm = \left\{(x,\xi) \in \mathbb{R}^{2d};\ \xi = \frac{\partial \phi_\pm}{\partial x}(x)\right\},$$

where the generating functions $\phi_\pm$ behave like

$$\phi_\pm(x) = \pm\sum_{j=1}^{d} \frac{\lambda_j}{4} x_j^2 + \mathcal{O}(|x|^3) \quad \text{as} \quad x \to 0. \tag{13}$$

Now suppose $(x^0,\xi^0) \in \Lambda_- \setminus \{(0,0)\}$. Then of course by definition $\exp tH_{p_0}(x^0,\xi^0) \to (0,0)$ as $t \to +\infty$. More precisely,

Proposition 3.1. *For $(x^0,\xi^0) \in \Lambda_-\setminus\{(0,0)\}$, one has*

$$\exp tH_{p_0}(x^0,\xi^0) \sim \sum_{k=1}^{\infty} \gamma_k(t) e^{-\mu_k t} \quad \text{as} \quad t \to +\infty,$$

where $0 < \mu_1 < \mu_2 < \cdots$ are linear combinations over $\mathbb{N}$ of $\{\lambda_j\}_{j=1}^d$, and in particular $\mu_1 = \lambda_1$. $\gamma_k(t)$ are vector valued polynomials in t, and in particular γ_1 is an eigenvector of F_{p_0} corresponding to $-\lambda_1$ and independent of t. Remark that $\gamma_1 e^{-\lambda_1 t}$ is a solution to (12).

For the proof, see Remark 4.2 at the end of section §4. In fact, we prove that $\exp tH_{p_0}(x^0,\xi^0)$ is expandible in the sense of Definition 4.2.

Remark 3.1. *If the remainder term of p_0 in* (10) *is independent of ξ, then p_0 is a classical Hamitonian associated to a Schrödinger equation* (1) *:*

$$p_0(x,\xi) = |\xi|^2 + V(x), \quad V(x) = -\sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}(|x|^3) \quad \textit{as } x \to 0. \tag{14}$$

The potential $V(x)$ attains its local non-degenerate maximum 0 at the origin. In this case, by the symmetry with respect to ξ, one has

$$\phi_-(x) = -\phi_+(x) \quad \textit{and} \quad \Lambda_- = \{(x,-\xi) \in \mathbb{R}^{2d};\ (x,\xi) \in \Lambda_+\}.$$

The vector γ_1 depends on (x^0,ξ^0). Let $X^-(x^0,\xi^0)$ be the x-component of γ_1. We assume that

$$X^-(x^0,\xi^0) \neq 0. \tag{A1}$$

Then we can assume, without loss of generality, that

$$X^-(x^0,\xi^0) = c(1,0,\cdots,0), \quad c > 0, \tag{15}$$

i.e. the Hamilton flow passing through (x^0,ξ^0) converges to the origin tangentially to the x_1-axis. Indeed it is necessarily the case when the eigenspace of $-\lambda_1$ is one-dimensional, and it suffices to rotate the coordinate axes when the eigenspace is multi-dimensional.

Example 3.1. Let us calculate the stable manifolds and the Hamilton flow in the case where p_0 is quadratic :

$$q_0(x,\xi) = \xi^2 - \sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2. \tag{16}$$

The canonical equation is (12) itself and the solution with the initial condition $(x(0),\xi(0)) = (x^0,\xi^0)$ is given by

$$\begin{aligned}\begin{pmatrix} x_j(t) \\ \xi_j(t) \end{pmatrix} &= \begin{pmatrix} \cosh\lambda_j t & \frac{2}{\lambda_j}\sinh\lambda_j t \\ \frac{\lambda_j}{2}\sinh\lambda_j t & \cosh\lambda_j t \end{pmatrix} \begin{pmatrix} x_j^0 \\ \xi_j^0 \end{pmatrix} \\ &= \begin{pmatrix} (\frac{x_j^0}{2} + \frac{\xi_j^0}{\lambda_j})e^{\lambda_j t} + (\frac{x_j^0}{2} - \frac{\xi_j^0}{\lambda_j})e^{-\lambda_j t} \\ (\lambda_j\frac{x_j^0}{4} + \frac{\xi_j^0}{2})e^{\lambda_j t} + (-\lambda_j\frac{x_j^0}{4} + \frac{\xi_j^0}{2})e^{-\lambda_j t} \end{pmatrix}\end{aligned}$$

for each $1 \le j \le d$. The stable manifolds are

$$\Lambda_\pm^0 = \{(x,\xi) \in \mathbb{R}^{2d};\ \xi_j = \pm\frac{\lambda_j}{2}x_j\ \ 1 \le j \le d\},$$

and for $(x^0,\xi^0) \in \Lambda_-^0$, i.e. $\xi_j^0 = -\lambda_j\frac{x_j^0}{2}$, one has

$$\begin{pmatrix} x_j(t) \\ \xi_j(t) \end{pmatrix} = e^{-\lambda_j t}\begin{pmatrix} x_j^0 \\ \xi_j^0 \end{pmatrix}, \quad j = 1,\cdots,d.$$

If there are m smallest eigenvalues of F_{q_0}; $\lambda_1 = \cdots = \lambda_m < \lambda_{m+1}$, then

$$\begin{aligned}(x(t),\xi(t)) = &\ (x_1^0,\ldots,x_m^0,0,\cdots,0;-\tfrac{\lambda_1}{2}x_1^0,\cdots,-\tfrac{\lambda_m}{2}x_m^0,0,\cdots,0) \\ + &\ \mathcal{O}(e^{-\lambda_{m+1}t}).\end{aligned}$$

3.2. **WKB solution on the incoming stable manifold.** In what follows, we assume that p_0 is of the form (14) and consider the corresponding Schrödinger equation (1) with $E_0 = 0$, i.e. writing now $E = hz$, consider

$$-h^2\Delta u + V(x)u = hzu. \tag{17}$$

Fix a point (x^0,ξ^0) on Λ_- sufficiently near the origin. Suppose we are given a WKB solution near (x^0,ξ^0).

$$u(x,h) = e^{\frac{i}{h}\psi(x)}b(x,h), \quad b(x,h) \sim \sum_{j\ge 0}(-ih)^j b_j(x). \tag{18}$$

This means that $\psi(x)$ and $b_j(x)$ are defined near $x = x^0$ and satisfy respectively, the eikonal equation (6) and the transport equation (7) with $E_0 = 0$, $E_1 = iz$, and that $\xi^0 = \nabla_x\psi(x^0)$.

