数理解析研究所講究録
第 1609 巻 2008 年 214-222

Realizations of duality between Gaussian and Poisson noises

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2000 AMS subject classification : 60H40

1 Introduction

Gaussian and Poisson noises can be discussed not only in parallel, but we can see dissimilarities between them, which are more interesting. In addition, dualities between the two noises are particularly significant and important. We are going to discuss the duality, in question, between Gaussian noise and Poisson noise which is realized in terms of unitary representations of the groups associated with them, respectively. To establish those unitary representations we use the space of quadratic Hida distributions. As is well known, Gaussian noise (white noise) can be characterized by the infinite dimensional rotation group. In reality, it plays an important roles in white noise analysis. In this note, we use a unitary representation of the symmetric group that characterises Poisson noise in order to show a relationship, that is a duality. Our idea will be reviewed quickly in Section 2 (see [13] for details) together with some notes.

In the paper [13] one of the realizations of duality appears in connection with the Lévy Laplacian. Now we shall focus our attention from another side, although we use spaces of quadratic generalized white noise functionals, i.e. quadratic Hida distributions.

Again we emphasize the significance of the Lévy construction (infact, it is nothing but an approximation) of Brownian motion and importance of the quadratic Hida distributions in Section 3.

In the last section we shall see a new recognition of the subgroup G_∞ which characterizes the white noise measure. It seems good to give a unitary representation of the group on the space of quadratic Hida distributions. Thus, we shall see some connection with the representation of the symmetric group; in this sense we can recognize a relationship between two noises, white noise and Poisson noise.

2 Review on Poison noise and quadratic forms of $\dot{B}(t)$'s

When we have discussed the characterization of Poisson noise in [11], we have defned the infinite symmetric group $S(\infty)$, as the projective limit of $S(n)$.

There we define a unitary representation $U_\pi^{(n)}$ of $S(n)$ on R^n for each $\pi \in S(n)$, in the following manner. We take $\{e_k, k = 1, \cdots, n\}$ as a base of R^n, then a simplest irreducible factor of the representation is formed by taking a subset R_1^n of R^n, and we obtain an irreducible unitary reprsentation $\widetilde{U}_\pi^{(n)}$ of $S(n)$ on R_1^n.

We now have a family of pairs $\{(S(n), \widetilde{U}^{(n)})\}$. Following Bochner's method (see [1]), the projective limit of $\{(S(n), \widetilde{U}^{(n)})\}$ is obtained. The limit is denoted by $\{(S(\infty), \widetilde{U}^{(\infty)})\}$.

There is a big freedom to choose a base $\{e_n\}$, we choose a base that is formed by quadratic Hida distributions. k,ilp@: We choose the base $\{: \Delta_k X_n(t)^2 :\}$ as for $\{e_n\}$, where the $X_n(t)$ is the process appearing in the Lévy's construction of Brownian motion, which we are going to show in what follows.

Prepare a sequence $Y_n, n \geq 1$, of independent identically distributed $N(0,1)$ random variables. ($N(0,1)$ stands for the standard Gaussian distribution.)

Define $X_1(t), t \in [0,1]$ by

$$X_1(t) = tY_1.$$

The sequence $X_n(t)$ can be defined inductivelyas follows:,

$$X_{n+1}(t) = \begin{cases} X_n(t),\ t \in T_n, \\ \dfrac{X_n(t+2^{-n}) + X_n(t-2^{-n})}{2} + 2^{-(n+1)/2}Y_k,\ t \in T_{n+1} - T_n, \\ \qquad\qquad k = k(t) = 2^{n-1} + \frac{2^n t+1}{2}, \\ (k+1-2^n t)X_{n+1}(k2^{-n}) + (2^n t - k)X_{n+1}((k+1)2^{-n}), \\ \qquad\qquad t \in [k2^{-n}, (k+1)2^{-n}], \end{cases} \tag{2.1}$$

where $k = k(t) = 2^{n-1} + \frac{1}{2}(2^n t + 1), t \in T_{n+1} - T_n$.

