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On the Hida product and QFT with interactions

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Abstract

By making use of the Hida product we construct a new reflection positive random field with the space time dimension $d=4$ which surely has a correspondence to a concrete QFT with interaction.

1 Introduction

In section 2 we give a concise guide of the mathematical structure of *quantum field theory* (QFT) through the arguments by means of Gaussian random fields (cf. e.g., [Si]) and stochastic integrals with respect to the Gaussian white noise (cf. e.g., [AFY], [AY1,2,3]). In section 3 by making use of the Hida product, of which definition has been introduced in [AY4], we present a new reflection positive random field with the space time dimension $d=4$ that surely has a correspondence to a concrete QFT with interaction.

2 Identification of Euclidean quantum fields on $\mathbb{R}^d$ with $\mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R})$ valued Markov processes

Throughout this paper, we denote by $d\in\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, the space-time dimension, and we understand that $d-1$ is the space dimension and 1 is the dimension of time. Correspondingly, we use the notations

$$\mathbf{x}\equiv(t,\vec{x})\in\mathbb{R}\times\mathbb{R}^{d-1}.$$

Let $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^{d-1})$) be the Schwartz space of rapidly decreasing test functions on the d dimensional Euclidean space $\mathbb{R}^d$ (resp. $d-1$ dimensional Euclidean space $\mathbb{R}^{d-1}$), equipped with the usual topology by which it is a Fréchet nuclear space. Let $\mathcal{S}'(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^{d-1})$) be the topological dual space of $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^{d-1})$).

The probability measures on $\mathcal{S}'(\mathbb{R}^d\to\mathbb{R})$ which are invariant with respect to the Euclidean transformations are called as **Euclidean random fields**. The

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Euclidean random fields which admit an analytic continuation to the quantum fields (**Wightman fields**) are called as **Euclidean quantum (random) fields**. Where the analytic continuation, very roughly speaking, means the analytic continuation of the time variable $t \in \mathbb{R}$ of Euclidean fields to $\sqrt{-1}t$, and **Wightaman fields** are the fields that are invariant with respect to the transformations keeping the Lorentz scalar product unchanged (i.e. the restricted Poincré invariance).

In this section we review how the Euclidean quantum random fields on $\mathbb{R}^d$, the probability measures on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$, are identified with the probability measures on the space $C\big(\mathbb{R} \to \mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})\big)$ which are generated by some $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$ valued Markov processes.

In order to simplify the notations, in the sequel, by the symbol D we denote both d and $d-1$. In each discussion we exactly explain the dimension (space-time or space) of the field on which we are working.

Now, suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ we are given an isonormal Gaussian process $W^D = \{W^D(h), h \in L^2(\mathbb{R}^D; \lambda^D)\}$, where λ^D denotes the Lebesgue measure on $\mathbb{R}^D$ (cf., e.g., [AY1,2]). Precisely, W^D is a centered Gaussian family of random variables such that

$$E[W^D(h)\, W^D(g)] = \int_{\mathbb{R}^D} h(\mathbf{x})\, g(\mathbf{x}) \lambda^D(d\mathbf{x}), \qquad h,\ g \in L^2(\mathbb{R}^D; \lambda^D). \tag{2.1}$$

We write

$$W^D_\omega(h) = \int_{\mathbb{R}^D} h(y) W^D_\omega(d\mathbf{y}), \qquad \omega \in \Omega$$

with $W^D_\omega(\cdot)$ a Gaussian generalized random variable (in the general notation of **Hida calculus** for the Gaussian white noise $W^D_\omega(d\mathbf{y})$ should be written as $\dot{W}^D(\mathbf{y})d\mathbf{y}$).