The last condition is equivalent to saying that (x^0, ξ^0) is on the Lagrangian manifold associatetd to ψ, $\Lambda_\psi = \{(x,\xi) \in \mathbb{R}^{2d};\ \xi = \nabla_x\psi(x)\}$.

The eikonal equation (6) means $\Lambda_\psi \subset p_0^{-1}(0)$.

The curve $\gamma = \{\exp tH_{p_0}(x^0,\xi^0);\ t \geq 0\}$ is included in Λ_- and Λ_ψ. We assume the following condition :

$$\Lambda_- \text{ and } \Lambda_\psi \text{ intersect transversally along } \gamma. \qquad (\mathbf{A}.\mathbf{2})$$

The phase ψ and the symbol b of the WKB solution u can be continued so long as the Lagrangian manifold Λ_ψ is defined, i.e. in a neighborhood of γ, but they have, in general, singularities at the origin $(0,0)$. We will then have to represent it in another way.

Example 3.2. We return back to the example (16). We take

$$(x^0,\xi^0) = (\epsilon, 0, \cdots, 0; -\frac{\lambda_1}{2}\epsilon, 0, \cdots, 0)$$

on Λ_-^0. Any phase function of the form $\psi(x) = -\frac{\lambda_1}{4}x_1^2 \pm \frac{\lambda_2}{4}x_2^2 \pm \cdots \pm \frac{\lambda_d}{4}x_d^2$ satisfies the eikonal equation $q_0(x, \nabla_x\psi) = 0$ and the condition $\xi^0 = \nabla_x\psi(x^0)$. But among these, only when

$$\psi(x) = \psi_0(x) = -\frac{\lambda_1}{4}x_1^2 + \sum_{j=1}^{d}\frac{\lambda_j}{4}x_j^2,$$

the two Lagrangian manifolds

$$\begin{aligned}\Lambda_{\psi_0} = &\ \{(x,\xi) \in \mathbb{R}^{2d};\ \xi_1 = -\tfrac{\lambda_1}{2}x_1,\ \xi_2 = \tfrac{\lambda_d}{2}x_2, \cdots, \xi_d = \tfrac{\lambda_d}{2}x_d\},\\ \Lambda_- = &\ \{(x,\xi) \in \mathbb{R}^{2d};\ \xi_1 = -\tfrac{\lambda_1}{2}x_1,\ \xi_2 = -\tfrac{\lambda_d}{2}x_2, \cdots, \xi_d = -\tfrac{\lambda_d}{2}x_d\}\end{aligned}$$

intersect along

$$\Lambda_{\psi_0} \cap \Lambda_- = \{(x,\xi) \in \mathbb{R}^{2d};\ \xi_1 = -\frac{\lambda_1}{2}x_1,\ x_2 = \xi_2 = \cdots = x_d = \xi_d = 0\},$$

which is the Hamilton flow $\gamma = \cup_t \exp tH_{q_0}(x^0,\xi^0)$ passing through (x^0,ξ^0), and the intersection is transversal.

Let us calculate the principal term b_0 of the symbol b. We take $\{x_1 = \epsilon\}$ as initial surface.

$$\Lambda_{\psi_0} \cap \{x_1 = \epsilon\} = \left\{\left(\epsilon, y'; -\frac{\lambda_1}{2}\epsilon, \frac{\lambda'}{2}\cdot y'\right) \in \mathbb{R}^{2d};\ y' \in \mathbb{R}^{d-1}\right\},$$

where $\lambda' = (\lambda_2, \cdots, \lambda_d)$. Then

$$x(t,y') = \left(\epsilon e^{-\lambda_1 t}, y_2 e^{\lambda_2 t}, \cdots, y_d e^{\lambda_d t}\right), \quad J(t,y') = -\lambda_1\epsilon e^{(-\lambda_1 + \sum_{j=2}^d \lambda_j)t}.$$

Remark that $\lim_{t\to+\infty} J(t,y') = 0$ when $d=1$ and $\lim_{t\to+\infty} |J(t,y')| = +\infty$ when $d=2$, $\lambda_2 > \lambda_1$ or $d \geq 3$.

The solution of the transport equation $\frac{d}{dt}b_0 + (\Delta\psi_0 - iz)b_0 = 0$ on the curve $x = x(t,y')$ is given by $b_0(x(t,y')) = e^{-(S-\lambda_1)t}b_0(\epsilon,y')$, where $S := \sum_{j=1}^{d} \frac{\lambda_j}{2} - iz$, remark that $S - \lambda_1 = \Delta\psi_0 - iz$. Putting $x(t,y') = x$, we obtain

$$b_0(x) = \left(\frac{x_1}{\epsilon}\right)^{\frac{S}{\lambda_1}-1} b_0\left(\epsilon, x_2\left(\frac{x_1}{\epsilon}\right)^{\frac{\lambda_2}{\lambda_1}}, \cdots, x_d\left(\frac{x_1}{\epsilon}\right)^{\frac{\lambda_d}{\lambda_1}}\right).$$

Hence $b_0(x)$ has a singularity along $\{x_1 = 0\}$, which is in fact the projection to $\mathbb{R}^d_x$ of the set $\Lambda_{\psi_0} \cap \Lambda_+ = \{(x,\xi) \in \mathbb{R}^{2d};\ x_1 = \xi_1 = 0\}$.

3.3. **Time evolution equation.** The main idea to continue the WKB solution to a neighborhood of the origin is to consider the corresponding time-dependent Schrödinger equation. Let us write u as inverse h-Fourier transform of a time-dependent function $v(t,x;h)$:

$$u(x;h) = \mathcal{F}^{-1}_{h,t\to E}v := \frac{1}{\sqrt{2\pi h}}\int_{-\infty}^{\infty} e^{itE/h}v(t,x;h)dt.$$

Then v should satisfy the time-dependent Schrödinger equation :

$$(hD_t - h^2\Delta + V(x))v = 0. \tag{19}$$

Since $E = hz$, we look for a time-dependent WKB solution

$$e^{izt}v = e^{\frac{i}{h}\varphi(t,x)}a(t,x;h), \tag{20}$$

with

$$a(t,x;h) \sim \sum_{l=0}^{\infty}\left(\frac{h}{i}\right)^l a_l(t,x).$$

Then u is represented in the integral form

$$u(x,h) = \frac{1}{\sqrt{2\pi h}}\int^{\infty} e^{\frac{i}{h}\varphi(t,x)}a(t,x;h)\,dt \tag{21}$$

and the WKB solution should now satisfy

$$(hD_t - h^2\Delta + V(x) - hz)(e^{\frac{i}{h}\varphi}a) = 0, \tag{22}$$

which leads, as in §2, to the following eikonal and transport equations respectively :

$$\partial_t\varphi + |\nabla_x\varphi|^2 + V(x) = 0, \tag{23}$$

$$\partial_t a_l + 2\nabla_x\varphi\cdot\nabla_x a_l + (\Delta\varphi - iz)a_l = -\Delta a_{l-1}, \quad l \geq 0. \tag{24}$$

Here we use the convention $a_{-1} = 0$.