There 2^n independent random variables $Y_1, \cdots, Y_{2^n}$ are involved to define $X_n(t)$. Note that $X_n(t)$ approximates $B(t)$ uniformly in t and the interpolations by Y_k's are done independently in different intervals. The conditional expectations under the conditions $\mathbf{B}_n(Y)$ define the projections which are consistent, where

$$\mathbf{B}_n(Y) = \sigma\text{-field generated by } Y(k), 1 \le k \le 2^n.$$

Note that $X_n(t)$ is $\mathbf{B}_{2^n}(Y)$ measurable. Also note that for binary point t we may identify $X_n(t) = B(t)$, $B(t)$ being a Brownian motion. So, in what follows we shall use these two notations.

Let

$$\mathcal{D}_n = \left\{ \Delta_k^n; \Delta_k^n = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right], k = 0, 1, \cdots, 2^n \right\}.$$

be the sequence of partitions of the unit time interval $I = [0,1]$.

Let $\mathbf{L}_{2^n}$ denote a linear space spanned by the $Y_k^p, 0 \le p \le 2, 1 \le k \le 2^n$, and let $P(\cdot | \mathbf{L}_{2^n})$ denote the projection down to the subspace $\mathbf{L}_{2^n}$.

Proposition 2.1 *The following relation holds.*

$$P\Big(:\Delta_{2k}X_{n+1}(t)^2: + :\Delta_{2k+1}X_{n+1}(t)^2: \mid \mathbf{L}_{2^n}\Big) =: \Delta_k X_n(t)^2 : . \tag{2.2}$$

This is the key proposition that guarantees the possibility of defining the projections $f_{n+1,n} : \mathbf{L}_{2^{n+1}} \to \mathbf{L}_{2^n}$ and the consistency of gthe family $\{f_{n+1,n}\}$.

Theorem 2.1 *The sequence* $\{(S(2^n), \widetilde{U}^{2^n})\}$ *defines a projective limit* $(S(\infty), \widetilde{U}^{(\infty)})$.

Space of quadratic Hida distributions.

Let us remind the Gel'fand triple to form (see [13]).

$$H_2^{(2)} \subset H_2 \subset H_2^{(-2)},$$

where

$$H_2^{(2)} = \{\int\int_{I^2} F(u,v) : \dot{B}(u)\dot{B}(v) : dudv, F \in \widehat{K}^{\frac{3}{2}}(I^2)\},$$

$$H_2 = \{\int\int_{I^2} F(u,v) : \dot{B}(u)\dot{B}(v) : dudv, F \in \widehat{K}^2(I^2)\},$$

and

$$H_2^{(-2)} = \{\int\int_{I^2} F(u,v) : \dot{B}(u)\dot{B}(v) : dudv, F \in \widehat{K}^{-\frac{3}{2}}(I^2)\},$$

(the quadratic Hida distribution space)
where $K^{\frac{3}{2}}(I^2)$ is a Sobolev space over I^2 of order $\frac{3}{2}$, $K^{-\frac{3}{2}}(I^2)$ is the dual space of $K^{\frac{3}{2}}(I^2)$ and $\widehat{\cdot}$ means symmetric.

It can be seen that the space $H_2^{(-2)}$ is the space of quadratic functionals $\dot{B}(t), t \in I = [0,1]$.

Define the new subspace of $H^{(-2)}$:

$$H_2^{(-2,1)} = \{\int_I f(u) : \dot{B}(u)^2 : du, F \in L^2(I)\}.$$

The function f is viewed as $f(\frac{u+v}{2})\delta(u-v) \equiv \tilde{f}(u,v)$.

$$\langle \int_I f(u) : \dot{B}(u)^2 : du, \int_I g(u) : \dot{B}(u)^2 : du \rangle = (\tilde{f}, \tilde{g})_{K^{-\frac{3}{2}}}.$$

Note that the null space of $H_2^{(-2,1)}$ is $\{0\}$. The Hilbert space $H_2^{(-2,1)}$ involves only diagonal elements of degree 2 and is defined as a subspace (in the ordinary sense) of $H_2^{(-2)}$.

It is surprising that the **dual space** $H^{(-2,2)}$ of $H_2^{(-2,1)}$ can be introduced, where the bilinear form that connects two spaces can come from the integral with respect to the white noise measure. Actually the new space is given by

$$H_2^{(-2,2)} = \{\int_I g(u) : \dot{B}(u)^2 : du^2, g \in L^2(I)\}.$$

can be defined in the same stage of approximation of Brownian motion due to P. Lévy.