Since, we are considering a massive scalar field, we suppose that we are given a mass $m > 0$. Let Δ_d and resp. Δ_{d-1} be the d, resp. $d-1$, dimensional Laplace operator, and define the pseudo differential operators $L_{-\frac{1}{2}}$ and $H_{-\frac{1}{4}}$ as follows:

$$L_{-\frac{1}{2}} = (-\Delta_d + m^2)^{-\frac{1}{2}}. \tag{2.2}$$

$$H_{-\frac{1}{4}} = (-\Delta_{d-1} + m^2)^{-\frac{1}{4}}, \tag{2.3}$$

By the same symbols as $L_{-\frac{1}{2}}$ and $H_{-\frac{1}{4}}$, we also denote the integral kernels of the corresponding pseudo differential operators, i.e., the Fourier inverse transforms of the corresponding symbols of the pseudo differential operators.

By making use of stochastic integral expressions, we define two extremely important random fields ϕ_N, the **Nelson's Euclidean free field**, and ϕ_0, the

sharp time free field, as follows:
For $d \geq 2$,

$$\phi_N(\cdot) \equiv \int_{\mathbb{R}^d} L_{-\frac{1}{2}}(\mathbf{x} - \cdot) W^d(d\mathbf{x}), \tag{2.4}$$

$$\phi_0(\cdot) \equiv \int_{\mathbb{R}^{d-1}} H_{-\frac{1}{4}}(\vec{x} - \cdot) W^{d-1}(d\vec{x}). \tag{2.5}$$

These definitions of ϕ_N and resp. ϕ_0 seems formal, but they are rigorously defined as $\mathcal{S}'(\mathbb{R}^d)$ and resp. $\mathcal{S}'(\mathbb{R}^{d-1})$ valued random variables through a limiting procedure (cf. [AY1,2]), more precisely it has been shown that

$$P(\phi_N(\cdot) \in B_d^{a,b}) = 1, \quad \text{for } a, b \text{ such that } \min(1, \frac{2a}{d}) + \frac{2}{d} > 1, \ b > d \tag{2.6}$$

$$P(\phi_0 \in B_{d-1}^{a',b'}) = 1, \quad \text{for } a', b' \text{ such that } \min(1, \frac{2a'}{d-1}) + \frac{1}{d-1} > 1, \ b' > d-1. \tag{2.7}$$

Here for each a, b, $D > 0$, the Hilbert spaces $B_d^{a,b}$, which is a linear subspace of $\mathcal{S}'(\mathbb{R}^D)$, is defined by

$$B_D^{a,b} = \{(|\mathbf{x}|^2 + 1)^{\frac{b}{4}}(-\Delta_D + 1)^{+\frac{a}{2}} f : \ f \in L^2(\mathbb{R}^D; \lambda^D)\}, \tag{2.8}$$

where $\mathbf{x} \in \mathbb{R}^D$ and λ denotes the Lebesgue measure on $\mathbb{R}$, the scalar product of $B_d^{a,b}$ is given by

$$\begin{aligned} < u \,|\, v > \ &= \ \int_{\mathbb{R}^D} \left\{(-\Delta_D + 1)^{\frac{a}{2}}((1 + |\mathbf{x}|^2)^{-\frac{b}{4}} u(\mathbf{x}))\right\} \\ &\times \ \left\{(-\Delta_D + 1)^{\frac{a}{2}}((1 + |\mathbf{x}|^2)^{-\frac{b}{4}} v(\mathbf{x}))\right\} d\mathbf{x}, \quad u, \, v \in B_D^{a,b}. \end{aligned} \tag{2.9}$$

The following definition of $< \phi_N, f >$ and $< \phi_0, \varphi >$ would give a good explanation of (2.4) and (2.5). We denote

$$< \phi_N, f > \equiv \int_{\mathbb{R}^d} \left(L_{-\frac{1}{2}} f\right)(\mathbf{x})\, W^d(d\mathbf{x}), \qquad f \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}), \tag{2.10}$$

$$< \phi_0, \varphi > \equiv \int_{\mathbb{R}^{d-1}} \left(H_{-\frac{1}{4}} \varphi\right)(\vec{x})\, W^{d-1}(d\vec{x}), \qquad \varphi \in \mathcal{S}(\mathbb{R}^{d-1} \to \mathbb{R}), \tag{2.11}$$

It may possible to say that every idea of probabilistic treatment of Euclidean quantum field theory are included in the **Nelson's Euclidean free field ϕ_N**.