4. Expandible solution

Let $\nu(x, \nabla_x)$ be a vector field of the form

$$\nu(x, \nabla_x) = A(x)x \cdot \nabla_x, \quad A(0) = \operatorname{diag}(\lambda_1, \dots, \lambda_d), \tag{25}$$

where $0 < \lambda_1 \leq \dots \leq \lambda_d$ are positive constants, and consider the Cauchy problem

$$\begin{cases} \partial_t u + \nu(x, \nabla_x)u = & v(t, x), \\ u_{|t=0} = & w(x). \end{cases} \tag{26}$$

We denote by $\exp(t\nu)(x_0)$ the solution to the system of ordinary differential equations

$$\begin{cases} \dot{x}(t) & = A\big(x(t)\big)x(t), \\ x_{|t=0} & = x_0. \end{cases} \tag{27}$$

Then

$$\begin{aligned} \tfrac{d}{dt}\Big[u\big(t, \exp(t\nu)(x_0)\big)\Big] &= \big[\partial_t + \nu(x, \nabla_x)\big]u\big(t, \exp(t\nu)(x_0)\big) \\ &= v\big(t, \exp(t\nu)(x_0)\big). \end{aligned}$$

Hence

$$u\big(t, \exp(t\nu)(x_0)\big) = w(x_0) + \int_0^t v\big(s, \exp(s\nu)(x_0)\big)\, ds.$$

Put now $x = \exp(t\nu)(x_0)$. Since $x_0 = \exp(-t\nu)(x)$, $\exp(s\nu)(x_0) = \exp(-(t-s)\nu)(x)$, we get

$$u(t, x) = w\big(\exp(-t\nu)(x)\big) + \int_0^t v\big(t - s, \exp(-s\nu)(x)\big)\, ds. \tag{28}$$

When $\nu = \nu_0 = \sum_{j=1}^{d} \lambda_j x_j \frac{\partial}{\partial x_j}$, in particular,

$$\exp(-t\nu_0)(x) = \big(e^{-\lambda_1 t}x_1, \dots, e^{-\lambda_d t}x_d\big).$$

Let Ω be a suitable neighborhood of 0 in $\mathbb{R}^d$.

Definition 4.1. *We write $u(t, x) \in \mathcal{O}^\infty\big(e^{-\mu t}|x|^M\big)$ if for every $\epsilon > 0$, $k \in \mathbb{N}, \alpha \in \mathbb{N}^d$,*

$$D_t^k D_x^\alpha u(t, x) = \mathcal{O}\big(e^{-(\mu-\epsilon)t}|x|^{(M-|\alpha|)_+}\big)$$

in $[0, \infty) \times \Omega$.

The map $\exp(-t\nu) : \Omega \to \Omega$ is well defined and

$$\big|\exp(-t\nu)(x)\big| = \mathcal{O}(e^{-\lambda_1 t}|x|), \quad |D_t^k D_x^\alpha \exp(-t\nu)(x)| = \mathcal{O}(e^{-\lambda_1 t})$$

for $x \in \Omega, t \geq 0$ and for all $k \in \mathbb{N}, \alpha \in \mathbb{N}^d$. It is easy to check the following lemmas:

Lemma 4.1. *If* $v \in \mathcal{O}^\infty(e^{-\lambda t}|x|^N)$, $w = 0$, *and if* $N\lambda_1 \geq \lambda$, *then* $u \in \mathcal{O}^\infty(e^{-\lambda t}|x|^N)$.

Lemma 4.2. *If* $w \in \mathcal{O}(|x|^N)$ *and* $v = 0$, *then* $u \in \mathcal{O}^\infty(e^{-N\lambda_1 t}|x|^N)$.

We will see that the solution u to the Cauchy problem (26) is expandible in the following sense:

Definition 4.2. *Let* $\mu_1 < \mu_2 < \cdots$ *be the series of linear combinations over* $\mathbb{N}$ *of* $\lambda_1, \cdots, \lambda_d$. *A function* $u(t,x) \in C^\infty([0,\infty)\times\Omega)$ *is said to be expandible if there exist* u_k $(k = 1, 2, \ldots)$ *polynomials in* t *with smooth coefficients in* $x \in \Omega$ *such that for any* $N \in \mathbb{N}$, *one has*

$$u(t,x) - \sum_{k=1}^{N} e^{-\mu_k t} u_k(t,x) = \mathcal{O}^\infty(e^{-\mu_{N+1} t})$$

First let us look for the homogeneous solution of the Cauchy problem

$$\left\{\begin{aligned} \partial_t u + \sum_{j=1}^{d} \lambda_j x_j \frac{\partial u}{\partial x_j} &= e^{-\mu t} \sum_{|\alpha|=N} c_\alpha(t) x^\alpha \\ u_{|t=0} &= 0, \end{aligned}\right. \tag{29}$$

where $c_\alpha(t)$ are polynomials in t.