We would like to note that both spaces $H_2^{(-2,1)}$ and $H_2^{(-2,2)}$ involve only quadratic, diagonal elements of the $\dot{B}(t)$, and they have clear identity as the spaces of Hida distributions. They are defined by using the Lévy approximation of Brownian nmotion.

3 Note on Lévy's approximation of Brownian motion

There are many ways to define a Brownian motion. We claim that Lévy's method is most powerful and convenient for stochastic calculus, as we have seen so far. His way of construction is at the same time a method of approximation. From these viewpoints, we summarize the characteristics of the Lévy's method.

1) Approximation by $X_n(t)$ is uniform in t, that runs through the unit time interval.

2) The time variable t is always involved and never changes as n is getting large.

3) White noise can be approximated only by taking the time derivative $\dot{X}_n(t)$ uniformly in t.

4) The renormalized squares (may be denoted by : $\dot{B}(t)^2$: can be approximated by those of $X(t)$. This enables us to define projections depending on n the step of approximation.

5) The limits $\dot{X}(t)$ or $\dot{B}(t)$, obtained by projective limit, form an independent system, so that they can be chosen as basic variable system of Brownian functionals (white noise functionals). Hence the partial derivative $\partial_t = \frac{\partial}{\partial \dot{B}(t)}$ can be defined. This operator corresponds to the Fréchet derivative in functional analysis. It can be apprfoximated in line with the Lévy method.

6) Those properties mentioned above enable us to consider certain limit of rotation group $SO(n)$ through the unitary representation.

We have only thought of advantages from thr viewpoint of white npoise analysis, so one may think of favorable properties from aother side.

4 Motion group M_∞

Take a nuclear space E which is dense in $L^2(R)$. The infinite dimensional rotation group $O(E)$ is a collection of such g's as

i) g is a linear homeomorphism of E,

ii) $||g\xi|| = ||\xi||$ for every $\xi \in E$, where $|| \cdot ||$ is the $L^2(R)$-norm.

Each g is a rotation of E and the $O(E)$ is often called an infinite dimensional rotation group if E is not specified.

There are two classes of rotations of E; Class I and Class II (see [5] Chapter 5, for detail). We are now concerned with a subgroup of

$O(E)$ involving members in Class I. Roughly speaking those members in Class I are determined by fixing a complete orthonormal system $\{\xi_n\}$ in $L^2(R)$ such that each ξ_n belongs to E.

Take finitely many members, say $\xi_1, \cdots, \xi_n$ from E. They span an n dimensional space denoted by E_n. Then, we can consider g such that the restriction of g to E_n is a rotation of E_n and the restriction to $E_n^\perp$ is the identity. The collection of all such g's form a subgroup G_n of $O(E)$.

In [5] we define the projective limit G_∞:

$$G_\infty = \text{proj } \lim_{n \to \infty} G_n.$$

We now consider the motion group M_n which is generated by rotations and translations. To concretize the theory, we take the space

$$E_n = \text{span}\{: (\Delta_k X_n(t))^2 :, 1 \leq k \leq 2^n\}.$$

The rotation group $SO(2^n)$ determines a subgroup G_{2^n}, now simply denoted by G_n. Combine translations by members of E_n with G_n to define a motion group M_n, which acts on E_n. In the usual manner we can define a unitary representation of M_n given on E_n.

We know the projective limit of E_n defines the Hilbert space $H_2^{(-2,2)}$ with a modification of the coefficients $(\Delta^n)^2$. We can finally conlude

Theorem 4.1 *The projective limit M_∞ of the motion group M_n can be defined together with that of E_n.*

As a consequence of this theorem we have

Proposition 4.1 *Unitary representation of motion group M_∞ is given on the space $H_2^{(-2,2)}$ of quadratic Hida distribution.*

The motion group M_∞ is related to Gaussian (See Hida [3] Section 5.7). On the other hand symmetric group is related to Poisson.

We have obtained the representation of the symmetric group by using the space $H_2^{(-2,2)}$ of quadratic Hida distribution. The representation of motion group also uses the space $H_2^{(-2,2)}$ of quadratic Hida distribution. In this sense we can see a kind of duality.

Further discussions will appear in the forthcoming paper.

Acknowledgement The author expresses her deep thanks to Professor I. Ojima who suggests the study of duality in stochastic analysis.

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