ϕ_N satisfies all the requirements under which it admits an analytic continuation to a quantum field that satisfies the Wightman axioms (cf.,e.g., [Si], [AY1,2] and references therein). In particular, ϕ_N satisfies the following important property:

N-1) ϕ_N is Markovian with respect to time in the sense that

$$E[<\phi_N, f_1> \cdots <\phi_N, f_k> |\mathcal{F}_{(-\infty,0]}] = E[<\phi_N, f_1> \cdots <\phi_N, f_k> |\mathcal{F}_0],$$

$$\text{for any} \quad k \in \mathbb{N}, \quad f_j \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}), \quad j = 1, \cdots, k, \quad \text{such that}$$

$$\text{supp}[f_j] \subset \{(t, \vec{x}) \,|\, t \geq 0, \, \vec{x} \in \mathbb{R}^{d-1}\}, \quad j = 1, \cdots, k,$$

$\mathcal{F}_{(-\infty,0]} \equiv$ the σ field generated by the random variables $<\phi_N, g>$ such that

$$\text{supp}[g] \subset \{(t, \vec{x}) \,|\, t \leq 0, \, \vec{x} \in \mathbb{R}^{d-1}\},$$

$\mathcal{F}_0 \equiv$ the σ field generated by the random variables $<\phi_N, \varphi \times \delta_{\{0\}}(\cdot)>$, where φ are functions having only the space variable $\vec{x}$, i.e., $\varphi(\vec{x})$ such that $\varphi \in \mathcal{S}(\mathbb{R}^{d-1} \to \mathbb{R})$ and $\delta_{\{0\}}(t)$ is the Dirac point measure at time $t = 0$, namely

$$\text{supp}[\varphi \times \delta_{\{0\}}(\cdot)] \subset \{(t, \vec{x}) \,|\, t = 0, \, \vec{x} \in \mathbb{R}^{d-1}\}.$$

Remark 1. For ϕ_N, the random variable $<\phi_N, \varphi \times \delta_{\{0\}}(\cdot)>$ is well defined (cf. [AY1,2]), precisely for any $t_0 \in \mathbb{R}$ and the Dirac point measure $\delta_{\{t_0\}}(\cdot)$ at time $t = t_0$

$$<\phi_N, \varphi \times \delta_{\{t_0\}}(\cdot)> \in \cap_{q \geq 1} L^q(\Omega; P).$$

□

Let θ be the time reflection operator:

$$(\theta f)(t, \vec{x}) = f(-t, \vec{x}),$$

then by N-1), for any $k \in \mathbb{N}$, $f_j \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R})$, $j = 1, \cdots, k$, such that $\text{supp}[f_j] \subset \{(t, \vec{x}) \,|\, t \geq 0, \, \vec{x} \in \mathbb{R}^{d-1}\}$, $j = 1, \cdots, k$, we easily see that (cf. e.g. [AY2])

$$E[<\phi_N, f_1> \cdots <\phi_N, f_k> \,|\, \mathcal{F}_0] = E[<\phi_N, \theta f_1> \cdots <\phi_N, \theta f_k> \,|\, \mathcal{F}_0] \quad P - a.s., \tag{2.12}$$

hence,

$$E\Big[\Big(E[<\phi_N, f_1> \cdots <\phi_N, f_k> \,|\, \mathcal{F}_0]\Big)^2\Big] \geq 0.$$

Consequently, we see that ϕ_N satifies the following :

$$E[(<\phi_N, f_1> \cdots <\phi_N, f_k>)\,(<\phi_N, \theta f_1> \cdots <\phi_N, \theta f_k>)] \geq 0 \tag{2.13}$$

The property (2.13) is refered as the **reflection positivity**, and Nelson's Euclidean free field ϕ_N is a **reflection positive** random field. But from the above discussion (cf. (2.12)) we see that the property of reflection positivity is a property of symmetric Markov processes.