First put $u_1(t,x) = e^{-\mu t} \sum_{|\alpha|=N} a_\alpha(t) x^\alpha$. Then u_1 satisfies

$$\partial_t u_1 + \sum_{j=1}^{d} \lambda_j x_j \frac{\partial u_1}{\partial x_j} = e^{-\mu t} \sum_{|\alpha|=N} \Big\{ a'_\alpha(t) + \big(\sum_{j=1}^{d} \lambda_j \alpha_j - \mu\big) a_\alpha(t) \Big\} x^\alpha.$$

Hence if u_1 satisfies the first equation of (29), $a_\alpha(t)$ should satisfy

$$a'_\alpha(t) + \delta_\alpha a_\alpha(t) = c_\alpha(t), \quad \delta_\alpha = \sum_{j=1}^{d} \lambda_j \alpha_j - \mu. \tag{30}$$

Equation (30) has a polynomial solution $a_\alpha(t)$ with

$$\deg a_\alpha = \left\{\begin{array}{ll} \deg c_\alpha & \text{if } \ \delta_\alpha \neq 0, \\ \deg c_\alpha + 1 & \text{if } \ \delta_\alpha = 0. \end{array}\right.$$

Now, set $u_2 := u - u_1$, u_2 satisfies

$$\left\{\begin{aligned} \partial_t u_2 + \sum_{j=1}^{d} \lambda_j x_j \frac{\partial u_2}{\partial x_j} &= 0 \\ u_2\big|_{t=0} &= -\sum_{|\alpha|=N} a_\alpha(0) x^\alpha, \end{aligned}\right.$$

which leads to

$$u_2 = -\sum_{|\alpha|=N} a_\alpha(0) x^\alpha e^{-(\sum_{j=1}^d \lambda_j \alpha_j)t}.$$

Thus the solution to (29) is given by

$$u(t,x) = \sum_{|\alpha|=N} \left\{ e^{-\mu t} a_\alpha(t) - e^{-\left(\sum_{j=1}^d \lambda_j \alpha_j\right)t} a_\alpha(0) \right\} x^\alpha.$$

Proposition 4.1. *Suppose $v(t,x)$ is expandible and $v = \mathcal{O}(|x|^N)$ with $N \geq 1$. Then the solution $u(t,x)$ of the Cauchy problem*

$$\left\{ \begin{aligned} \partial_t u + \nu(x, \nabla_x) u =&\ v(t,x), \\ u_{|t=0} =&\ 0 \end{aligned} \right. \tag{31}$$

is also expandible.

Proof. Put $v \sim v^{(N)} + v^{(N+1)} + \cdots$, where $v^{(M)}$ is homogeneous of order M:

$$v^{(M)} = \sum_{k=1}^{\infty} e^{-\mu_k t} \sum_{|\alpha|=M} c_\alpha^k(t) x^\alpha.$$

In the case where $\nu = \nu_0$, Proposition 4.1 holds by the preceding argument.

In the general case, let $\nu \sim \nu^{(0)} + \nu^{(1)} + \cdots$, where $\nu^{(k)}$ is homogeneous of order $k+1$. Expanding also $u \sim u^{(N)} + u^{(N+1)} + \cdots$, the equation becomes

$$\sum_{M=N}^{\infty} \partial_t u^{(M)} + \left(\nu^{(0)} + \nu^{(1)} + \cdots\right)\Big[\sum_{M=N}^{\infty} u^{(M)}\Big] = \sum_{M=N}^{\infty} v^{(M)},$$

which leads to

$$\begin{aligned} \partial_t u^{(N)} + \nu^{(0)} u^{(N)} \quad &= v^{(N)}, \\ \partial_t u^{(N+1)} + \nu^{(0)} u^{(N+1)} \quad &= v^{(N+1)} - \nu^{(1)} u^{(N)}, \end{aligned}$$

and in general for $M \geq N$,

$$\partial_t u^{(M)} + \nu^{(0)} u^{(M)} = v^{(M)} - \nu^{(1)} u^{(M-1)} - \cdots - \nu^{(M-N)} u^{(N)}.$$

Hence we can check inductively that

$$u^{(M)} = \sum_{k=1}^{\infty} e^{-\mu_k t} \sum_{|\alpha|=M} a_\alpha^k(t) x^\alpha,$$

with

$$\deg a_\alpha^k(t) \leq \max\left(\deg c_\alpha^k(t), \max_{|\alpha|<M} \deg a_\alpha^k(t)(+1)\right)$$

where $(+1)$ occurs only for a finite number of α for each k. Therefore, for each k, $\deg a_\alpha^k$ is uniformly bounded with respect to M, since it is so for $c_\alpha^k(t)$.

For each M, we have

$$u^{(M)} \sim \sum_{k=1}^{\infty} e^{-\mu_k t} \sum_{|\alpha|=M} a_\alpha^k(t) x^\alpha.$$

There exists d_k independent of M such that $\deg a_\alpha^k \leq d_k$ for all α.

We can construct a realization $\bar{u}$ such that

$$\bar{u} \sim u^{(N)} + u^{(N+1)} + \cdots, \quad \bar{u}_{|t=0} = 0.$$

Letting $\tilde{u} := u - \bar{u}$, it remains to show the existence of an expandible solution $\tilde{u} = \mathcal{O}(|x|^\infty)$ such that

$$\begin{cases} \partial_t \tilde{u} + \nu(x, \nabla_x)\tilde{u} & = \tilde{v}, \\ \tilde{u}_{|t=0} & = 0. \end{cases}$$

This is done by proving the following proposition by induction in N :

$$\tilde{u} = \tilde{u}_N + \tilde{v}_N$$

with expandible and $\mathcal{O}(|x|^\infty)$ function $\tilde{u}_N$ and $\mathcal{O}^\infty(e^{-\mu_N t}|x|^\infty)$ function $\tilde{v}_N$. We omit it. □

Theorem 4.1. *Proposition 4.1 holds for time-dependent vector field*

(32) $$\tilde{\nu}(t, x, \nabla_x) = A(t, x)x \cdot \nabla_x, \quad A(t, x) = A(x) + \tilde{A}(t, x),$$

where $A(x)$ is as in (25) and $\tilde{A}(t, x)$ is expandible.

Remark 4.1. *If we add $\mu_0 = 0$ in the definition of expandibility, Theorem 4.1 holds without the assumption $N \geq 1$.*

Corollary 4.1. *Suppose that a function $s(t, x)$ is expandible; $s(t, x) \sim \sum_{k=0}^{\infty} e^{-\mu_k t} s_k(t, x)$ and that $s_0(x)$ is independent of t. If $v(t, x)$ is expandible in the form:*

$$v(t, x) \sim \sum_{k=0}^{\infty} e^{-(\mu_k + s_0(0))t} v_k(t, x),$$

then the solution of the Cauchy problem

$$\begin{cases} \partial_t u + \big(\tilde{\nu}(t, x, \nabla_x) + s(t, x)\big)u & = v, \\ u_{|t=0} & = 0 \end{cases}$$

is also expandible in the same form:

$$u(t, x) \sim \sum_{k=0}^{\infty} e^{-\big(\mu_k + s_0(0)\big)t} u_k(t, x).$$

Remark 4.2. *The solution to the homogeneous equation*

$$\begin{cases} \partial_t u + \tilde{\nu}(t,x,\nabla_x)u &= 0, \\ u_{|t=0} &= w, \end{cases}$$

is also expandible since $\bar{u} := u - \chi(t)w(x)$, *where* $\chi(t)$ *is a cutoff function near* $t = 0$, *satisfies*

$$\begin{cases} \partial_t \bar{u} + \tilde{\nu}(t,x,\nabla_x)\bar{u} &= -\chi(t)\tilde{\nu}w, \\ \bar{u}_{|t=0} &= 0, \end{cases}$$

which means by Theorem 4.1 *that* $\bar{u}$ *is expandible.*

Recall that $u = w\big(\exp t\nu(x)\big)$ *when the vector field is independent of* t, $\tilde{\nu} = \nu$. *Taking* x_j *as the initial data* w, *we see that* $\exp t\nu(x)$ *is expandible. This fact also implies that the Hamilton flow* $\big(x(t),\xi(t)\big) = \exp tH_{p_0}(x^0,\xi^0)$ *on the incoming stable manifold* Λ_- *is expandible. In fact,* $x(t)$ *satisfies*

$$\begin{cases} \dot{x}(t) &= \nabla_\xi p_0\big(x, \nabla_x\phi_-(x)\big), \\ x(0) &= x^0, \end{cases}$$

where $\nabla_\xi p_0(x,\nabla_x\phi_-) = -\mathrm{diag}(\lambda_1,\dots,\lambda_d)x + \mathcal{O}(|x|^2)$.

5. Connection at the fixed point

The aim of this section is to construct an asymptotic solution u to the Schrödinger equation (17) in a small neighborhood W of the origin $x = 0$, whose asymptotic expansion in a neighborhood V of the curve γ coincide with the given WKB solution (18). Recall that we are looking for such a solution in the form (21) and that the problem is reduced to the construction of a WKB solution (20) to the time-dependent Schrödinger equation (22).

5.1. **Construction of the time-dependent phase.** Let Γ_0 be the submanifold of Λ_ψ:

$$\Gamma_0 = \{(x,\xi) \in \Lambda_\psi;\ \psi(x) = \psi(x^0)\},$$

and Λ_0 a Lagrangian manifold intersecting Λ_ψ cleanly along Γ_0. Put

$$\Lambda_t := \exp tH_{p_0}(\Lambda_0), \quad \Gamma_t := \exp tH_{p_0}(\Gamma_0).$$

We have the following proposition :

Proposition 5.1. *There exist a neighborhood* Ω *of* $x = 0$, *a positive number* T_0 *and* $\varphi \in C^\infty\big((T_0,\infty)\times\Omega\big)$ *such that*

$$\Lambda_t = \{(x,\xi) \in \mathbb{R}^{2d};\ \xi = \nabla_x\varphi(t,x)\}. \tag{33}$$

Moreover, φ can be choosen so that it satisfies the eikonal equation (23) and the estimate

$$\varphi(t,x) - \phi_+(x) = \mathcal{O}^\infty(e^{-\lambda_1 t}). \tag{34}$$

Proof. Here we only show how to choose φ so that it satisfies (23).

We construct a Lagrangian manifold $\widetilde{\Lambda}$ in the phase space $T^*\mathbb{R}^{d+1}_{(t,x)}$ by taking first a d-dimensional submanifold

$$\Lambda' = \{(t,x,\tau,\xi) \in \mathbb{R}^{2(d+1)};\ t=0,\ (x,\xi)\in\Lambda_0,\ \tau + p_0(x,\xi) = 0\},$$

and putting

$$\widetilde{\Lambda} = \bigcup_t \exp tH_{\tau+p_0}(\Lambda').$$

We see that the projection of $\widetilde{\Lambda}$ to the (t,x)-space is diffeomorphic, and hence there exists a generating function $\varphi(t,x)$ such that

$$\widetilde{\Lambda} = \{(t,x,\tau,\xi)\in\mathbb{R}^{2(d+1)};\ \tau = \frac{\partial\varphi}{\partial t}(t,x),\ \xi = \frac{\partial\varphi}{\partial x}(t,x)\}$$

This φ is determined modulo constant. Since $\widetilde{\Lambda} \subset (\tau+p_0)^{-1}(0)$, φ satisfies the eikonal equation (23). □

Proposition 5.2. *The function $\varphi(t,x) - \phi_+(x)$ is expandible in the sense of §4:*

$$\varphi(t,x) - \phi_+(x) \sim \phi_1(x)e^{-\lambda_1 t} + \sum_{k=2}^{\infty}\phi_k(t,x)e^{-\mu_k t}. \tag{35}$$

In particular, ϕ_1 is independent of t and given by

$$\phi_1(x) = -\lambda_1 X^-(x^0,\xi^0)\cdot x + \mathcal{O}(|x|^2), \tag{36}$$

where $X^-(x^0,\xi^0)$ is a non-zero eigenvector associated to the eigenvalue $-\lambda_1$ of the fundamental matris F_{p_0}, see assumption **(A1)**.

Proof. Let us introduce new symplectic local coordinates (x,ξ) centered at $(0,0)$ such that Λ_- is given by $x=0$ and Λ_+ is given by $\xi = 0$. Then

$$p_0(x,\xi) = A(x,\xi)x\cdot\xi,$$

where the matrix $A(0,0)$ has the eigenvalues $\lambda_1,\dots,\lambda_d$ and we may assume that

$$A(0,0) = \mathrm{diag}(\lambda_1\cdots,\lambda_d).$$

The curve γ now becomes $(0,\xi_0(t))$, where $\xi_0(t) = \mathcal{O}(e^{-\mu_1 t})$.