Remark 2. By N-1) and (2.12), $\{\phi_N(t,\cdot)\}_{t\in\mathbb{R}}$ can be understood as a symmetric "Markov process", moreover since it satisfies the property of Euclidean invariance, $\phi_N(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^d$ is a **Markov field** (cf. more precisely, e.g., [Si], [AY1,2] and references therein).

□

Let μ_0 be the probability measure on $\mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R})$ which is the probability law of the sharp time free field ϕ_0 on $(\Omega,\mathcal{F},P)$ (cf. (2.7)), and μ_N be the probability measure on $\mathcal{S}'(\mathbb{R}^d\to\mathbb{R})$ which is the probability law of the Nelson's Euclidean free field on $\mathbb{R}^d$ (cf. (2.6)).

We denote

$$\phi_0(\varphi)\equiv<\phi_0,\varphi>\equiv\int_{\mathbb{R}^{d-1}}\left(H_{-\frac{1}{4}}\varphi\right)(\vec{x})\,W^{d-1}(d\vec{x}),$$

and

$$\begin{aligned}
&:\phi_0(\varphi_1)\cdots\phi_0(\varphi_n):\\
&\quad=\int_{\mathbb{R}^{k(d-1)}}H_{-\frac{1}{4}}\varphi_1(\vec{x}_1)\cdots H_{-\frac{1}{4}}\varphi_1(\vec{x}_k)W^{d-1}(d\vec{x}_1)\cdots W^{d-1}(d\vec{x}_k)\in\cap_{q\geq1}L^q(\mu_0)\\
&\qquad\text{for}\quad\varphi,\varphi_j\in\mathcal{S}(\mathbb{R}^{d-1}\to\mathbb{R}),\quad j=1,\cdots,k,\quad k\in\mathbb{N},
\end{aligned}\tag{2.14}$$

where (2.14) is the k-th multiple stochastic integral with respect to the isonormal Gaussian process W^{d-1} on $\mathbb{R}^{d-1}$.

Since, $:\phi_0(\varphi_1)\cdots\phi_0(\varphi_n):$ is nothing more than an element of the n-th Wiener chaos of $L^2(\mu_0)$, it also adomits an expression by means of the **Hermite polynomial** of $\phi_0(\varphi_j)$, $j=1,\cdots,k$ (cf., e.g., [AY1,2] and references therein).

Remark 3. From the view point of the notational rigorousness, ϕ_0 and ϕ_N are the distribution valued random variables on the probability space $(\Omega,\mathcal{F},P)$, hence the notation such as

$$:\phi_0(\varphi_1)\cdots\phi_0(\varphi_n):\in\bigcap_{q\geq1}L^q(\mu_0)$$

is incorrect. However in the above and in the sequel, since there is no ambiguity, for the simplicity of the notations we use the notations ϕ_0 and ϕ_N (with an obvious interpretation) to indicate the measurable functions X and resp. Y on the measure spaces $(\mathcal{S}'(\mathbb{R}^{d-1}), \mu_0, \mathcal{B}(\mathcal{S}'(\mathbb{R}^{d-1})))$ and resp. $(\mathcal{S}'(\mathbb{R}^d), \mu_N, \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)))$ such that

$$P\Big(\{\omega : \phi_0(\omega) \in A\}\Big) = \mu_0\Big(\{\phi : X(\phi) \in A\}\Big), \quad A \in \mathcal{B}(\mathcal{S}'(\mathbb{R}^{d-1})),$$

$$P\Big(\{\omega : \phi_N(\omega) \in A'\}\Big) = \mu_N\Big(\{\phi : Y(\phi) \in A'\}\Big), \quad A' \in \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)),$$

respectively, where $\mathcal{B}(S)$ denotes the Borel σ-field of the topological space S.