We check the following proposition by induction :

$\mathbf{(H)}_N$: $\varphi = \psi_N + r_N$; ψ_N expandible $\mathcal{O}^\infty(e^{-\lambda_1 t}|x|)$,

$$r_N = \mathcal{O}^\infty(e^{-N\lambda_1 t}|x|^{N+1}),\ r_{N|t=0} = 0.$$

By Taylor expansion with respect to r_N, we get

$$\partial_t r_N + \tilde{\nu}_N r_N = f_N + \mathcal{O}^\infty\big(e^{-2N\lambda_1 t}|x|^{2N+1}\big),$$

where

$$\tilde{\nu}_N := \nabla_\xi p_0(x, \nabla_x \psi_N) \cdot \nabla_x,$$
$$\nabla_\xi p_0(x, \nabla_x \psi_N) = A(x,0)x + \mathcal{O}^\infty\big(|x|^2 e^{-\lambda_1 t}\big), \quad \text{expandible,}$$

and

$$f_N = -\big(\partial_t \psi_N + p_0(x, \nabla_x \psi_N)\big)$$

is expandible and $\mathcal{O}^\infty\big(e^{-N\lambda_1 t}|x|^{N+1}\big)$. Let ρ_N be the solution to

$$\begin{cases} \partial_t \rho_N + \tilde{\nu}_N \rho_N &= f_N, \\ \rho_{N|t=0} &= 0. \end{cases}$$

then, by Theorem 4.1 and Lemma 4.1, which holds also for t-dependent $\tilde{\nu}$, $\rho_N = \mathcal{O}\big(e^{-N\lambda_1 t}|x|^{N+1}\big)$ is expandible. Now we put

$$\varphi = (\psi_N + \rho_N) + (r_N - \rho_N) =: \psi_{2N} + r_{2N}.$$

We see that $r_{2N} = \mathcal{O}^\infty\big(e^{-2N\lambda_1 t}|x|^{2N+1}\big)$ since it satisfies

$$\begin{cases} \partial_t r_{2N} + \tilde{\nu}_N r_{2N} &= \mathcal{O}^\infty\big(e^{-2N\lambda_1 t}|x|^{2N+1}\big), \\ r_{2N|t=0} &= 0. \end{cases}$$

Hence $(\mathrm{H})_N$ implies $(\mathrm{H})_{2N}$.

It remains to prove $(\mathrm{H})_1$. The estimate (34) implies

$$\varphi(t,x) = \varphi(t,0) + x \cdot \nabla_x \varphi(t,0) + \mathcal{O}^\infty\big(e^{-\lambda_1 t}|x|^2\big).$$

Differentiating the eikonal equation

$$\partial_t \varphi + A(x, \nabla_x \varphi)x \cdot \nabla_x \varphi = 0$$

with respect to x, and substituting $x = 0$, $\xi(t) := \nabla_x \varphi(t,0)$ satisfies

$$\dot{\xi}(t) + {}^t A\big(0, \xi(t)\big)\xi(t) = 0.$$

Then $\xi(t)$ is expandible by Remark 4.2 since ${}^t A(0,0) = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$. Hence $(\mathrm{H})_1$ holds.

It is not difficult to see that $\varphi - \phi_+$ is expandible also for the original coordinates. □

Let $V \subset \mathbb{R}^d$ be a small open neighborhood of $\Pi_x \gamma$.

Proposition 5.3. *For each $x \in V$, there is unique $t = t(x)$ such that $\Pi_\psi^{-1} x \in \Lambda_t$ ($\Pi_\psi^{-1} x$ is the lift of x on Λ_ψ). Then $t(x)$ is a critical point of $\varphi(\cdot, x)$ and the critical value is $\psi(x)$:*

$$\partial_t \varphi\big(t(x), x\big) = 0, \quad \varphi\big(t(x), x\big) = \psi(x).$$

Proof. Put $\Pi_\psi^{-1}x = (x,\xi)$. Since $(x,\xi) \in \Lambda_\psi \cap \Lambda_{t(x)} \subset p_0^{-1}(0) \cap \Lambda_{t(x)}$, one has

$$p_0(x,\xi) = 0 \quad \text{and} \quad \xi = \nabla_x\varphi(t(x),x),$$

i.e. $p_0\Big(x, \nabla_x\varphi\big(t(x),x\big)\Big) = 0$. Hence by the eikonal equation, we have $\partial_t\varphi\big(t(x),x\big) = 0$.

From this, we see that $\nabla_x\Big[\varphi\big(t(x),x\big)\Big] = (\nabla_x\varphi)\big(t(x),x\big) = \xi$. On the other hand, $\xi = \nabla_x\psi$ because $(x,\xi) \in \Lambda_\psi$.

Hence $\nabla_x\Big[\varphi\big(t(x),x\big)\Big] = \nabla_x\psi$. □

5.2. **Construction of the time-dependent symbol.** Let $V \subset \mathbb{R}^d$ be a small open neighborhood of $\Pi_x\gamma$. By the stationary phase method at the critical point $t(x)$, we have an asymptotic expansion of the following form:

$$\frac{1}{\sqrt{2\pi h}}\int^{\infty} e^{i\varphi(t,x)/h}a(t,x;h)\chi(t-t(x))dt \sim e^{i\psi(x)/h}\sum_{l=0}^{\infty}\bar{b}_l(x)h^l, \tag{37}$$

where χ is a cutoff function near $t=0$. In particular, if we fix t_0, then for $x \in V \cap \Gamma_{t_0}$, one has

$$\bar{b}_0(x) = \frac{e^{\pi i/4}}{\sqrt{\varphi_{tt}(t_0,x)}}a_0(t_0,x). \tag{38}$$

We define $a_0(t_0,x)$ so that $\bar{b}_0(x) = b_0(x)$ on $\Gamma_{t_0} \cap V$. Since $\bar{b}_0$ and b_0 satisfy the same transport equation, they coincide in V. Continuing this way, we can determine the formal symbol a so that (37) holds formally near 0 in V, with b_l instead of $\bar{b}_l$.

Proposition 5.4. *The function a_l is expandible in the sense of §4 and*

$$a_l(t,x) \sim e^{-St}\sum_{k=0}^{\infty}a_{lk}(t,x)e^{-\mu_k t}, \tag{39}$$

where $S = \frac{1}{2}\sum_{j=1}^{d}\lambda_j - iz$.

In particular, $a_{0,0}(t,x) = a_{0,0}(x)$ is independent of t and

$$a_{0,0}(0) = ce^{\pi i/4}\lambda_1^{3/2}\lim_{t\to+\infty}e^{(S-\lambda_1)t}\sqrt{\frac{J(0,x^{0\prime})}{J(t,x^{0\prime})}}. \tag{40}$$

Here c is the constant given in (15).

Proof. By the change of variable $y = x - x(t)$, the transport equation becomes, for each $l \geq 0$,

$$\partial_t a_l + 2\big[\nabla_x\varphi(t,\cdot)\big]_{x(t)}^{x(t)+y}\cdot\nabla_x a_l + \Big(\Delta\varphi\big(t,x(t)+y\big) - iz\Big)a_l = -\Delta a_{l-1}.$$

By convention $a_{-1}=0$. We define $\big[f(\cdot)\big]_a^b = f(b)-f(a)$.

By (35), we have

$$2\big[\nabla_x\varphi(t,\cdot)\big]_{x(t)}^{x(t)+y} = 2\big[\nabla_x\phi_+(\cdot)\big]_{x(t)}^{x(t)+y} + \tilde{A}(t,y)y,$$

where $\tilde{A}(t,y)$ is an expandible matrix. Moreover,

$$2\big[\nabla_x\phi_+(\cdot)\big]_{x(t)}^{x(t)+y} = 2\nabla_x^2\phi_+\big(x(t)\big)y + \mathcal{O}\big(|y|^2\big),$$

and $2\nabla_x^2\phi_+\big(x(t)\big) = \operatorname{diag}(\lambda_1,\dots,\lambda_d) +$ expandible. Hence the vector field $2\big[\nabla_x\varphi(t,\cdot)\big]_{x(t)}^{x(t)+y}\cdot\nabla_x$ is of the form (32).

On the other hand, $s(t,y) := \Delta\varphi\big(t,x(t)+y\big) - iz$ is expandible, $s(t,y) \sim \sum_{k\geq 0} e^{-\mu_k t}s_k(t,y)$, with $s_0(y) = \Delta\phi_+(y) - iz$ is independent of t and $s_0(0)=S$.

Then it follows from Corollary 4.1 that a_l is expandible of the form (39).

Next we calculate $a_{0,0}(0)$ as the limit of $e^{St}a_0\big(t,x(t)\big)$ when $t\to+\infty$.

First, in (38), recall that

$$b_0\big(x(t)\big) = e^{izt}\sqrt{\frac{J(0,x^{0\prime})}{J(t,x^{0\prime})}}b_0(x^0) = e^{izt}e^{-\int_0^t \Delta\psi(x(\tau))d\tau}b_0(x^0). \tag{41}$$

Lemma 5.1. *The functions ψ and φ satisfy*

$$\Delta\psi\big(x(t)\big) = \sum_{j=1}^{d}\frac{\lambda_j}{2} - \lambda_1 + \mathcal{O}\big(e^{-\hat{\mu}_1 t}\big), \tag{42}$$

$$\varphi_{tt}\big(t,x(t)\big) = c^2\lambda_1^3 e^{-2\lambda_1 t}\Big(1+\mathcal{O}\big(e^{-\hat{\mu}_1 t}\big)\Big). \tag{43}$$

The formula (42) means the existence of the limit as $t\to\infty$ of the function $e^{(\sum_{j=1}^d \frac{\lambda_j}{2}-\lambda_1)t}\sqrt{\frac{J(0,x^{0\prime})}{J(t,x^{0\prime})}}$. On the other hand, (38) and (41) with (43) give

$$e^{St}a_0\big(t,x(t)\big) \sim ce^{\pi i/4}\lambda_1^{3/2}e^{(\sum_{j=1}^d\frac{\lambda_j}{2}-\lambda_1)t}\sqrt{\frac{J(0,x^{0\prime})}{J(t,x^{0\prime})}}b_0(x^0).$$

Letting $t\to\infty$, we get (40). □

5.3. **Asymptotic expansion on the outgoing stable manifold.** To conclude this text, we calculate the asymptotic expansion of u, constructed in the previous section, near a given point on the outgoing stable manifold Λ_+. Since Λ_t converges to Λ_+ as $t\to+\infty$, The asymptotic expansion of u of the integral form (21) comes from the aymptotic behavior of the time-dependent WKB solution (20) as $t\to+\infty$, which was already studied in §4.1 and §4.2.

Fix a point (x,ξ) on Λ_+ and put

$$\big(x(t),\xi(t)\big) = \exp tH_{p_0}(x,\xi).$$

As on Λ_-, the curve $x(t)$ has the asymptotic behavior of the type

$$x(t) = X^+(x,\xi)e^{\lambda_1 t} + \mathcal{O}^\infty(e^{\mu_2 t}) \quad \textit{as} \quad t \to -\infty, \tag{44}$$

where $X^+(x,\xi)$ is a vector independent of t, see Proposition 3.1. We assume

$$X^-(x^0,\xi^0)\cdot X^+(x,\xi) \neq 0. \tag{\textbf{A3}}$$

We also assume

$$S + \mu_k \neq 0 \quad \forall k \in \mathbb{N}. \tag{\textbf{A4}}$$

This is equivalent to the condition that the energy z does not belong to the discrete set $-i\zeta_0$ where

$$\zeta_0 = \left\{ \sum_{j=1}^d \left(\alpha_j + \frac{1}{2}\right)\lambda_j;\ \alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d \right\}$$

is the set of eigenvalues of the harmonic oscillator $-\Delta + \sum_{j=1}^d \frac{\lambda_j^2}{4}x_j^2$.

Theorem 5.1. *On Λ_+, we have*

$$\int_0^\infty e^{\frac{i}{h}\varphi(t,x)}a(t,x,;h)\,dt = h^{\frac{S}{\lambda_1}}e^{\frac{i}{h}\phi_+(x)}c(x;h), \tag{45}$$

$$c(x;h) \sim \sum_{k=0}^\infty c_k(x,\ln h)h^{\frac{\hat\mu_k}{\lambda_1}}.$$

Here, $0 = \hat\mu_0 < \hat\mu_1 < \hat\mu_2 < \cdots$ is a numbering of the linear combinations of $\{\mu_k - \mu_1\}_{k=1}^\infty$ over $\mathbb{N}$, and c_0 is independent of $\ln h$ and given by

$$c_0(x) = \Gamma\left(\frac{S}{\lambda_1}\right)\frac{\exp(i\frac{S\pi}{2\lambda_1}\ \mathrm{sgn}\phi_1)}{\lambda_1|\phi_1(x)|^{\frac{S}{\lambda_1}}}a_{0,0}(x). \tag{46}$$