□

Let

$$H_{\frac{1}{2}} \equiv (-\Delta_{d-1} + m^2)^{\frac{1}{2}}, \tag{2.15}$$

and define the operator $d\Gamma(H_{\frac{1}{2}})$ on $L^2(\mu_0)$ such that (for the notations cf. Remark 3.)

$$\begin{aligned} d\Gamma(H_{\frac{1}{2}})(: \phi_0(\varphi_1) \cdots \phi_0(\varphi_n) :) &=: \phi_0(H_{\frac{1}{2}}\varphi_1)\phi_0(\varphi_2) \cdots \phi_0(\varphi_n) : + \cdots \\ \cdots + : \phi_0(\varphi_1) \cdots \phi_0(\varphi_{n-1})\phi_0(H_{\frac{1}{2}}\varphi_k) : \end{aligned} \tag{2.16}$$

Only for the next two propositions, suppose that $\mathbf{d} = \mathbf{2}$. For each $p \in \mathbb{N}$, $T \geq 0$ and $r \in \mathbb{N}$ we define the random variables $v^{2p}(r)$ and $V^{2p}(r, T)$, which are potential terms on the sharp time free field and Nelson's Euclidean free field respectively, as follows:

$$\begin{aligned} v^{2p}(r) &=<: \phi_0^{2p} :, \Lambda_r > \\ &\equiv \int_{\mathbb{R}^{2p}} \left\{ \int_{-\infty}^{\infty} \Lambda_r(x) \prod_{k=1}^{2p} H_{-\frac{1}{4}}(x - x_k)\, dx \right\} W^1(dx_1) \cdots W^1(dx_{2p}) \\ &\equiv \int_{-\infty}^{\infty} \Lambda_r(x) \, : \phi_0^{2p} : (x)\, dx \in \bigcap_{q \geq 1} L^q(\mu_0), \end{aligned} \tag{2.17}$$

$$\begin{aligned} V^{2p}(r, T) &= \int_{-T}^{T} <: \phi_N^{2p} : (t, \cdot), \Lambda_r > dt \\ &\equiv \int_{-T}^{T} \int_{(\mathbb{R}^2)^{2p}} \left\{ \int_{-\infty}^{\infty} \Lambda_r(x) \prod_{k=1}^{2p} L_{-\frac{1}{2}}((t, x) - (t_k, x_k))\, dx \right\} W^2(d(t_1, x_1)) \\ &\quad \times \cdots \times W^2(d(t_{2p}, x_{2p}))\, dt \\ &\equiv \int_{-T}^{T} \int_{-\infty}^{\infty} \Lambda_r(x) : \phi_N^{2p} : (t, x)\, dx\, dt \in \bigcap_{q \geq 1} L^q(\mu_N), \end{aligned} \tag{2.18}$$

where for $r \in \mathbb{N}$, $\Lambda_r \in C_0^\infty(\mathbb{R} \to \mathbb{R}_+)$ is a given function such that $0 \le \Lambda_r(x) \le 1$ $(x \in \mathbb{R})$, $\Lambda_r \equiv 1$ $(|x| \le r)$, $\Lambda_r \equiv 0$ $(|x| \ge r+1)$ (for the notations cf. Remark 3.).

We have the following important estimates (cf. eg., [Si]).

Proposition 2.1 *The operator $d\Gamma(H_{\frac{1}{2}})$ given by (2.16) defines a positive self adjoint operator on $L^2(\mu_0)$. For each $p \in \mathbb{N}$ there exists some $\mathcal{S}(\mathbb{R})$ norm $|||\cdot|||$ and the following holds:*

$$|v^{2p}(r)| \le (d\Gamma(H_{\frac{1}{2}}) + 1)|||\Lambda_r|||, \qquad \forall r \in \mathbb{N}. \tag{2.19}$$

For each $p \in \mathbb{N}$, $\lambda \ge 0$ and $r \in \mathbb{N}$ the operator $d\Gamma(H_{\frac{1}{2}}) + \lambda v^{2p}(r)$ on $L^2(\mu_0)$ is essentially self adjoint on the natural domain and bounded bellow:
There exists the smallest Eigenvalue $\alpha = \alpha_{2p,r,\lambda} > -\infty$ and the corresponding Eigenfunction $\rho = \rho_{2p,r,\lambda}$ such that

$$\left(d\Gamma(H_{\frac{1}{2}}) + v^{2p}(r)\right)\rho = \alpha \cdot \rho, \tag{2.20}$$

$$\rho(\phi) > 0, \quad \mu_0 \text{ a.e. } \phi \in \mathcal{S}'(\mathbb{R}); \qquad d\Gamma(H_{\frac{1}{2}}) + v^{2p}(r) \ge \alpha. \tag{2.21}$$

For each $p \in \mathbb{N}$, $\lambda \ge 0$, $r \in \mathbb{N}$ and $T \ge 0$

$$e^{-\lambda V^{2p}(r,T)} \in \bigcap_{q \ge 1} L^q(\mu_N). \tag{2.22}$$

Here, all the way of using notations follow the rule given by Remark 3.

Because $v^{2p}(r)$ is defined through $H_{-\frac{1}{4}}$ (cf. (2.17)), (2.19) holds for $d\Gamma(H_{\frac{1}{2}})$ with $H_{\frac{1}{2}}$. (2.21) can be shown by crucially use of the **hypercontractivity** of $e^{-td\Gamma(H_{\frac{1}{2}})}$ and (2.19). (2.22) is also a consequence of the Nelson's hypercontractive bound on $L^q(\mu_N)$, $q \ge 1$.

Proposition 2.2 *Let $\alpha_{2p,r,\lambda}$ and $\rho_{2p,r,\lambda} > 0$ be the Eigenvalue and function in Prop.2.1 respectively, and suppose that $\rho_{2p,r,\lambda}$ is normalized in order that*

$$E^{\mu_0}\left[\rho_{2p,r,\lambda}(\cdot))^2\right] = 1.$$

Let $\nu_{2p,r,\lambda}$ be the probability measure on $\mathcal{S}'(\mathbb{R})$ such that

$$\nu_{2p,r,\lambda} \equiv \left(\rho_{2p,r,\lambda}\right)^2 \mu_0,$$

and define a mapping $U : L^2(\mu_0) \to L^2(\nu_{2p,r,\lambda})$ as follows:

$$UX \equiv \frac{X}{\rho_{2p,r,\lambda}}, \qquad X \in L^2(\mu_0).$$

Then the operator $\tilde{T}_t$, $t \geq 0$, on $L^q(\nu_{2p,r,\lambda})$, $q \geq 0$, defined by

$$\tilde{T}_t \equiv U \exp\Big\{-t\big(d\Gamma(H_{\frac{1}{2}}) + \lambda v^{2p}(r) - \alpha_{2p,r,\lambda}\big)\Big\}U^{-1}, \quad t \geq 0, \tag{2.23}$$

is Markovian contraction semigroup. By taking $\nu_{2p,r,\lambda}$ the initial distribution, $\tilde{T}_{|t|}$, $t \in \mathbb{R}$, generats a random field on $\mathcal{S}'(\mathbb{R}^2)$ of which probability law is identical to

$$d\mu_{V^{2p}(r,\infty)} = \lim_{T\to\infty} \frac{e^{-\lambda V^{2p}(r,T)} d\mu_N}{E^{\mu_N}[e^{-\lambda V^{2p}(N,T)}]}, \tag{2.24}$$

more precisely, for any φ_1, $\varphi_2 \in \mathcal{S}(\mathbb{R} \to \mathbb{R})$, and any t_1, $t_2 \geq 0$

$$\begin{aligned} &\int_{\mathcal{S}'(\mathbb{R}\to\mathbb{R})} \tilde{T}_{t_1}\Big(\Big(\tilde{T}_{t_2} < \cdot, \varphi_2 >_{\mathcal{S}',\mathcal{S}}\Big)(\cdot) < \cdot, \varphi_1 >_{\mathcal{S}',\mathcal{S}}\Big)(\phi)\ \nu_{2p,r,\lambda}(d\phi) \\ &\quad = E^{\mu_{V^{2p}(r,\infty)}}[< \phi, \varphi_1 \times \delta_{\{t_1\}}(\cdot) >< \phi, \varphi_2 \times \delta_{\{t_1+t_2\}}(\cdot) >], \end{aligned} \tag{2.25}$$

where $E^{\mu_{V^{2p}(r,\infty)}}[\ \cdot\ ]$ denots the expaectation taken with respect to the measure $\mu_{V^{2p}(r,\infty)}$, and all the way of using notations follow the rule given by Remark 3.