Proof. Put $e^{-t} = s$, $\varphi - \phi_+ = \phi_1\sigma^{\mu_1}$, then, by (35), σ has an expansion with respect to s of the form

$$\begin{aligned}\sigma \sim &\ s\left(1 + \sum_{k=2}^\infty \frac{\phi_k(-\log s,x)}{\phi_1(x)}s^{\mu_k-\mu_1}\right)^{1/\mu_1} \\ \sim &\ s\left(1 + \sum_{k=2}^\infty \rho_k(-\log s,x)s^{\hat\mu_k}\right).\end{aligned}$$

It is not difficult to see that, conversely, s has the asymptotic expansion with respect to σ of the same form :

$$s \sim \sigma\left(1 + \sum_{k=2}^\infty f_k(-\log\sigma,x)\sigma^{\hat\mu_k}\right).$$

By this change of variable, the left hand side of (45) becomes

$$\int_0^\infty e^{\frac{i}{h}\varphi} a_0\, dt = e^{\frac{i}{h}\phi_+} \int_0^1 e^{\frac{i}{h}\phi_1 \sigma^{\mu_1}} s^S \sum a_{0,k}(-\log s, x) s^{\mu_k - 1}\, ds$$

$$= e^{\frac{i}{h}\phi_+} \int_0^{\alpha(x)^{\frac{1}{\mu_1}}} e^{\frac{i}{h}\phi_1 \sigma^{\mu_1}} \sum b_k(-\log\sigma, x)\sigma^{S+\hat{\mu}_k - 1}\, d\sigma,$$

and moreover by $\sigma^{\mu_1} = \tau$

$$= e^{\frac{i}{h}\phi_+} \int_0^{\alpha(x)} e^{\frac{i}{h}\phi_1 \tau} \sum c_k(-\log\tau, x)\tau^{\frac{S+\hat{\mu}_k}{\mu_1} - 1}\, d\tau,$$

where $\alpha(x) = \dfrac{\varphi(0,x) - \phi_+(x)}{\phi_1(x)}$. In particular, the coefficients $c_0(x)$, $b_0(x)$ depend only on x and $c_0(x) = \frac{1}{\mu_1} b_0(x) = \frac{1}{\mu_1} a_{0,0}(x)$.

The last integral is not well-defined when $\frac{S+\hat{\mu}_k}{\mu_1}$ is a negatif integer, that is, $z = z_{\alpha,N} = -i\big(\sum_{j=1}^d (\alpha_j + \frac{1}{2})\lambda_j - N\lambda_1\big)$, for some $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ and $N \in \mathbb{N}$. Recall that $S := \sum_{j=1}^d \frac{\lambda_j}{2} - iz$. If $\alpha_1 \geq N$, this z is excluded by (**A4**). On the contrary, the other cases corresponding to $\alpha_1 < N$ never occur because we already know that our solution u is holomorphic outside $-ih\zeta_0$. For details see [BFRZ], page 111, identity (5.82) and proposition 5.11, and identities (5.97), (5.98) page 113.

Now Proposition follows from the following facts:

$$\text{(47)} \qquad \int_0^\infty e^{\frac{i}{h}as} s^{p-1}\, ds = e^{i\frac{p\pi}{2}\operatorname{sgn} a}\Gamma(p)\big(\frac{h}{|a|}\big)^p, \quad a \in \mathbb{R}\backslash\{0\},\ 0 < \operatorname{Re} p < 1$$

and

$$\int_0^1 e^{\frac{i}{h}\phi_1\tau}\tau^{\mu-1}(\log\tau)^m\, d\tau = -\left(\frac{\partial}{\partial\mu}\right)^m \int_0^1 e^{i\phi_1\tau/h}\tau^{\mu-1}\, d\tau.$$

□

It remains to write $c_0(x)$ in terms of the original WKB solution on Λ_- instead of $a_{0,0}(x)$. Thanks to the formula (40), it suffices to write it in terms of $a_{0,0}(0)$.

The function $c_0(x)$ satisfies the transport equation

$$\text{(48)} \qquad 2\nabla_x\phi_+ \cdot \nabla_x c_0 + (\Delta\phi_+ - iz)c_0 = 0.$$

This is an ordinary differential equation on Λ_+ along the Hamilton flow $x = x(t)$ along which (48) is written as

$$\frac{d}{dt}c_0(x(t)) + \big(\Delta\phi_+(x(t)) - iz\big)c_0 = 0,$$

and the solution is $c_0(x(t)) = e^{izt-\int_0^t \Delta\phi_+(x(\tau))d\tau} c_0(x)$.

Together with (46), we obtain $c_0(x) = e^{-izt+\int_0^t \Delta\phi_+(x(\tau))\,d\tau} c_0(x(t))$, hence

$$c_0(x) = e^{-izt+\int_0^t \Delta\phi_+(x(\tau))d\tau}\Gamma\left(\frac{S}{\lambda_1}\right)\frac{\exp(i\frac{S\pi}{2\lambda_1}\ \mathrm{sgn}\phi_1)}{\lambda_1|\phi_1(x(t))|^{\frac{S}{\lambda_1}}} a_{0,0}(x(t)). \tag{49}$$

Recalling (44) and (36), we get

$$\phi_1(x(t)) = -\lambda_1 X^-(x^0,\xi^0)\cdot X^+(x,\xi)e^{\lambda_1 t} + \mathcal{O}\big(e^{(\mu_2-\delta)t}\big) \quad \text{as } t \to -\infty, \tag{50}$$

Here $\delta > 0$ is arbitrary. Taking the limit $t \to -\infty$ in (49), we obtain:

Proposition 5.5.

$$c_0(x) = \Gamma\left(\frac{S}{\lambda_1}\right)\frac{\exp(-i\frac{S\pi}{2\lambda_1}\,\sigma)}{\lambda_1|\lambda_1 X^-(x^0,\xi^0)\cdot X^+(x,\xi)|^{\frac{S}{\lambda_1}}} e^{I_\infty(x)} a_{0,0}(0), \tag{51}$$

where

$$\sigma = \mathrm{sgn}\left[X^-(x^0,\xi^0)\cdot X^+(x,\xi)\right],$$

and $a_{0,0}(0)$ *is given in* (40)*:*

$$a_{0,0}(0) = e^{\frac{\pi i}{4}}\lambda_1^{\frac{3}{2}}|X^-(x^0,\xi^0)|\lim_{t\to+\infty} e^{(S-\lambda_1)t}\sqrt{\frac{J(0,x^{0\prime})}{J(t,x^{0\prime})}}.$$

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