3 The Hida product on 4-space time dimensions and the corresponding results to Prop. 2.2 and 2.3

Firstly, we remark that if we substitute the potentials in (2.17) resp. (2.18) by the finite linear combinations of $:\phi_0^{2p}:$ resp. $:\phi_N^{2p}:$, then Propositions 2.1 and 2.2 also true. In particular these Proposisions hold for $(:\phi_0^4: - :\phi_0^2:)$ together with $(:\phi_N^4: - :\phi_N^2:)$.

Secondly, for such a substitution in the definition of $\tilde{T}_t$ given by (2.23) we have the term such that

$$e^{-\lambda(v^4(r)+v^2(r))}. \tag{3.1}$$

Thirdly, $e^{-\lambda(v^4(r)+v^2(r))}$ is in $L^2(\Omega, P)$ when $\mathbf{d} = \mathbf{2}$, but we have to stress that by performing the formal Taylor expansion to (3.1) and then applying the Hida product argument (cf. [AY3]), even for the space time dimension $\mathbf{d} = \mathbf{4}$, we can find several integrable random variables in it, in particular we are able to find the following random variable included in (3.1) with $\mathbf{d} = \mathbf{4}$:

$$
\begin{aligned}
v(r) \equiv \int_{(\mathbb{R}^3)^4} \Big\{ \int_{\mathbb{R}^3} \Lambda_r(\vec{x}) \prod_{k=1}^{4} H_{-\frac{1}{4}}(\vec{x}-\vec{x}_k) \\
\times \Big(\int_{\mathbb{R}^3} \Lambda_r(\vec{x'}_k) H_{-\frac{1}{4}}(\vec{x}-\vec{x'}_k)\, H_{-\frac{1}{4}}(\vec{x'}_k-\vec{x}_k) d\vec{x'}_k \Big) d\vec{x} \Big\} \\
\times W^3(d\vec{x}_1)\cdots W^3(d\vec{x}_4) \ \in \bigcap_{q\geq 1} L^q(\mu_0).
\end{aligned}
\tag{3.2}
$$

Correspondingly we can define

$$
\begin{aligned}
V(r,T) \equiv \int_{-T}^{T} \int_{(\mathbb{R}^4)^4} \Big\{ \int_{\mathbb{R}^3} \Lambda_r(\vec{x}) \prod_{k=1}^{4} L_{-\frac{1}{2}}((t,\vec{x})-(t,\vec{x}_k)) \\
\times \Big(\int_{\mathbb{R}^3} \Lambda_r(\vec{x'}_k) L_{-\frac{1}{2}}((t,\vec{x})-(t,\vec{x'}_k))\, L_{-\frac{1}{2}}((t,\vec{x'}_k)-(t_k,\vec{x}_k)) d\vec{x'}_k \Big) d\vec{x} \Big\} \\
\times W^4(d(t_1,\vec{x}_1))\cdots W^4(d(t_4,\vec{x}_4))\, dt \ \in \bigcap_{q\geq 1} L^q(\mu_N).
\end{aligned}
\tag{3.3}
$$

The main result of the present paper is the following:

Theorem 3.1 *By substituting the terms $v^{2p}(r)$ resp. $V^{2p}(r,T)$ in Propsitions 2.1 and 2.2 for $\mathbf{d}=\mathbf{2}$ by $v(r)$ resp. $V(r,T)$ given by (3.2) resp. (3.3), then all the corresponding statements of Propsitions 2.1 and 2.2 hold for the case $\mathbf{d}=\mathbf{4}$ with such changes.*

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